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Singular perturbation and initial layer for the abstract Moore-Gibson-Thompson equation

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A B S T R A C T

We investigate the singular limit of a third-order abstract equation in time, in relation to the complete second-order Cauchy problem on Banach spaces, where the principal operator is the generator of a strongly continuous cosine family. Assuming that an initial datum is ill prepared, the initial layer problem is studied. We show convergence, which is uniform on compact sets that stay away from zero, as long as initial data are sufficiently smooth. Our method employs suitable results from the theory of general resolvent families of operators. The abstract formulation of the third-order in time equation is inspired by the Moore-Gibson-Thompson equation, which is the linearization of a model that currently finds applications in the propagation of ultrasound waves, displacement of certain viscoelastic materials, flexible structural systems that possess internal damping and the theory of thermoelasticity.

1. Introduction

1.1. State of the art and objectives

In recent times a good deal of attention has been devoted to studies of the singular limit for vanishing relaxation time of the Jordan-Moore-Gibson-Thompson equation, a third order in time wave equation describing the nonlinear propagation of sound that avoids the infinite signal speed paradox of classical second order in time strongly damped models of nonlinear acoustics, such as the Westervelt and Kuznetsov equation [33,34]. The singular relaxation limit for the linearized version of the Jordan-Moore-Gibson-Thompson equation (JMGT), called the Moore-Gibson-Thompson (MGT) equation

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$$\begin{cases} \tau\psi^\tau_{ttt} + \psi^\tau_{tt} - c^2\Delta\psi^\tau - b\Delta\psi^\tau_t = 0, \quad t \geq 0; \\ \psi^\tau(0) = u^0; \\ \psi^\tau_t(0) = u^1; \\ \psi^\tau_{tt}(0) = u^2, \end{cases} \tag{1.1}$$

where ψ^τ denotes the acoustic velocity potential, τ is a positive constant accounting for relaxation (the relaxation time), c is the speed of the sound, $b = \delta + \tau c^2$, δ is the diffusivity of sound, has been studied very recently by Bongarti, Charoenphon and Lasiecka [5,11].

The MGT equation has received a lot of attention in recent years. Several papers have appeared in the literature on this topic. Well posedness and exponential decay rates were studied in the seminal articles [36,37] by Kaltenbacher and co-authors. See also [15]. In the important reference [49] an abstract semigroup approach was carried out that studies structural decomposition, spectral analysis and exponential stability. Regularity and asymptotic behavior were analyzed in the papers [1,15,25,26], whereas chaotic behavior in [13]. The singular thermal relaxation limit for (1.1) (linear and nonlinear) is first studied by Bongarti, Charoenphon, Kaltenbacher, Lasieka and Nikolić in references [5,6,11,33,34], stability and controllability in references [7,8] and [48]. Nonexistence of global solutions is analyzed in [12]. Some generalizations of the (1.1) model are studied in [14] and [40] where delay and memory terms are incorporated, along with applications to inverse problems [42]. A numerical analysis based on the finite element method and the backward Euler scheme were developed in the reference [4]. See also [9,17,18] for related works. Recently, the study of certain non-local variants of the model (1.1) has been the subject of research [35,47]. The monograph [32] provides a useful background on the subject of the MGT equation.

The JMGT equation was originally introduced in connection with fluid mechanics [57] as a model for the acoustic velocity potential in thermally relaxing fluids. In addition, the same equation arises as a model for the displacement in certain viscoelastic materials (see [19,52] and references therein), as a model for flexible structural systems possessing internal damping [7,8], and as a model for the temperature displacement in a type of heat conduction with a relaxation parameter [55].

We observe that the so-called singular relaxation limit for the MGT equation studied recently in the references [5,6,11] belongs to the framework of the more general theory of singular perturbation problems [3,51,58]. We should recall that by the term singular perturbation of a given Partial Differential Equation (PDE) we refer to cases when its nature formally changes. For instance, one of the higher derivatives may formally disappear, or the order of the equation formally drops when a certain parameter is set to zero, hence the order of the PDE becomes lower, or its space dimensionality, or its type changes.

In this way, the singular perturbation problem for the MGT equation is that of showing that the solution ψ^τ of (1.1) converges, as $\tau \to 0$, to the solution ψ of the linearized Kuznetsov equation

$$\begin{cases} \psi_{tt} - c^2\Delta\psi - \delta\Delta\psi_t = 0, \quad t \geq 0; \\ \psi^\tau(0) = u^0; \\ \psi^\tau_t(0) = u^1, \end{cases} \tag{1.2}$$

where convergence can be understood in various senses. This problem arises naturally when one tries to quantify the sensitivity of the relaxation parameter τ on a variety of materials. In fact, it has been proved that a number of experiments found this parameter to be small in several mediums, although not all (see [5, p. 150] and references therein).

The main objective of this article is to investigate the singular perturbation problem for an abstract version of the equation (1.1) in the context of *Banach spaces*. This allows us to generalize this problem for a broader class of operators than the Laplacian, and somewhat improve the results in [5], by including the case of non-constants initial conditions. We also consider in our analysis the study of the presence of *initial layers*. To our knowledge, the present work is the first to explore such qualitative behavior.

More precisely, let $A : D(A) \subset X \to X$ be a closed and densely defined linear operator on a complex Banach space X. We ask ourselves under what conditions the solution $u_\epsilon(t)$ of the abstract singular perturbation problem

$$\begin{cases} \epsilon u_\epsilon'''(t) + u_\epsilon''(t) - \beta A u_\epsilon(t) - b_\epsilon A u_\epsilon'(t) = 0, \quad t \geq 0, \quad \epsilon > 0; \\ u_\epsilon(0) = u^0(\epsilon); \\ u_\epsilon'(0) = u^1(\epsilon); \\ u_\epsilon''(0) = u^2(\epsilon), \end{cases} \tag{1.3}$$

where $b_\epsilon = \delta + \epsilon\beta$, converges to the solution of the problem

$$\begin{cases} u_0''(t) - \delta A u_0'(t) - \beta A u_0(t) = 0, \quad t \geq 0; \\ u_0(0) = u^0; \\ u_0'(0) = u^1, \end{cases} \tag{1.4}$$

as $\epsilon \to 0$, incorporating the presence of an initial layer by assuming that $u^0(\epsilon) \to u^0$ and $u^1(\epsilon) \to v \in X$ in place of $u^1(\epsilon) \to u^1$. This could also be interpreted, to some extent, as an ill-prepared initial datum.

When an initial datum is ill-prepared, the so-called *initial layer* is created at $t = 0$. The initial layer can be understood physically as an impulsively started motion at $t = 0$ near the boundary [27, Section 2.3.3]. This interesting phenomenon has been discussed in several papers [28,59]. For instance, the study of the singular perturbation problem with initial layer for the heat equation appears in [27, Section 2.3.3]. Notably, Fattorini [23, Theorem 3.2] was among the first to give an operator theoretic approach to the initial layer problem for the second order linear Cauchy problem. Note that even when convergence near $t = 0$ cannot be expected, convergence can be attained through addition of correctors (solutions of a different approximating equation) [23, Section 7]. In such abstract setting, typically A is the generator of a C_0-semigroup or strongly continuous cosine family in a Banach space.

In the case of complete second order abstract Cauchy problems, this question is referred as the abstract singular perturbation problem, and has been studied long time ago. The abstract singular perturbation problem was first considered in 1963 by Kisynski [38] in the case where A is a self adjoint, positive definite operator on a Hilbert space. Later, in 1970, Sova [56] studied the problem under the assumptions that A is the generator of a strongly continuous cosine function. The most precise results are those by Kisynski [39] who applied the theory of monotonic functions and gave explicit solutions. See also [41], [21], [45] and [24] for other developments. The treatment of the non homogeneous equation is due to Fattorini [22, Chapter VI]; see also the references therein. Lately, the singular perturbation for abstract non-densely defined Cauchy problems has been studied by Ducrot et al. [20]. An excellent monograph on the subject on singular perturbation is provided by Verhulst [58].

In the setting of Hilbert spaces, and using reduction of order, the authors in [5] studied convergence of the semigroup $T^\tau(t)$ governing (1.1) to the semigroup $T(t)$ associated with (1.2) when $\tau \to 0^+$. They considered the Dirichlet Laplacian, i.e. $A = -\Delta$ with $D(A) := H_0^1(\Omega) \cap H^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ and have shown that (in a formal sense) for initial data in $\mathbb{H}_0 := D(A^{1/2}) \times D(A^{1/2}) \times L^2(\Omega)$ there is a strong convergence for the projection of the semigroups $T^\tau(t)$ defined over the phase space $\mathbb{H}_0$ to the semigroup $T(t)$ defined over the phase space $\mathbb{H}_0^0 := D(A^{1/2}) \times D(A^{1/2})$, and that the rate of convergence is τ [5, Theorem 2.4]. Moreover, uniform asymptotic stability properties and asymptotic behavior of the spectrum were also analyzed.

Since the Laplacian, with appropriate boundary conditions, is the generator of a strongly continuous cosine family on $X = L^2(\Omega)$ (see e.g. [2, Example 7.2.1]) the study of [5] suggests that we should require A to be the infinitesimal generator of a cosine family if we want to work in the general context of Banach spaces. We note that the fractional powers $A = -(-\Delta)^\nu$, $0 < \nu \leq 1$ are also generators of cosine families in a Hilbert space [31, Theorem 2], as well as the negative bi-Laplacian operator $A = -\Delta^2$ (see [2, Example 3.14.15] and [29]).

1.2. Methodology and results

Using a generalization of the Trotter-Kato Theorem [44] valid for generalized resolvent families of operators, and under the assumption that A is the generator of a cosine family, we are able to solve the singular perturbation problem with initial layer for the MGT equation (1.3) as follows:

Theorem 1.1. *Suppose that A is the generator of a strongly continuous cosine family on a Banach space X. Let $u^0(\epsilon), u^1(\epsilon), u^0 \in D(A^2)$, $u^2(\epsilon), u^1 \in D(A)$ be such that, for some $v \in X$,*

(a) $\|u^0(\epsilon) - u^0\|_{[D(A)]} \to 0$ *as* $\epsilon \to 0^+$;
(b) $\|u^1(\epsilon) - v\| \to 0$ *as* $\epsilon \to 0^+$;
(c) $\|\epsilon u^2(\epsilon) - (u^1 - v)\| \to 0$ *as* $\epsilon \to 0^+$;
(d) *the sets* $\{A^2 u^0(\epsilon)\}_{\epsilon>0}$ *and* $\{Au^1(\epsilon)\}_{\epsilon>0}$ *are bounded,*

and let $u_\epsilon(t)$ be the solution of (1.3). Finally, let $t(\epsilon) > 0$ be such that $t(\epsilon)/\epsilon \to \infty$ $(\epsilon \to 0)$. Then

$$u_\epsilon(t) \to u_0(t),$$

uniformly on compacts of $t \geq t(\epsilon)$ (see Definition 3.8 below), where $u_0(t)$ is the solution of (1.4).

Remark 1.2. We note that even though

$$\tau\psi^\tau_{ttt} + \psi^\tau_{tt} + c^2\Delta^2\psi^\tau + b\Delta^2\psi^\tau_t = 0, \quad t \geq 0 \tag{1.5}$$

has the same structure as the actual MGT equation, it does not have necessarily the same nature - from a PDE viewpoint - as the MGT equation. Indeed, while the MGT equation is a hyperbolic PDE, as clarified in [10], the equation (1.5) is not, owing to the changed principal part of the differential operator.

In the particular case of $u^0(\epsilon) = u^0, u^1(\epsilon) = u^1$ and $u^2(\epsilon) = u^2$ and choosing $v = u^1$ we have that all the conditions in Theorem 1.1 are automatically satisfied. In contrast with previous work [5], we do not require that $u^0(\epsilon), u^1(\epsilon)$ and $u^2(\epsilon)$ be constant, and A could be other than the Dirichlet Laplacian. Convergence can only be assured outside of an initial layer at $t = 0$.

Our study is not focused in the special case of Hilbert spaces and questions of optimal regularity of initial data, because we are mainly interested in non-constant initial conditions and the study of initial layers as it has been considered from the beginning of the singular perturbation theory in abstract spaces. In fact, in this paper, we follow the line of work of Sova [56], Fattorini [22], Engel [21], among others.

The main tool normally used to study the MGT equation (1.1) is to reduce it to a first order problem defined in a suitable space (phase space). Unfortunately, we cannot use this method to deal with the MGT equation (1.3) when we are in a Banach space setting since, for a closed operator A, the matrix of unbounded operators is in general not a closed operator on the product space. Therefore, one of the main novelties of this work is that we incorporate a new strategy based on a direct representation of the solution in terms of general resolvent families of operators to show convergence, see Theorem 3.2 below. However, this strategy may require more regularity in the initial data than when working in the case of Hilbert space and the Laplacian operator, since the optimal regularity of the initial data depends on the representation of the solution of (1.3) by the family of resolvent operators chosen and by their regularity.

More precisely, after representing the solution of (1.3), for each $\epsilon > 0$, by families of operators, we can prove that such family of operators is uniformly stable with respect to the parameter ϵ and then, using a generalized version of the Trotter-Kato Theorem, we show that the convergence of the solution of (1.3) to the solution of (1.4) can be guaranteed, under suitable hypotheses.

We notice that the main tasks to be accomplished, in order to employ the generalized version of the Trotter-Kato Theorem are to prove two conditions:

(i) To show an uniform boundedness property with respect to the parameter ϵ of the resolvent families $R_\epsilon(t)$ associated with the MGT equation (1.3) under appropriate convergence requirements on the initial conditions.
(ii) To guarantee the convergence of certain resolvents operators associated to $R_\epsilon(t)$ (formally, the corresponding Laplace transforms).

In order to overcome these difficulties, we notice that the resolvent family $R_\epsilon(t)$ satisfies:

$$\frac{\hat{k}_\epsilon(\lambda)}{\hat{a}_\epsilon(\lambda)}\left(\frac{1}{\hat{a}_\epsilon(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} R_\epsilon(t) x dt, \quad x \in X,$$

for all λ sufficiently large and each $\epsilon \geq 0$, where

$$a_\epsilon(t) := \delta(1 - e^{-\frac{t}{\epsilon}}) + \beta t \qquad k_\epsilon(t) := t - \epsilon(1 - e^{-\frac{t}{\epsilon}}), \qquad t \geq 0,$$

and

$$a_0(t) := \delta + \beta t \qquad k_0(t) := t, \qquad t \geq 0.$$

Then, the first condition on uniform boundedness with respect to ϵ will be guaranteed essentially due to the following properties:

$$0 \leq a_\epsilon(t) \leq a_0(t) \qquad \text{and} \qquad 0 \leq k_\epsilon(t) \leq k_0(t), \qquad t \geq 0.$$

Then, a new and original subordination argument, exploiting the fact that $a_\epsilon(t)$ is nonnegative, nondecreasing and concave, shows the existence of uniformly bounded family $R_\epsilon(t)$. The second condition can be proved thanks to the Weierstrass formula, which asserts that if A is the generator of a cosine family, then A is also the generator of an analytic semigroup.

We observe that $a_\epsilon(t) \to a_0(t)$ as $\epsilon \to 0$ for all $t > 0$, except in case $t = 0$. The same happens with $k'_\epsilon(t)$ which converges to $k'_0(t)$ except for $t = 0$. The presence of these singularities is revealed by the convergence of the solution $u_\epsilon(t)$ to $u_0(t)$ uniformly on compacts of $t \geq t(\epsilon)$ as long as $t(\epsilon)/\epsilon \to \infty$ $(\epsilon \to 0)$.

1.3. Overview

This paper is organized as follows: Section 2 is devoted to construct the (a_ϵ, k_ϵ)-regularized resolvent family associated with (1.3) and the (a_0, k_0)-regularized resolvent family governing (1.4). In section 3, we prove the uniform convergence of resolvents $R_\epsilon(t)$ on compacts subsets of $\mathbb{R}_+$ under the assumption that A is the generator of a strongly continuous cosine family. We also prove the convergence of the derivatives $R'_\epsilon(t)$ to $R'_0(t)$ as $\epsilon \to 0$ for $t > 0$; see Theorem 3.9. Section 4 is mainly devoted to the proof of Theorem 1.1. Finally, we mention some well-known results on (a, k)-regularized resolvent families in an Appendix.

2. Preliminaries

In this section we construct appropriate (a_ϵ, k_ϵ)-regularized resolvent families governing (1.3) as well as (a_0, k_0)-regularized resolvent families associated with the formal limit (1.4).

For each $\epsilon > 0$ be fixed, we will consider the following functions (see [16, Section 2]):

$$a_\epsilon(t) := \beta k_\epsilon(t) + \frac{b_\epsilon}{\epsilon}\int_0^t e^{-s/\epsilon}ds = \delta(1 - e^{-\frac{t}{\epsilon}}) + \beta t, \quad t \geq 0, \tag{2.1}$$

where $b_\epsilon := \delta + \epsilon\beta$ and

$$k_\epsilon(t) := \frac{1}{\epsilon}\int_0^t (t-s)e^{-s/\epsilon}ds = t - \epsilon(1 - e^{-\frac{t}{\epsilon}}), \quad t \geq 0, \tag{2.2}$$

and

$$a_0(t) := \delta + \beta t \qquad k_0(t) := t, \qquad t \geq 0.$$

In such case, and for each $\epsilon \geq 0$, we denote by $\{R_\epsilon(t)\}_{t\geq 0}$ the (a_ϵ, k_ϵ)-regularized resolvent family generated by A, if it exists (see the Appendix). For further use, we note that

$$a'_\epsilon(t) = \beta + \frac{\delta}{\epsilon}e^{-t/\epsilon}, \qquad k'_\epsilon(t) = 1 - e^{-t/\epsilon} \qquad t \geq 0, \tag{2.3}$$

and

$$a'_0(t) = \beta, \qquad k'_0(t) = 1, \qquad t \geq 0,$$

as well as

$$\hat{a}_\epsilon(\lambda) = \frac{\beta + b_\epsilon\lambda}{\lambda^2(\epsilon\lambda + 1)}, \qquad \hat{k}_\epsilon(\lambda) = \frac{1}{\lambda^2(\epsilon\lambda + 1)}, \tag{2.4}$$

and

$$\hat{a}_0(\lambda) = \frac{\beta + \delta\lambda}{\lambda^2}, \qquad \hat{k}_0(\lambda) = \frac{1}{\lambda^2}, \tag{2.5}$$

for all λ sufficiently large.

For the explanation of the meaning of regularized resolvent in the following result, which is a direct consequence of Proposition 5.3, see the Appendix section.

Proposition 2.1. *Let A be a closed linear operator defined on a Banach space X. Suppose that for each $\epsilon \geq 0$ given, A is the generator of an (a_ϵ, k_ϵ)-regularized resolvent family $\{R_\epsilon(t)\}_{t\geq 0}$ on X. Then the following assertions hold true:*

1. *$R_\epsilon(t)$ is strongly continuous and $R_\epsilon(0) = 0$.*
2. *For each $\epsilon > 0$*

$$R_\epsilon(t)x = [t - \epsilon(1 - e^{-\frac{t}{\epsilon}})]x + A\int_0^t [\beta(t-s) + \delta(1 - e^{-\frac{t-s}{\epsilon}})]R_\epsilon(s)x ds, \quad x \in X, \quad t \geq 0 \tag{2.6}$$

and, for $\epsilon = 0$

$$R_0(t)x = tx + A\int_0^t [\delta + \beta(t-s)]R_0(s)xds, \quad x \in X, \quad t \geq 0.$$

3. *For all* $x \in D(A^2)$ *and* $\epsilon \geq 0$ *we have* $R_\epsilon(\cdot)x \in C^2(\mathbb{R}_+; X)$. *Moreover,*

$$R'_\epsilon(t)x = (1 - e^{-\frac{t}{\epsilon}})x + \int_0^t \left[\frac{\delta}{\epsilon}e^{-\frac{t-s}{\epsilon}} + \beta\right] R_\epsilon(s)Axds, \quad x \in D(A), \quad t \geq 0,$$

in case $\epsilon > 0$ *and*

$$R'_0(t)x = x + \delta AR_0(t)x + \beta\int_0^t R_0(s)Axds, \quad x \in D(A), \quad t \geq 0,$$

in case $\epsilon = 0$. *Finally, we have*

$$R''_\epsilon(t)x = \frac{1}{\epsilon}e^{-\frac{t}{\epsilon}}x + \int_0^t \left[\frac{\delta}{\epsilon}e^{-\frac{t-s}{\epsilon}} + \beta\right] R'_\epsilon(s)Axds, \quad x \in D(A^2), \quad t \geq 0,$$

in case $\epsilon > 0$, *and* $R''_0(t)x = \delta R'_0(t)Ax + \beta R_0(t)Ax$, $x \in D(A^2), t \geq 0$ *in case* $\epsilon = 0$.

3. Stability and convergence of resolvent families

We start with the analysis of the strongly damped second order problem

$$\begin{cases} u''_0(t) - \delta Au'_0(t) - \beta Au_0(t) = 0, \quad t \geq 0, \\ u_0(0) = u^0, \\ u'_0(0) = u^1. \end{cases} \tag{3.1}$$

Assume that A is the generator of a strongly continuous semigroup on X. Then, according to a result of Neubrander [50, Corollary 13] there is an exponentially bounded, strongly continuous and differentiable family $\{R_0(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ and $\omega_0 \in \mathbb{R}$ such that $\{\frac{\lambda^2}{\delta\lambda+\beta}\}_{Re(\lambda)>\omega_0} \subset \rho(A)$, the resolvent set of A, that satisfies

$$\frac{1}{\delta\lambda + \beta}\left(\frac{\lambda^2}{\delta\lambda + \beta} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t}R_0(t)xdt,$$

for all $x \in X$ and every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0$. Moreover, by [50, Corollary 18] we have that

$$u_0(t) = R'_0(t)u^0 + R_0(t)(u^1 - \delta Au^0), \quad t \geq 0, \tag{3.2}$$

is the unique strong solution of (3.1), i.e. u_0 belongs to $C^1(\mathbb{R}_+; [D(A)]) \cap C^2(\mathbb{R}_+, X)$ and satisfies (3.1), whenever $u^0, u^1 \in D(A)$.

Taking into account (2.5) we note that

$$\frac{1}{\delta\lambda + \beta}\left(\frac{\lambda^2}{\delta\lambda + \beta} - A\right)^{-1} x = \frac{\hat{k}_0(\lambda)}{\hat{a}_0(\lambda)}\left(\frac{1}{\hat{a}_0(\lambda)} - A\right)^{-1} x, \qquad x \in X.$$

It means, by Definition 5.2 (Appendix), that A is the generator of an (a_0, k_0)-regularized family $\{R_0(t)\}_{t\geq 0}$. Observe that the family depends only on the parameters δ and β. Also we observe that for $u^0, u^1 \in D(A)$, the inhomogenous equation

$$\begin{cases} u_0''(t) - \delta A u_0'(t) - \beta A u_0(t) = f(t), \quad t \geq 0, \\ u_0(0) = u^0, \\ u_0'(0) = u^1 \end{cases} \tag{3.3}$$

has a unique solution u_0 in $C^2(\mathbb{R}_+; X)$ given by

$$u_0(t) = R_0'(t)u^0 + R_0(t)(u^1 - \delta A u^0) + \int_0^t R_0(t-s)f(s)ds, \quad t \geq 0, \tag{3.4}$$

whenever $f(\cdot)$ is continuously differentiable or $f : [0, \infty) \to D(A)$ and $Af(\cdot)$ is integrable, see [50, Corollary 18]. This formula will be useful later.

We now analyze the problem:

$$\begin{cases} \epsilon u_\epsilon'''(t) + u_\epsilon''(t) - b_\epsilon A u_\epsilon'(t) - \beta A u_\epsilon(t) = 0, \quad t \geq 0, \quad \epsilon > 0, \\ u_\epsilon(0) = u^0(\epsilon), \\ u_\epsilon'(0) = u^1(\epsilon), \\ u_\epsilon''(0) = u^2(\epsilon), \end{cases} \tag{3.5}$$

where $b_\epsilon = \delta + \epsilon\beta$.

Definition 3.1. Let $\epsilon > 0$ be fixed. By a strong solution of the problem (3.5) we mean a function $u_\epsilon \in C^1(\mathbb{R}_+; [D(A)]) \cap C^3(\mathbb{R}_+; X)$ that verifies (3.5).

Recall the definitions of a_ϵ and k_ϵ given in (2.1) and (2.2), respectively. The following result was proved in [16, Proposition 3.1].

Theorem 3.2. *Let A be a closed linear operator defined on a Banach space X and $\epsilon > 0$ be given. Assume that A is the generator of (a_ϵ, k_ϵ)-regularized resolvent families $\{R_\epsilon(t)\}_{t\geq 0}$. If $u^0(\epsilon) \in D(A^3)$ and $u^1(\epsilon), u^2(\epsilon) \in D(A^2)$, then the unique strong solution of the problem (3.5) is given by*

$$u_\epsilon(t) = \epsilon R_\epsilon''(t)u^0(\epsilon) + R_\epsilon'(t)[u^0(\epsilon) + \epsilon u^1(\epsilon)] + R_\epsilon(t)[u^1(\epsilon) - b_\epsilon A u^0(\epsilon) + \epsilon u^2(\epsilon)], \tag{3.6}$$

for all $t \geq 0$, where $b_\epsilon := \delta + \epsilon\beta$.

Remark 3.3. Formally, comparing the representations (3.2) with (3.6) it is apparent that, in general, there could be a gap near $t = 0$. This gap might be hidden when $\epsilon \to 0$ for $t > 0$, but note that at $t = 0$ the initial value u^0 in (3.2) originates - as $\epsilon \to 0$ - from the term $\epsilon R_\epsilon''(t)u^0(\epsilon)$ rather than from the term $R_\epsilon'(t)[u^0(\epsilon) + \epsilon u^1(\epsilon)]$ in (3.6). Therefore, in general Banach spaces, we can expect a singular behavior at $t = 0$ instead of the regular behavior proved in the case of Hilbert spaces.

In order to show that u_ϵ given in (3.6) converges as $\epsilon \to 0$ to u_0 given in (3.2) using the extension of the Trotter-Kato theorem given by Theorem 5.4 stated in the Appendix, we need to prove the stability condition (5.1). This is the objective of the following results.

We start recalling the following definition due to Prüss [54, Definition 4.4, p.94].

Definition 3.4. A function $a : (0, \infty) \to \mathbb{R}$ is called a creep function if $a(t)$ is nonnegative, nondecreasing and concave.

A creep function $a(t)$ has the standard form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(s)ds, \quad t > 0, \tag{3.7}$$

where $a_0 = a(0+) \geq 0$, $a_\infty = \lim_{t\to\infty} \frac{a(t)}{t} \geq 0$, and $a_1(t) := a'(t) - a_\infty$ is nonnegative, nonincreasing and $\lim_{t\to\infty} a_1(t) = 0$.

Lemma 3.5. *Let $\epsilon > 0$ be given. Then $a_\epsilon(t)$ is a creep function, $a_\infty = \beta$ and $a_1^\epsilon(t) := a'_\epsilon(t) - \beta$ is log-convex.*

Proof. Note from (2.1) and (2.2) that $a_\epsilon(t) \geq 0$ and $k_\epsilon(t) \geq 0$ since $\epsilon, \beta, \delta > 0$. Moreover, by definition, $a_\epsilon(0) = 0$ and $k_\epsilon(0) = 0$. Now, from (2.3) we obtain that a_ϵ is non-decreasing. On the other hand, the identity $a''_\epsilon(t) = -\frac{\delta}{\epsilon^2} e^{-\frac{t}{\epsilon}} < 0$, implies that a_ϵ is concave. It shows that $a_\epsilon(t)$ is a creep function. Next, observe that $\lim_{t\to\infty} \frac{a_\epsilon(t)}{t} = \beta$. Then, equation (3.7) and (2.3) implies that $a_1^\epsilon(t) := a'_\epsilon(t) - \beta = \frac{\delta}{\epsilon} e^{-\frac{t}{\epsilon}}$. Let $f_\epsilon(t) := \ln(a_1^\epsilon(t))$. Then an easy calculation shows that $f''_\epsilon(t) = 0$ and hence a_1^ϵ is log-convex, proving the Lemma. □

We recall that an infinitely differentiable function $f : (0, \infty) \to \mathbb{R}$ is called completely monotone (CM) if

$$(-1)^n f^{(n)}(\lambda) \geq 0,$$

for all $\lambda > 0$ and for all $n = 0, 1, 2, ...$

Lemma 3.6. *For each $\epsilon > 0,\ t > 0$ the function*

$$h_\epsilon(\lambda, t) = \frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}} t}, \quad \lambda > 0,$$

is completely monotone in λ.

Proof. Let $\epsilon > 0$ be fixed. Note that

$$h_\epsilon(\lambda, t) = \frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}} t} = \lambda^2 \widehat{k}_\epsilon(\lambda) \frac{1}{\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}} \frac{1}{\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}} e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}} t} =: \frac{1}{\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}} g_\epsilon(\lambda, t),$$

where in view of (2.4) we have

$$g_\epsilon(\lambda, t) = \frac{1}{(\epsilon\lambda + 1)\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}} e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}} t}, \quad \lambda > 0.$$

Since, by Lemma 3.5 we have that $a_\epsilon(t)$ is a creep function with $a_1^\epsilon(t)$ log-convex, we obtain by [54, Lemma 4.2] that the function

$$\psi(\lambda) := \frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}},$$

is positive with $\psi'(\lambda)$ CM (i.e. a Bernstein function, see [54, Definition 4.3, p.91]). By [54, Proposition 4.5, p.96] we have that the function

$$\psi_t(\lambda) := e^{-\frac{1}{\sqrt{\hat{a}_\epsilon(\lambda)}}t}, \quad \lambda > 0, \ t > 0$$

is CM, too. We claim that the function $\phi(\lambda) := \frac{1}{(\epsilon\lambda+1)\lambda\sqrt{\hat{a}_\epsilon(\lambda)}}$, $\lambda > 0$, is completely monotone. In fact, the function ϕ can be written as $\phi(\lambda) = \phi_1(\lambda)\phi_2(\lambda)$, where $\phi_1(\lambda) = \frac{1}{\epsilon\lambda+1}$ and $\phi_2(\lambda) = \frac{1}{\lambda\sqrt{\hat{a}_\epsilon(\lambda)}}$. We can directly check that ϕ_1 is CM because $\epsilon > 0$. Let us prove that ϕ_2 is CM. Indeed, since $a_1^\epsilon(t) = a_\epsilon'(t) - \beta$ and $a_\epsilon(0) = 0$, then from $\widehat{a_1^\epsilon}(\lambda) = \widehat{a_\epsilon'}(\lambda) - \frac{\beta}{\lambda}$ we obtain $\lambda\widehat{a_1^\epsilon}(\lambda) = \lambda(\lambda\hat{a}_\epsilon(\lambda) - a_\epsilon(0)) - \beta = \lambda^2\hat{a}_\epsilon(\lambda) - \beta$, and using that $\widehat{\dot{a}_1^\epsilon}(\lambda) = \lambda\widehat{a_1^\epsilon}(\lambda) - a_1^\epsilon(0) = \lambda\widehat{a_1^\epsilon}(\lambda) - \frac{\delta}{\epsilon}$ we have

$$\lambda^2\widehat{a}_\epsilon(\lambda) = \beta + \lambda\widehat{a_1^\epsilon}(\lambda) = \frac{b_\epsilon}{\epsilon} + \widehat{\dot{a}_1^\epsilon}(\lambda) = \frac{b_\epsilon}{\epsilon}\left(1 - \left(-\frac{\epsilon}{b_\epsilon}\widehat{\dot{a}_1^\epsilon}(\lambda)\right)\right),$$

since $b_\epsilon := \delta + \epsilon\beta$. This implies that

$$\frac{1}{\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}} = \frac{\epsilon\sqrt{\epsilon}}{\sqrt{b_\epsilon}\sqrt{1 - \left(-\frac{\epsilon}{b_\epsilon}\widehat{\dot{a}_1^\epsilon}(\lambda)\right)}}.$$

Note that ϕ_2 is a composition between $\frac{C_1}{\sqrt{1-x}}$ which can be directly proved that is CM and $\varphi(\lambda) = -\frac{\epsilon}{b_\epsilon}\widehat{\dot{a}_1^\epsilon}(\lambda)$. But, $\dot{a}_1^\epsilon(t) = a_\epsilon''(t) < 0$ for all $t > 0$, then $-\frac{\epsilon}{b_\epsilon}\widehat{\dot{a}_1^\epsilon}(\lambda)$ is CM. Hence $\phi_2(\lambda) = \frac{1}{\lambda\sqrt{\widehat{a}_\epsilon(\lambda)}}$ is CM and therefore ϕ, as the product of completely monotone functions, too. So the claim is proved. The conclusion follows from the fact that $h_\epsilon(\lambda, t) = \phi_2(\lambda)g_\epsilon(\lambda, t) = \phi_2(\lambda)\phi(\lambda)\psi_t(\lambda)$ and, once again, the property that the product of completely monotone functions is completely monotone. This proves the Lemma. □

We are in position to prove the following important result, which is the key in order to apply Theorem 5.4 in Appendix.

Theorem 3.7. *Let $\epsilon > 0$ be given. Let A be the generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ on a Banach space X. Then A generates an (a_ϵ, k_ϵ)-regularized family $\{R_\epsilon(t)\}_{t\geq 0}$ satisfying*

$$\|R_\epsilon(t)\| \leq Me^{\omega t}, \quad t \geq 0,$$

for some constant $M > 0$ and $\omega \in \mathbb{R}$ independent of $\epsilon > 0$. Moreover, for each $J := [a, b] \subset \mathbb{R}_+$, $0 \leq a \leq b$, and $x \in X$, we have

$$\lim_{\epsilon\to 0}\sup_{t\in J}\|R_\epsilon(t)x - R_0(t)x\| = 0.$$

Proof. Since A generates a cosine family, it is exponentially bounded [2, Lemma 3.14.3]. It follows that A generates a sine family $\{S(t)\}_{t\geq 0}$, such that [23, II.5 (5.2)]

$$\|S(t)\| \leq \frac{M}{\omega_1}\sinh(\omega_1 t), \quad t \geq 0, \tag{3.8}$$

for some constants $M > 0$ and $\omega_1 > 0$. Note that $S(t)x := \int_0^t C(s)xds$, $x \in X$ [2, Section 3.14]. Then, for all $\mu > \omega_1$ and all $x \in X$ we have

$$(\mu^2 - A)^{-1}x = \int_0^\infty e^{-\mu t} S(t)x dt.$$

Since $\widehat{a}_\epsilon(\lambda) \to 0$ as $\lambda \to \infty$, we have that $\frac{1}{\widehat{a}_\epsilon(\lambda)} > \omega_1$ for all λ large enough. Then $\frac{1}{\widehat{a}_\epsilon(\lambda)} \in \rho(A)$ and

$$H_\epsilon(\lambda)x = \frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} \int_0^\infty e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}}t} S(t)x dt = \int_0^\infty h_\epsilon(\lambda, t)S(t)x dt$$

for all $x \in X$ and λ large enough, say $\lambda > \omega_1$. Here $h_\epsilon(\lambda, t)$ was defined in Lemma 3.6.

By Lemma 3.6 we have that $h_\epsilon(\lambda, t)$ is completely monotone for each fixed $\epsilon > 0$. Let $L_\lambda^n := \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n}$. Then

$$L_\lambda^n H_\epsilon(\lambda)x = \int_0^\infty L_\lambda^n h_\epsilon(\lambda, t)S(t)x dt.$$

Since $h_\epsilon(\lambda, t)$ is completely monotone, we obtain from (3.8)

$$\begin{aligned}
\|L_\lambda^n H_\epsilon(\lambda)\| &\le \frac{M}{\omega_1} \int_0^\infty \sinh(\omega_1 t) L_\lambda^n h_\epsilon(\lambda, t) dt = \frac{M}{\omega_1} L_\lambda^n \int_0^\infty \sinh(\omega_1 t) h_\epsilon(\lambda, t) dt \\
&= M L_\lambda^n \left[\frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} \int_0^\infty \frac{\sinh(\omega_1 t)}{\omega_1} e^{-\frac{1}{\sqrt{\widehat{a}_\epsilon(\lambda)}}t} dt \right] \\
&= M L_\lambda^n \left[\frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} \left(\frac{1}{\widehat{a}_\epsilon(\lambda)} - \omega_1^2 \right)^{-1} \right] = M L_\lambda^n \, \widehat{r}_\epsilon(\lambda, -\omega_1^2),
\end{aligned}$$

where $r_\epsilon(t, -\omega_1^2)$ is the solution of

$$r_\epsilon(t, -\omega_1^2) = k_\epsilon(t) + \omega_1^2 \int_0^t r_\epsilon(\tau - t, -\omega_1^2) a_\epsilon(\tau) d\tau. \tag{3.9}$$

Note that because $\sinh(\omega_1 t) \ge 0$ and $h_\epsilon(\lambda, t)$ is completely monotone, Berstein's theorem [54, Section 4.1] implies that $r_\epsilon(t, -\omega_1^2)$ is nonnegative. We claim that there exist constants $C_0 > 0$ and $\omega_2 \in \mathbb{R}$ which are not depending on ϵ such that

$$r_\epsilon(t, -\omega_1^2) \le C_0 e^{\omega_2 t}, \quad t \ge 0. \tag{3.10}$$

Indeed, since $0 \le 1 - e^{-\frac{t}{\epsilon}} \le 1$, from (2.1) and (2.2) we have that

$$0 \le a_\epsilon(t) \le a_0(t) \quad \text{and} \quad 0 \le k_\epsilon(t) \le k_0(t)$$

for all $t \ge 0$ and for all $\epsilon > 0$. It follows that

$$r_\epsilon(t, -\omega_1^2) = k_\epsilon(t) + \omega_1^2 (r_\epsilon * a_\epsilon)(t) \le k_0(t) + \omega_1^2 (r_\epsilon * a_0)(t).$$

Therefore there exists a continuous and nonnegative function $\alpha_\epsilon(t)$ $(:= k_0(t) + \omega_1^2 (r_\epsilon * a_0)(t) - r_\epsilon(t, -\omega_1^2))$ such that

$$r_\epsilon(t, -\omega_1^2) = k_0(t) - \alpha_\epsilon(t) + \omega_1^2(r_\epsilon * a_0)(t), \quad t \geq 0. \tag{3.11}$$

By variation of constants formula for linear convolution integral equations [30, Chapter 2] we get that

$$r_\epsilon(t, -\omega_1^2) = k_\epsilon(t) - \alpha_\epsilon(t) - (s * (k_\epsilon - \alpha_\epsilon))(t), \quad t \geq 0, \tag{3.12}$$

where $s(t)$ is the nonnegative solution of

$$s(t) = \omega_1^2 a_0(t) + \omega_1^2(a_0 * s)(t), \quad t \geq 0.$$

The function $s(t)$ can be explicitly written as

$$s(t) = \sum_{j=1}^{\infty} (\omega_1^2)^j (a_0)^{*j}(t), \quad t \geq 0,$$

where $(a_0)^{*j}$ means convolution j-times. Since $a_0(t) = \gamma + \beta t$ is exponentially bounded, then $s(t)$ is exponentially bounded (independent of ϵ) too. Then, by (3.12)

$$r_\epsilon(t, -\omega_1^2) \leq t - (s(t) * t), \quad t \geq 0.$$

Hence there exist $C_0 > 0$ and $\omega_2 \in \mathbb{R}$ such that (3.10) holds. Since r_ϵ is nonnegative

$$L_\lambda^n \, \widehat{r}_\epsilon(\lambda, -\omega_1^2) \leq C_0 \int_0^\infty L_n^\lambda e^{-(\lambda - \omega_2)t} dt \leq C_0(\lambda - \omega_2)^{-n-1}.$$

This implies that

$$\|L_n^\lambda H_\epsilon(\lambda)\| \leq C(\lambda - \omega_2)^{-n-1}.$$

In view of Theorem 5.5 in the Appendix, the operator A generates an (a_ϵ, k_ϵ)-regularized family $\{R_\epsilon(t)\}_{t\geq 0}$ satisfying

$$\|R_\epsilon(t)\| \leq M e^{\omega t}, \quad t \geq 0,$$

for some constants $M > 0$ and $\omega \in \mathbb{R}$ independent of $\epsilon > 0$. This proves the first part of the Theorem.

For the second part, note that $\hat{a}_\epsilon(\lambda) \to \hat{a}_0(\lambda)$ and $\hat{k}_\epsilon(\lambda) \to \hat{k}_0(\lambda)$ as $\epsilon \to 0$. Since A generates a strongly continuous cosine family then A is the generator of a strongly continuous semigroup (in fact, analytic of angle $\pi/2$ and given by Weierstrass formula, see e.g. [2, Theorem 3.14.17]). Therefore, by the comments at the beginning of this section, we have that A is also the generator of a (a_0, k_0)-regularized family $\{R_0(t)\}_{t\geq 0}$.

On the other hand, a computation using the resolvent identity

$$R(\lambda, A) - R(\mu, B) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \lambda, \mu \in \rho(A),$$

shows that

$$H_\epsilon(\mu) - H_0(\mu) = \frac{1}{\hat{a}_0(\mu)}(\hat{k}_\epsilon(\mu)\hat{a}_0(\mu) - \hat{k}_0(\mu)\hat{a}_\epsilon(\mu))\frac{1}{\hat{a}_\epsilon(\mu)}\left(\frac{1}{\hat{a}_\epsilon(\mu)} - A\right)^{-1} +$$

$$= \frac{k_0(\mu)}{\hat{a}_0(\mu)}\left[(a_\epsilon(\mu) - \hat{a}_0(\mu))\frac{1}{\hat{a}_\epsilon(\mu)}\left(\frac{1}{\hat{a}_\epsilon(\mu)} - A\right)^{-1}\frac{1}{\hat{a}_0(\mu)}\left(\frac{1}{\hat{a}_0(\mu)} - A\right)^{-1}\right]$$

holds. Since A generates an analytic semigroup, there exists a constant $M > 0$ such that $\|\lambda(\lambda - A)^{-1}\| \leq M$. Therefore, from the above identity, we obtain

$$\|\hat{k}_\epsilon(\mu)(I - \hat{a}_\epsilon(\mu)A)^{-1} - \hat{k}_0(\mu)(I - \hat{a}_0(\mu)A)^{-1}\| \leq |\frac{1}{\hat{a}_0(\mu)}(\hat{k}_\epsilon(\mu)\hat{a}_0(\mu) - \hat{k}_0(\mu)\hat{a}_\epsilon(\mu))|M$$
$$+ |\frac{k_0(\mu)}{\hat{a}_0(\mu)}(a_\epsilon(\mu) - \hat{a}_0(\mu))|M,$$

where the right hand side converges to 0 as $\epsilon \to 0$. By Theorem 5.4, we get the second conclusion. □

The following definition was introduced by Fattorini [22, p.175] (see also [23]).

Definition 3.8. Let $t(\epsilon) > 0$ for each $\epsilon > 0$. We say that a family of vector valued functions $g_\epsilon : \mathbb{R}_+ \to X$ converges to $g : \mathbb{R}_+ \to X$ uniformly on compacts of $t \geq t(\epsilon)$ if for each $a > 0$:

$$\lim_{\epsilon \to 0} \sup_{t(\epsilon) \leq t \leq a} \|g_\epsilon(t) - g(t)\| = 0.$$

We next prove the following result revealing that the convergence of the first derivative of the resolvent families i.e. $\{R'_\epsilon(t)\}_{t\geq 0}$, possesses a singular behavior, as $\epsilon \to 0$, when $t = 0$.

Theorem 3.9. *Let $\epsilon \geq 0$ be given. Under the hypothesis of Theorem 3.7 we have*

(a) *If $y \in D(A)$, then there exist $M_1 > 0$ and $\omega_1 > 0$ such that*

$$\|R'_\epsilon(t)y\| \leq M_1 e^{\omega_1 t}\|y\|_{[D(A)]}, \quad t \geq 0,$$

where $\|\cdot\|_{[D(A)]}$ denotes the graph norm of A.

(b) *For each $w \in D(A^2)$ we have that $R'_\epsilon(t)w$ converges to $R'_0(t)w$ uniformly on compacts of $t \geq t(\epsilon)$, as long as*

$$t(\epsilon)/\epsilon \to \infty \qquad (\epsilon \to 0). \tag{3.13}$$

Proof. Recall that

$$a'_\epsilon(t) = \beta + \frac{\delta}{\epsilon}e^{-\frac{t}{\epsilon}}, \quad k'_\epsilon(t) = 1 - e^{-\frac{t}{\epsilon}},$$

and

$$a'_0(t) = \beta, \quad k'_0(t) = 1.$$

Let us prove part (a). We start with the case $\epsilon = 0$. Since $y \in D(A)$ and $a_0(0) = \delta$, we have that

$$R'_0(t)y = k'_0(t)y + \delta R_0(t)Ay + (a'_0 * R_0)(t)Ay, \quad t \geq 0. \tag{3.14}$$

Since $\{R_0(t)\}_{t\geq 0}$ is exponentially bounded, there exist $M_0 > 0$ and $\omega_0 > 0$ such that $\|R_0(t)x\| \leq M_0 e^{\omega_0 t}\|x\|$ for all $t \geq 0$ and for all $x \in X$. Then, (3.14) implies that

$$\|R_0'(t)y\| \le \|y\| + \delta\|R_0(t)Ay\| + \beta\int_0^t \|R_0(s)Ay\|ds$$
$$\le \|y\| + \delta M_0 e^{\omega_0 t}\|Ay\| + \beta M_0 \int_0^t e^{\omega_0 s}\|Ay\|ds \le M_0 e^{\omega_0 t}\|y\|_{[D(A)]}, \quad t \ge 0.$$

Next, we continue with the case $\epsilon > 0$. Since $y \in D(A)$,

$$R_\epsilon(t)y = k_\epsilon(t)y + (a_\epsilon * R_\epsilon)(t)Ay, \quad t \ge 0,$$

and $a_\epsilon(0) = 0$, we have that

$$R_\epsilon'(t)y = k_\epsilon'(t)y + (a_\epsilon' * R_\epsilon)(t)Ay, \quad t \ge 0. \tag{3.15}$$

By the uniform exponential boundedness of $R_\epsilon(t)$ and the fact that $|k_\epsilon'(t)| \le 1$, we deduce from (3.15) the following uniform estimate

$$\|R_\epsilon'(t)y\| \le |k_\epsilon'(t)|\|y\| + \int_0^t |a_\epsilon'(t-s)|\|R_\epsilon(s)Ay\|ds$$
$$\le \|y\| + \int_0^t \left(\beta + \frac{\delta}{\epsilon}e^{-(t-s)/\epsilon}\right) Me^{\omega s}\|Ay\|ds$$
$$\le \|y\| + \beta M e^{\omega t}\|Ay\| + \frac{M\delta}{\epsilon}e^{-t/\epsilon}\int_0^t e^{s/\epsilon}e^{\omega s}\|Ay\|ds$$
$$\le \|y\| + \beta M e^{\omega t}\|Ay\| + \frac{M\delta}{\epsilon}e^{-t/\epsilon}\frac{\epsilon}{\omega\epsilon + 1}e^{t(1/\epsilon+\omega)}\|Ay\|$$
$$\le \|y\| + \beta M e^{\omega t}\|Ay\| + M\delta\frac{1}{\omega\epsilon+1}e^{\omega t}\|Ay\|$$
$$\le \overline{M}e^{\omega t}\|y\|_{[D(A)]}, \quad t \ge 0,$$

where $\overline{M}$ and $\omega > 0$ are independent of $\epsilon > 0$. Taking $M_1 := \max\{M_0, \overline{M}\}$ and $\omega_1 := \max\{\omega_0, \omega\}$, the conclusion follows.

Let us show part (b). Let $w \in D(A^2)$ be given. Integration by parts gives

$$\delta\int_0^t \frac{1}{\epsilon}e^{-s/\epsilon}R_\epsilon(t-s)Awds = \delta R_\epsilon(t)Aw + \delta\int_0^t e^{-s/\epsilon}R_\epsilon'(t-s)Awds. \tag{3.16}$$

Since $a_0'(t) = \beta$ and $a_\epsilon'(s) = \beta + \frac{\delta}{\epsilon}e^{-\frac{s}{\epsilon}}$, from (3.15) and (3.14) we get

$$\|R_\epsilon'(t)w - R_0'(t)w\| = \|k_\epsilon'(t)w - k_0'(t)w + \int_0^t a_\epsilon'(s)R_\epsilon(t-s)Awds$$
$$- \delta R_0(t)Aw - \int_0^t a_0'(s)R_0(t-s)Awds\|$$

$$
\begin{aligned}
&= \|k_\epsilon'(t)w - k_0'(t)w + \beta \int_0^t [R_\epsilon(s)Aw - R_0(s)Aw]ds \\
&- \delta R_0(t)Aw + \delta \int_0^t \frac{e^{-s/\epsilon}}{\epsilon} R_\epsilon(t-s)Awds\| \\
&\le |k_\epsilon'(t) - k_0'(t)|\|w\| + \beta \int_0^t \|R_\epsilon(s)Aw - R_0(s)Aw\|ds \\
&+ \delta\|R_\epsilon(t)Aw - R_0(t)Aw\| + \delta \int_0^t e^{-s/\epsilon}\|R_\epsilon'(t-s)Aw\|ds \\
&=: \bar{I}_1(t,\epsilon) + \bar{I}_2(t,\epsilon) + \bar{I}_3(t,\epsilon) + \bar{I}_4(t,\epsilon),
\end{aligned}
$$

where we have used (3.16) in the last inequality. Let $t(\epsilon) > 0$ for each $\epsilon > 0$ and define $J_\epsilon := [t(\epsilon), b] \subset \mathbb{R}_+$, where $b > 0$. Recalling that $k_\epsilon'(t) = 1 - e^{-\frac{t}{\epsilon}}$ and $k_0'(t) = 1$, we obtain

$$
\lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} \bar{I}_1(t,\epsilon) = \lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} |k_\epsilon'(t) - k_0'(t)| = \lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} e^{-t/\epsilon} \le \lim_{\epsilon\to 0} e^{-t(\epsilon)/\epsilon} = 0, \tag{3.17}
$$

as long as $t(\epsilon)/\epsilon \to \infty$. Moreover, we obtain

$$
\bar{I}_2(t,\epsilon) = \beta \int_0^t \|R_\epsilon(s)Aw - R_0(s)Aw\|ds \le \beta t \sup_{\tau\in J_\epsilon} \|R_\epsilon(\tau)Aw - R_0(\tau)Aw\|.
$$

Therefore, using Theorem 3.7 we obtain

$$
\lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} \bar{I}_2(t,\epsilon) \le \lim_{\epsilon\to 0} \sup_{\tau\in[0,b]} \|R_\epsilon(\tau)Aw - R_0(\tau)Aw\|\beta b = 0.
$$

Analogously, Theorem 3.7 proves that $\lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} \bar{I}_3(t,\epsilon) = 0$. Finally, using part (a) note that

$$
\begin{aligned}
\bar{I}_4(t,\epsilon) &= \delta \int_0^t e^{-s/\epsilon}\|R_\epsilon'(t-s)Aw\|ds \le \overline{M}\delta e^{\omega t}\|Aw\|_{D(A)} \int_0^t e^{-s/\epsilon}ds \\
&= \overline{M}\delta e^{\omega t}\|w\|_{D(A^2)}\epsilon(1 - e^{-t/\epsilon}).
\end{aligned}
$$

Therefore

$$
\lim_{\epsilon\to 0} \sup_{t\in J_\epsilon} \bar{I}_4(t,\epsilon) \le \lim_{\epsilon\to 0} \overline{M}\delta e^{\omega b}\|w\|_{D(A^2)}\epsilon\|Aw\| = 0.
$$

Hence, we have proved that $\lim_{\epsilon\to 0+} \sup_{t\in J_\epsilon} \|R_\epsilon'(t)w - R_0'(t)w\| = 0$ as long as $t(\epsilon)/\epsilon \to \infty$. □

Remark 3.10. The notion of uniform convergence on compacts of $t \ge t(\epsilon)$ as long as $\frac{t(\epsilon)}{\epsilon} \to \infty$ $(\epsilon \to 0)$ is taken from Fattorini [23].

We will need the following result in order to prove our main result.

Lemma 3.11. *Let $\epsilon > 0$ be given. Under the hypothesis of Theorem 3.7 we have: There exist constants $\overline{M} > 0$ and $\omega > 0$ independent of $\epsilon > 0$ such that*

$$\|tR''_\epsilon(t)w\| \leq \overline{M}e^{\omega t}\|w\|_{[D(A^2)]} \quad \text{for all} \quad t \geq 0, \quad w \in D(A^2)$$

where $\|\cdot\|_{[D(A^2)]}$ denotes the graph norm of $D(A^2)$.

Proof. Since $w \in D(A^2)$ and $R_\epsilon(0) = 0$, we obtain from (3.15) that

$$R''_\epsilon(t)w = k''_\epsilon(t)w + (a'_\epsilon * R'_\epsilon)(t)Aw, \quad t > 0.$$

By Theorem 3.9 we have that $R'_\epsilon(t)$ is exponentially bounded for all $t \geq 0$. Note that $k''(t) = \frac{1}{\epsilon}e^{-t/\epsilon}$ and hence $|tk''_\epsilon(t)| \leq 1$. It follows that

$$\begin{aligned}
\|tR''_\epsilon(t)w\| &\leq |tk''_\epsilon(t)|\|w\| + \int_0^t |a'_\epsilon(t-s)|\|R'_\epsilon(s)Aw\|ds \\
&\leq \|w\| + \int_0^t \left(\beta + \frac{\delta}{\epsilon}e^{-(t-s)/\epsilon}\right) Me^{\omega s}\|Aw\|ds \\
&\leq \|w\| + \beta Me^{\omega t}\|Aw\| + \frac{M\delta}{\epsilon}e^{-t/\epsilon}\int_0^t e^{s/\epsilon}e^{\omega s}\|Aw\|ds \\
&\leq \|w\| + \beta Me^{\omega t}\|Aw\| + M\delta\frac{1}{\omega\epsilon + 1}e^{\omega t}\|Aw\| \leq \overline{M}e^{\omega t}\|w\|_{[D(A)]}, \quad t \geq 0. \quad \square
\end{aligned}$$

4. Proof of Theorem 1.1

Let $\epsilon > 0$ be given. With respect to the estimates in Theorem 3.7, Theorem 3.9 and Lemma 3.11, we will use in what follows the same symbols $K > 0$ and $\omega > 0$ to denote several distinct positive constants (which are independent of ϵ). In this way, the estimates can now be read as follows

(E1) $\|R_0(t)x\| \leq Ke^{\omega t}\|x\|$ for all $t \geq 0$ and for all $x \in X$.
(E2) $\|R_\epsilon(t)x\| \leq Ke^{\omega t}\|x\|$ for all $t \geq 0$ and for all $x \in X$.
(E3) $\|R'_0(t)y\| \leq Ke^{\omega t}\|y\|_{[D(A)]}$ for all $t \geq 0$ and for all $y \in D(A)$.
(E4) $\|R'_\epsilon(t)y\| \leq Ke^{\omega t}\|y\|_{[D(A)]}$ for all $t \geq 0$ and for all $y \in D(A)$.
(E5) $\|tR''_\epsilon(t)w\| \leq Ke^{\omega t}\|w\|_{[D(A^2)]}$ for all $t \geq 0$ and for all $w \in D(A^2)$.

Proof. Since A generates a strongly continuous cosine family, Theorem 3.7, Theorem 3.9 and Lemma 3.11 hold. So we can use relations (E1)-(E5). Furthermore, hypotheses (a)-(d) imply that there exists constants $M_1 > 0$ and $M_2 > 0$ independent of $\epsilon > 0$ such that

(E6) $\|u^0(\epsilon)\|_{[D(A^2)]} \leq M_1$.
(E7) $\|u^1(\epsilon)\|_{[D(A)]} \leq M_2$.

Now, by the representation of $u_\epsilon(t)$ (see Theorem 3.2) and $u_0(t)$ (see Theorem 3.2), we obtain

$$\|u_\epsilon(t) - u_0(t)\| \leq \epsilon\|R''_\epsilon(t)u^0(\epsilon)\| + \|(R'_\epsilon(t) - R'_0(t))(u^0(\epsilon) + \epsilon u^1(\epsilon))\|$$

$$
\begin{aligned}
&+ \|R_0'(t)(u^0(\epsilon) - u^0 + \epsilon u^1(\epsilon))\| \\
&+ \|(R_\epsilon(t) - R_0(t))[u^1(\epsilon) - b_\epsilon A u^0(\epsilon) + \epsilon u^2(\epsilon)]\| \\
&+ \|R_0(t)[u^1(\epsilon) - u^1 + b_\epsilon A u^0(\epsilon) - \delta A u^0 + \epsilon u^2(\epsilon)]\| \\
&=: I_1(t,\epsilon) + ... + I_5(t,\epsilon).
\end{aligned}
$$

Let $J_\epsilon := [t(\epsilon), b] \subset \mathbb{R}_+$ with $b > 0$ and where $t(\epsilon) > 0$ is such that $t(\epsilon)/\epsilon \to \infty$ as $\epsilon \to 0$. Let us prove that $\lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} I_m(t,\epsilon) = 0$ for each $m = 1, ..., 5$. Since $u^0(\epsilon) \in D(A^2)$, (E5) and (E6) imply that

$$
\begin{aligned}
\lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} I_1(t,\epsilon) &= \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \frac{\epsilon}{t}\|tR_\epsilon''(t)u^0(\epsilon)\| \\
&\le \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \left\{\frac{\epsilon}{t} K e^{\omega t}\|u^0(\epsilon)\|_{[D(A^2)]}\right\} \\
&\le K M_1 e^{\omega b} \lim_{\epsilon\to 0^+} \frac{\epsilon}{t(\epsilon)} = 0.
\end{aligned} \tag{4.1}
$$

Next, since $u^0(\epsilon), u^1(\epsilon), u^0 \in D(A^2)$, we deduce from (E3) and (E4) that

$$
\begin{aligned}
I_2(t,\epsilon) &\le \|(R_\epsilon'(t) - R_0'(t))(u^0(\epsilon) - u^0)\| + \|(R_\epsilon'(t) - R_0'(t))u^0\| + \epsilon\|(R_\epsilon'(t) - R_0'(t))u^1(\epsilon)\| \\
&\le 2Ke^{\omega t}\|u^0(\epsilon) - u^0\|_{[D(A)]} + \|(R_\epsilon'(t) - R_0'(t))u^0\| + 2\epsilon K e^{\omega t}\|u^1(\epsilon)\|_{[D(A)]}.
\end{aligned}
$$

Therefore, by (E7), Theorem 3.9 part (b) and hypothesis (a)

$$
\begin{aligned}
\lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} I_2(t,\epsilon) &\le 2K \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} e^{\omega t}\|u^0(\epsilon) - u^0\|_{[D(A)]} + \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \|(R_\epsilon'(t) - R_0'(t))u^0\| \\
&+ 2K \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \epsilon e^{\omega_0 t}\|u^1(\epsilon)\|_{[D(A)]} \\
&\le 2Ke^{\omega b} \lim_{\epsilon\to 0^+} \|u^0(\epsilon) - u^0\|_{[D(A)]} + \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \|(R_\epsilon'(t) - R_0'(t))u^0\| \\
&+ 2KM_2\, e^{\omega b} \lim_{\epsilon\to 0^+} \epsilon = 0.
\end{aligned}
$$

For $I_3(t,\epsilon)$, observe that since $u^0(\epsilon), u^0$ and $u^1(\epsilon)$ belong to $D(A)$, then (E3), (E7) and hypothesis (a) give

$$
\begin{aligned}
\lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} I_3(t,\epsilon) &= \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \{\|R_0'(t)(u^0(\epsilon) - u^0)\| + \epsilon\|R_0'(t)u^1(\epsilon)\|\} \\
&\le \lim_{\epsilon\to 0^+} \sup_{t\in J_\epsilon} \left\{Ke^{\omega t}(\|u^0(\epsilon) - u^0\|_{[D(A)]} + \epsilon\|u^1(\epsilon)\|_{[D(A)]})\right\} \\
&\le Ke^{\omega b} \lim_{\epsilon\to 0^+} \|u^0(\epsilon) - u^0\|_{[D(A)]} + KM_2 e^{\omega b} \lim_{\epsilon\to 0^+} \epsilon \\
&= 0.
\end{aligned}
$$

We continue with $I_4(t,\epsilon)$. By (E1) and (E2), we obtain

$$
\begin{aligned}
I_4(t,\epsilon) &\le \|(R_\epsilon(t) - R_0(t))(u^1(\epsilon) - v)\| + b_\epsilon\|(R_\epsilon(t) - R_0(t))Au^0\| \\
&+ b_\epsilon\|(R_\epsilon(t) - R_0(t))(Au^0(\epsilon) - Au^0)\| + \|(R_\epsilon(t) - R_0(t))(\epsilon u^2(\epsilon) - (u^1 - v)\| \\
&+ \|(R_\epsilon(t) - R_0(t))u^1\| \\
&\le 2Ke^{\omega t}\|u^1(\epsilon) - v\| + b_\epsilon\|(R_\epsilon(t) - R_0(t))Au^0\| + 2b_\epsilon K e^{\omega t}\|Au^0(\epsilon) - Au^0\| \\
&+ 2Ke^{\omega t}\|\epsilon u^2(\epsilon) - (u^1 - v)\| + \|(R_\epsilon(t) - R_0(t))u^1\|.
\end{aligned}
$$

It follows from Theorem 3.7 and hypotheses $(a), (b)$ and (c) that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} I_4(t,\epsilon) \le & \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} 2Ke^{\omega t}\|u^1(\epsilon)-v\| + \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} b_\epsilon\|(R_\epsilon(t)-R_0(t))Au^0\| \\ & + \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} 2b_\epsilon Ke^{\omega t}\|Au^0(\epsilon)-Au^0\| + \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} 2Ke^{\omega t}\|\epsilon u^2(\epsilon)-(u^1-v)\| \\ & + \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon}\|(R_\epsilon(t)-R_0(t))u^1\| = 0,\end{aligned}$$

where we have used that $b_\epsilon \to \delta$ as $\epsilon \to 0^+$.

Finally, let us see $I_5(t,\epsilon)$. Theorem 3.7 implies that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} I_5(t,\epsilon) &= \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon}\{\|R_0(t)[u^1(\epsilon)-u^1+b_\epsilon Au^0(\epsilon)-\delta Au^0+\epsilon u^2(\epsilon)]\|\} \\ &\le \lim_{\epsilon\to 0^+}\sup_{t\in J_\epsilon} Ke^{\omega t}\{\|u^1(\epsilon)-v\|+|b_\epsilon-\delta|\|Au^0\| \\ &+ b_\epsilon\|Au^0(\epsilon)-Au^0\|+\|\epsilon u^2(\epsilon)-(u^1-v)\|\} \\ &\le Ke^{\omega b}\{\lim_{\epsilon\to 0^+}\|u^1(\epsilon)-v\|+\lim_{\epsilon\to 0^+}|b_\epsilon-\delta|\|Au^0\| \\ &+ \lim_{\epsilon\to 0^+} b_\epsilon\|Au^0(\epsilon)-Au^0\|+\lim_{\epsilon\to 0^+}\|\epsilon u^2(\epsilon)-(u^1-v)\|\} = 0,\end{aligned}$$

where we have used $(a), (b)$, (c) and the fact that $b_\epsilon \to \delta$ as $\epsilon \to 0^+$.

Therefore, for each $\epsilon > 0$ the solution $u_\epsilon(t)$ of (1.1) converges to the unique solution $u_0(t)$ of (1.2) as $\epsilon \to 0$ uniformly on compacts of $t \ge t(\epsilon)$. $\square$

Remark 4.1. Uniform convergence in $t \ge 0$ cannot be expected since in general $u^1(\epsilon)$ does not converge to u^1, as $\epsilon \to 0$. This means that there is a initial layer near zero where u_ϵ is not a good approximation to u_0.

5. Appendix: (a, k)-regularized resolvent families and approximation

The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \lim_{T\to\infty}\int_0^T e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

when the limit exists. In particular if f is such that $\int_0^t f(s)ds$ is exponentially bounded, i.e., there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|\int_0^t f(s)ds\| \le Me^{\omega t}$ for all $t \ge 0$, then $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for $\mathrm{Re}(\lambda) > \omega$, and the integral is absolutely convergent. This remains true if we make the stronger assumption that f is exponentially bounded (see [2, Chapter I]).

We recall from [43] the following definition.

Definition 5.2. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and $a \in L^1_{loc}(\mathbb{R}_+)$, $k \in C(\mathbb{R}_+)$ such that $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$ exist. The operator A is called the generator of an (a, k)-regularized resolvent family if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{1}{\hat{a}(\lambda)} : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, the resolvent set of A, and

$$H(\lambda)x := \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} R(t)x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In such case we say that $\{R(t)\}_{t\geq 0}$ is the resolvent family generated by A.

We recall that the concept of (a,k)-regularized resolvent family generalizes - and therefore includes - the concepts of strongly continuous semigroup, integrated semigroup, strongly continuous cosine family, integrated cosine family, resolvent operator associated to fractional order problems and resolvent operator associated to Volterra equations, among others [43]. Several properties and applications can be found in [46,53] and references therein.

From [43, Proposition 3.1 and Lemma 2.2] we obtain directly the following properties.

Proposition 5.3. *Let A be a closed linear operator defined on a Banach space X and $a \in L^1_{loc}(\mathbb{R}_+), k \in C(\mathbb{R}_+)$. Suppose that A is the generator of an (a,k)-regularized resolvent family $\{R(t)\}_{t\geq 0}$ on X. Then the following assertions hold true:*

1. *For each $x \in X$ we have $t \to R(t)x$ is continuous in $\mathbb{R}_+$ and $R(0) = k(0)$.*
2. *For all $x \in D(A)$ and $t \geq 0$ we have $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$.*
3. *For each $x \in X$ and $t \geq 0$ we have $\int_0^t a(t-s)R(s)xds \in D(A)$ and*

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds.$$

4. *For all $x \in D(A)$ we have*

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Axds.$$

The following result is taken from [44, Theorem 2.5].

Theorem 5.4. *Let $\{a_\epsilon\}_{\epsilon\geq 0}, \{k_\epsilon\}_{\epsilon\geq 0} \subset C^1(\mathbb{R}_+)$ be Laplace transformable functions and assume that there exists $\omega_0 \geq 0$ such that $\widehat{a}_\epsilon(\mu) \neq 0$ for all $\mu > \omega_0$, and $\int_0^\infty e^{-\omega_0 s}|a'_\epsilon(s)|ds < \infty$. Suppose that A is densely defined and, for all $\epsilon \geq 0$, the generator of (a_ϵ, k_ϵ)-regularized families $\{R_\epsilon(t)\}_{t\geq 0}$ satisfying the following stability property: there exist constants $M, \omega \geq 0$, independent of $\epsilon \geq 0$, such that*

$$\sup_{\epsilon\geq 0} \|R_\epsilon(t)\| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+. \tag{5.1}$$

Assume $\hat{a}_\epsilon(\lambda) \to \hat{a}_0(\lambda)$ and $\hat{k}_\epsilon(\lambda) \to \hat{k}_0(\lambda)$ as $\epsilon \to 0$. Then the following statements are equivalent:

(i) $\lim_{\epsilon\to 0} \hat{k}_\epsilon(\mu)(I - \hat{a}_\epsilon(\mu)A)^{-1}x = \hat{k}_0(\mu)(I - \hat{a}_0(\mu)A)^{-1}x$ *for all $\mu > \omega$, and for all $x \in X$.*

(ii) $\lim_{\epsilon\to 0} R_\epsilon(t)x = R_0(t)x$ *for all $x \in X$ and for all $t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.*

Recall that a one-parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded and linear operators on X is called a strongly continuous cosine family if $C(0) = I$, $2C(t)C(s) = C(t+s) + C(t-s)$ and $\lim_{t\to 0} C(t)x = x$ for all $x \in X$. Notice that a strongly continuous cosine family is an (a,k)-regularized resolvent family in the special case $k(t) \equiv 1$ and $a(t) = t$. For further literature on cosine families we refer to the monographs of Fattorini [22] and Arendt-Batty-Hieber-Neubrander [2] and the references therein.

The following characterization of generators of (a_ϵ, k_ϵ)-regularized resolvent families follows directly from [54] or [43, Theorem 3.4].

Theorem 5.5. *Let A be a closed linear densely defined operator in a Banach space X. Then the following assertions are equivalent.*

(i) *The operator A is the generator of an (a_ϵ, k_ϵ)-regularized resolvent family $\{R_\epsilon(t)\}_{t\geq 0}$ satisfying $||R_\epsilon(t)|| \leq M_\epsilon e^{\omega_\epsilon t}$ for all $t \geq 0$ and for some constants $M_\epsilon > 0$ and $\omega_\epsilon \in \mathbb{R}$.*

(ii) *There exist constants $\omega_\epsilon \in \mathbb{R}$ and $M_\epsilon > 0$ such that*
 (P1) $\frac{1}{\widehat{a}_\epsilon(\lambda)} \in \rho(A)$ *for all λ with $Re(\lambda) > \omega_\epsilon$ and*
 (P2) $H_\epsilon(\lambda) := \frac{\widehat{k}_\epsilon(\lambda)}{\widehat{a}_\epsilon(\lambda)} \left(\frac{1}{\widehat{a}_\epsilon(\lambda)} - A\right)^{-1}$ *satisfies the estimates*

$$||H_\epsilon^{(n)}(\lambda)|| \leq \frac{M_\epsilon n!}{(\lambda - \omega_\epsilon)^{n+1}}, \quad \lambda > \omega_\epsilon, \quad n = 0, 1, 2\dots.$$

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