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A characterization of L^p-maximal regularity for time-fractional systems in UMD spaces and applications

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Abstract

In this article we provide new insights into the well-posedness and maximal regularity of systems of abstract evolution equations, in the framework of periodic Lebesgue spaces of vector-valued functions. Our abstract model is flexible enough as to admit time-fractional derivatives in the sense of Liouville-Grünwald. We characterize the maximal regularity property solely in terms of R-boundedness of a block operator-valued symbol, and provide corresponding estimates. In addition, we show practical criteria that imply the R-boundedness part of the characterization. We apply these criteria to show that the Keller-Segel system, as well as a reactor model system, have $L^q - L^p$ maximal regularity.

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1. Introduction

It is very rare that a real life phenomenon can be modeled by a single partial differential equation. Usually, a system of coupled partial differential equations is needed to produce a complete model. Systems of equations are common in the theory of heat and mass transfer of reacting media, the theory of chemical reactors, the theory of combustion, mathematical biology, and biophysics. A typical example is the Keller-Segel system given by

$$\partial_t u = \nabla \cdot (d_1 \nabla u) - \nabla \cdot (\chi u \nabla v) \tag{1.1}$$

$$\partial_t v = d_2 \Delta v + au - bv. \tag{1.2}$$

Equation (1.1) represents the cell density variation over time, and equation (1.2) represents the chemical attractant concentration variation over time.

In (1.1), d_1 is the diffusion coefficient of cell, χ is the chemotactic sensitivity, and the function f regulates the cell die/divide, which controls the gross cell number in observations. In equation (1.2), d_2 represents the diffusion coefficient of chemical attractant, a regulates the production rate of chemical attractant, and b regulates the degradation rate of chemical attractant.

A very important mathematical question that needs to be solved for a good understanding of the behavior of *systems* of equations is the knowledge of the well-posedness and regularity property of such systems in various function spaces. For example, for the simplified Keller-Segel system of parabolic-elliptic type, also called Jäger-Luckhaus system or Nagai model [36], the maximal regularity property appears well studied by Ogawa and Shimizu [36] and Takeuchi [43] obtaining maximal regularity in the Besov and the Lorentz spaces, respectively.

Let X be a Banach space and $\mathbb{T}$ the one dimensional torus. In this article, we are concerned with the property of well-posedness and maximal regularity in vector-valued periodic Lebesgue spaces $L^p(\mathbb{T}; X)$ for general abstract systems that can be modeled in the form

$$\partial_t^\alpha u = Au + Bv + f, \tag{1.3}$$

$$\partial_t^\beta v = Cu + Dv + g, \tag{1.4}$$

where $0 < \alpha, \beta < 2$, ∂_t^α and ∂_t^β are the fractional derivatives in the sense of Liouville-Grünwald, A, B, C and D are closed linear operators and $f, g \in L^p(\mathbb{T}; X)$ are given functions. Note that if d_1 in (1.1) is assumed to be constant, then the Keller-Segel system (1.1)-(1.2) is a particular case of (1.3)-(1.4) with $\alpha = \beta = 1$, $A = d_1\Delta$, $B \equiv 0$, $f(u, v) = -\nabla \cdot (\chi u \nabla v)$, $C = aI$, $D = d_2\Delta - bI$ and $g \equiv 0$.

We note that systems of partial differential equations with time-fractional derivatives have appeared in recent years as a way to study the influence of memory effects on the dynamics of models. Fractional time derivatives for systems are discussed, for example, in references [1] and [24] for the Keller-Segel and reactor models, respectively, which we will consider in the last section of this article.

As far as the authors know, the problem of well posedness and maximal regularity, in the vector-valued Lebesgue space of periodic functions, directly for linear *systems* in the general form (1.3)-(1.4) has not been previously studied in the literature and remains open. We are only aware of the recent preprint [2] by Agresti and Hussein, whose interesting research on the properties of sectoriality and the boundedness of the H^∞-calculus for block operator-valued matrices of the form

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

leads naturally with maximal regularity on the vector-valued Lebesgue space $L_p(\mathbb{R}_+; X)$. However, in addition to the different vector-valued Lebesgue spaces considered by Agresti and Hussein, we observe that the results of [2] can only include the case $0 < \alpha = \beta < 2$, as for fractional evolution equations maximal L^p-regularity can be proven by using R-sectoriality with an appropriate angle condition, see either [45] or [40, Theorem 4.5.15]. In fact, finding a method to deal with the non-local case, i.e. $0 < \alpha, \beta < 2$ is one of the main problems we will overcome in this article.

Maximal regularity in Lebesgue spaces has been a very important issue in the last thirty years and there are numerous works dealing with the subject, see e.g. [3], [18], [19], [22], [25], and its references. However, criteria for maximal regularity of evolution equations involving a matrix $\mathcal{A}$ of unbounded operators are an exception.

One of the main difficulties is that most of the time, maximal regularity implies that $\mathcal{A}$ must be the generator of an analytic semigroup [19], which could be very hard to verify for matrices of unbounded operators [39, Section 13.6]. However, things change drastically in the case of the Lebesgue space of vector-valued periodic functions, $L^p(\mathbb{T}; X)$, where only closedness of $\mathcal{A}$ is necessary [5, Theorem 2.3]. In this article, we will take advantage of this last fact.

Previous research of maximal regularity for the first order abstract Cauchy problem on $L^p(\mathbb{T}; X)$, where X is a UMD space, appear first studied by Arendt and Bu [5]. These authors develop a technique based on operator-valued multipliers theorems that were later extended to the context of Hölder vector-valued spaces [6] by Arendt, Batty and Bu. After that, extensions to the case of vector-valued Lebesgue spaces on the d-dimensional torus $(0, 2\pi)^d$ where considered by Bu and Kim [10], to the scale of periodic vector-valued Besov spaces by Arendt and Bu [7], to the scale of periodic vector-valued Triebel spaces by Bu and Kim [11] and to vector-valued Hardy spaces by Bu and Le Merdy [12].

First studies on well posedness and maximal regularity on periodic Lebesgue spaces for more general abstract evolution equations than the abstract Cauchy problem, and using the technique of operator-valued theorems, or Fourier multipliers, appeared in studies of Keyantuo and Lizama [28–31], Poblete [37,38], Bu and Fang [13]. Lately, maximal regularity for abstract evolution equations with a fractional time-derivative have been studied by Bu [9] and Bu and Cai [14].

Note that the Laplacian operator with Dirichlet or Neumann boundary often appears in applications. An interesting analysis of maximal regularity for the Laplacian with mixed boundary conditions in domains carrying a cylindrical structure that includes periodic initial conditions was carry out by Nau [34], Nau and Saal [35] and Denk and Nau [20].

In this article, we succeed not only solving the maximal regularity problem for the system (1.3)-(1.4) but also to *characterize* such property uniquely in terms of the R-boundedness of the set $\{\mathrm{Diag}_{\alpha,\beta}(ik)\left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right)^{-1}\}_{k\in\mathbb{Z}}$. See Theorem 3.10 below. Here, we introduce the matrix

$$\mathrm{Diag}_{\alpha,\beta}(ik) := \begin{pmatrix} (ik)^\alpha & 0 \\ 0 & (ik)^\beta \end{pmatrix}, \qquad k \in \mathbb{Z}. \tag{1.5}$$

We also show criteria that ensure the R-boundedness assuming that the same property holds for the sets $\{(ik)^\alpha R((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ and $\{(ik)^\beta R((ik)^\beta, D)\}_{k\in\mathbb{Z}}$ as well as the condition:

$$R_p(\{BR((ik)^{\beta}, D)CR((ik)^{\alpha}, A)\}_{k\in\mathbb{Z}}) < \gamma < 1, \tag{1.6}$$

where R_p denotes the R-bound of the set. See Theorem 4.1. We show a second criterion if, instead of R-boundedness, we assume that X is a UMD space, and that $-A, -D \in \mathcal{RH}^{\infty}(X)$ with an appropriate angle depending on the values of α, β, plus the condition (1.6). See Theorem 4.2. With these criteria in hand, and denoting $H^{\delta,q}(\mathbb{T}; L^p(\Omega)) = \{u \in L^q(\mathbb{T}; L^p(\Omega)) : \partial_t^{\delta} u \in L^q(\mathbb{T}; L^p(\Omega))\}$, $0 < \delta < 2$, $1 < q, p < \infty$, we prove the following $L^q - L^p$ maximal regularity result for the non-local linear Keller-Segel system (1.3)-(1.4) with Dirichlet boundary conditions.

Theorem 1.1. *Let $1 < p, q < \infty$, $0 < \alpha \le \beta < 2$ and $\Omega \subset \mathbb{R}^N$ a bounded C^2 domain. Then, for all $f \in L^q(\mathbb{T}; L^p(\Omega))$ there exist unique functions $u \in L^q(\mathbb{T}; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap H^{\alpha,q}(\mathbb{T}; L^p(\Omega))$ and $v \in L^q(\mathbb{T}; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap H^{\beta,q}(\mathbb{T}; L^p(\Omega))$ satisfying the time-fractional system*

$$\begin{cases} \partial_t^{\alpha} u(x,t) = d_1 \Delta_p u(x,t) + f(x,t), \text{ for } (x,t) \in \Omega \times (0, 2\pi); \\ \partial_t^{\beta} v(x,t) = au(x,t) + d_2 \Delta_p v(x,t) - bv(x,t), \text{ for } (x,t) \in \Omega \times (0, 2\pi); \\ u(x,t) = v(x,t) = 0, \text{ for } (x,t) \in \partial\Omega \times (0, 2\pi); \\ u(x,0) = u(x,2\pi), \ v(x,0) = v(x,2\pi), \ x \in \Omega. \end{cases} \tag{1.7}$$

Moreover the following estimate

$$\left(\int_0^{2\pi} \left(\int_{\Omega} |\partial_t^{\alpha} u(x,t)|^p dx \right)^{q/p} + \left(\int_{\Omega} |\partial_t^{\beta} v(x,t)|^p dx \right)^{q/p} dt \right)^{1/q}$$

$$+ \left(\int_0^{2\pi} \left(\int_{\Omega} |d_1 \Delta_p u(x,t)|^p dx \right)^{q/p} + \left(\int_{\Omega} |au(x,t) + d_2 \Delta_p v(x,t) - bv(x,t)|^p dx \right)^{q/p} dt \right)^{1/q}$$

$$\le C \left(\int_0^{2\pi} \left(\int_{\Omega} |f(x,t)|^p dx \right)^{q/p} dt \right)^{1/q} \tag{1.8}$$

holds.

We note that our result also includes the local case $\alpha = \beta = 1$ which, as far as we know, have not been studied previously. For the proof, see subsection 5.1. We finish this article with an application to a system that considers a reactor model, see subsection 5.2 and a fractional in time Beris-Edwards type model, see subsection 5.3.

2. Preliminaries

Let X and Y be complex Banach spaces. $\mathcal{B}(X, Y)$ will denote the space of bounded operators from X to Y; and $\mathcal{B}(X)$ those from X to X. The resolvent of a linear operator A with domain $D(A)$ will be denoted by $\rho(A)$ and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda - A)^{-1}$.

Let us denote by $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ (the one dimensional torus). There exists an obvious identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.

The space of vector-valued functions defined on $[0, 2\pi]$ will be identified frequently to their periodic extensions to $\mathbb{R}$. In consequence, we consider the space $L^p(\mathbb{T}; X)$, $1 \le p \le \infty$ of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$). Given $f \in L^p(\mathbb{T}; X)$, $(1 \le p < \infty)$, the Riemann difference

$$\Delta_t^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj)$$

exists almost everywhere and

$$\|\Delta_t^\alpha f\|_{L^p(\mathbb{T};X)} \le \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \|f\|_{L^p(\mathbb{T};X)} = \mathcal{O}(1)$$

since $\binom{\alpha}{j} = \mathcal{O}(j^{-\alpha-1})$ as $j \to \infty$.

Let us recall the definition of Liouville-Grünwald fractional derivative [16, Definition 2.1].

Definition 2.1. Let X be a complex Banach space, $\alpha > 0$ and $1 \le p < \infty$. If for $f \in L^p(\mathbb{T}; X)$ there exists $g \in L^p(\mathbb{T}; X)$ such that

$$\|t^{-\alpha} \Delta_t^\alpha f - g\|_{L^p(\mathbb{T};X)} \to 0, \quad t \to 0^+,$$

then g is called the α^{th}-Liouville-Grünwald derivative of f in the mean of order p. In this case, we write $g = \partial_t^\alpha f$.

The Liouville-Grünwald fractional derivative was defined by Butzer and Westphal in the reference [16]. Several properties are given in [16, Sections 4 and 5] as well as their connection with the classical Weyl derivative [16, Section 6]. Also, the connection of the Liouville-Grünwald fractional derivative with the fractional powers $(-A)^\gamma$ of the operator $Af = -f'$ and the C_0-semigroup of translations $T(t)f(x) = f(x - t)$, appear in [17, Section 2.3, p. 28]. Note that here the fractional powers are defined by $(-A)^\gamma f = \lim_{t\to 0} \dfrac{[I - T(t)]^\gamma f}{t^\gamma}$, see [17, Formula (2.20) p. 21].

Next, we recall that the Fourier transform of $f \in L^p(\mathbb{T}; X)$ for $(1 \le p < \infty)$ is given by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)\, dt, \quad k \in \mathbb{Z}.$$

We introduce the following definition.

Definition 2.2. Let $1 \le p < \infty$. We say that a sequence $\{\mathcal{M}_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_1 \times X_2, Y_1 \times Y_2)$ is an (L^p, L^p)-multiplier, if for each $F \in L^p(\mathbb{T}; X_1 \times X_2)$ there exists $U \in L^p(\mathbb{T}; Y_1 \times Y_2)$ such that

$$\widehat{U}(k) = \mathcal{M}_k \widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}. \tag{2.1}$$

The following Lemma follows from the definition. We omit the easy proof.

Lemma 2.3. *We have that*

$$\mathcal{M}(ik) = \begin{pmatrix} M_{11}(ik) & M_{12}(ik) \\ M_{21}(ik) & M_{22}(ik) \end{pmatrix}, \quad k \in \mathbb{Z}$$

is an (L^p, L^p)-multiplier if and only if $\{M_{11}(ik)\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_1, Y_1)$, $\{M_{12}(ik)\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_2, Y_1)$, $\{M_{21}(ik)\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_1, Y_2)$ and $\{M_{22}(ik)\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_2, Y_2)$ are (L^p, L^p)-multipliers.

Let $(ik)^\gamma = |k|^\gamma e^{\frac{\gamma i\pi}{2}\operatorname{sgn} k}$. We shall denote by $H^{\gamma,p}(\mathbb{T}; X)$ the vector-valued function space

$$\left\{u \in L^p(\mathbb{T}; X) : \text{there exists } v \in L^p(\mathbb{T}; X) \text{ such that } \widehat{v}(k) = (ik)^\gamma \widehat{u}(k) \text{ for all } k \in \mathbb{Z}\right\}.$$

Note that $H^{\gamma,p}(\mathbb{T}; X)$ becomes a Banach space with the sum norm

$$\|u\|_{H^{\gamma,p}} := \|u\|_{L^p} + \|\partial_t^\gamma u\|_{L^p}.$$

For further properties, see [9]. Given $\alpha, \beta \ge 0$, let us define:

$$\operatorname{Diag}_{\alpha,\beta}(ik) := \begin{pmatrix} (ik)^\alpha & 0 \\ 0 & (ik)^\beta \end{pmatrix}, \qquad k \in \mathbb{Z}, \tag{2.2}$$

and

$$H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2) := \Big\{U \in L^p(\mathbb{T}; X_1 \times X_2) : \text{there exists } V \in L^p(\mathbb{T}; X_1 \times X_2) \\ \widehat{V}(k) = \operatorname{Diag}_{\alpha,\beta}(ik)\, \widehat{U}(k) \text{ for all } k \in \mathbb{Z}\Big\}.$$

It can be proved that

$$H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2) = H^{\alpha,p}(\mathbb{T}; X_1) \times H^{\beta,p}(\mathbb{T}; X_2).$$

Let us introduce the following definition of operator-valued Fourier multipliers.

Definition 2.4. Let $1 \le p < \infty$. We say that a sequence $\{\mathcal{M}_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_1 \times X_2)$ is an $(L^p, H^{\alpha,\beta,p})$-multiplier, if for each $F \in L^p(\mathbb{T}; X_1 \times X_2)$ there exists $U \in H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{U}(k) = \mathcal{M}_k \widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}. \tag{2.3}$$

Using analogous arguments to [32, Lemma 2.6], we obtain the following result.

Lemma 2.5. *Let* $1 \le p < \infty$, $\alpha, \beta \ge 0$ *and* $\{\mathcal{M}_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X_1 \times X_2)$. *The following assertions are equivalent:*

(i) $\{\mathcal{M}_k\}_{k\in\mathbb{Z}}$ *is an* $(L^p, H^{\alpha,\beta,p})$*-multiplier;*
(ii) $\left\{Diag_{\alpha,\beta}(ik)\mathcal{M}_k\right\}_{k\in\mathbb{Z}}$ *is an* (L^p, L^p)*-multiplier.*

Proof. Assume that $\{\mathcal{M}_k\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,\beta,p})$-multiplier and let $F \in L^p(\mathbb{T}; X_1 \times X_2)$. Then there exists $U \in H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{U}(k) = \mathcal{M}_k \widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

By definition of $H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$, there exists $V \in L^p(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\,\widehat{U}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Combining these two identities, we get

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\,\widehat{U}(k) = \left(\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}_k\right)\widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Hence $\left\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}_k\right\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

Conversely, assume that $\left\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}_k\right\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier and let $F \in L^p(\mathbb{T}; X_1 \times X_2)$. Then there exists $V \in L^p(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}_k\,\widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Let $\widehat{U}(k) := \mathrm{Diag}_{-\alpha,-\beta}(ik)\widehat{V}(k)$. Then $U \in H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ and

$$\widehat{U}(k) = \mathcal{M}_k \widehat{F}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

It follows that $\{\mathcal{M}_k\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,\beta,p})$-multiplier. □

Let us recall the concept of R-boundedness.

Definition 2.6. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded, if there is a constant $C > 0$ and $1 \le p < \infty$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$, the inequality

$$\left\|\sum_{j=1}^{N} r_j T_j x_j\right\|_{L^p(\Omega,Y)} \le C \left\|\sum_{j=1}^{N} r_j x_j\right\|_{L^p(\Omega,X)}$$

holds. The smallest such C is called R-bound of $\mathcal{T}$ and it will be denoted by $R_p(\mathcal{T})$.

Some useful properties of R-bounded families are the following (see [19]).

(a) If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded, then it is uniformly bounded, with

$$\sup\{\|T\| : T \in \mathcal{T}\} \leq R_p(\mathcal{T}). \tag{2.4}$$

(b) The definition of R-boundedness is independent of $p \in [1, \infty)$.
(c) If X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.
(d) Let X and Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X, Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T},\ S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.
(e) Let X, Y and Z be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded. Then the set $\mathcal{S}\mathcal{T} = \{ST :\ S \in \mathcal{S},\ T \in \mathcal{T}\}$ is R-bounded, and

$$R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S}) R_p(\mathcal{T}). \tag{2.5}$$

Remark 2.7. It is well-known that every singleton $\{T\}$ in $\mathcal{B}(X, Y)$ is R-bounded and $R_p(T) = \|T\|$. Then, if $\mathcal{S} \subset \mathcal{B}(X, Y)$ is R-bounded, we obtain from part (e) of Definition 2.6 that

$$T\mathcal{S} = \{TS :\ S \in \mathcal{S}\}$$

is R-bounded and $R_p(T\mathcal{S}) \leq \|T\| R_p(\mathcal{S})$.

We remind that the norm of

$$\mathcal{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

is given, equivalently, by

$$\|\mathcal{Q}\| := \max\{\|Q_{ij}\| : 1 \leq i, j \leq 2\} \quad \text{or} \quad \|\mathcal{Q}\| := \sum_{i,j=1}^{2} \|Q_{ij}\|.$$

Denote $\mathcal{Q}(k) = \begin{pmatrix} Q_{11}(k) & Q_{12}(k) \\ Q_{21}(k) & Q_{22}(k) \end{pmatrix}$, $k \in \mathbb{Z}$. We need the following result. The proof follows by an adequate use of the definition and therefore we omit it.

Theorem 2.8. *The set $\{\mathcal{Q}(k)\}_{k \in \mathbb{Z}}$ is R-bounded if and only if $\{Q_{ij}(k)\}_{k \in \mathbb{Z}}$ is R-bounded for $1 \leq i, j \leq 2$.*

Next, we remember the concept of UMD spaces. For $f \in L^2(\mathbb{R}, X)$ and $0 < \epsilon < R$, let

$$(H_{\epsilon,R} f)(t) := \frac{1}{\pi} \int\limits_{\epsilon \le |t-s| \le R} \frac{f(s)}{t-s} ds = (\psi_{\epsilon R} * f)(t), \qquad t \in \mathbb{R},$$

where

$$\psi_{\epsilon R}(t) := \begin{cases} \frac{1}{\pi t}, & \text{if } \epsilon \le |t| \le R, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\psi_{\epsilon R} \in L^1(\mathbb{R}, X)$, it can be verified that $H_{\epsilon,R} \in \mathcal{B}(L^2(\mathbb{R}, X), L^2(\mathbb{R}, X))$ (see for example [8, Proposition 1.3.2]).

The Banach space X is said to be a *UMD space* if

$$Hf := \lim_{\substack{\epsilon \downarrow 0 \\ R \to \infty}} H_{\epsilon,R} f$$

exists in $L^2(\mathbb{R}, X)$ for each $f \in L^2(\mathbb{R}, X)$.

The following result can be found in [26, Proposition 4.2.17 (4)]. The proof uses only the definition.

Proposition 2.9. *Assume that X_1 and X_2 are UMD spaces. Then $X_1 \times X_2$ is a UMD space.*

The next theorem was proved by Arendt and Bu in the reference [5, Theorem 1.3].

Theorem 2.10. *Let X and Y be UMD spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$. If the sets $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier for $1 < p < \infty$.*

We also need to recall some preliminaries on sectorial operators. Let $\Sigma_\phi \subset \mathbb{C}$ denote the open sector $\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$. We define the following spaces of functions as follows: $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$, and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}$$

which is endowed with the norm $||f||_\infty^\phi = \sup_{|\arg \lambda| < \phi} |f(\lambda)|$. We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \{f \in \mathcal{H}(\Sigma_\phi) : ||f||_{\alpha,\beta}^\phi < \infty\},$$

with $||f||_{\alpha,\beta}^\phi = \sup_{|\lambda|\le 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)|$.

Definition 2.11. Given a closed linear operator A in X, we say that A is sectorial if the following conditions hold:

(i) $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$;

(ii) $||t(t + A)^{-1}|| \le M$ for all $t > 0$ and some $M > 0$.

A is called R-sectorial if the set $\{t(t+A)^{-1}\}_{t>0}$ is R-bounded.

If A is sectorial then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and

$$\sup_{|\arg\lambda|<\phi} ||\lambda(\lambda+A)^{-1}|| < \infty.$$

We denote the *spectral angle* of a sectorial operator A by

$$\phi_A = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \quad \sup_{\lambda\in\Sigma_{\pi-\phi}} ||\lambda(\lambda+A)^{-1}|| < \infty\}.$$

Definition 2.12. Given a sectorial operator A, we say that it admits a bounded $\mathcal{H}^\infty$–calculus if there exist $\phi > \phi_A$ and a constant $K_\phi > 0$ such that

$$||f(A)|| \le K_\phi ||f||_\infty^\phi \text{ for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{2.6}$$

The class of sectorial operators A which admit a bounded $\mathcal{H}^\infty$–calculus is denoted by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty$-angle is defined by $\phi_A^\infty = \inf\{\phi > \phi_A : (2.6) \text{ holds }\}$. When $A \in \mathcal{H}^\infty(X)$ we say that A *admits an R-bounded $\mathcal{H}^\infty$–calculus* if the set

$$\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta), ||h||_\infty^\theta \le 1\}$$

is R-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The corresponding angle is defined in an obvious way and denoted by $\theta_A^{R_\infty}$.

Remark 2.13. If A is a sectorial operator on a Hilbert space, Lebesgue spaces $L^p(\Omega)$, $1<p<\infty$, Sobolev spaces $W^{s,p}(\Omega)$, $1<p<\infty$, $s\in\mathbb{R}$ or Besov spaces $B^s_{p,q}(\Omega)$, $1<p,q<\infty$, $s\in\mathbb{R}$ and A admits a bounded $\mathcal{H}^\infty$–calculus of angle β, then A admits a $\mathcal{RH}^\infty$–calculus on the same angle β on each of the above described spaces (see Kalton and Weis [27]). More generally, this property is true whenever X is a UMD space with the so called property (α) (see [27]).

There exist well known examples for general classes of closed linear operators with a bounded $\mathcal{H}^\infty$ calculus such as: normal sectorial operators in a Hilbert space; m-accretive operators in a Hilbert space; generators of bounded C_0-groups on L^p-spaces and negative generators of positive contraction semigroups on L^p-spaces.

Remark 2.14. Let $1<q<\infty$ and denote by Δ the Laplacian operator in $\mathbb{R}^n$. By [19, Theorem 7.2] we obtain that the $L^q(\mathbb{R}^n)$ realization Δ_q of the Laplacian operator admits an R-bounded $\mathcal{H}^\infty$–calculus for each $0<\theta_A^{\mathcal{R}_\infty}<\pi$. Moreover, by [19, Corollary 7.3] the same is true for $-\Delta_D$, the negative Dirichlet Laplacian in $\mathbb{R}^{n+1}_+$.

We also remind the following result [19, Proposition 4.10], that will be needed for our characterization, which shows under suitable conditions of uniform boundedness the R-boundedness of certain sets of operators.

Proposition 2.15. *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R_\infty}$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda\in\Lambda}$ is R-bounded.*

3. A characterization of maximal regularity

3.1. Block operator matrices

Along the first part of this section we consider the case of diagonally dominant block operator matrices, that is, the case where the operator $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ can be seen as a relatively bounded perturbation of its diagonal part with $D(\mathcal{A}) = D(A) \times D(D)$ though with possibly large relative bound. In other words, let X_1 and X_2 be Banach spaces and suppose that

(i) the operator $A : D(A) \subset X_1 \to X_1$ has non-empty resolvent set $\rho(A)$ in X_1,
(ii) the operator $D : D(D) \subset X_2 \to X_2$ has non-empty resolvent set $\rho(D)$ in X_2,
(iii) $C : D(C) \subset X_1 \to X_2$ is relatively A-bounded and $B : D(B) \subset X_2 \to X_1$ is relatively D-bounded, i.e., $D(A) \subset D(C)$, $D(D) \subset D(B)$ and there exist $c_A, c_D, L \geq 0$ such that

$$\|Cx\|_{X_2} \leq c_A\|Ax\|_{X_1} + L\|x\|_{X_1}, \quad \text{for all} \quad x \in D(A),$$
$$\|Bx\|_{X_1} \leq c_D\|Dx\|_{X_2} + L\|x\|_{X_2}, \quad \text{for all} \quad x \in D(D),$$

(iv) the operator $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with domain $D(A) \times D(D)$ is closed in $X_1 \times X_2$.

For such operators the properties of sectoriality, R-sectoriality and the boundedness of the H^∞-calculus are admissible. Most of this material is contained in the recent article [2]. For further use, we note the following:

Remark 3.1. As a consequence of (iii) we have that $BR(\mu, D) \in \mathcal{B}(X_2, X_1)$ for $\mu \in \rho(D)$ and $CR(\lambda, A) \in \mathcal{B}(X_1, X_2)$ for $\lambda \in \rho(A)$.

Remark 3.2. If A and D are closed operators and (iii) holds with $c := \max\{c_A, c_D\} < 1$, then (iv) follows from [44, Theorem 2.2.7 (i)].

Assuming (i)-(iv), let $\lambda \in \rho(A)$ and $\mu \in \rho(D)$ be given and consider the operators

$$\Delta_{X_1}(\lambda, \mu) = \lambda - A - BR(\mu, D)C \tag{3.1}$$

and

$$\Delta_{X_2}(\lambda, \mu) = \mu - D - CR(\lambda, A)B \tag{3.2}$$

with domains $D(A)$ in X_1 and $D(D)$ in X_2, respectively. Then, we obtain the next result which is a consequence of [33, Theorem 2.4].

Theorem 3.3. *Consider an operator matrix* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *for which (i)-(iv) holds. For* $\lambda \in \rho(A)$ *and* $\mu \in \rho(D)$ *the following assertions are equivalent:*

(a) $\mathcal{M} = \begin{pmatrix} \lambda - A & -B \\ -C & \mu - D \end{pmatrix}^{-1}$ *exists,*
(b) $\Delta_{X_1}^{-1}(\lambda, \mu)$ *exists,*
(c) $\Delta_{X_2}^{-1}(\lambda, \mu)$ *exists.*

Moreover, for $\lambda \in \rho(A)$, $\mu \in \rho(D)$ *and* (a), *we have*

$$\mathcal{M} = \begin{pmatrix} \Delta_{X_1}^{-1}(\lambda, \mu) & \Delta_{X_1}^{-1}(\lambda, \mu) B R(\mu, D) \\ R(\mu, D) C \Delta_{X_1}^{-1}(\lambda, \mu) & R(\mu, D)\left(I + C\Delta_{X_1}^{-1}(\lambda, \mu) B R(\mu, D)\right) \end{pmatrix} \tag{3.3}$$

or the analogous expression using $\Delta_{X_2}^{-1}(\lambda, \mu)$*:*

$$\mathcal{M} = \begin{pmatrix} R(\lambda, A)\left(I + B\Delta_{X_2}^{-1}(\lambda, \mu) C R(\lambda, A)\right) & R(\lambda, A) B \Delta_{X_2}^{-1} \\ \Delta_{X_2}^{-1}(\lambda, \mu) C R(\lambda, A) & \Delta_{X_2}^{-1}(\lambda, \mu) \end{pmatrix}. \tag{3.4}$$

Proof. For $\lambda \in \rho(A)$ and $\mu \in \rho(D)$ the matrix $\mathcal{M}$ can be written as

$$\begin{pmatrix} \lambda - A & -B \\ -C & \mu - D \end{pmatrix} = \begin{pmatrix} I & -BR(\mu, D) \\ -CR(\lambda, A) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \mu - D \end{pmatrix} =: \mathscr{B}_{\lambda,\mu} \circ \mathscr{A}_{\lambda,\mu}.$$

Note that $\mathscr{A}_{\lambda,\mu}$ is a bijection from $D(A) \times D(D)$ to $X_1 \times X_2$ while $\mathscr{B}_{\lambda,\mu}$ is bounded in $X_1 \times X_2$. Therefore $\mathcal{M}$ is invertible if and only if $\mathscr{B}_{\lambda,\mu}$ is invertible. Since $\mathscr{B}_{\lambda,\mu}$ has invertible diagonal entries, the assertions follow from [33, Lemma 2.1] and the fact that $\Delta_1 := I - BR(\mu, D)CR(\lambda, A)$ is invertible if and only if $\Delta_{X_1}(\lambda, \mu)$ is invertible. Moreover, $\Delta_{X_1}^{-1} = R(\lambda, A)\Delta_1^{-1}$. The matrix representation of $\mathcal{M}$ is a consequence of [33, Equation (2.1)] observing that

$$\begin{aligned} \mathscr{A}_{\lambda,\mu}^{-1} \circ \mathscr{B}_{\lambda,\mu}^{-1} &= \begin{pmatrix} R(\lambda, A) & 0 \\ 0 & R(\mu, D) \end{pmatrix} \begin{pmatrix} \Delta_1^{-1} & \Delta_1^{-1} BR(\mu, D) \\ CR(\lambda, A)\Delta_1^{-1} & \left(I + CR(\lambda, A)\Delta_1^{-1} BR(\mu, D)\right) \end{pmatrix} \\ &= \begin{pmatrix} \Delta_{X_1}^{-1}(\lambda, \mu) & \Delta_{X_1}^{-1}(\lambda, \mu) BR(\mu, D) \\ R(\mu, D)C\Delta_{X_1}^{-1}(\lambda, \mu) & R(\mu, D)\left(I + C\Delta_{X_1}^{-1}(\lambda, \mu) BR(\mu, D)\right) \end{pmatrix}. \quad \square \end{aligned}$$

Remark 3.4. Note that condition (b) is equivalent to $\Delta_1 = I - BR(\mu, D)CR(\lambda, A)$ be invertible and condition (c) is equivalent to $\Delta_2 = I - CR(\lambda, A)BR(\mu, D)$ be invertible. Moreover, $\Delta_{X_1}^{-1} = R(\lambda, A)\Delta_1^{-1}$ and $\Delta_{X_2}^{-1} = R(\mu, D)\Delta_2^{-1}$. In such case,

$$\mathcal{M} = \begin{pmatrix} R(\lambda, A)\Delta_1^{-1} & R(\lambda, A)\Delta_1^{-1} BR(\mu, D) \\ R(\mu, D)CR(\lambda, A)\Delta_1^{-1} & R(\mu, D)\left(I + CR(\lambda, A)\Delta_1^{-1} BR(\mu, D)\right) \end{pmatrix}, \tag{3.5}$$

or the analogous expression using $\Delta_{X_2}^{-1} = R(\mu, D)\Delta^{-1}$:

$$\mathcal{M} = \begin{pmatrix} R(\lambda, A)\left(I + BR(\mu, D)\Delta_2^{-1}CR(\lambda, A)\right) & R(\lambda, A)BR(\mu, D)\Delta_2^{-1} \\ R(\mu, D)\Delta_2^{-1}CR(\lambda, A) & R(\mu, D)\Delta_2^{-1} \end{pmatrix}. \tag{3.6}$$

Remark 3.5. From the representations of $\mathcal{M}$, we have the following relations

(a) $\Delta_1^{-1} = I + BR(\mu, D)\Delta_2^{-1}CR(\lambda, A)$;
(b) $\Delta_1^{-1}BR(\mu, D) = BR(\mu, D)\Delta_2^{-1}$;
(c) $\Delta_2^{-1}CR(\lambda, A) = CR(\lambda, A)\Delta_1^{-1}$;
(d) $\Delta_2^{-1} = I + CR(\lambda, A)\Delta_1^{-1}BR(\mu, D)$.

In what follows, we particularize the situation presented above by assuming the following condition.

Assumption (B). Let X_1 and X_2 be Banach spaces, $0 < \alpha, \beta < 2$ and assume that

(B1) $(ik)^\alpha \in \rho(A)$ for all $k \in \mathbb{Z}$, where $A : D(A) \subset X_1 \to X_1$,
(B2) $(ik)^\beta \in \rho(D)$ for all $k \in \mathbb{Z}$, where $D : D(D) \subset X_2 \to X_2$,
(B3) The assumptions (iii) and (iv) hold.

3.2. *A characterization*

Let us consider functions $u : [0, 2\pi) \to X_1$ and $v : [0, 2\pi) \to X_2$. We deal with systems of the form

$$\begin{cases} \partial_t^\alpha u(t) = Au(t) + Bv(t) + f(t), & a.e.\ 0 \le t \le 2\pi, \\ \\ \partial_t^\beta v(t) = Cu(t) + Dv(t) + g(t), & a.e.\ 0 \le t \le 2\pi \end{cases} \tag{3.7}$$

where $0 < \alpha, \beta < 2$, ∂_t^α and ∂_t^β are the fractional derivatives in the sense of Liouville-Grünwald, A, B, C and D are suitable closed linear operators (bounded or unbounded) and $f \in L^p(\mathbb{T}; X_1)$, $g \in L^p(\mathbb{T}; X_2)$ are given functions.

We introduce the following operator notation:

$$\partial_t^{\alpha,\beta} := \begin{pmatrix} \partial_t^\alpha & 0 \\ 0 & \partial_t^\beta \end{pmatrix}. \tag{3.8}$$

This means that

$$\partial_t^{\alpha,\beta}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_t^\alpha & 0 \\ 0 & \partial_t^\beta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_t^\alpha u \\ \partial_t^\beta v \end{pmatrix}.$$

Moreover, its Fourier transform is given by

$$\widehat{\partial_t^{\alpha,\beta}\begin{pmatrix} u \\ v \end{pmatrix}}(k) = \widehat{\begin{pmatrix} \partial_t^\alpha u \\ \partial_t^\beta v \end{pmatrix}}(k) = \begin{pmatrix} (ik)^\alpha \widehat{u}(k) \\ (ik)^\beta \widehat{v}(k) \end{pmatrix}, \qquad k \in \mathbb{Z},$$

whenever it exists.

Then, we can write (3.7) in the form

$$\partial_t^{\alpha,\beta}\begin{pmatrix} u \\ v \end{pmatrix}(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(t) + \begin{pmatrix} f \\ g \end{pmatrix}(t), \qquad \text{a.e. } t \in [0, 2\pi]. \tag{3.9}$$

Let us introduce the following notion of strong solution and well-posedness to (3.9) (or, equivalently, strong solution of (3.7)).

Definition 3.6. Let $1 \le p < \infty$. The function $U = \begin{pmatrix} u \\ v \end{pmatrix} \in X_1 \times X_2$ is called a strong L^p-solution of (3.9) if $U \in L^p(\mathbb{T}; D(A) \times D(D)) \cap H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ and equation (3.9) holds for almost all $t \in [0, 2\pi)$.

Definition 3.7. Let $1 \le p < \infty$. We say that the problem (3.9) is strongly L^p-well posed (or has maximal regularity) if for every $F = \begin{pmatrix} f \\ g \end{pmatrix} \in L^p(\mathbb{T}; X_1 \times X_2)$ there exists a unique strong solution U of (3.9).

If we take formally Fourier transform in both sides of (3.9), we obtain

$$\left(\begin{pmatrix} (ik)^\alpha & 0 \\ 0 & (ik)^\beta \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)\widehat{\begin{pmatrix} u \\ v \end{pmatrix}}(k) = \widehat{\begin{pmatrix} f \\ g \end{pmatrix}}(k), \quad k \in \mathbb{Z}. \tag{3.10}$$

Let

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \; U := \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} f \\ g \end{pmatrix}. \tag{3.11}$$

Then we can rewrite (3.10) as

$$\left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right)\widehat{U}(k) = \widehat{F}(k), \qquad k \in \mathbb{Z}. \tag{3.12}$$

Under Assumption **(B)**, we have that Theorem 3.3 and Remark 3.4 hold. Now, putting $\Delta \equiv \Delta(ik) := I - BR((ik)^\beta, D)CR((ik)^\alpha, A)$ and using Theorem 3.3 we have that

$$\begin{aligned}\mathcal{M} \equiv \mathcal{M}(ik) &:= \left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right)^{-1} = \left(\begin{pmatrix} (ik)^\alpha & 0 \\ 0 & (ik)^\beta \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} (ik)^\alpha - A & -B \\ -C & (ik)^\beta - D \end{pmatrix}^{-1},\end{aligned} \tag{3.13}$$

exists if and only if $\Delta(ik)$ is invertible. Moreover, taking into account (3.5) we obtain

$$\begin{aligned}&\mathcal{M}(ik) \\ &= \begin{pmatrix} R((ik)^\alpha, A)\Delta^{-1} & R((ik)^\alpha, A)\Delta^{-1}BR((ik)^\beta, D) \\ R((ik)^\beta, D)CR((ik)^\alpha, A)\Delta^{-1} & R((ik)^\beta, D)\left(I + CR((ik)^\alpha, A)\Delta^{-1}BR((ik)^\beta, D)\right) \end{pmatrix}\end{aligned}$$

$$=: \begin{pmatrix} M_{11}(ik) & M_{12}(ik) \\ M_{21}(ik) & M_{22}(ik) \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{3.14}$$

Thus, we have that system (3.10) has the solution

$$\begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} = \mathcal{M}(ik) \begin{pmatrix} \widehat{f}(k) \\ \widehat{g}(k) \end{pmatrix}, \quad k \in \mathbb{Z},$$

if and only if $\Delta(ik) = I - BR((ik)^\beta, D)CR((ik)^\alpha, A)$ is invertible.

With the above preliminaries, we are ready to prove the following result which establishes the equivalence between L^p-well-posedness and the notion of (L^p, L^p)-multiplier.

Theorem 3.8. *Let $1 \le p < \infty$, and suppose that $\mathcal{A}$ is closed. The following assertions are equivalent.*

(a) *The problem (3.9) is strongly L^p-well posed.*
(b) $\mathcal{M}(ik) = \big(Diag_{\alpha,\beta}(ik) - \mathcal{A}\big)^{-1}$ *exists in $\mathcal{B}(X_1 \times X_2)$ for each $k \in \mathbb{Z}$ and the sequence*

$$\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$$

is an (L^p, L^p)-multiplier.

Proof. $(a) \Longrightarrow (b)$. Let $k \in \mathbb{Z}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in X_1 \times X_2$. We will use the notation (2.2). Define

$$F(t) := \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} := \begin{pmatrix} e^{ikt} y_1 \\ e^{ikt} y_2 \end{pmatrix}.$$

Note that $\widehat{F}(k) = y$ and $\widehat{F}(j) = 0$ for $j \neq k$. Since the problem (3.9) is strongly L^p-well posed, then there exists a unique $U \in L^p(\mathbb{T}; D(A) \times D(D)) \cap H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ such that equation (3.9) holds for almost all $t \in [0, 2\pi)$. Taking Fourier transform on both sides of (3.9) and using that the operator $\mathcal{A}$ is closed, we get that $\widehat{U}(k) \in D(\mathcal{A}) = D(A) \times D(D)$ and

$$\big(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\big)\,\widehat{U}(k) = \widehat{F}(k) = y, \qquad k \in \mathbb{Z}.$$

It follows that the operator $\big(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\big)$ is surjective.

Let us see that $\big(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\big)$ is injective. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(\mathcal{A})$. If

$$\big(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\big)\, x = 0,$$

then $U(t) := \begin{pmatrix} e^{ikt} x_1 \\ e^{ikt} x_2 \end{pmatrix}$ defines a periodic solution of equation (3.9) with $F = 0$. Indeed,

$$\begin{aligned}
\partial_t^{\alpha,\beta} U(t) &= \partial_t^{\alpha,\beta} \begin{pmatrix} e^{ikt}x_1 \\ e^{ikt}x_2 \end{pmatrix} = \begin{pmatrix} \partial_t^{\alpha} e^{ikt}x_1 \\ \partial_t^{\beta} e^{ikt}x_2 \end{pmatrix} = \begin{pmatrix} (ik)^{\alpha} e^{ikt}x_1 \\ (ik)^{\beta} e^{ikt}x_2 \end{pmatrix} \\
&= \begin{pmatrix} (ik)^{\alpha} & 0 \\ 0 & (ik)^{\beta} \end{pmatrix} \begin{pmatrix} e^{ikt}x_1 \\ e^{ikt}x_2 \end{pmatrix} = e^{ikt} \begin{pmatrix} (ik)^{\alpha} & 0 \\ 0 & (ik)^{\beta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= e^{ikt} \mathrm{Diag}_{\alpha,\beta}(ik)x \\
&= e^{ikt} \mathcal{A}x = \mathcal{A}(e^{ikt}x) \\
&= \mathcal{A}U(t).
\end{aligned}$$

It follows from the uniqueness of solution that $x = 0$. Since $\mathcal{A}$ is closed, we conclude that $\mathcal{M}(ik)$ exists in $\mathcal{B}(X_1 \times X_2)$.

Now, we see that $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. Let $F \in L^p(\mathbb{T}; X_1 \times X_2)$. By hypothesis, there exists a unique $U \in L^p(\mathbb{T}; D(\mathcal{A})) \cap H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ such that $U(t) \in D(\mathcal{A})$ and equation (3.9) is valid. Taking Fourier transform, we deduce that $\widehat{U}(k) \in D(\mathcal{A})$ and

$$\left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right) \widehat{U}(k) = \widehat{F}(k), \qquad k \in \mathbb{Z}.$$

Therefore,

$$\mathrm{Diag}_{\alpha,\beta}(ik)\widehat{U}(k) = \mathrm{Diag}_{\alpha,\beta}(ik) \left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right)^{-1} \widehat{F}(k), \qquad k \in \mathbb{Z}.$$

Since $U \in H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ there exists $V \in L^p(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\widehat{U}(k).$$

Hence

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\widehat{F}(k), \qquad k \in \mathbb{Z}.$$

It follows that $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

$(b) \Longrightarrow (a)$. Let $F \in L^p(\mathbb{T}; X_1 \times X_2)$. Since $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, there exists $V \in L^p(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{V}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\,\widehat{F}(k), \qquad k \in \mathbb{Z}. \tag{3.15}$$

Lemma 2.5 implies that $\{\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,\beta,p})$-multiplier. From here we deduce that there exists $U \in H^{\alpha,\beta,p}(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{U}(k) = \mathcal{M}(ik)\widehat{F}(k). \tag{3.16}$$

In particular, by definition of $\mathcal{M}(ik)$, we have that $\widehat{U}(k) \in D(\mathcal{A})$ and $\mathrm{Diag}_{\alpha,\beta}(ik)\widehat{U}(k) = \widehat{V}(k)$. It follows that $\partial_t^{\alpha,\beta} U = V$.

From the identity

$$I = \left(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\right)\mathcal{M}(ik) = \mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik) - \mathcal{A}\mathcal{M}(ik), \quad k \in \mathbb{Z}, \tag{3.17}$$

we get that $\{\mathcal{A}\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. Then there exists $Z \in L^p(\mathbb{T}; X_1 \times X_2)$ such that

$$\widehat{Z}(k) = \mathcal{A}\mathcal{M}(ik)\widehat{F}(k) = \mathcal{A}\widehat{U}(k). \tag{3.18}$$

Now, by the closedness of $\mathcal{A}$ we get that $U \in L^p(\mathbb{T}; D(\mathcal{A}))$ and $\mathcal{A}U = Z$. Using (3.16) and (3.17), we obtain

$$\begin{aligned}\mathcal{A}\widehat{U}(k) &= \mathcal{A}\mathcal{M}(ik)\widehat{F}(k) = (\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik) - I)\widehat{F}(k)\\ &= \mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\widehat{F}(k) - \widehat{F}(k) = \mathrm{Diag}_{\alpha,\beta}(ik)\widehat{U}(k) - \widehat{F}(k).\end{aligned}$$

From here we deduce that U is a strong L^p-solution of (3.9). Let us prove the uniqueness. Let U_1 and U_2 strongly L^p-solutions of (3.9). By the linearity of ∂_t^γ, we have that $W := U_1 - U_2$ is a strong L^p-solution of $\partial_t^{\alpha,\beta} W = \mathcal{A}W$. Taking Fourier transform, we get $(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A})\widehat{W}(k) = 0$. Since $\mathcal{M}(ik) = (\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A})^{-1}$ exists in $\mathcal{B}(X_1 \times X_2)$, we have that $\widehat{W}(k) = 0$ for all $k \in \mathbb{Z}$. It follows that $W = 0$. Hence $U_1 = U_2$. □

Our next result establishes the equivalence between (L^p, L^p)-multiplier and R-boundedness.

Theorem 3.9. *Let X_1 and X_2 be UMD spaces. The following assertions are equivalent:*

(a) *$\mathcal{M}(ik)$ exists in $\mathcal{B}(X_1 \times X_2)$ for each $k \in \mathbb{Z}$ and the sequence $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier for $1 < p < \infty$.*
(b) *$\mathcal{M}(ik)$ exists in $\mathcal{B}(X_1 \times X_2)$ for each $k \in \mathbb{Z}$ and the sequence $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. $(a) \Longrightarrow (b)$ Follows from [5, Proposition 1.11].

$(b) \Longrightarrow (a)$ Since X_1 and X_2 are UMD spaces, by Proposition 2.9 we get that $X_1 \times X_2$ is a UMD space.

Next, let us define

$$d_k^{-\gamma} := \frac{1}{(ik)^\gamma},\ 0 < \gamma < 1 \quad \text{and} \quad D_k := \begin{pmatrix} d_k^{-\alpha} & 0 \\ 0 & d_k^{-\beta} \end{pmatrix}, \qquad k \neq 0.$$

We see that $\{k(d_{k+1}^{-\gamma} - d_k^{-\gamma}\}_{k\in\mathbb{Z}\setminus\{-1,0\}}$ is uniformly bounded. Indeed,

$$\sup_{k\in\mathbb{Z}\setminus\{-1,0\}} |k(d_{k+1}^{-\gamma} - d_k^{-\gamma})| = \sup_{k\in\mathbb{Z}\setminus\{-1,0\}} \frac{|k^\gamma - (k+1)^\gamma|}{|k|^{\gamma-1}|k+1|^\gamma}.$$

By the Mean Value Theorem, for each $k \in \mathbb{Z}$ there exists $k < c_k < k+1$ such that

$$|k^\gamma - (k+1)^\gamma| = \gamma |c_k|^{\gamma-1} < |c_k|^{\gamma-1}.$$

Hence

$$\sup_{k\in\mathbb{Z}\setminus\{-1,0\}} |k(d_{k+1}^{-\gamma} - d_k^{-\gamma})| < \sup_{k\in\mathbb{Z}\setminus\{-1,0\}} \frac{1}{|c_k|^{1-\gamma}|k|^{\gamma-1}|k+1|^\gamma} < 1.$$

This fact together with the property (c) in Definition 2.6 means that $\{k(d_{k+1}^{-\gamma} - d_k^{-\gamma})\}_{k\in\mathbb{Z}\setminus\{-1,0\}}$ is R-bounded. This in turn implies that $\{k(D_{k+1} - D_k)\}_{k\in\mathbb{Z}\setminus\{-1,0\}}$ is R-bounded because

$$\begin{aligned}\sup_{k\in\mathbb{Z}\setminus\{-1,0\}} \|k(D_{k+1} - D_k)\| &= \sup_{k\in\mathbb{Z}\setminus\{-1,0\}} \Big\{\max\{|k(d_{k+1}^{-\alpha} - d_k^{-\alpha})|, |k(d_{k+1}^{-\beta} - d_k^{-\beta})|\}\Big\} \\ &= \max\left\{\sup_{k\in\mathbb{Z}\setminus\{-1,0\}} |k(d_{k+1}^{-\alpha} - d_k^{-\alpha})|, \sup_{k\in\mathbb{Z}\setminus\{-1,0\}} |k(d_{k+1}^{-\beta} - d_k^{-\beta})|\right\} \\ &< 1.\end{aligned}$$

Next, let $H_k := D_k\mathcal{M}(ik)$. Then

$$H_k = (I - D_k\mathcal{A})^{-1}, \qquad k \neq 0.$$

Now, note that

$$\begin{aligned}k(H_{k+1} - H_k) &= k[(I - D_{k+1}\mathcal{A})^{-1} - (I - D_k\mathcal{A})^{-1}] \\ &= k\mathcal{M}(i(k+1))[(I - D_k\mathcal{A}) - (I - D_{k+1}\mathcal{A})]\mathcal{M}(ik) \\ &= k\mathcal{M}(i(k+1))[(D_{k+1} - D_k)\mathcal{A}]\mathcal{M}(ik).\end{aligned}$$

Since $\mathcal{A}\mathcal{M}(ik) = I - D_k\mathcal{M}(ik)$, we have that

$$\begin{aligned}k(H_{k+1} - H_k) &= \mathcal{M}(i(k+1))k(D_{k+1} - D_k)\mathcal{M}(ik) \\ &\quad - \mathcal{M}(i(k+1))k(D_{k+1} - D_k)\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik).\end{aligned}$$

By hypothesis, $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. Furthermore, since the sets $\{\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ and $\{\mathcal{M}(i(k+1))\}_{k\in\mathbb{Z}}$ are bounded and by the discussion above $\{k(D_{k+1} - D_k)\}_{k\in\mathbb{Z}}$ is R-bounded, it follows that $\{k(H_{k+1} - H_k)\}_{k\in\mathbb{Z}}$ is R-bounded. The conclusion follows from Theorem 2.10 (Marcinkiewicz multiplier theorem, see [5, Theorem 1.3]). □

Using the previous results, we obtain the following equivalence between L^p-well-posedness and R-boundedness that configures the main result of this section.

Theorem 3.10. *Let X_1 and X_2 be UMD spaces, $1 < p < \infty$, $0 < \alpha, \beta < 2$. Assume that A and C are closed linear operators defined on X_1, B and D are closed linear operators defined on X_2 and the operator matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with domain $D(A) \times D(D)$ is closed in $X_1 \times X_2$. The following assertions are equivalent:*

(a) *For every* $f \in L^p(\mathbb{T}; X_1)$, $g \in L^p(\mathbb{T}; X_2)$ *there exist* $u \in L^p(\mathbb{T}; D(A)) \cap H^{\alpha,p}(\mathbb{T}; X_1)$ *and* $v \in L^p(\mathbb{T}; D(D)) \cap H^{\beta,p}(\mathbb{T}; X_2)$ *such that*

$$\begin{cases} \partial_t^{\alpha} u(t) = Au(t) + Bv(t) + f(t), & a.e.\ 0 \le t \le 2\pi, \\ \partial_t^{\beta} v(t) = Cu(t) + Dv(t) + g(t), & a.e.\ 0 \le t \le 2\pi \end{cases} \tag{3.19}$$

hold.

(b) $\mathcal{M}(ik) := \big(Diag_{\alpha,\beta}(ik) - \mathcal{A}\big)^{-1}$ *exists in* $\mathcal{B}(X_1 \times X_2)$ *for each* $k \in \mathbb{Z}$ *and the sequence* $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Moreover, under the equivalent conditions above, there exists a constant $C > 0$ *such that the following estimate*

$$\left(\int_0^{2\pi} \|\partial_t^{\alpha} u(t)\|_{X_1}^p + \|\partial_t^{\beta} v(t)\|_{X_2}^p dt\right)^{1/p} + \left(\int_0^{2\pi} \|Au(t) + Bv(t)\|_{X_1}^p + \|Cu(t) + Dv(t)\|_{X_2}^p dt\right)^{1/p}$$
$$\le C \left(\int_0^{2\pi} \|f(t)\|_{X_1}^p + \|g(t)\|_{X_2}^p dt\right)^{1/p} \tag{3.20}$$

holds.

Recall that in the context of a Hilbert space, R-boundedness is equivalent to uniform boundedness.

Under the assumption **(B)** we known that $\mathcal{A}$ is closed and, by (3.13), that $\mathcal{M}(ik)$ exists for each $k \in \mathbb{Z}$ if and only if $\Delta(ik) = I - BR((ik)^{\beta}, D)CR((ik)^{\alpha}, A) : X_1 \to X_1$ is invertible for all $k \in \mathbb{Z}$.

In view of Theorem 3.10, the following questions arise:

(i) Under what conditions the operators $\Delta(ik) = I - BR((ik)^{\beta}, D)CR((ik)^{\alpha}, A)$ are invertible for all $k \in \mathbb{Z}$?,

(ii) Under what conditions the set $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded?

The answer to these questions will be the object of the next section.

4. Sufficient conditions for L^p well-posedness

We remind that $\mathcal{M}(ik) := \big(\mathrm{Diag}_{\alpha,\beta}(ik) - \mathcal{A}\big)^{-1}$ exists in $\mathcal{B}(X_1 \times X_2)$ for each $k \in \mathbb{Z}$ if and only if $\Delta^{-1}(ik) := \big(I - BR((ik)^{\beta}, D)CR((ik)^{\alpha}, A)\big)^{-1}$ exists in $\mathcal{B}(X_1)$ for each $k \in \mathbb{Z}$, and in such case the representation (3.14) for $\mathcal{M}(ik)$ holds.

In what follows we will require that the off diagonal of the operator $\mathcal{A}$ has a small coupling. We observe that an analogous condition was assumed in the recent article [2, Proposition 4.8].

The following results answer the questions posed at the end of the previous section.

Theorem 4.1. *Suppose that Assumption **(B)** holds. Assume the following:*

(i) *The sets* $\{(ik)^\alpha R((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ *and* $\{(ik)^\beta R((ik)^\beta, D)\}_{k\in\mathbb{Z}}$ *are* R*-bounded;*
(ii) $R_p(\{BR((ik)^\beta, D)CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}) < \gamma < 1$.

Then

(a) $\Delta^{-1}(ik)$ *exists in* $\mathcal{B}(X_1)$ *for each* $k \in \mathbb{Z}$.
(b) *The set* $\{\Delta^{-1}(ik)\}_{k\in\mathbb{Z}}$ *is* R*-bounded.*
(c) $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ *is* R*-bounded.*

Proof. We first observe that by assumption (B) (see also Remark 3.1) we get $CR((ik)^\alpha, A) \in \mathcal{B}(X_1, X_2)$ and $BR((ik)^\alpha, D) \in \mathcal{B}(X_2, X_1)$ for all $k \in \mathbb{Z}$. In particular, $0 \in \rho(A)\cap\rho(D)$, $CA^{-1} \in \mathcal{B}(X_1, X_2)$ and $BD^{-1} \in \mathcal{B}(X_2, X_1)$. This in turn implies that $BR((ik)^\beta, D)CR((ik)^\alpha, A) \in \mathcal{B}(X_1)$ for all $k \in \mathbb{Z}$. Hence, the identity

$$\begin{aligned} CR((ik)^\alpha, A) &= CA^{-1}AR((ik)^\alpha, A) = CA^{-1}[(ik)^\alpha R((ik)^\alpha, A) - I] \\ &= CA^{-1}(ik)^\alpha R((ik)^\alpha, A) - CA^{-1}, \end{aligned}$$

the hypothesis (i) and Remark 2.7 show that the set $\{CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ is R-bounded too. Analogously, the hypothesis (i) shows that the set $\{BR((ik)^\alpha, D)\}_{k\in\mathbb{Z}}$ is R-bounded. Consequently the set $\{BR((ik)^\beta, D)CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ is R-bounded and therefore (ii) is well defined.

Let us prove (a). By hypothesis (ii) and using (2.4) we obtain

$$\|BR((ik)^\beta, D)CR((ik)^\alpha, A)\| \le \gamma < 1, \quad \forall k \in \mathbb{Z}. \tag{4.1}$$

It follows that $\Delta^{-1}(ik) = \big(I - BR((ik)^\beta, D)CR((ik)^\alpha, A)\big)^{-1}$ exists and

$$\Delta^{-1}(ik) = \sum_{n=0}^{\infty} \big(BR((ik)^\beta, D)CR((ik)^\alpha, A)\big)^n \tag{4.2}$$

converges in $\mathcal{B}(X_1)$. This proves the first item.

Next, we show (b). From (4.1) and (4.2), we obtain

$$\left\|\Delta^{-1}(ik)\right\| \le \sum_{n=0}^{\infty} \left\|BR((ik)^\beta, D)CR((ik)^\alpha, A)\right\|^n < \sum_{n=0}^{\infty} \gamma^n = \frac{1}{1-\gamma}.$$

Since the set $\{BR((ik)^\beta, D)CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ is R-bounded, we can use repeatedly (2.5) and obtain

$$R_p\left(\big(BR((ik)^\beta, D)CR((ik)^\alpha, A)\big)^n\right) \le \gamma^n.$$

Since R-boundedness preserves convergence in $\mathcal{B}(X_1)$, we get

$$R_p(\Delta^{-1}(ik)) \le \sum_{n=0}^{\infty} R_p\left(\big(BR((ik)^\beta, D)CR((ik)^\alpha, A)\big)^n\right) \le \frac{1}{1-\gamma}.$$

This shows that the set $\{\Delta^{-1}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded.

Next, let us show (c). By hypothesis (i) the sets $\{(ik)^\alpha R((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ and $\left\{(ik)^\beta R((ik)^\beta, D)\right\}_{k\in\mathbb{Z}}$ are R-bounded as well as the sets $\{CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$ and $\left\{BR((ik)^\beta, D)\right\}_{k\in\mathbb{Z}}$. Using that and part (b) we obtain that the sets

$$\begin{aligned}
\{Q_{11}(k)\}_{k\in\mathbb{Z}} &:= \{(ik)^\alpha R((ik)^\alpha, A)\Delta^{-1}(ik)\}_{k\in\mathbb{Z}},\\
\{Q_{12}(k)\}_{k\in\mathbb{Z}} &:= \{(ik)^\alpha R((ik)^\alpha, A)\Delta^{-1}(ik)BR((ik)^\beta, D)\}_{k\in\mathbb{Z}},\\
\{Q_{21}(k)\}_{k\in\mathbb{Z}} &:= \{(ik)^\beta R((ik)^\beta, D)CR((ik)^\alpha, A)\Delta^{-1}(ik)\}_{k\in\mathbb{Z}},\\
\{Q_{22}(k)\}_{k\in\mathbb{Z}} &= \{(ik)^\beta R((ik)^\beta, D)\Big(I + CR((ik)^\alpha, A)\Delta^{-1}(ik)BR((ik)^\beta, D)\Big)\}_{k\in\mathbb{Z}}
\end{aligned}$$

are R-bounded since each of them are products and sums of R-bounded sets. Finally, the claim follows from the matrix representation of $\{\mathrm{Diag}_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ using (3.14), and Theorem 2.8. This finishes the proof. □

Theorem 4.2. *Let X_1 and X_2 be UMD spaces, $1 < p < \infty$ and assume that* **(B)** *holds. Suppose that $-A \in \mathcal{RH}^\infty(X_1)$ and $-D \in \mathcal{RH}^\infty(X_2)$ with either angle $\theta_D^{R_\infty} \in (\theta_A^{R_\infty}, \pi(1-\beta/2))$ if $0 < \alpha \le \beta < 2$ or $\theta_A^{R_\infty} \in (\theta_D^{R_\infty}, \pi(1-\alpha/2))$ if $0 < \beta \le \alpha < 2$. If*

$$R_p(\{BR((ik)^\beta, D)CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}) < \gamma < 1, \tag{4.3}$$

then $\mathcal{M}(ik)$ exists in $\mathcal{B}(X_1 \times X_2)$ for each $k \in \mathbb{Z}$ and the set $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. Without loss of generality, suppose that $0 < \alpha \le \beta < 2$. Since $0 < \theta_A^{R_\infty} < \theta_D^{R_\infty} < \pi(1 - \beta/2)$ there exists $\theta > 0$ such that $\theta_D^{R_\infty} < \theta < \pi(1-\beta/2)$. For each $z \in \Sigma_\theta$ and $\delta \in \{\alpha, \beta\}$ we define

$$S_\delta(ik, z) := (ik)^\delta((ik)^\delta + z)^{-1} = \left(1 + \frac{z}{(ik)^\delta}\right)^{-1}, \quad k \in \mathbb{Z},\ k \neq 0.$$

Note that the fraction $\dfrac{z}{(ik)^\delta}$ belongs to the sector $\Sigma_{\theta+\delta\pi/2}$ where $\theta + \delta\pi/2 \le \theta + \beta\pi/2 < \pi$. Hence, for any value of δ, the distance from $\Sigma_{\theta+\delta\pi/2}$ to -1 is always positive. Then, there exists a constant $C_0 > 0$ independent of k and $z \in \Sigma_\theta$ such that $|S_\delta(ik, z)| < C_0$. Since by hypothesis $-A \in \mathcal{RH}^\infty(X_1)$, and $-D \in \mathcal{RH}^\infty(X_2)$ and by assumption **(B)** we have $0 \in \rho(A) \cap \rho(D)$ it follows from Proposition 2.15 that the sets $\{(ik)^\alpha((ik)^\alpha - A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{(ik)^\beta((ik)^\beta - D)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded. The conclusion follows from Theorem 4.1. □

We can apply the previous theorems, to state the following main result for this section.

Theorem 4.3. *Let X_1 and X_2 be UMD spaces and assume the hypotheses of Theorem 4.1 (resp. Theorem 4.2). Then, for all $f \in L^p(\mathbb{T}; X_1)$ and $g \in L^p(\mathbb{T}; X_2)$ there exist unique functions $u \in L^p(\mathbb{T}; D(A)) \cap H^{\alpha,p}(\mathbb{T}; X_1)$ and $v \in L^p(\mathbb{T}; D(D)) \cap H^{\beta,p}(\mathbb{T}; X_2)$ such that*

$$\begin{cases} \partial_t^\alpha u(t) = Au(t) + Bv(t) + f(t), & a.e.\ 0 \le t \le 2\pi,\\ \partial_t^\beta v(t) = Cu(t) + Dv(t) + g(t), & a.e.\ 0 \le t \le 2\pi. \end{cases} \tag{4.4}$$

Proof. By assumption **(B)**, the operator-valued matrix $\mathcal{A}$ is closed. By Theorem 4.1 (resp. Theorem 4.2) we have that $\mathcal{M}(ik)$ exists in $\mathcal{B}(X_1 \times X_2)$ and $\{Diag_{\alpha,\beta}(ik)\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. Theorem 3.9 implies that $\{\mathcal{M}(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. Then, from Theorem 3.8 we get that the problem (3.9) is strongly L^p-well posed. The conclusion follows from Definition 3.6. □

5. Applications

5.1. Keller-Segel model

Keller-Segel equations arise in the mathematical modeling of chemotaxis, see e.g. [15,23] for surveys and further literature. We consider the classical minimal Keller-Segel system given by

$$\partial_t u = \nabla \cdot (d_1 \nabla u) - \nabla \cdot (\chi u \nabla v) + f \tag{5.1}$$

$$\partial_t v = d_2 \Delta v + au - bv. \tag{5.2}$$

Assuming that f does not depend on either u or v, the linearized part of the Keller-Segel system reads

$$\begin{cases} \partial_t u = d_1 \Delta u + f \\ \partial_t v = d_2 \Delta v + au - bv \end{cases} \tag{5.3}$$

which admits the form of the system (4.4) with $\alpha = \beta = 1$, $A = d_1\Delta$, $B \equiv 0$, $C = aI$ and $D = d_2\Delta - bI$.

Next, we consider the time-fractional Keller-Segel system in a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$

$$\begin{cases} \partial_t^\alpha u(x,t) = d_1 \Delta_p u(x,t) + f(x,t), \text{ for } (x,t) \in \Omega \times (0, 2\pi); \\ \partial_t^\beta v(x,t) = au(x,t) + d_2\Delta_p v(x,t) - bv(x,t), \text{ for } (x,t) \in \Omega \times (0,2\pi); \\ u(x,t) = v(x,t) = 0, \text{ for } (x,t) \in \partial\Omega \times (0, 2\pi); \\ u(x,0) = u(x,2\pi), \ v(x,0) = v(x,2\pi), \ x \in \Omega, \end{cases} \tag{5.4}$$

where Δ_p denotes the Dirichlet Laplacian operator on $L^p(\Omega)$, $1 < p < \infty$, with domain $D(\Delta_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

We prove Theorem 1.1 as follows:

Proof. We apply Theorem 4.3 with $A = d_1\Delta_p$, $B \equiv 0$, $C = aI$ and $D = d_2\Delta_p - bI$. Now, we check the hypothesis of Theorem 4.1.

It is well-known that $L^p(\Omega)$ is a UMD space.

We observe that applying [34, Proposition 5.1 (a)-(i)] we have that $-\Delta_p \in \mathcal{RH}^\infty(L^p(\Omega))$ and by [34, Theorem 4.2] that $0 \in \rho(\Delta_p)$. Moreover, since Ω is a bounded C^2 domain, then $\theta_{-\Delta_p}^{\mathcal{R}_\infty} = 0$, see [34, Remark 4.7].

We now verify assumption **(B)**. Since the spectrum of the Dirichlet Laplacian Δ_p is independent of p (see [4]), we have $\sigma(\Delta_p) \subset (-\infty, 0)$. This fact, together with the condition $\alpha < 2$ implies that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(\Delta_p)$ for all $k \in \mathbb{Z}$, proving (B1).

Analogously, since $b/d_2 > 0$ and $\beta < 2$ we obtain that $(ik)^\beta \in \rho(d_2\Delta_p - bI)$ for all $k \in \mathbb{Z}$. This proves (B2).

Since $0 \in \rho(\Delta_p)$ and $C = aI$ we have $CA^{-1} \in \mathcal{B}(L^p(\Omega))$. And, since $B \equiv 0$, we obtain (iii).

The closedness of $\mathcal{A}$ follows easily by definition, from the fact that $B \equiv 0$ and Δ_p is closed. This proves (iv) and consequently (B3) holds. This finishes the proof of Assumption **(B)**.

We now check part (i) of Theorem 4.1.

First, note that the thanks to $-\Delta_p \in \mathcal{RH}^\infty(L^q(\Omega))$ the proof of Theorem 4.2 shows that the set $\{(ik)^\alpha((ik)^\alpha + \Delta_p)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

In order to prove that the set $\{(ik)^\beta((ik)^\beta - (d_2\Delta_p - bI))^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded we use Proposition 2.15 as follows. Fix $\theta > 0$ such that $0 < \theta < \pi - \beta\pi/2$ and define

$$F(k,z) := (ik)^\beta((ik)^\beta + b + d_2 z)^{-1} = \frac{(ik)^\beta}{(ik)^\beta + b}\left(1 + \frac{d_2 z}{(ik)^\beta + b}\right)^{-1}, \; k \in \mathbb{Z}, \; z \in \Sigma_\theta. \quad (5.5)$$

Since $b > 0$ and $\beta < 2$ the number $(ik)^\beta + b$ has angle less than $\frac{\beta\pi}{2}$. Therefore, the fraction $\dfrac{d_2 z}{(ik)^\beta + b}$ belongs to the sector $\Sigma_{\theta+\beta\pi/2}$ where $\theta + \beta\pi/2 < \pi$. Hence, for any value of β, the distance from $\Sigma_{\theta+\beta\pi/2}$ to -1 is always positive. It implies that there exists a constant $M > 0$ independent of k and $z \in \Sigma_\theta$ such that $|F(k,z)| \le M$. Since $-\Delta_p \in \mathcal{RH}^\infty(L^q(\Omega))$ it follows from Proposition 2.15 that the set $\{F(k, -\Delta_p)\}_{k\in\mathbb{Z}}$ is R-bounded, proving the claim.

Finally, again since $B \equiv 0$, it immediately follows that part (ii) of Theorem 4.1 holds (take $\gamma = 1/2$ for example). This finishes the proof of the theorem. □

Remark 5.4. We note that the fractional derivatives for the Keller-Segel system have been previously studied by other authors. For example, Salem [42] studied the Keller-Segel system of parabolic-elliptic type and Escudero [23] for (5.1)-(5.2) with the fractional Laplacian, while Acevedo, Cuevas and Henriquez [1] studied the existence and asymptotic behavior of the time-fractional model with the Caputo fractional derivative in the case $d_1 = 1$ and $0 < \alpha = \beta < 1$.

5.2. *Reactor model*

We consider the following system

$$\begin{cases} \partial_t u = d\Delta u + bu - cuv + f \\ \partial_t v = au - hv + g. \end{cases} \quad (5.6)$$

This model is a reactor model that has been proposed by Rumble and Kastenberg [41], [24, Section 4.3]. In (5.6), u represents the neutron flux and v the fuel temperature. The constants are as follows: d represents the average diffusion coefficient, $b = \nu\Sigma_f - \Sigma_a$ where ν is the average number of neutrons per fission, Σ_f is the fission cross section and Σ_a is the absorption cross section; c is the fuel feedback coefficient, $a = \epsilon\Sigma_f$ where ϵ is the fission to heat energy conversion factor, and h is the average cool enthalpy. The functions f and g represent nonlinear terms, in general.

The linear part of (5.6) is

$$\begin{cases} \partial_t u = d\Delta u + bu + f \\ \partial_t v = au - hv + g \end{cases} \tag{5.7}$$

which admits the form of the system (4.4) with $\alpha = \beta = 1$, $A = d\Delta + bI$, $B \equiv 0$, $C = aI$ and $D = -hI$.

Assuming the condition $\nu\Sigma_f < \Sigma_a$ we arrive at the following result:

Theorem 5.5. *Let $1 < p, q < \infty$, $0 < \alpha \leq \beta < 2$ and $\Omega \subset \mathbb{R}^N$ a bounded C^2 domain. Suppose $b < 0$. Then, for all $f, g \in L^q(\mathbb{T}; L^p(\Omega))$ there exist unique functions $u \in L^q(\mathbb{T}; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap H^{\alpha,q}(\mathbb{T}; L^p(\Omega))$ and $v \in H^{\beta,q}(\mathbb{T}; L^p(\Omega))$ satisfying the time-fractional system*

$$\begin{cases} \partial_t^\alpha u(x,t) = d\Delta_p u(x,t) + bu(x,t) + f(x,t), \ \text{for } (x,t) \in \Omega \times (0,2\pi); \\ \partial_t^\beta v(x,t) = au(x,t) - hv(x,t) + g(x,t), \ \text{for } (x,t) \in \Omega \times (0,2\pi); \\ u(x,t) = v(x,t) = 0, \ \text{for } (x,t) \in \partial\Omega \times (0,2\pi); \\ u(x,0) = u(x,2\pi), \ v(x,0) = v(x,2\pi), \ x \in \Omega. \end{cases} \tag{5.8}$$

Moreover the following estimate

$$\begin{aligned} &\left(\int_0^{2\pi} \left(\int_\Omega |\partial_t^\alpha u(x,t)|^p dx \right)^{q/p} + \left(\int_\Omega |\partial_t^\beta v(x,t)|^p dx \right)^{q/p} dt \right)^{1/q} \\ &+ \left(\int_0^{2\pi} \left(\int_\Omega |d\Delta_p u(x,t) + bu(x,t)|^p dx \right)^{q/p} + \left(\int_\Omega |au(x,t) - hv(x,t)|^p dx \right)^{q/p} dt \right)^{1/q} \\ &\leq C \left(\int_0^{2\pi} \left(\int_\Omega |f(x,t)|^p dx \right)^{q/p} + \left(\int_\Omega |g(x,t)|^p dx \right)^{q/p} dt \right)^{1/q} \end{aligned} \tag{5.9}$$

holds.

Proof. The proof is very similar to the proof of Theorem 1.1, taking into account that $b < 0$ in the analog of (5.6) for the operator $d\Delta_p + bI$, to obtain that the set $\{(ik)^\beta((ik)^\beta - (d\Delta_p + bI))^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. □

Remark 5.6. The time-fractional reactor model has been previously studied and physically justified by Gal and Warma [24, Section 4.3]. These authors also consider fractional powers of the Laplacian, namely $(-\Delta)^s$, $s \in (0,1)$. We note that we can also include this situation in our previous results, with a slight modification in the proof that consists of replacing the variable z in (5.5) by the power z^s. We leave the details to the interested reader.

5.3. Fractional Beris-Edwards type model for liquid crystals

Let us consider the following fractional in time Beris-Edwards type model

$$\begin{cases} \partial_t^\alpha u(x,t) = \Delta_p^w u(x,t) + \epsilon \,\mathrm{div}\Delta_p^s v(x,t) + f(x,t), \text{ for } (x,t) \in \mathbb{R}^n \times (0,2\pi); \\ \partial_t^\beta v(x,t) = -\rho \nabla u(x,t) + \Delta_p^s v(x,t) + g(x,t), \text{ for } (x,t) \in \mathbb{R}^n \times (0,2\pi); \\ u(x,t) = v(x,t) = 0, \text{ for } (x,t) \in \mathbb{R}^n \times (0,2\pi); \\ u(x,0) = u(x,2\pi),\ v(x,0) = v(x,2\pi),\ x \in \mathbb{R}^n, \end{cases} \tag{5.10}$$

where $0 < \alpha, \beta < 2,\ 0 < \rho < 1,\ \epsilon > 0,\ \Delta_p^w = -\Delta$ denotes the realization of the Laplace operator on $W^{-1,p}(\mathbb{R}^n)$ and $\Delta_p^s = -\Delta$ denotes the realization of the Laplace operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with domains $D(\Delta_p^w) = W^{1,p}(\mathbb{R}^n)$ and $D(\Delta_p^s) = W^{2,p}(\mathbb{R}^n;\mathbb{R}^n)$, respectively. In case $\alpha = \beta = 1$, the corresponding operator-valued matrix defined on $W^{-1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ was considered in [2, Section 8.6]. It has a similar structure to the studied in [2, Section 8.5] which arises in the analysis of Beris-Edwards model for liquid crystals.

In what follows, we will verify the hypotheses of Theorem 4.2. Let $X_1 := W^{-1,p}(\mathbb{R}^n)$ and $X_2 := L^p(\mathbb{R}^n)$. It is well-known that X_1 and X_2 are UMD spaces.

Let us verify assumption **(B)**. Since the spectrum of Δ_p^w and Δ_p^s are independent of p (see [4]), we have $\sigma(\Delta_p^w)$, $\sigma(\Delta_p^s) \subset (-\infty, 0)$. This fact, together with the condition $0 < \alpha, \beta < 2$ implies that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(\Delta_p^w)$ and $\{(ik)^\beta\}_{k\in\mathbb{Z}} \subset \rho(\Delta_p^s)$ for all $k \in \mathbb{Z}$, proving (B1) and (B2).

We next prove the first part of (B3), i.e. that $\rho\nabla u$ is relatively Δ_p^w-bounded and $\epsilon\,\mathrm{div}\Delta_p^s v$ is relatively Δ_p^s-bounded. Indeed, since $(1-\Delta)^{s/2} : W^{\beta+s,p}(\mathbb{R}^n) \to W^{\beta,p}(\mathbb{R}^n)$ is an isomorphism (see the proof of Proposition 8.16 in [2]) for all $s, \beta \in \mathbb{R}$, in particular, we have $1-\Delta : W^{1,p}(\mathbb{R}^n) \to W^{-1,p}(\mathbb{R}^n)$ is an isomorphism and

$$\|(1-\Delta)u\|_{W^{-1,p}(\mathbb{R}^n)} = \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in W^{1,p}(\mathbb{R}^n)$. Therefore

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \le \|u\|_{W^{1,p}(\mathbb{R}^n)} = \|(1-\Delta)u\|_{W^{1,p}(\mathbb{R}^n)} \le \|u\|_{W^{-1,p}(\mathbb{R}^n)} + \|\Delta u\|_{W^{-1,p}(\mathbb{R}^n)}.$$

Hence, for $\rho > 0$ we have

$$\|\rho\nabla u\|_{L^p(\mathbb{R}^n)} \le \rho\|\Delta_p^w u\|_{W^{-1,p}(\mathbb{R}^n)} + \rho\|u\|_{W^{-1,p}(\mathbb{R}^n)}, \quad \text{for all} \quad u \in W^{1,p}(\mathbb{R}^n). \tag{5.11}$$

Next,

$$\begin{aligned} \|\mathrm{div}(\Delta_p^s u)\|_{W^{-1,p}(\mathbb{R}^n)} &= \|\mathcal{F}^{-1}[(1+|\xi|^2)^{-1/2}\widehat{\mathrm{div}(\Delta_p^s u)}(\xi)](\cdot)\|_{L^p(\mathbb{R}^n)} \\ &= \|\mathcal{F}^{-1}[i\xi\cdot(1+|\xi|^2)^{-1/2}\widehat{\Delta_p^s u}(\xi)](\cdot)\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Since $M(\xi) := i\xi \cdot (1+|\xi|^2)^{-1/2} \in L^1_{loc}(\mathbb{R}^n;\mathcal{B}(\mathbb{R}^n))$ verifies that the sets $\{M(\xi)\}_{\xi\in\mathbb{R}^n\setminus\{0\}}$ and $\{|\xi|M'(\xi)\}_{\xi\in\mathbb{R}^n\setminus\{0\}}$ are uniformly bounded, it follows from [19, Theorem 3.25] (Mikhlin Fourier multiplier theorem) that there exists a constant $C > 0$ such that

$$\|\mathrm{div}(\Delta_p^s u)\|_{W^{-1,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[M(\xi)\widehat{\Delta_p^s u}(\xi)](\cdot)\|_{L^p(\mathbb{R}^n)} \le C\|\Delta_p^s u\|_{L^p(\mathbb{R}^n)},$$

for all $u \in W^{2,p}(\mathbb{R}^n;\mathbb{R}^n)$. Hence there exists $\delta_1 := 1/C$ such that for all $0<\epsilon<\delta_1$ we have

$$\|\epsilon\,\mathrm{div}(\Delta_p^s u)\|_{W^{-1,p}(\mathbb{R}^n)} \le \epsilon C\|\Delta_p^s u\|_{L^p(\mathbb{R}^n)} + \rho\|u\|_{L^p(\mathbb{R}^n)}, \text{ for all } u \in W^{2,p}(\mathbb{R}^n;\mathbb{R}^n). \quad (5.12)$$

This proves the first part of (B3) with $L=\rho, c_A=\rho$ and $c_D=\epsilon C$. For the second part, note that by hypothesis $c:=\max\{\rho,\epsilon C\}<1$ and the operators Δ_p^w and Δ_p^s are closed in X_1 and X_2 respectively. Therefore, it follows from Remark 3.2 that

$$\mathcal{A} = \begin{pmatrix} \Delta_p^w & \epsilon\,\mathrm{div}\Delta_p^s \\ -\rho\nabla & \Delta_p^s \end{pmatrix}$$

is closed in $X_1 \times X_2$. This proves assumption **(B)**.

Now, from [19, Theorem 4.11] we have that $\Delta_p^s \in \mathcal{RH}^\infty(L^p(\mathbb{R}^n))$. Analogously, by [21, Proposition 2.9] we get $\Delta_p^w \in \mathcal{RH}^\infty(W^{-1,p}(\mathbb{R}^n))$. Also, $0 \in \rho(\Delta_p^w) \cap \rho(\Delta_p^s)$, $\theta_{\Delta_p^w}^{\mathcal{R}_\infty}=0$ and $\theta_{\Delta_p^s}^{\mathcal{R}_\infty}=0$.

Finally, we check condition (4.3) of Theorem 4.2 as follows: Let $R_p(\mathcal{T})$ be the R-bound of the R-bounded set

$$\mathcal{T} := \{BR((ik)^\beta, D)CR((ik)^\alpha, A)\}_{k\in\mathbb{Z}}$$

where $A=\Delta_p^w$, $B=\mathrm{div}\Delta_p^s$, $C=\nabla$ and $D=\Delta_p^s$. Then, there exist $\delta_2 := \frac{1}{R_p(\mathcal{T})} > 0$ such that for $0<\epsilon<\delta_2$

$$R_p(\epsilon BR((ik)^\beta, D)\rho CR((ik)^\alpha, A)) = \epsilon\rho R_p(\mathcal{T}) < \epsilon R_p(\mathcal{T}) < 1.$$

Therefore, there exists $\delta := \min\{\delta_1,\delta_2\}$ such that for all $0<\epsilon<\delta$ all the hypotheses of Theorem 4.2 are satisfied. Hence, from Theorem 4.3, we conclude the following:

Theorem 5.7. *Let $1<p,q<\infty$, $0<\alpha\le\beta<2$ and $0<\rho<1$. There is $\delta>0$ such that for each $\varepsilon\in(0,\delta)$ and for all $f\in L^q(\mathbb{T};W^{-1,p}(\mathbb{R}^n))$ and $g\in L^q(\mathbb{T};L^p(\mathbb{R}^n))$ there exist unique functions $u\in L^q(\mathbb{T};W^{1,p}(\mathbb{R}^n))\cap H^{\alpha,q}(\mathbb{T};W^{-1,p}(\mathbb{R}^n))$ and $v\in L^p(\mathbb{T};W^{2,p}(\mathbb{R}^n;\mathbb{R}^n))\cap H^{\beta,p}(\mathbb{T};L^p(\mathbb{R}^n))$ satisfying the time-fractional system (5.10).*

Data availability

No data was used for the research described in the article.

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