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POISSON EQUATION AND DISCRETE ONE-SIDED HILBERT TRANSFORM FOR (C,α)-BOUNDED OPERATORS

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ABSTRACT

We characterize the solutions of the Poisson equation and the domain of its associated one-sided Hilbert transform for (C,α)-bounded operators, $\alpha > 0$. This extends known results for power bounded operators to the setting of Cesàro bounded operators of fractional order, thus generalizing the results substantially. In passing, we obtain a generalization of the mean ergodic theorem in our framework. Examples are given to illustrate the theory.

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1. Introduction

Let X be a complex Banach space and let $\mathcal{B}(X)$ denote the Banach algebra of bounded linear operators on X. Let $T \in \mathcal{B}(X)$. Then T is said to be power-bounded if $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$. Put

$$M_T^1(n) := (n+1)^{-1} \sum_{j=0}^{n} T^j x.$$

The operator T is called Cesàro-mean bounded if

$$\sup\{\|M_T^1(n)\| : n \in \mathbb{N}\} < \infty,$$

and mean-ergodic if there exists

$$P_1 x := \lim_{n\to\infty} M_T^1(n)x$$

for all $x \in X$ (in norm). In this case P_1 is in fact a bounded projection onto the closed subspace $\mathrm{Ker}(I-T)$ of X. Clearly, mean-ergodicity implies mean-boundedness.

Mean ergodic theorems form an important, classical, area of study since the beginning of the operator theory; see [17, p. 657 and subseq.]. There are different versions of that type of theorems. The following result is well known. Let T be a power-bounded operator on X. Then, for a given $x \in X$, there exists $P_1 x$ if and only if x belongs to the (topological) direct sum $\mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$ in X; see [33, Th. 1.3]. Thus T is mean-ergodic if and only if

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T),$$

which happens, for instance, when X is reflexive [33, Th. 1.2]. The study of ergodic theorems for power-bounded operators is related, in particular, to probability theory via ergodic Markov chains and the associated transition operators $\mathcal{P}$; e.g., it is possible to obtain central limit theorems for elements from $\mathrm{Ran}(\mathrm{I}-\mathcal{P})$.

With this kind of applications in mind and also looking at ergodicity in itself, it is of importance to characterize elements $x \in X$ for which $M_T^1(n)x$ converges to $P_1 x$ as $n \to \infty$ at a specific rate, e.g., with a polynomial rate. It turns out that the above question is closely related with a suitable description of elements y in range spaces $(I-T)^s X$, $0 < s \le 1$, which takes us to the study of the so-called fractional Poisson equation

$$(I-T)^s x = y, \quad x, y \in X. \tag{1.1}$$

As a matter of fact, given $y \in X$, the equation (1.1) for a power-bounded and mean ergodic operator T has a solution x if and only if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-s}} T^n y$$

converges in the norm of X, whence

$$\lim_{N\to\infty} N^{s-1} \left\| \sum_{n=1}^{N} T^n y \right\| = 0.$$

Moreover, x is obtained as a series representation $x = \sum_{n=0}^{\infty} c_n(s) T^n y$ where

$$c_n(s) \sim n^{s-1} \quad \text{as } n \to \infty.$$

For the above facts and other pertinent remarks, we refer to the introductions of [16, 28] and [24].

In [16], it is observed that $((I-T)^s)_{\mathfrak{Re}\, s>0}$ is a (holomorphic) semigroup in $\mathcal{B}(X)$, and a C_0-semigroup if we further assume that $(I-T)X$ is dense in X. Let $\log(I-T)$ denote the infinitesimal generator of this semigroup. A natural question is whether or not $-\log(I-T)$ coincides with the one-sided ergodic Hilbert transform $\mathcal{H}_T$ for T, given by

$$\mathcal{H}_T x := \sum_{n=1}^{\infty} \frac{T^n}{n} x,$$

whenever $x \in X$ is such that the series converges in X. Also in [16], it was shown that $\mathcal{H}_T \subseteq \log(I-T)$ as (generally) unbounded operators on X. The equality $\mathcal{H}_T = -\log(I-T)$ has been established in full generality in [12] and [28] independently and with different proofs (see also [9, 13, 14] for particular T). While the arguments used in [12] are specific for series concerned with that one defining $\mathcal{H}_T$, the approach carried out in [28] relies on the usage of the functional calculus. Looking at power operators $(I-T)^s$, it is clear that the analysis of $\operatorname{Ran}(I-T)^s$ is equivalent to the study of the domain $\operatorname{Dom}(I-T)^{-s}$ of the inverse operator $(I-T)^{-s}$, $s > 0$, where one is assuming that $I-T$ is injective. Also, the study of the plausible equality $\mathcal{H}_T = -\log(I-T)$ entails the study of $\operatorname{Dom}\log(I-T)$. Following the idea proposed in [28] one can use a functional calculus for suitable analytic functions $\mathfrak{f}$, involving $\log(1-z)$ and $(1-z)^{-s}$, so that the elements of $\operatorname{Dom}\mathfrak{f}(T)$ can be identified with $x \in X$ giving rise to norm (or weakly) convergent series $\sum_{n=1}^{\infty} a_n T^n x$, where a_n, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, are the Taylor coefficients of $\mathfrak{f}$.

Initially, the basic domain of that calculus is the sequence space ℓ^1, or its alternative version as (holomorphic) Wiener algebra $A^1_+(\mathbb{D})$ on the unit disc $\mathbb{D}$. Then the domain is extended to so-called regularizable functions $\mathfrak{f}$, with respect to $A^1_+(\mathbb{D}) \equiv \ell^1$, in the way introduced in [27]; see [28, Def. 2.4]. But this class of $\mathfrak{f}$ appears to be too large for obtaining reasonable characterisations of domains $\operatorname{Dom}\mathfrak{f}(T)$. A sufficient condition on $\mathfrak{f}$ is provided in [28] by introducing the notion of admissibility, so that $\operatorname{Dom}\mathfrak{f}(T)$ can be identified if $\mathfrak{f}$ is both regularizable and admissible on $\mathbb{D}$. By a theorem due to Th. Kaluza in 1928 (see [32]), remarkable examples of admissible functions are those whose Taylor coefficients form logarithmically convex sequences; see [28, Prop. 4.4] (which gives a new proof of Kaluza's theorem). Within this framework, the domains of $(I-T)^{-s}$, $s>0$, and $\log(I-T)$ are characterized in [28, Th. 6.1, Th. 6.2].

The aim of this paper is to give a new impetus to that topic by noticing that the Poisson equation (1.1) can be also thought in the setting of fractional difference equations. This new point of view appears to be fruitful and leads to a number of new results. Some authors have realized that the semigroup $(I-T)^s$ is a useful tool for modeling differential equations of fractional order, and it is well known that discretization techniques are useful in problems on differential equations; see [10, 35] and references therein. As a sample, let T be the backward shift operator on $\ell^2(\mathbb{N}_0)$. Then $I-T=D$, where D is the first order finite difference given by $Da(n) := a(n)-a(n+1)$, $n \geq 0$, for every sequence $(a(n)) \in \ell^2(\mathbb{N}_0)$. Thus D^2 is the discretization of the one-dimensional Laplace operator. The discrete Poisson problem $D^{2s}u=f$ arises in the Markov chains theory, so that the corresponding operator T is the transition matrix. The solution u can be expressed as the asymptotic variance, which is an important parameter in central limit theorems ([8, 23, 30, 42]). On the other hand, maximum and comparison principles for fractional differences D^s, as well as the issue of uniqueness in corresponding Dirichlet problems, have been recently established in connection with the problem $D^s u = f$ in [4]; a probabilistic interpretation of equation $D^s u = 0$ is also given in [4, Remark 2.5].

In the preceding discussion the operators T were assumed to be power bounded. However, there are other important and natural classes of operators with weaker assumptions on growth of their powers for which the problems discussed above are of interest. This is the case of the (C,α)-bounded operators defined below, in Section 2. The connection of these operators and ergodicity dates back to the fourties of last century; see [11] and [29]. In the latter, E. Hille studies (C,α)-convergence in terms of Abel convergence (that is, via the resolvent operator). As an application, the well known mean ergodic von Neumann's theorem for unitary groups on Hilbert spaces is extended to (C,α)-convergence for every $\alpha > 0$ [29, p. 255]. Also, the (C,α)-ergodicity on $L_1(0,1)$ of fractional (Riemann–Liouville) integrals is elucidated in [29, Th. 11]. In particular, if V is the Volterra operator then $T_V := I - V$, as operator on $L_1(0,1)$, is not power-bounded, and it is (C,α)-ergodic if and only if $\alpha > 1/2$ [29, Th. 11]. As a matter of fact, growth properties and ergodicity of (C,α) means of operators have been extensively studied over the years (see [3, 15, 19, 20, 34, 38, 40, 41, 43] and references therein). Very recently, in connection with operator inequalities and models, it has been shown that the shift operator on weighted Bergman spaces is (C,α)-ergodic, for $\alpha > 0$ depending on the weight. See [2] and Section 9 below.

Summarizing the above, there is a well established literature on (C,α)-bounded operators and ergodicity, which explores quite a number of properties and their interplays. However, neither the Poisson equation $(I-T)^s x = y$ nor the relation between the one-sided Hilbert transform and the logarithm operator $\log(I-T)$ have been studied for (C,α)-bounded operators T with $\alpha > 0$ so far. Here, we extend results of [16, 12, 28] to the setting of (C,α)-bounded operators. To do this, we follow the methodology introduced in [28] and use recent tools associated with fractional differences [4, 36]. The paper is organized accordingly.

After this introduction, Section 2 is devoted to preliminaries on (C,α)-bounded operators and fractional differences, which are defined in terms of Cesàro numbers. It is to be noticed that the Weyl difference operator W^α and its partner D^α, $\alpha > 0$, defined on sequence spaces, coincide sometimes but not always. This fact is one of the subtle difficulties to circumvent in the article. The section contains examples of further application.

The decomposition $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$ plays a key role in ergodic theorems, and is also relevant in the treatment of the (fractional) Poisson equation and the one-sided (discrete) Hilbert transform. Thus the decomposition $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$ in the setting of (C,α)-bounded operators T is discussed in Section 3. In Theorem 3.3, it is shown that for such an operator T the above splitting of X occurs if and only if T is (C,β)-ergodic for every $\beta > \alpha$. This result is a fairly general extension of the mean ergodic theorem for power-bounded operators.

In order to set our discussion within the framework built in [28], we need to replace the ℓ^1-calculus with another one which characterizes (C,α)-bounded operators. The domain of this calculus is the convolution subalgebra $\tau^\alpha(\mathbb{N}_0)$ of ℓ^1 formed by sequences $f=(f(n))$ in ℓ^1 such that the series

$$\|f\|_{1,(\alpha)} := \sum_{n=0}^{\infty} |[W^\alpha f](n)| k_{\alpha+1}(n)$$

is finite (see [5]); this is the meaning of $[W^\alpha f](n)$ above.

Let $A^\alpha(\mathbb{D})$ denote the range of the Gelfand transform of $\tau^\alpha(\mathbb{N}_0)$, which is to say the space of holomorphic functions on $\mathbb{D}$ whose Taylor coefficients are in $\tau^\alpha(\mathbb{N}_0)$. Hence, $A^\alpha(\mathbb{D})$ is a Banach algebra isomorphic to $\tau^\alpha(\mathbb{N}_0)$, for the pointwise multiplication and the norm obtained from $\|\cdot\|_{1,(\alpha)}$ by transference (note that the algebra $A^1_+(\mathbb{D})$ of [28] is $A^0(\mathbb{D})$ here). Section 4 contains basic properties of the $\tau^\alpha(\mathbb{N}_0)$-calculus (or $A^\alpha(\mathbb{D})$-calculus, equivalently), its relation with fractional difference operators and the compatibility of this calculus with that one for sectorial operators via the transformation $z\mapsto(1-z)$. Then we call α-regularizable function any function which is regularizable with respect to the algebra $A^\alpha(\mathbb{D})$ in the sense defined in [27]; see Definition 4.3 below. We show in particular that the functions $(1-z)^{-s}$, $0<s<1$, and $\log(1-z)$ are α-regularizable. Finally, the identity $\overline{\mathrm{Ran}}(I-T)^s=\overline{\mathrm{Ran}}(I-T)$ is proved for (C,α)-bounded operators T. A generalization of admissibility, called here α-admissibility, is implemented in Section 5. Its relation with fractional differences is analyzed, as well as the possibility to construct certain approximate identities, for specific elements in $A^\alpha(\mathbb{D})$, out of so-called α-admissible functions, see Definition 5.5 for the concept of α-admissibility.

Having laid the technical groundwork of our paper, a large number of examples of α-admissible functions, including $-z^{-1}\log(1-z)$ and $(1-z)^{-s}$, $0<s<1$, are given at the end of Section 5. The above concrete examples in particular could have been presented in the setting of a generalized version of Kaluza's theorem for log-convex sequences of higher order in terms of D^α and W^α. However, we do not go on this way here, due to the length and very technical character of the proof of such a generalization. Instead, we follow the indication of an anonymous referee who suggested considering Hausdorff moment sequences. That quoted generalized version, as well as its relationship with other extensions of Kaluza's theorem like those relevant results of [31], will be the subject of a forthcoming paper.

Using the functional calculi approach described above, we give in Section 7 a characterization of $x \in \operatorname{Dom} \mathfrak{f}(T)$ by convergence of the series

$$\sum_{n=0}^{\infty} [D^\alpha f](n)[\Delta^{-\alpha}\mathcal{T}](n)x,$$

where

$$[\Delta^{-\alpha}\mathcal{T}](n) := \sum_{j=0}^{n} k_\alpha(n-j)T^j$$

and $(k_\alpha(n))$ is the sequence of Cesàro numbers generated by $(1-z)^{-\alpha}$; see Section 2. Moreover, we prove that $\mathfrak{f}(T)$ is given by

$$\mathfrak{f}(T)x = \sum_{n=0}^{\infty} [D^\alpha f](n)[\Delta^{-\alpha}\mathcal{T}](n)x, \quad x \in \operatorname{Dom} \mathfrak{f}(T);$$

see Theorem 7.3. In particular, in Section 8 it is shown that $x \in \operatorname{Dom}(I-T)^{-s}$ if and only if

$$\sum_{n=1}^{\infty} n^{s-\alpha-1}[\Delta^{-\alpha}\mathcal{T}](n)x$$

converges for $0 < s < 1$ (Theorem 8.1). As a consequence, one obtains convergence rates (to 0) of $M_T^\beta(n)$, for $\beta > \alpha$, in Corollary 8.2, which generalize previous results about power-bounded operators. Also, we prove that $x \in \operatorname{Dom} \log(I-T)$ if and only if the series

$$\sum_{n=1}^{\infty} n^{-(1+\alpha)}[\Delta^{-\alpha}\mathcal{T}](n)x$$

converges. In this case, we obtain the formula

$$\log(I-T)x = (\psi(\alpha+1) - \psi(1))x - \sum_{n=1}^{\infty} B(\alpha+1, n)[\Delta^{-\alpha}\mathcal{T}](n)x,$$

where $\psi(x) := \frac{d}{dx}\ln(\Gamma(x))$ is the digamma function, Γ is the Gamma function and B is the Beta function (Theorem 8.7). This latter result suggests a possibility to define the one-sided α-ergodic Hilbert transform for a (C,α)-bounded operator T by

$$H_T^{(\alpha)} := (\psi(1) - \psi(\alpha+1)) + \sum_{n=1}^{\infty} B(\alpha+1, n)[\Delta^{-\alpha}\mathcal{T}](n)$$

(for $\alpha = 0$ it equals the usual one-sided Hilbert transform H_T). This issue is elaborated in Section 8.

Furthermore, in the case where $0 < s < 1$ and $0 < \alpha < 1 - s$, the arguments concerning coefficients $[W^\alpha f](n)$ and $[D^\alpha f](n)$ in Section 5 can be refined in order to handle Taylor coefficients. This is done in Section 6; see Theorem 6.2 and Theorem 6.4.

Thus we obtain, for $0 < s < 1$ and $0 < \alpha < 1 - s$ (Theorem 8.4), the interesting characterisation

$$x \in \mathrm{Dom}(I - T)^{-s} \Longleftrightarrow \sum_{n=1}^{\infty} n^{s-1} T^n x \ \text{ converges}$$

and the representation

$$(I - T)^{-s} x = \sum_{n=0}^{\infty} k_s(n) T^n x,$$

where $(k_s(n))$ are the Taylor coefficients of $(1 - z)^{-s}$.

Similarly, for $0 < \alpha < 1$, in Theorem 8.10 we show that

$$x \in \mathrm{Dom}\log(I - T) \Longleftrightarrow \sum_{n=1}^{\infty} n^{-1} T^n x \ \text{ converges}$$

and that $\log(I - T) = -\mathcal{H}_T$, that is,

$$\log(I - T) = -\sum_{n=1}^{\infty} \frac{1}{n} T^n.$$

Theorem 8.4 extends [16, Th. 2.11] and [28, Th. 6.1], and Theorem 8.10 extends [12, Prop. 3.3] and [28, Th. 6.2], where the case $\alpha = 0$ was treated. In Corollary 8.5, convergence rates of $M_T^\beta(n)$ are given for $0 < \alpha < 1 - s \leq \beta \leq 1$, which in particular provides the well known estimate

$$M_T^1(n)x = o(n^{-s}) \quad \text{as } n \to \infty;$$

see [16] and [28] for the case of power-bounded operators.

In Section 9 we give two examples to illustrate and apply the preceding results. These examples are about the Volterra operator on L_p spaces, and about the shift operator on Bergman spaces.

2. Preliminaries

2.1. Cesàro numbers. For $\alpha \in \mathbb{C}$, let $(k_\alpha(n))$ denote the sequence of Taylor coefficients of the generating function $(1-z)^{-\alpha}$, that is,

$$\sum_{n=0}^{\infty} k_\alpha(n) z^n = \frac{1}{(1-z)^\alpha}, \quad |z|<1. \tag{2.2}$$

The elements of sequences $(k_\alpha(n))$ are called Cesàro numbers, and are given by $k_\alpha(0)=1$ and

$$k_\alpha(n) := \binom{n+\alpha-1}{\alpha-1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \quad n \in \mathbb{N}; \tag{2.3}$$

see [44, Vol. I, p. 77], where $k_\alpha(n)$ is denoted by $A_n^{\alpha-1}$. For $\alpha \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$ one has

$$k_\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

Sequences $(k_\alpha(n))$ play an important role in summability theory [44]. Recently they found nontrivial applications in the theory of fractional difference equations; see [26, 35, 36]. Next, we give several properties of such sequences that will be used throughout the present paper.

From (2.2), one gets immediately the identity

$$\sum_{n=0}^{\infty} k_\alpha(n) = 0, \quad \text{for } \alpha<0, \tag{2.4}$$

which will be used several times in the paper to simplify calculations.

Assume $\alpha \in \mathbb{R}$. As a function of n, $k_\alpha(n)$ is increasing for $\alpha>1$, decreasing for $0<\alpha<1$, and $k_1(n)=1$ for all $n \in \mathbb{N}_0$ ([44, Th. III.1.17]). Furthermore, $0 \le k_\alpha(n) \le k_\beta(n)$ for $\beta \ge \alpha > 0$ and $n \in \mathbb{N}_0$. For $m \in \mathbb{N}_0$,

$$k_{-m}(n) = \begin{cases} (-1)^n \binom{m}{n}, & 0 \le n \le m; \\ 0, & n \ge m+1, \end{cases} \tag{2.5}$$

and, if $m<\alpha<m+1$,

$$\operatorname{sign} k_{-\alpha}(n) = \begin{cases} (-1)^n, & 0 \le n \le m; \\ (-1)^{m+1}, & n \ge m+1. \end{cases} \tag{2.6}$$

As regards the asymptotic behavior of the sequence $(k_\alpha(n))$ we have

$$k_\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + O\Big(\frac{1}{n}\Big)\Big), \quad \text{as } n \to \infty, \tag{2.7}$$

for every $\alpha \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$; see [44, Vol. I, p. 77 (1.18)] or [21, Eq. (1)].

It follows from (2.2) that the sequence $(k_\alpha(n))$ satisfies the group property, $k_\alpha * k_\beta = k_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{C}$, where the notation " $*$ " stands for the convolution of sequences on $\mathbb{N}_0$. Recall that for $a = (a(n)) \subset \mathbb{C}$ and $b = (b(n)) \subset \mathbb{C}$ the convolution is defined by

$$(a * b)(n) = \sum_{j=0}^{n} a(n-j)b(j), \quad n \geq 0.$$

Note that k_0 is the Dirac mass δ_0 on $\mathbb{N}_0$ and so it is the unit for the convolution. The following lemma is to be applied in the next subsection.

Lemma 2.1: *For every $\alpha > 0$ there exists $M_\alpha > 0$ such that*

$$(|k_{-\alpha}| * k_\alpha)(q) \leq M_\alpha k_\alpha(q).$$

Proof. Take the integer part $[\alpha]$ of α and $q > [\alpha]$. If $0 < \alpha < 1$ then, by (2.6),

$$\begin{aligned}(|k_{-\alpha}| * k_\alpha)(q) &= \sum_{p=0}^{q} |k_{-\alpha}(p)| k_\alpha(q-p) \\ &= 2k_\alpha(q) - \sum_{p=0}^{q} k_{-\alpha}(p)k_\alpha(q-p) = 2k_\alpha(q) - k_0(q) = 2k_\alpha(q)\end{aligned}$$

and, if $\alpha \geq 1$,

$$\begin{aligned}&(|k_{-\alpha}| * k_\alpha)(q) \\ &\quad = \sum_{p=0}^{[\alpha]} (-1)^p k_{-\alpha}(p)k_\alpha(q-p) + \sum_{p=[\alpha]+1}^{q} (-1)^{[\alpha]+1} k_{-\alpha}(p)k_\alpha(q-p) \\ &\quad = \sum_{p=0}^{[\alpha]} ((-1)^p + (-1)^{[\alpha]})k_{-\alpha}(p)k_\alpha(q-p) + (-1)^{[\alpha]+1} \sum_{p=0}^{q} k_{-\alpha}(p)k_\alpha(q-p) \\ &\quad \leq 2\sum_{p=0}^{[\alpha]} |k_{-\alpha}(p)| k_\alpha(q-p) + (-1)^{[\alpha]+1} k_0(q) \leq M_\alpha k_\alpha(q),\end{aligned}$$

for some positive constant M_α. ∎

2.2. Fractional differences involving Cesàro numbers. For a sequence $f = (f(n))$, define

$$[Wf](n) := f(n) - f(n+1), \quad n \in \mathbb{N}_0,$$

and subsequently, $W^1 := W$, $W^{m+1} := W^m W$ for $m \in \mathbb{N}$. Then one has

$$[W^m f](n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(n+j), \quad n \in \mathbb{N}_0,\, m \in \mathbb{N}.$$

Differences W^m are extended to the fractional case in [5, Def. 2.2] as follows.

Definition 2.2: Let $f = (f(n))_{n\geq 0}$ and $\alpha > 0$ be given. The **Weyl sum** $W^{-\alpha} f$ of **order** α of f is defined by

$$[W^{-\alpha} f](n) := \sum_{j=n}^{\infty} k_\alpha(j-n) f(j), \quad n \in \mathbb{N}_0, \tag{2.8}$$

whenever the series converges. The **Weyl difference** $W^\alpha f$ of **order** α of f is defined by

$$[W^\alpha f](n) = [W^m [W^{-(m-\alpha)} f]](n), \quad n \in \mathbb{N}_0,$$

for $m = [\alpha] + 1$, with $[\alpha]$ the integer part of α, whenever the corresponding series converges.

The fractional difference W^α admits another very useful description on specific spaces of sequences. Let $(f(n))$ be an arbitrary sequence. Then note that, for $m \in \mathbb{N}$,

$$[W^m f](n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(j+n) = \sum_{j=n}^{\infty} k_{-m}(j-n) f(j),$$

so for any $\alpha > 0$ we define the operator D^α by

$$[D^\alpha f](n) := \sum_{j=n}^{\infty} k_{-\alpha}(j-n) f(j), \quad n \in \mathbb{N}_0, \tag{2.9}$$

whenever the series converges; see [4]. The operators W^α and D^α are different, in general. However, they coincide on ℓ^1-spaces with appropriate weights, as the proposition below shows.

Let $\ell^1(\omega)$ denote the space of absolutely summable sequences on $\mathbb{N}_0$ with respect to a weight $\omega\colon \mathbb{N}_0 \to \mathbb{C}$, endowed with its usual norm $\|\cdot\|_{|\omega|}$. In the folllowing proposition we gather several properties of $W^{-\alpha}$, W^α and D^α on $\ell^1(\omega)$-spaces for $\omega = k_\mu$, $\mu \in \mathbb{R}$. Notice that for $\alpha > 0$ and $m := [\alpha]+1$ one has $m \le 2\alpha$ if and only if $\alpha \ge 1/2$, so that $\ell^1(k_\alpha) \hookrightarrow \ell^1(k_{m-\alpha})$ if $\alpha \ge 1/2$, and $\ell^1(k_{m-\alpha}) \hookrightarrow \ell^1(k_\alpha)$ if $0<\alpha<1/2$.

Proposition 2.3: *Let $\alpha > 0$ and $m := [\alpha]+1$. Then:*

(i) *The Weyl sum $W^{-\alpha}$ defines a bounded linear operator*

$$W^{-\alpha}\colon \ell^1(k_\alpha) \to \ell^1(k_{-\alpha}).$$

As a consequence, W^α is well defined on $\ell^1(k_{m-\alpha})$.

(ii) *For every $\beta > 0$, the Weyl sum $W^{-\alpha}$ defines a bounded linear operator*

$$W^{-\alpha}\colon \ell^1(k_{\alpha+\beta}) \to \ell^1(k_\beta).$$

(iii) *The operator D^α is well defined on the space $\ell^1(k_{-\alpha})$. Moreover, its restriction $D^\alpha\mid_{\ell^1(k_\alpha)}$ satisfies $D^\alpha\mid_{\ell^1(k_\alpha)}(\ell^1(k_\alpha)) \subset \ell^1(k_\alpha)$ and $D^\alpha\mid_{\ell^1(k_\alpha)}\colon \ell^1(k_\alpha) \to \ell^1(k_\alpha)$ is bounded.*

(iv) *Operators W^α and D^α coincide on $\ell^1(k_{m-\alpha})$.*

(v) *For $f \in \ell^1(k_\alpha)$, we have $D^\alpha(W^{-\alpha}f) = f$ and $W^{-\alpha}(D^\alpha f) = f$.*

Proof. (i) It is clear from (2.8) that $W^{-\alpha}$ is well defined on $\ell^1(k_\alpha)$ for every $\alpha > 0$. Further, for $f \in \ell^1(k_\alpha)$ and $n \in \mathbb{N}_0$ one has

$$\begin{aligned}\sum_{n=0}^\infty |k_{-\alpha}(n)||(W^{-\alpha}f)(n)| &\le \sum_{n=0}^\infty\sum_{j=n}^\infty |k_{-\alpha}(n)|\ k_\alpha(j-n)|f(j)| \\ &= \sum_{j=0}^\infty (|k_{-\alpha}| * k_\alpha)(j)\ |f(j)| \le M_\alpha \|f\|_{k_\alpha},\end{aligned}$$

for some constant $M_\alpha > 0$, by Lemma 2.1. Hence, $W^{-\alpha}\colon \ell^1(k_\alpha) \to \ell^1(k_{-\alpha})$ is well defined and bounded. Now, the consequence of the statement follows readily since $W^\alpha = W^m W^{-(m-\alpha)}$.

(ii) For $\alpha > 0$, $\beta > 0$, one can show the inclusion $W^{-\alpha}(\ell^1(k_{\alpha+\beta})) \subset \ell^1(k_\beta)$, together with the corresponding boundedness of the operator

$$W^{-\alpha}\colon \ell^1(k_{\alpha+\beta}) \to \ell^1(k_\beta),$$

by mimicking the above argument in part (i), with k_β instead $|k_{-\alpha}|$, and applying that $k_\beta * k_\alpha = k_{\beta+\alpha}$.

(iii) According to (2.9), $[D^\alpha g](n)$ is well defined for $g \in \ell^1(k_{-\alpha})$ and $n \in \mathbb{N}_0$. If now f belongs to $\ell^1(k_\alpha)$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} k_\alpha(n)|[D^\alpha f](n)| &= \sum_{j=0}^{\infty}\sum_{n=0}^{j} k_\alpha(n)|k_{-\alpha}(j-n)||f(j)| \\ &= \sum_{j=0}^{\infty}(k_\alpha * |k_{-\alpha}|)(j)|f(j)| \leq M_\alpha \|f\|_{k_\alpha},\end{aligned}$$

for some constant $M_\alpha > 0$, by Lemma 2.1. This proves (iii).

(iv) First note that $k_{-m}(q) = 0$, for $q > m$, and therefore there exists a constant $M_\alpha > 0$ such that

$$\sup_{q\in\mathbb{N}_0}(|k_{-m}(q)||k_{-(m-\alpha)}(q)|^{-1}) < M_\alpha.$$

Hence, for $f \in \ell^1(k_{m-\alpha})$ and $n \in \mathbb{N}_0$,

$$\begin{aligned}&\sum_{j=n}^{\infty}|k_{-m}(j-n)|\sum_{l=j}^{\infty}k_{m-\alpha}(l-j)|f(l)| \\ &\qquad = \sum_{p=0}^{\infty}(\sum_{q=0}^{p}|k_{-m}(q)|k_{m-\alpha}(p-q))|f(p+n)| \\ &\qquad \leq M_\alpha\sum_{p=0}^{\infty}\left(\sum_{q=0}^{p}|k_{-(m-\alpha)}(q)||k_{m-\alpha}(p-q)\right)|f(p+n)| \\ &\qquad = M_\alpha\sum_{p=0}^{\infty}(|k_{-(m-\alpha)}| * k_{m-\alpha})(p)|f(p+n)| \\ &\qquad \leq M'_\alpha\sum_{p=0}^{\infty}k_{m-\alpha}(p)|f(p+n)| < \infty,\end{aligned}$$

for some constant M'_α, where we have applied Lemma 2.1 in the last inequality.

Thus we can exchange series in $W^m(W^{-(m-\alpha)}$ to obtain

$$\begin{aligned}[W^m(W^{-(m-\alpha)})f](n) &= \sum_{p=0}^{\infty}(k_{-m} * k_{m-\alpha})(p)f(p+n) \\ &= \sum_{p=0}^{\infty}k_{-\alpha}(p)f(p+n) = [D^\alpha(f)](n),\end{aligned}$$

as we wanted to show.

(v) Let $f \in \ell^1(k_\alpha)$ and $n \in \mathbb{N}_0$. Similarly to previous calculations,

$$\sum_{q=n}^{\infty} |k_{-\alpha}(q-n)| \sum_{p=q}^{\infty} k_\alpha(p-q)|f(p)| = \sum_{p=0}^{\infty}(|k_{-\alpha}| * k_\alpha)(p)|f(p+n)| < \infty$$

and therefore

$$[D^\alpha(W^{-\alpha}f)](n) = \sum_{p=0}^{\infty}(k_{-\alpha} * k_\alpha)(p)f(p+n) = \sum_{p=0}^{\infty} k_0(p)f(p+n) = f(n).$$

Finally, to prove the equality $[W^{-\alpha}(D^\alpha f)](n) = f(n)$ it is enough to exchange the places of $|k_{-\alpha}|$ (and of $k_{-\alpha}$) and k_α in the above argument. ∎

Let now $\alpha > 0$ and define

$$\tau^\alpha(\mathbb{N}_0) := W^{-\alpha}(\ell^1(k_{\alpha+1})).$$

By Proposition 2.3 (v), $W^{-\alpha}$ is injective on $\ell^1(k_\alpha)$ and so the linear operator $W^{-\alpha}\,|_{\ell^1(k_{\alpha+1})}\colon \ell^1(k_{\alpha+1}) \to \tau^\alpha(\mathbb{N}_0)$ is bijective. By part (ii) of the proposition one has $\tau^\alpha(\mathbb{N}_0) \subset \ell^1$. Also,

$$\ell^1(k_{\alpha+1}) \subset \ell^1(k_{m-\alpha}), \quad \ell^1 \subseteq \ell^1(k_{m-\alpha})$$

since $\alpha + 1 \geq 1 \geq m - \alpha$. Then it follows readily by parts (iv) and (v) that $W^\alpha = D^\alpha$ on $\tau^\alpha(\mathbb{N}_0)$ and that the mappings $W^{-\alpha}\colon \ell^1(k_{\alpha+1}) \to \tau^\alpha(\mathbb{N}_0)$, $W^\alpha\colon \tau^\alpha(\mathbb{N}_0) \to \ell^1(k_{\alpha+1})$ are inverse one of the other. We endow the space $\tau^\alpha(\mathbb{N}_0)$ with the norm given by the finite series

$$\|f\|_{1,(\alpha)} := \sum_{n=0}^{\infty} |[W^\alpha f](n)| k_{\alpha+1}(n),$$

obtained by transference of the norm $\|\|_{\ell^1(k_{\alpha+1})}$ in $\ell^1(k_{\alpha+1})$ through $W^{-\alpha}$ (for $\alpha = 0$, the notation $\|f\|_{1,(0)}$ corresponds to the usual norm $\|f\|_1$ in ℓ^1), and so the space $\tau^\alpha(\mathbb{N}_0)$ can be described as the space of sequences $f = (f(n))$ in ℓ^1 such that $\|f\|_{1,(\alpha)} < \infty$. We also have

$$\tau^\beta(\mathbb{N}_0) \hookrightarrow \tau^\alpha(\mathbb{N}_0) \hookrightarrow \ell^1, \quad \text{for } \beta > \alpha > 0, \tag{2.10}$$

since $k_\beta(n) > k_\alpha(n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ if $\beta > \alpha > 0$.

Note that $W^{-\alpha}$ takes the space c_{00} of eventually null sequences onto itself, whence one gets that c_{00} is dense in $\tau^\alpha(\mathbb{N}_0)$. The space $\tau^\alpha(\mathbb{N}_0)$ will be considered in Section 4 in connection with the functional calculus. It will be used frequently in the remainder of the paper.

Example 2.4: The following fractional differences will be applied in the sequel.

(i) For $\mu \in \mathbb{C}\backslash\{0\}$ define $p_\mu(n) := \mu^{-(n+1)}$, $n \in \mathbb{N}_0$. It is proven in [5, Ex. 2.5] that functions p_μ are eigenfunctions (considered in a purely algebraic sense) of the operator W^α for $\alpha \in \mathbb{R}$ and $|\mu| > 1$, namely,

$$W^\alpha p_\mu = \Big(\frac{\mu-1}{\mu}\Big)^\alpha p_\mu, \quad |\mu| > 1.$$

(ii) Let $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$. Then

$$[D^m k_s](n) = [W^m k_s](n) = (-1)^m k_{s-m}(n+m), \quad n \in \mathbb{N}_0;$$

see [6, Ex. 3.4]. Also, if $\alpha > 0$ and $s \in (0,1)$, then by [6, Lemma 1.1] one gets

$$[D^\alpha k_s](n) = \frac{\mathrm{B}(1-s+\alpha, s+n)}{\Gamma(s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi}\mathrm{B}(1-s+\alpha, s+n), \quad n \in \mathbb{N}_0, \tag{2.11}$$

where B is the Beta function. Therefore, by [21, Eq. (1)],

$$[D^\alpha k_s](n) = \frac{\Gamma(1-s+\alpha)}{\Gamma(s)\Gamma(1-s)} n^{s-\alpha-1}\Big(1 + O\Big(\frac{1}{n}\Big)\Big), \quad \text{as } n \to \infty. \tag{2.12}$$

From the above, one obtains that $(k_\beta(n)) \in \tau^\alpha(\mathbb{N}_0)$ if $\beta \in \mathbb{C}$ with $\mathfrak{Re}\,\beta < 0$ or $\beta = 0$, for all $\alpha \geq 0$.

(iii) Let $m \in \mathbb{N}_0$ and let L be the sequence defined by $L(n) = \frac{1}{n+1}$ for $n \in \mathbb{N}_0$. Then, for $\alpha > 0$, by [6, Lemma 1.1] we have

$$D^\alpha L(n) = \frac{\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+2)}, \quad n \in \mathbb{N}_0, \tag{2.13}$$

and by [21, Eq. (1)],

$$D^\alpha L(n) = \frac{\Gamma(\alpha+1)}{n^{\alpha+1}}(1 + O(\frac{1}{n})), \quad \text{as } n \to \infty. \tag{2.14}$$

2.3. Cèsaro Operators. We now introduce (C,α)-bounded operators and ergodicity in terms of Cesàro numbers. Let X be a Banach space. For T in $\mathcal{B}(X)$, let $\mathcal{T}$ denote the discrete semigroup associated with T, given by $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$, with T^0 the identity operator on X. Take $\alpha \geq 0$ and set, for $x \in X$ and $n \in \mathbb{N}_0$,

$$[\Delta^{-\alpha}\mathcal{T}](n)x := (k_\alpha * \mathcal{T})(n) = \sum_{j=0}^n k_\alpha(n-j)T^j x,$$

$$M_T^\alpha(n)x := \frac{1}{k_{\alpha+1}(n)}[\Delta^{-\alpha}\mathcal{T}](n)x.$$

The operators $[\Delta^{-\alpha}\mathcal{T}](n)$ and $M_T^\alpha(n)$ in $\mathcal{B}(X)$ are called the n-th Cesàro sum and Cesàro mean of T of order α, respectively. The operator T is called Cesàro bounded of order α, or simply (C,α)-bounded, if it satisfies $\sup_n \|M_T^\alpha(n)\| < \infty$. Thus $(C,0)$-boundedness is the same as power-boundedness, that is,

$$\sup_n \|T^n\| < \infty.$$

For $\alpha = 1$ the operator T is called Cesàro mean bounded (or Cesàro bounded). Notice that, for arbitrary $\alpha > 0$, the (C,α)-boundedness of an operator T implies $\|T^n\| = O(n^\alpha)$ as $n \to \infty$. If T is (C,α)-bounded then it is (C,β)-bounded for every $\beta > \alpha$ but the converse does not hold true in general; e.g., [20, Section 4.7] , [40, Remark 2.3] and [3, Section 2]. Moreover, a bounded linear operator T is said to be (C,α)-ergodic if there exists $P_\alpha x := \lim_{n\to\infty} M_T^\alpha(n)x$ for all $x \in X$, in the norm of X (in this case P_α is in fact a bounded projection onto the closed subspace $\mathrm{Ker}(I-T)$ of X). For $\alpha = 1$, T is called mean ergodic. Clearly, (C,α)-ergodicity implies (C,α)-boundedness.

There is a number of papers addressing ergodicity of (C,α)-bounded operators, which look for extending to fractional order the main results and features of the operator ergodic theory. We focus here on the particular line of research explained in the introduction, Section 1, which seems to have not been considered before. Nonetheless, we start with establishing several ergodic results on (C,α)-bounded operators in the next section, being quite instructive for the subsequent considerations.

3. (C,α)-mean ergodic results

A version of the mean ergodic theorem says that a power-bounded operator T is Cesàro mean ergodic if and only if X splits as $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$ [33, Th. 1.3]. Our aim here is to give an extension of that result for (C,α)-bounded operators for every $\alpha > 0$.

Lemma 3.1: *Let T be a (C,α)-bounded operator on a Banach space X, and $\beta > \alpha$. Put $X_\beta := \{x \in X :$ there exists $P_\beta x := \lim_{n\to\infty} M_T^\beta(n)x$ in $X\}$. Then P_β is a projection onto $\mathrm{Ker}(I-T)$ along $\overline{\mathrm{Ran}}(I-T)$, that is,*

$$\mathrm{Ran}P_\beta = \mathrm{Ker}(I-T) \quad \textit{and} \quad \mathrm{Ker}P_\beta = \overline{\mathrm{Ran}}(I-T),$$

so that

$$X_\beta = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T).$$

Proof. First, note that $T\Delta^{-\beta}\mathcal{T}(n) = [\Delta^{-\beta}\mathcal{T}](n+1) - k_\beta(n+1)I$, which implies

$$TM_T^\beta(n) = \frac{\beta+n+1}{n+1}M_T^\beta(n+1) - \frac{\beta}{n+1}I.$$

For a given $x \in X_\beta$, letting $n \to \infty$ one obtains that there exists

$$P_\beta Tx = TP_\beta x = P_\beta x$$

since X_β is a closed subspace of X. Hence,

$$\begin{aligned} P_\beta M_T^\beta(n)x = M_T^\beta(n)P_\beta x &= \frac{1}{k_{\beta+1}(n)}\sum_{j=0}^{n} k_\beta(n-j)T^j P_\beta x \\ &= \frac{1}{k_{\beta+1}(n)}(k_\beta * k_1)(n)(P_\beta x) = P_\beta x, \end{aligned}$$

which implies that $P_\beta^2 x = P_\beta x$. Therefore, P_β is a (linear) bounded projection on X_β.

Now, for $x \in X_\beta$, $(I-T)P_\beta x = 0$ so $\mathrm{Ran}P_\beta \subseteq \mathrm{Ker}(I-T)$. Conversely, if $Tx = x$ then $M_T^\beta(n)x = x$ for all n, and therefore there exists $P_\beta x = x$ in X_β. In short, $\mathrm{Ran}P_\beta = \mathrm{Ker}(I-T)$.

To see that $\mathrm{Ker}P_\beta = \overline{\mathrm{Ran}}(I-T)$, note that by Example 2.4 (ii) with $m = 1$ one gets

$$\begin{aligned} &(I-T)[\Delta^{-\beta}\mathcal{T}](n) \\ &\quad=(I-T)(k_{\beta-\alpha} * k_\alpha * \mathcal{T})(n) \\ &\quad=\sum_{j=0}^{n} k_{\beta-\alpha}(n-j)[\Delta^{-\alpha}\mathcal{T}](j) - \sum_{j=0}^{n} k_{\beta-\alpha}(n-j)([\Delta^{-\alpha}\mathcal{T}](j+1) - k_\alpha(j+1)I) \\ &\quad=k_{\beta-\alpha}(n) + \sum_{j=0}^{n} k_{\beta-\alpha}(n-j)k_\alpha(j+1) - \sum_{j=1}^{n+1} k_{\beta-\alpha-1}(n+1-j)[\Delta^{-\alpha}\mathcal{T}](j) \\ &\quad=k_{\beta-\alpha}(n) + k_\beta(n+1) - k_{\beta-\alpha}(n+1) + k_{\beta-\alpha-1}(n+1) \\ &\qquad-\sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)[\Delta^{-\alpha}\mathcal{T}](j) \\ &\quad=k_\beta(n+1) - \sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)[\Delta^{-\alpha}\mathcal{T}](j). \end{aligned}$$

If $\beta \geq \alpha+1$, then one has

$$\begin{aligned}\frac{\|(I-T)[\Delta^{-\beta}\mathcal{T}](n)\|}{k_{\beta+1}(n)} &\leq \frac{k_\beta(n+1)}{k_{\beta+1}(n)} + \frac{1}{k_{\beta+1}(n)}\left\|\sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)[\Delta^{-\alpha}\mathcal{T}](j)\right\| \\ &\leq \frac{k_\beta(n+1)}{k_{\beta+1}(n)} + \frac{M_\alpha}{k_{\beta+1}(n)}\sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)k_{\alpha+1}(j) \\ &= (1+M_\alpha)\frac{k_\beta(n+1)}{k_{\beta+1}(n)} = \frac{\beta(1+M_\alpha)}{n+1} \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

where $M_\alpha > 0$, since T is (C,α)-bounded.

If $\beta-\alpha \in (0,1)$, by (2.6) we have

$$\begin{aligned}&\frac{1}{k_{\beta+1}(n)}\left\|\sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)[\Delta^{-\alpha}\mathcal{T}](j)\right\| \\ &\qquad \leq \frac{1}{k_{\beta+1}(n)}\sum_{j=0}^{n+1} |k_{\beta-\alpha-1}(n+1-j)|k_{\alpha+1}(j) \\ &\qquad = \frac{1}{k_{\beta+1}(n)}\left(-\sum_{j=0}^{n+1} k_{\beta-\alpha-1}(n+1-j)k_{\alpha+1}(j) + 2k_{\alpha+1}(n+1)\right) \\ &\qquad = \frac{1}{k_{\beta+1}(n)}(-k_\beta(n+1) + 2k_{\alpha+1}(n+1)) \\ &\qquad = \mathrm{O}(n^{\alpha-\beta}) \to 0, \quad n\to\infty.\end{aligned}$$

So, we conclude that $M_T^\beta(n)(I-T) \to 0$ strongly as $n\to\infty$ for any $\beta > \alpha$. It follows that $\mathrm{Ran}(I-T) \subseteq X_\beta$, or in other words $P_\beta(I-T) = 0$. That is, $\mathrm{Ran}(I-T) \subseteq \mathrm{Ker}P_\beta$ and then $\overline{\mathrm{Ran}}(I-T) \subseteq \mathrm{Ker}P_\beta$. Conversely, given x in X_β such that $P_\beta x = 0$ we have

$$x = \lim_{n\to\infty}\frac{1}{k_{\beta+1}(n)}\sum_{j=0}^{n} k_\beta(n-j)(x-T^jx) = \lim_{n\to\infty}(I-T)\left(\sum_{j=1}^{n}\frac{k_\beta(n-j)}{k_{\beta+1}(n)}\sum_{l=0}^{j-1}T^lx\right),$$

that is, $\mathrm{Ker}P_\beta \subseteq \overline{\mathrm{Ran}}(I-T)$ and the proof is complete. ∎

Remark 3.2: In accordance with Lemma 3.1, for a given (C,α)-bounded operator T and $\beta > \alpha$, the direct sum $\mathrm{Ker}(I-T)\oplus\overline{\mathrm{Ran}}(I-T)$ is the largest subspace of X on which there exists $\lim_{n\to\infty} M_T^\beta(n)$.

The following result is our version of the mean ergodic theorem for (C,α)-bounded operators.

THEOREM 3.3: *Let T be a (C,α)-bounded operator, and $\beta > \alpha$. Then T is (C,β)-ergodic if and only if $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$.*

Proof. It is a direct consequence of Lemma 3.1. ∎

The following immediate corollary shows the significance of the above theorem.

COROLLARY 3.4: *Let T be a bounded operator on a Banach space X.*

(i) *If T is (C,α)-ergodic on X for some $\alpha > 0$ then*

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T).$$

(ii) *Let $\beta > 0$ and assume T is power-bounded. Then T is (C,β)-ergodic if and only if $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$.*

(iii) *Let T be a (C,α)-bounded operator with $0 < \alpha < 1$. Then T is Cesàro mean ergodic if and only if $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$.*

When $\beta = 1$, Corollary 3.4 (ii) is the well known mean ergodic theorem cited in the beginning of this section.

Next, we extend [40, Theorem 5.1]. The proof runs parallel to the case of arbitrary $\beta \geq 1$ though it needs [1, Th. 4.3] in our case.

THEOREM 3.5: *Let $\beta \geq 1$ and $T \in \mathcal{B}(X)$ such that $\sigma(T) \subset \mathbb{D} \cup \{1\}$. Then T is a (C,β)-ergodic operator if and only if T is (C,β)-bounded and $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$.*

Proof. Let T be a (C,β)-bounded operator on $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$ with $\sigma(T) \subset \mathbb{D} \cup \{1\}$. By assumption, every $x \in X$ can be written as $x = y + z$ with $y \in \overline{\mathrm{Ran}}(I-T)$ and $Tz = z$. Then it is enough to show that $M_T^\beta(n)y \to 0$ as $n \to \infty$ to prove the theorem.

Take $y = a - Ta$, $a \in X$. Then

$$M_T^\beta(n)(a - Ta) = \frac{\beta}{n+1}(I - M_T^{\beta-1}(n+1))a \to 0, \quad \text{as } n \to \infty$$

since $\sigma(T) \subset \mathbb{D} \cup \{1\}$ and therefore $\|M_T^{\beta-1}(n+1)\| = o(n)$, as $n \to \infty$; see [1, Th. 4.3]. By density, one obtains

$$\lim_{n\to\infty} M_T^\beta(n)y = 0$$

for all $y \in \overline{\mathrm{Ran}}(I-T)$. The opposite implication is clear. ∎

Remark 3.6: It can be shown directly from definitions that if T is a (C,α)-ergodic operator then T is (C,β)-ergodic for every $\beta > \alpha$. (This result is usually proved as a consequence of mean ergodic results involving the resolvent function of the operator; see [19, Cor. 3.1] for example.) In this case, the projection operators P_α and P_β are the same.

4. Functional calculus for Cesàro bounded operators

The Banach space $\tau^\alpha(\mathbb{N}_0)$ defined in Subsection 2.2 is actually a Banach algebra, for the convolution product, in the sense that there exists a (not necessarily equal to one) constant M_α such that

$$\|f * g\|_{1,(\alpha)} \leq M_\alpha \|f\|_{1,(\alpha)} \|g\|_{1,(\alpha)}$$

for $f, g \in \tau^\alpha(\mathbb{N}_0)$; see [5, Th. 2.11]. The algebras $\tau^\alpha(\mathbb{N}_0)$, $\alpha > 0$, were introduced in [22] for $\alpha \in \mathbb{N}$. Their extensions to $\alpha > 0$ have been defined in [5] and [1, Section 2], though with a slightly different presentation.

For $f = (f(n)) \in \tau^\alpha(\mathbb{N}_0)$, let $\mathfrak{f}$ be the holomorphic function on the unit disc $\mathbb{D}$ (and continuous on $\overline{\mathbb{D}}$) given by $\mathfrak{f}(z) := \sum_{n=0}^\infty f(n) z^n$. Define

$$A^\alpha(\mathbb{D}) := \{\mathfrak{f} : f \in \tau^\alpha(\mathbb{N}_0)\},$$

endowed with pointwise multiplication and the norm $\|\mathfrak{f}\|_{A^\alpha(\mathbb{D})} := \|f\|_{1,(\alpha)}$. Thus $A^\alpha(\mathbb{D})$ and $\tau^\alpha(\mathbb{N}_0)$ are Banach algebras isometrically isomorphic and the correspondence $f \in \tau^\alpha(\mathbb{N}_0) \mapsto \mathfrak{f} \in A^\alpha(\mathbb{D})$ is the Gelfand transform of $\tau^\alpha(\mathbb{N}_0)$. It can be given in terms of (scalar) Cesàro sums $\Delta^{-\alpha}\mathcal{Z}$ and Weyl differences, as we see next:

For $\alpha \geq 0$ and $\mathcal{Z} = (z^n)_{n \in \mathbb{N}_0}$ set

$$[\Delta^{-\alpha}\mathcal{Z}](n) := \sum_{j=0}^n k_\alpha(n-j) z^j, \quad z \in \overline{\mathbb{D}}, \ n \in \mathbb{N}_0.$$

Clearly,

$$|[\Delta^{-\alpha}\mathcal{Z}](n)| \leq \sum_{j=0}^n k_\alpha(n-j)|z|^j \leq \sum_{j=0}^n k_\alpha(n-j) = k_{\alpha+1}(n)$$

uniformly on $\overline{\mathbb{D}}$. Regarding estimates on compact subsets Q of $\mathbb{D}$, we have

$$(4.1)\quad |[\Delta^{-\alpha}\mathcal{Z}](n)| \leq \sum_{j=0}^n k_\alpha(n-j)|z|^j \leq k_\alpha(n) \frac{1-|z|^{n+1}}{1-|z|} \leq M_Q k_\alpha(n), \quad z \in Q,$$

if $\alpha \geq 1$ since k_α is increasing in this case, and

$$|[\Delta^{-\alpha}\mathcal{Z}](n)| \leq \sum_{j=0}^{n} k_\alpha(n-j)|z|^j \leq k_\alpha(0)\frac{1-|z|^{n+1}}{1-|z|} \leq M_Q \tag{4.2}$$

when $0 < \alpha < 1$, since k_α is decreasing now. Here M_Q is a constant depending on Q.

Now, let $\mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n)z^n$ be a holomorphic function in $A^\alpha(\mathbb{D})$. Using Fubini's theorem (for series) in the standard way, it is readily seen that

$$\mathfrak{f}(z) = \sum_{n=0}^{\infty} [W^\alpha f](n)[\Delta^{-\alpha}\mathcal{Z}](n), \quad z \in \overline{\mathbb{D}}, \tag{4.3}$$

where the series converges absolutely in $\overline{\mathbb{D}}$. Thus in particular

$$\mathfrak{f}(1) = \sum_{n=0}^{\infty} [W^\alpha f](n) k_{\alpha+1}(n).$$

The uniqueness of coefficients $[W^\alpha f](n)$ in (4.3) is a consequence of the following result.

Lemma 4.1: *For $\alpha > 0$ and $n \in \mathbb{N}_0$ set $\omega_\alpha(n) := k_\alpha(n)$ if $\alpha \geq 1$, and $\omega_\alpha(n) := 1$ when $0 < \alpha < 1$. Let $g = (g(n))$ be a sequence such that*

$$\sum_{n=0}^{\infty} |g(n)|\omega_\alpha(n) < \infty.$$

Assume that

$$\sum_{n=0}^{\infty} g(n)[\Delta^{-\alpha}\mathcal{Z}](n) = 0, \quad z \in \mathbb{D}.$$

Then $g(n) = 0$ for all $n \in \mathbb{N}_0$.

Proof. Set $M_{\alpha,g} := \sum_{n=0}^{\infty} \omega_\alpha(n)|g(n)|$. For $z \in \mathbb{D}$, we have

$$\sum_{n=0}^{\infty} |g(n)| \sum_{j=0}^{n} k_\alpha(n-j)|z|^j = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} k_\alpha(n-j)|g(n)||z|^j \leq \frac{M_{\alpha,g}}{1-|z|} < \infty,$$

whence

$$0 = \sum_{n=0}^{\infty} g(n)[\Delta^{-\alpha}\mathcal{Z}](n) = \sum_{j=0}^{\infty} z^j \sum_{n=j}^{\infty} k_\alpha(n-j)g(n) = \sum_{j=0}^{\infty} [W^{-\alpha}g](j)z^j$$

for every $z \in \mathbb{D}$, so that $[W^{-\alpha}g](j) = 0$ for all j. Now, $g \in \ell^1(k_\alpha)$ for every $\alpha > 0$ by hypothesis and therefore $g = 0$ by Proposition 2.3, part (v). ∎

Remark 4.2: When $\alpha = m \in \mathbb{N}$, the coefficients $[W^m f](n)$ in (4.3), for $\mathfrak{f} \in A^\alpha(\mathbb{D})$, can be given in integral form by

$$[W^m f](n) = \frac{1}{2\pi} \int_{|\lambda|=r} \frac{(\lambda-1)^m}{\lambda^{m+n+1}}\, \mathfrak{f}(\lambda)\, d\lambda, \quad 0<r<1,\ n \in \mathbb{N}.$$

This formula is obtained by applying W^m to the Cauchy integral formula for the Taylor coefficients

$$f(n) = \mathfrak{f}^{(n)}(0)/n!$$

of $\mathfrak{f}$.

It is maybe in order to notice that, for $\alpha = m \in \mathbb{N}$, Lemma 4.1 can be extended to any sequence g such that $0 = \sum_{n=0}^\infty g(n)\Delta^{-m}\mathcal{Z}(n)$, $z \in \mathbb{D}$; that is, if the series is zero, without any other condition on g, then g must be the null sequence. To see this, first note that

$$\begin{aligned}[\Delta^{-m}\mathcal{Z}](n) &:= \sum_{j=0}^{m} k_m(n-j) z^j \\ &= \frac{1}{(z-1)^m}\left(z^{m+n} + \sum_{j=1}^{m-1} P_{m,j}(n) z^{m-j} + (-1)^m k_m(n)\right)\end{aligned}$$

for all $z \in \mathbb{D}$, where $P_{m,j}$ are polynomials of degree $m-1$ at most. This equality can be obtained by induction in m, on account of the identity

$$[\Delta^{-p}\mathcal{Z}](n) = \sum_{j=0}^{n} [\Delta^{-(p-1)}\mathcal{Z}](j); \quad p, n \in \mathbb{N}.$$

Evaluating at $z=0$ one gets $0 = \sum_{n=0}^\infty g(n) k_{m+1}(n)$ and therefore

$$0 = \sum_{n=0}^{\infty} g(n)\left(z^{m+n} + \sum_{j=1}^{m-2} P_{m,j}(n) z^{m-j} + P_{m,m-1}(n) z\right)$$

for all $z \in \mathbb{D}$. Dividing by $z \neq 0$ and then evaluating the resulting polynomial at $z=0$ once again, one obtains $0 = \sum_{n=0}^\infty g(n) P_{m,m-1}(n)$. By repetition of the argument we eventually arrive at $\sum_{n=0}^\infty g(n) z^n = 0$ $(z \in \mathbb{D})$, whence obviously $g(n) = 0$.

We next introduce the functional calculus which is needed to accomplish our aims.

Let $\alpha > 0$. A linear bounded operator $T \in \mathcal{B}(X)$ is (C, α)-bounded if and only if there exists a Banach algebra bounded homomorphism $\Theta_\alpha : \tau^\alpha(\mathbb{N}_0) \to \mathcal{B}(X)$, which furthermore is given by

$$\Theta_\alpha(f)x = \sum_{n=0}^{\infty} [W^\alpha f](n)[\Delta^{-\alpha}\mathcal{T}](n)x, \quad x \in X,\, f \in \tau^\alpha(\mathbb{N}_0);$$

see [5, Th. 3.5 and Cor. 3.7]. Then we define the functional calculus

$$\mathfrak{f} \in A^\alpha(\mathbb{D}) \mapsto \mathfrak{f}(T) \in \mathcal{B}(X)$$

by

$$\mathfrak{f}(T) := \Theta_\alpha(f),$$

for $\mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n) z^n$. If

$$K_\alpha(T) := \sup_{n \in \mathbb{N}_0} \|M_T^\alpha(n)\|,$$

we have

$$\|\mathfrak{f}(T)\| \le K_\alpha(T) \|\mathfrak{f}\|_{A^\alpha(\mathbb{D})}. \tag{4.4}$$

The calculus $\mathfrak{f} \mapsto \mathfrak{f}(T)$ just introduced will be called here the primary functional calculus on $A^\alpha(\mathbb{D})$, or primary $A^\alpha(\mathbb{D})$-calculus, for short. One can extend such a calculus by means of the regularization process considered in [28] for $\alpha = 0$; see [27] for a general background.

Definition 4.3: Let $\alpha > 0$ and let T be a (C, α)-bounded operator. We say that a function $\mathfrak{f}$ holomorphic in $\mathbb{D}$ is **α-regularizable** if there is an element $\mathfrak{e} \in A^\alpha(\mathbb{D})$ such that $\mathfrak{e}\mathfrak{f} \in A^\alpha(\mathbb{D})$ and $\mathfrak{e}(T)$ is an injective operator. In such a case, put

$$\mathfrak{f}(T) := \mathfrak{e}(T)^{-1}(\mathfrak{e}\mathfrak{f})(T).$$

The so-defined $\mathfrak{f}(T)$ does not depend on the α-regularizer $\mathfrak{e}$ (see [27, Lemma 1.2.1]) and it is a closed operator, generally unbounded.

We now consider sectorial operators. For $\theta \in (0, \pi)$ let S_θ denote the sector of angle 2θ in the complex plane, which is symmetric with respect to the half-line $(0, \infty)$. An operator $A \in \mathcal{B}(X)$ is said to be **sectorial** of angle θ if the spectrum $\sigma(A)$ is contained in $\overline{S}_\theta$ and, for every $\omega \in (\theta, \pi)$ and a constant K_ω,

$$\|\lambda(\lambda - A)^{-1}\| \le K_\omega, \quad \lambda \in \mathbb{C} \setminus \overline{S}_\omega.$$

It is well known that sectorial operators enjoy a remarkable functional calculus, see [27, Chapter 2]. Namely, for $\omega \in (\theta, \pi)$ let $\mathcal{E}_0(S_\omega)$ denote the space of holomorphic functions $\mathfrak{h}$ on S_ω such that $|\mathfrak{h}(z)| \le K|z|^s$ for all $z \in S_\omega \cap \mathbb{D}$ for some constants $K, s > 0$. Take γ the oriented counterclockwise path parameterized as

$$\gamma := \{re^{-i\varphi} : r \in [0, r_0]\} \cup \{r_0 e^{i\psi} : \psi \in (-\varphi, \varphi)\} \cup \{re^{i\varphi} : r \in [0, r_0]\},$$

with r_0 greater than the spectral radius $\rho(A)$ of A and $\theta < \varphi < \omega$.

Then the Bochner integral, so-called Dunford–Riesz formula,

$$\mathfrak{h}(A) = \frac{1}{2\pi i} \int_\gamma \mathfrak{h}(\lambda)(\lambda - A)^{-1}\, d\lambda, \tag{4.5}$$

is well defined for every $\mathfrak{h} \in \mathcal{E}_0(S_\omega)$, does not depend on the choice of φ and the mapping $\mathfrak{h} \to \mathfrak{h}(A)$ defines a functional calculus on $\mathcal{E}_0(S_\omega)$, independent of $\omega \in (\theta, \pi)$ (see [27, p. 46]) which we call here the primary $\mathcal{E}_0$-calculus for sectorial operators. A function $\mathfrak{h}$ holomorphic in S_ω is regularizable within this calculus if there is an element $\mathfrak{b} \in \mathcal{E}_0(S_\omega)$ such that $\mathfrak{b}\mathfrak{h} \in \mathcal{E}_0(S_\omega)$ and $\mathfrak{b}(T)$ is an injective operator. If so, we put

$$\mathfrak{h}(T) := \mathfrak{b}(T)^{-1}(\mathfrak{b}\mathfrak{h})(T).$$

The formula is independent of the regularizer $\mathfrak{b}$ and defines a (generally unbounded) closed operator on X.

Next, we connect (C, α)-bounded operators with sectorial operators following [28]. Details for arbitrary $\alpha > 0$ are included for the sake of completeness.

Lemma 4.4: *Let $\alpha > 0$ and let T be a (C, α)-bounded operator. Then the operator $A := I - T$ satisfies*

$$\|(\lambda - A)^{-1}\| \le K_\alpha(T) \frac{|\lambda|^\alpha}{(|\lambda - 1| - 1)^{\alpha+1}} \le K_\alpha(T) \frac{|\lambda|^\alpha}{|\mathfrak{Re}\, \lambda|^{\alpha+1}}, \quad \mathfrak{Re}\, \lambda < 0.$$

In consequence, A is a sectorial operator of angle $\pi/2$.

Proof. We notice that the spectral radius of T is less than or equal to 1; see [1, Lemma 1.1]. Hence, the spectrum $\sigma(A)$ is contained in the closed disc $\{z \in \mathbb{C} : |z - 1| \le 1\}$ and therefore in $\overline{S}_{\pi/2}$.

Let μ be a complex number such that $|\mu|>1$. By [5, Th. 4.4] and Example 2.4(i),

$$(\mu-T)^{-1}=\sum_{n=0}^{\infty}[W^{\alpha}p_{\mu}](n)[\Delta^{-\alpha}\mathcal{T}](n)=\Big(\frac{\mu-1}{\mu}\Big)^{\alpha}\sum_{n=0}^{\infty}\mu^{-n-1}[\Delta^{-\alpha}\mathcal{T}](n).$$

Then for $\mathfrak{Re}\,\lambda<0$ we have

$$\begin{aligned}\|(\lambda-A)^{-1}\| &\le K_{\alpha}(T)\frac{|\lambda|^{\alpha}}{|\lambda-1|^{\alpha}}\sum_{n=0}^{\infty}|\lambda-1|^{-n-1}k_{\alpha+1}(n)\\ &= K_{\alpha}(T)\frac{|\lambda|^{\alpha}}{(|\lambda-1|-1)^{\alpha+1}}\\ &\le K_{\alpha}(T)\frac{|\lambda|^{\alpha}}{|\mathfrak{Re}\,\lambda|^{\alpha+1}},\end{aligned}$$

where we have used the identity (2.2).

From the above estimate we obtain that A is sectorial, since

$$|\lambda|/|\mathfrak{Re}\,\lambda|\le|\cos(\pi-\omega)|^{-1}$$

for every $\lambda\in\mathbb{C}\setminus\overline{S}_{\omega}$ and $\omega\in(\pi/2,\pi)$. ∎

The primary $A^{\alpha}(\mathbb{D})$-calculus and the primary $\mathcal{E}_0$-calculus are compatible, through the change of variable $z\mapsto(1-z)$, as the following result shows (see [28, Prop. 3.2] for $\alpha=0$).

Proposition 4.5: *Let $\omega\in(\pi/2,\pi)$, $\mathfrak{h}\in\mathcal{E}_0(S_{\omega})$ and $\mathfrak{f}(z):=\mathfrak{h}(1-z)$ for $z\in\mathbb{D}$. Let $\alpha>0$ and let T be a (C,α)-bounded operator. Then*

$$\mathfrak{f}\in A^{\alpha}(\mathbb{D})\quad\textit{and}\quad \mathfrak{f}(T)=\mathfrak{h}(A).$$

Proof. Take γ as given in (4.5). For every $\lambda\in\operatorname{supp}(\gamma)\setminus\{0\}$ the function

$$\mathfrak{p}_{1-\lambda}(z):=-\frac{1}{z-(1-\lambda)}=\sum_{n=0}^{\infty}p_{1-\lambda}(n)z^n,\quad z\in\mathbb{D},$$

lies in $A^{\alpha}(\mathbb{D})$ with

$$\|\mathfrak{p}_{1-\lambda}\|_{A^{\alpha}(\mathbb{D})}=\sum_{n=0}^{\infty}|[W^{\alpha}p_{1-\lambda}](n)|k_{\alpha+1}(n)\le\frac{|\lambda|^{\alpha}}{(|\lambda-1|-1)^{\alpha+1}};\tag{4.6}$$

see Example 2.4(i). Thus one has $\mathfrak{p}_{1-\lambda}(T)=-(T-(1-\lambda)I)^{-1}=-(\lambda-A)^{-1}$; see [5, Cor. 3.6] for the first equality.

Moreover, the mapping $\lambda \in \operatorname{supp}(\gamma) \setminus \{0\} \mapsto \mathfrak{p}_{1-\lambda} \in \mathrm{A}^\alpha(\mathbb{D})$ is continuous. This is readily seen by estimating $\|\mathfrak{p}_{1-\lambda} - \mathfrak{p}_{1-\mu}\|_{A^\alpha(\mathbb{D})}$ in a similar manner to (4.6), and letting $\lambda \to \mu$. Also, by (4.6),

$$\int_\gamma \|\mathfrak{h}(\lambda)\mathfrak{p}_{1-\lambda}\|_{A^\alpha(\mathbb{D})}\ |d\lambda| \leq K \int_\gamma |\lambda|^s \frac{|\lambda|^\alpha}{(|\lambda - 1| - 1)^{\alpha+1}}\ |d\lambda| < \infty,$$

where $K, s > 0$ are such that $|\mathfrak{h}(\lambda)| \leq K|\lambda|^s$, $\lambda \in \operatorname{supp}(\gamma) \setminus \{0\}$. It follows that the function $\lambda \in \operatorname{supp}(\gamma) \mapsto \mathfrak{h}(\lambda)\mathfrak{p}_{1-\lambda} \in \mathrm{A}^\alpha(\mathbb{D})$ is Bochner integrable. Then, by the Cauchy formula one gets

$$\mathfrak{f}(z) = \mathfrak{h}(1-z) = \frac{-1}{2\pi i}\int_\gamma \mathfrak{h}(\lambda)\mathfrak{p}_{1-\lambda}(z)\, d\lambda, \quad z \in \mathbb{D},$$

since point evaluations are continuous on the space $A^\alpha(\mathbb{D})$ and separate elements on $A^\alpha(\mathbb{D})$.

Finally, by (4.5) we conclude that

$$\mathfrak{f}(T) = \frac{-1}{2\pi i}\int_\gamma \mathfrak{h}(\lambda)\mathfrak{p}_{1-\lambda}(T)\, d\lambda = \frac{1}{2\pi i}\int_\gamma \mathfrak{h}(\lambda)(\lambda - A)^{-1}\, d\lambda = \mathfrak{h}(A). \qquad \blacksquare$$

Remark 4.6: Let T be a (C,α)-bounded operator, $\alpha > 0$. Let $\mathfrak{h}$ be a regularizable function for the sectorial calculus with regularizer $\mathfrak{b}$, so that

$$\mathfrak{b}\mathfrak{h} \in \bigcup_{(\pi/2)<\omega<\pi} \mathcal{E}_0(S_\omega)$$

and $\mathfrak{b}(A)$ is injective. Then, for $\mathfrak{f}(z) := \mathfrak{h}(1-z)$ and $\mathfrak{e}(z) := \mathfrak{b}(1-z)$, $z \in \mathbb{D}$, applying the previous proposition to $\mathfrak{b}\mathfrak{h}$ and $\mathfrak{b}$ one gets that $\mathfrak{f}$ is regularizable in the $A^\alpha(\mathbb{D})$-calculus (with regularizer $\mathfrak{e}$) and $\mathfrak{f}(T) = \mathfrak{h}(A)$. This simple observation will be used in the two following examples, which are of key importance in the paper.

Example 4.7: (i) Let $r \in \mathbb{R}$ and set

$$\mathfrak{q}_r(z) := (1-z)^{-r} = \sum_{n=0}^\infty k_r(n) z^n, \quad z \in \mathbb{D}.$$

Let $\alpha > 0$. Clearly, $\mathfrak{q}_r \in A^\alpha(\mathbb{D})$ if $r \leq 0$ and $\mathfrak{q}_r \notin A^\alpha(\mathbb{D})$ for $r > 0$; see (2.7). In particular, for $r < 0$ one gets

$$0 = \mathfrak{q}_r(1) = \sum_{n=0}^\infty [W^\alpha k_r](n) k_{\alpha+1}(n). \tag{4.7}$$

Assume that T is a (C,α)-bounded operator on X such that $\mathrm{Ker}(I-T)=\{0\}$. Set $A := I-T$. Put $\mathfrak{h}_r(\lambda) := \lambda^{-r}$ for $r \in \mathbb{R}$. Let $s>0$ and take $n \in \mathbb{N}_0$ such that $n < s \le n+1$. Then $\mathfrak{b}(\lambda) = \lambda^{n+1}$ is a regularizer for the function $\mathfrak{h}_s(\lambda)$ in the sectorial calculus and therefore, by Remark 4.6, $\mathfrak{e} = \mathfrak{q}_{-(n+1)}$ is a regularizer for $\mathfrak{f} = \mathfrak{q}_s$ in the $A^\alpha(\mathbb{D})$-calculus, having moreover

$$(I-T)^{-s} := \mathfrak{q}_s(T) = (\mathfrak{e}(T))^{-1}\mathfrak{q}_{s-n-1}(T) = (\mathfrak{b}(A))^{-1}\mathfrak{h}_{s-n-1}(A) = \mathfrak{h}_s(A) = A^{-s}.$$

In other words, the unbounded closed operator $(I-T)^{-s}$ on X is the same if obtained, by regularization, either from the $A^\alpha(\mathbb{D})$-calculus or from the sectorial calculus.

(ii) Looking again at Remark 4.6, take now

$$\mathfrak{h}(\lambda) = \log(\lambda), \quad \mathfrak{b}(\lambda) = \lambda;$$
$$\mathfrak{e}(z) = (1-z), \quad \mathfrak{f}(z) = \log(1-z).$$

Similarly to the example in (i), we have that $\log(1-z)$ is α-regularizable by $1-z$ and that

$$\log(I-T) := (I-T)^{-1}(\mathfrak{e}\mathfrak{f})(T) = A^{-1}\mathfrak{h}(A) = \log(A)$$

is the same (unbounded and closed) operator on X independently of using either the sectorial calculus or the $A^\alpha(\mathbb{D})$-calculus.

The two properties reflected in the next proposition will be used in Corollary 7.2.

Proposition 4.8: *Let $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator with $\alpha > 0$. Then*

$$\lim_{s\to 1} \|(I-T)^s - (I-T)\| = 0, \tag{4.8}$$

and, for $0 < s < 1$,

$$\overline{(I-T)^s X} = \overline{(I-T)X}. \tag{4.9}$$

Proof. To show the limit, it is enough to use the Dunford–Riesz formula in the sectorial calculus applied to the function $\lambda^s - \lambda$ and then letting $s \to 1$.

With regard to (4.9), it is clear that $(I-T)X \subseteq (I-T)^s X$ whence

$$\overline{(I-T)X} \subseteq \overline{(I-T)^s X}.$$

To prove the converse, note that

$$\sum_{n=1}^{\infty}[W^\alpha k_{-s}](n)k_{\alpha+1}(n)=\sum_{n=0}^{\infty}[W^\alpha k_{-s}](n)k_{\alpha+1}(n)-[W^\alpha k_{-s}](0) = -[W^\alpha k_{-s}](0)$$

by (4.7). Furthermore, taking $\beta = \alpha + 1$ in the equality

$$(I-T)[\Delta^{-\beta}\mathcal{T}](n-1) = k_\beta(n) - \sum_{j=0}^{n} k_{\beta-\alpha-1}(n-j)[\Delta^{-\alpha}\mathcal{T}](j)$$

given in the proof of Lemma 3.1, one gets

$$[\Delta^{-(\alpha+1)}\mathcal{T}](n-1)(T-I) = [\Delta^{-\alpha}\mathcal{T}](n) - k_{\alpha+1}(n)I, \quad n \in \mathbb{N}.$$

Therefore, for $x \in X$ and $y = (I-T)^s x$, one has

$$\begin{aligned} y = \Theta_\alpha(\mathfrak{q}_{-s})x &= \sum_{n=0}^{\infty}[W^\alpha k_{-s}](n)[\Delta^{-\alpha}\mathcal{T}](n)x \\ &= \sum_{n=1}^{\infty}[W^\alpha k_{-s}](n)[\Delta^{-\alpha}\mathcal{T}](n)x + [W^\alpha k_{-s}](0)x \\ &= \sum_{n=1}^{\infty}[W^\alpha k_{-s}](n)([\Delta^{-\alpha}\mathcal{T}](n) - k_{\alpha+1}(n)I)x \\ &= \sum_{n=1}^{\infty}[W^\alpha k_{-s}](n)[\Delta^{-(\alpha+1)}\mathcal{T}](n-1)(T-I)x, \end{aligned}$$

so that $y \in \overline{(I-T)X}$. Then $\overline{(I-T)^s X} \subseteq \overline{(I-T)X}$ as we wanted to show. ∎

Remark 4.9: For a (C,α)-bounded operator T as above it can be also shown that $(I-T)^s X$ is closed if and only if $(I-T)X$ is closed. We do not include the proof of this result since it is not needed in the paper. Such a property and Proposition 4.8 are proved for power-bounded operators (case $\alpha = 0$) in [16, Prop. 2.1].

5. Admissibility and fractional differences

Here we generalize the notion of admissible function introduced in [28] and show how to approximate the key function $\mathfrak{q}_{-1}(z) = 1 - z$ in $A^\alpha(\mathbb{D})$ using (specific for $\mathfrak{q}_{-1}$) approximations of the identity. Our first result gives a representation of analytic functions by fractional differences.

Lemma 5.1: *Let $\alpha > 0$ and let $\mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n) z^n$ be a holomorphic function in $\mathbb{D}$ such that $D^\alpha f$ exists, $[D^\alpha f](j) \geq 0$ for all j and $W^{-\alpha}(D^\alpha f) = f$. Then*

$$\mathfrak{f}(z) = \sum_{j=0}^{\infty} [D^\alpha f](j)[\Delta^{-\alpha} \mathcal{Z}](j), \quad z \in \mathbb{D},$$

where the series converges absolutely and uniformly in compact subsets of $\mathbb{D}$.

Proof. Let $z \in [0,1)$. By Fubini's Theorem one gets

$$\sum_{j=0}^{\infty} [D^\alpha f](j)[\Delta^{-\alpha} \mathcal{Z}](j) = \sum_{l=0}^{\infty} z^l \sum_{j=l}^{\infty} k_\alpha(j-l)[D^\alpha f](j) = \mathfrak{f}(z).$$

Furthermore, it is clear that the series converges uniformly and absolutely on compact subsets of $\mathbb{D}$ since the representation holds in particular for $z \in [0,1)$ and $D^\alpha f \geq 0$. ∎

Let $\mathfrak{f}$ be a holomorphic function on $\mathbb{D}$ with positive Taylor coefficients. We will put $\mathfrak{f}(1)$ to denote the limit

$$\mathfrak{f}(1) := \lim_{0<z\nearrow 1} \mathfrak{f}(z)$$

in $[0, +\infty) \cup \{+\infty\}$.

Proposition 5.2: *Let $\alpha \geq 0$ and let $\mathfrak{f}$ be a zero-free holomorphic function on $\mathbb{D}$ with non-negative Taylor coefficients and $\mathfrak{f}(1) \neq 0$. Let $\frac{1}{\mathfrak{f}}$ be given by*

$$\frac{1}{\mathfrak{f}(z)} = \sum_{n=0}^{\infty} g(n) z^n, \quad z \in \mathbb{D},$$

and assume that there exists $W^\alpha g$, with $g(0) \geq 0$, $[W^\alpha g](0) \geq 0$ and $g(j) \leq 0$, $[W^\alpha g](j) \leq 0$, for $j \geq 1$.

Then $(g(n)) \in \tau^\alpha(\mathbb{N}_0)$ and therefore $\frac{1}{\mathfrak{f}}$ extends continuously to $\overline{\mathbb{D}}$, $\mathfrak{f}$ does not have zeros in $\overline{\mathbb{D}}$ and $\frac{1}{\mathfrak{f}} \in A^\alpha(\mathbb{D})$. Moreover,

$$\left\| \frac{1}{\mathfrak{f}} \right\|_{A^\alpha(\mathbb{D})} = 2[W^\alpha g](0) - \frac{1}{\mathfrak{f}(1)},$$

where $1/\infty := 0$ if $\mathfrak{f}(1) = \infty$.

Proof. Since $-W^\alpha g$ is nonnegative it follows that

$$\begin{aligned}\sum_{j=1}^{\infty}[W^\alpha g](j)k_{\alpha+1}(j) &= \lim_{0<z\nearrow 1}\sum_{j=1}^{\infty}[W^\alpha g](j)[\Delta^{-\alpha}\mathcal{Z}](j)\\ &= \lim_{0<z\nearrow 1}\sum_{j=1}^{\infty} g(j)z^j\\ &= \lim_{0<z\nearrow 1}\frac{1}{\mathfrak{f}(z)} - [W^\alpha g](0).\end{aligned}$$

This implies that $(g(n)) \in \tau^\alpha(\mathbb{N}_0)$ with

$$\|(g(n))\|_{1,(\alpha)} = [W^\alpha g](0) - \sum_{j=1}^{\infty}[W^\alpha g](j)k_{\alpha+1}(j) = 2[W^\alpha g](0) - \frac{1}{\mathfrak{f}(1)}.$$

In particular, $1/\mathfrak{f}$ admits a continuous extension at every $z \in \mathbb{C}$ with $|z| = 1$, since $A^\alpha(\mathbb{D}) \subset A^0(\mathbb{D})$, where $A^0(\mathbb{D}) \equiv \ell^1$. Thus $\mathfrak{f}$ does not have zeros on $\overline{\mathbb{D}}$. In other words, $\frac{1}{\mathfrak{f}} \in A^\alpha(\mathbb{D})$ with norm

$$\Big\|\frac{1}{\mathfrak{f}}\Big\|_{A^\alpha(\mathbb{D})} = 2[W^\alpha g](0) - \frac{1}{\mathfrak{f}(1)}. \qquad \blacksquare$$

The formula given in Proposition 5.4 below is established in [5, Lemma 2.7] for f and h in the Banach algebra $\tau^\alpha(\mathbb{N}_0)$. Here we will need the formula for $h \in \tau^\alpha(\mathbb{N}_0)$ but with f not necessarily in $\tau^\alpha(\mathbb{N}_0)$. Its proof is rather involved and needs the following lemma.

Lemma 5.3: *Let $\alpha > 0$, $q \in \mathbb{N}$ and $v \in \mathbb{N}_0$. Then*

$$k_\alpha(q+v) = -\sum_{p=0}^{q-1} k_\alpha(p) \sum_{j=q-p}^{v+q-p} k_{-\alpha}(j)k_\alpha(v+q-p-j).$$

Proof. We apply the induction method. First note that for $v = 0$ and $q \in \mathbb{N}$,

$$\sum_{p=0}^{q-1} k_\alpha(p) \sum_{j=q-p}^{q-p} k_{-\alpha}(j)k_\alpha(q-p-j) = \sum_{p=0}^{q-1} k_\alpha(p)k_{-\alpha}(q-p) = -k_\alpha(q).$$

Now, suppose that the identity of the statement is true for $v = m$. Then, for $v = m + 1$,

$$\begin{aligned}
&\sum_{p=0}^{q-1} k_\alpha(p) \sum_{j=q-p}^{m+1+q-p} k_{-\alpha}(j)k_\alpha(m+1+q-p-j)\\
&\quad=\sum_{p=0}^{q-1} k_\alpha(p)\Big(k_\alpha(m+1)k_{-\alpha}(q-p)+\sum_{j=q-p+1}^{m+1+q-p} k_{-\alpha}(j)k_\alpha(m+1+q-p-j)\Big)\\
&\quad=-k_{-\alpha}(m+1)k_\alpha(q)+\sum_{p=0}^{q-1} k_\alpha(p)\sum_{j=q-p+1}^{m+1+q-p} k_{-\alpha}(j)k_\alpha(m+1+q-p-j)\\
&\quad=k_\alpha(q)\sum_{j=1}^{m+1} k_{-\alpha}(j)k_\alpha(m+1-j)\\
&\qquad+\sum_{p=0}^{q-1} k_\alpha(p)\sum_{j=q-p+1}^{m+1+q-p} k_{-\alpha}(j)k_\alpha(m+1+q-p-j)\\
&\quad=\sum_{p=0}^{q} k_\alpha(p)\sum_{j=q-p+1}^{m+1+q-p} k_{-\alpha}(j)k_\alpha(m+1+q-p-j)\\
&\quad=-k_\alpha(m+(q+1))=-k_\alpha((m+1)+q), \quad q\in\mathbb{N}.
\end{aligned}$$

Thus we have completed the induction process and this finishes the proof. ∎

The following proposition will play a key role in the subsequent considerations.

Proposition 5.4: *Let $\alpha > 0$ and let f, h be sequences such that*

(i) *f is a bounded sequence, $f(j) \geq 0, [D^\alpha f](j) \geq 0$ for $j \in \mathbb{N}_0$ and $W^{-\alpha}(D^\alpha f) = f$;*
(ii) *$h \in \tau^\alpha(\mathbb{N}_0)$ with $[D^\beta h](0) \geq 0$ and $[D^\beta h](j) \leq 0$ $(j \geq 1)$, for $\beta = 0$ and $\beta = \alpha$;*
(iii) *$f * h \in \tau^\alpha(\mathbb{N}_0)$.*

Then

$$[W^\alpha(f*h)](v) := \Big(\sum_{j=0}^{v}\sum_{l=v-j}^{v} - \sum_{j=v+1}^{\infty}\sum_{l=v+1}^{\infty}\Big)k_\alpha(l+j-v)[D^\alpha f](j)[W^\alpha h](l),$$

for $v \in \mathbb{N}_0$.

Proof. Note that

$$|[W^\alpha(f*h)](v)| = |[D^\alpha(f*h)](v)| \le \sum_{j=0}^{\infty} |k_{-\alpha}(j)| \sum_{l=0}^{j+v} f(j+v-l)|h(l)| < \infty$$

since f is bounded and $k_{-\alpha}$ and h are in ℓ^1. Thus we can exchange the summation order in $W^\alpha(f*h)(v)$ to find that

$$\begin{aligned}
&[W^\alpha(f*h)](v)\\
&\quad = \sum_{l=0}^{v} h(l) \sum_{j=0}^{\infty} k_{-\alpha}(j) f(j+v-l) + \sum_{l=v+1}^{\infty} h(l) \sum_{j=l-v}^{\infty} k_{-\alpha}(j) f(j+v-l)\\
&\quad = (h*D^\alpha f)(v) + \sum_{l=v+1}^{\infty} [W^{-\alpha}(W^\alpha h)](l) \sum_{j=l-v}^{\infty} k_{-\alpha}(j)[W^{-\alpha}(D^\alpha f)](j+v-l).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\Big| \sum_{l=v+1}^{\infty} [W^{-\alpha}(W^\alpha h)](l) \sum_{j=l-v}^{\infty} k_{-\alpha}(j)[W^{-\alpha}(D^\alpha f)](j+v-l)\Big|\\
&\quad \le - \sum_{l=v+1}^{\infty} \sum_{p=l}^{\infty} k_\alpha(p-l)[W^\alpha h](p) \sum_{j=l-v}^{\infty} |k_{-\alpha}(j)| \sum_{q=j+v-l}^{\infty} k_\alpha(q-j-v+l)[D^\alpha f](q)\\
&\quad = - \sum_{l=v+1}^{\infty} h(l) \sum_{j=l-v}^{\infty} |k_{-\alpha}(j)| f(j+v-l) < \infty.
\end{aligned}$$

Therefore, rearranging the above series and using Lemma 5.3 one obtains

$$\begin{aligned}
&\sum_{l=v+1}^{\infty} h(l) \sum_{j=l-v}^{\infty} k_{-\alpha}(j) f(j+v-l)\\
&\quad = \sum_{q=0}^{\infty} \sum_{p=v+1}^{\infty} [D^\alpha f](q)[W^\alpha h](p) \sum_{l=v+1}^{p} k_\alpha(p-l) \sum_{j=l-v}^{q+l-v} k_{-\alpha}(j) k_\alpha(q-j-v+l)\\
&\quad = \sum_{q=0}^{\infty} \sum_{p=v+1}^{\infty} [D^\alpha f](q)[W^\alpha h](p) \sum_{m=0}^{p-v-1} k_\alpha(m) \sum_{j=p-v-m}^{q+p-v-m} k_{-\alpha}(j) k_\alpha(q+p-v-j-m)\\
&\quad = - \sum_{q=0}^{\infty} \sum_{p=v+1}^{\infty} [D^\alpha f](q)[W^\alpha h](p) k_\alpha(p-v+q).
\end{aligned}$$

On the other hand,

$$(h * D^\alpha f)(v) = \sum_{j=0}^{v}[D^\alpha f](j)h(v-j) = \sum_{j=0}^{v}[D^\alpha f](j)\sum_{l=v-j}^{\infty} k_\alpha(l+j-v)[W^\alpha h](l).$$

Altogether, the result follows. ∎

Assumptions in the above results suggest the following definition.

Definition 5.5: Let $\alpha \geq 0$, β be either 0 or α, and $\mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n)z^n$ be a holomorphic function on $\mathbb{D}$. Then $\mathfrak{f}$ is said to be α-**admissible** if:

(i) The sequence $(f(n))$ is bounded, $[D^\beta f](n) \geq 0$ for $n \in \mathbb{N}_0$, and

$$W^{-\beta}(D^\beta f) = f.$$

(ii) The function $\mathfrak{f}$ does not have zeros in $\mathbb{D}$ and, if $\mathfrak{g}(z) := \frac{1}{\mathfrak{f}(z)} = \sum_{n=0}^{\infty} g(n)z^n$, $z \in \mathbb{D}$, then the differences $W^\beta g$ exist and satisfy $[W^\beta g](n) \leq 0$ for $n \geq 1$ and $[W^\beta g](0) \geq 0$.

Remark 5.6: (a) If $\alpha = m \in \mathbb{N}$ then condition $W^{-m}(D^m f) = f$ in (i) above is redundant. (b) From the assumptions in the above definition and Proposition 5.2, it follows that $\mathfrak{g} \in A^\alpha(\mathbb{D})$.

Next, we discuss α-admissibility in relation with the algebra $A^\alpha(\mathbb{D})$. Let $\mathfrak{f}(z) = \sum_{j=0}^{\infty} f(j)z^j$ be an α-admissible function on $\mathbb{D}$ with reciprocal $\frac{1}{\mathfrak{f}(z)} = \sum_{j=0}^{\infty} g(j)z^j$. The fact that $\mathfrak{f}\frac{1}{\mathfrak{f}} = 1$ means that $f * g = k_0 (= \delta_0)$. For every $n \in \mathbb{N}$, let $\mathfrak{g}_n$ be the function

$$\mathfrak{g}_n(z) = \frac{1}{\mathfrak{f}(z)}\sum_{j=0}^{n-1}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{Z}](j) = \sum_{j=0}^{\infty} g_n(j)z^j, \tag{5.1}$$

and put $\mathfrak{f}_n := \mathfrak{g}_n\mathfrak{f}$. Note that $\mathfrak{g}_n \in A^\alpha(\mathbb{D})$ since $\frac{1}{\mathfrak{f}}, \mathfrak{f}_n \in A^\alpha(\mathbb{D})$. Note also that $\mathfrak{g}_n(1) = 0$ if $\mathfrak{f}(1) = \infty$, and that $\mathfrak{g}_n(1) \leq 1$ if $\mathfrak{f}(1) < \infty$, by Lemma 5.1. We proceed with finding a suitable estimate of the norm of $\mathfrak{g}_n$ in $A^\alpha(\mathbb{D})$.

THEOREM 5.7: *Let $\alpha > 0$. Let $\mathfrak{f}$ be an α-admissible function on $\mathbb{D}$ and $\mathfrak{g}_n$ be defined by* (5.1). *Then*

$$\|\mathfrak{g}_n\|_{A^\alpha(\mathbb{D})} \leq 2 - \mathfrak{g}_n(1) \quad \textit{for all } n \geq 1.$$

Proof. By Lemma 5.1 one has $\mathfrak{f}(z) = \sum_{j=0}^{\infty}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{Z}](j)$ for all $z \in \mathbb{D}$. This series converges absolutely and uniformly on compact subsets. Set

$$\mathfrak{F}_n(z) := \sum_{j=n}^{\infty}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{Z}](j).$$

It is readily seen that the sequence of coefficients $(f_n(j))_{j\geq 0}$ of $\mathfrak{f}_n = \mathfrak{g}_n\mathfrak{f}$ is such that

$$[D^\alpha f_n](j) = [D^\alpha f](j), \text{ if } 0 \leq j < n; \quad [D^\alpha f_n](j) = 0, \text{ if } j \geq n.$$

Therefore the sequence of coefficients $(\varphi_n(j))_{j\geq 0}$ of $\mathfrak{F}_n$ satisfies

$$[D^\alpha \varphi_n](j) = 0, \text{ if } 0 \leq j < n; \quad [D^\alpha \varphi_n](j) = [D^\alpha f](j), \text{ if } j \geq n,$$

since $D^\alpha\varphi_n = D^\alpha f - D^\alpha f_n$. Thus applying Proposition 5.4 to the polynomial $\mathfrak{f}_n$ we have

$$[W^\alpha g_n](v) = [W^\alpha(f_n * g)](v) := \sum_{j=0}^{n-1}\sum_{l=v-j}^{v} k_\alpha(l+j-v)[D^\alpha f](j)[W^\alpha g](l) \leq 0$$

for every $v \geq n$. Furthermore

$$[W^\alpha g_n](0) = [D^\alpha f](0)[W^\alpha g](0) - \sum_{j=1}^{n-1}\sum_{l=1}^{\infty} k_\alpha(l+j)[D^\alpha f](j)[W^\alpha g](l) \geq 0.$$

On the other hand $\varphi_n * g = f * g - f_n * g = \delta_0 - f_n * g \in \tau^\alpha(\mathbb{N}_0)$ whence

$$[W^\alpha g_n](v) = [W^\alpha \delta_0](v) - [W^\alpha(\varphi_n * g)](v) = \delta_0(v) - [W^\alpha(\varphi_n * g)](v).$$

Now, since $\delta_0(0) = 1$ and $\delta_0(v) = 0$ for $v \geq 1$, applying Proposition 5.4 to $\varphi_n * g$ one gets

$$[W^\alpha g_n](v) = \sum_{j=n}^{\infty}\sum_{l=v+1}^{\infty} k_\alpha(l+j-v)[D^\alpha f](j)[W^\alpha g](l) \leq 0, \quad 1 \leq v \leq n-1,$$

and

$$[W^\alpha g_n](0) = 1 + \sum_{j=n}^{\infty}\sum_{l=1}^{\infty} k_\alpha(l+j)[D^\alpha f](j)[W^\alpha g](l) \leq 1,$$

since the sums of both double series are nonpositive; see conditions (i) and (ii) on $D^\alpha f$, $W^\alpha g$ in Definition 5.5.

All in all,

$$\begin{aligned}\|\mathfrak{g}_n\|_{A^\alpha(\mathbb{D})} &= \sum_{j=0}^{\infty} |[W^\alpha g_n](j)| k_{\alpha+1}(j) \\ &= [W^\alpha g_n](0) - \sum_{j=1}^{\infty} [W^\alpha g_n](j) k_{\alpha+1}(j) \\ &= 2[W^\alpha g_n](0) - \sum_{j=0}^{\infty} [W^\alpha g_n](j) k_{\alpha+1}(j) \\ &= 2[W^\alpha g_n](0) - \mathfrak{g}_n(1) \le 2 - \mathfrak{g}_n(1),\end{aligned}$$

as we wanted to prove. ∎

The function $\mathfrak{q}_{-1}(z) = 1 - z$ in $A^\alpha(\mathbb{D})$ plays a central role in our discussion. We say that a sequence $(\mathfrak{a}_n)_{n\ge 1} \subset A^\alpha(\mathbb{D})$ is a $(1-z)$-bounded approximate identity if

$$\lim_{n\to\infty} (1-z)\mathfrak{a}_n(z) = 1 - z$$

in the norm of $A^\alpha(\mathbb{D})$. Theorem 5.8 shows in particular that the sequence $(\mathfrak{g}_n)_{n\ge 1}$ is a $(1-z)$-bounded approximate identity. If $\mathfrak{f}(1) < \infty$ then $(\mathfrak{g}_n)_{n\ge 1}$ converges in the norm to the identity element in the algebra $A^\alpha(\mathbb{D})$.

Theorem 5.8: *Let $\alpha > 0$. Let $\mathfrak{f}$ be an α-admissible function and $\mathfrak{g}_n$ be defined by* (5.1).

(i) *If $\mathfrak{f}(1) < \infty$ then $\lim_{n\to\infty} \mathfrak{g}_n(z) = 1$ in the norm in $A^\alpha(\mathbb{D})$.*
(ii) *If $\mathfrak{f}(1) = \infty$ then $\|\mathfrak{g}_n\|_{A^\alpha(\mathbb{D})} \le 2$ for every n.*
(iii) *If $(1-z)\mathfrak{f}(z) \in A^\alpha(\mathbb{D})$ and $[D^\alpha f](j) j^\alpha \to 0$ as $j \to \infty$, then*

$$\lim_{n\to\infty} (1-z)\mathfrak{g}_n(z) = 1 - z \quad \text{in } A^\alpha(\mathbb{D}).$$

Proof. (i) If $\mathfrak{f}(1) < \infty$, by Lemma 5.1 we have

$$\mathfrak{f}(1) = \lim_{0<z\nearrow 1} \sum_{j=0}^{\infty} [D^\alpha f](j)[\Delta^{-\alpha}\mathcal{Z}](j) = \sum_{j=0}^{\infty} [D^\alpha f](j) k_{\alpha+1}(j).$$

Therefore

$$\|\mathfrak{g}_n - 1\|_{A^\alpha(\mathbb{D})} = \frac{1}{\mathfrak{f}(1)} \sum_{j=n}^{\infty} [D^\alpha f](j) k_{\alpha+1}(j) \to 0, \quad n \to \infty.$$

(ii) If $\mathfrak{f}(1)=\infty$, then the proof of Theorem 5.7 gives

$$\|\mathfrak{g}_n\|_{A^\alpha(\mathbb{D})} = 2[W^\alpha g_n](0) - \mathfrak{g}_n(1) = 2[W^\alpha g_n](0) - \frac{1}{\mathfrak{f}(1)}\sum_{j=0}^{n-1}[D^\alpha f](j)k_{\alpha+1}(j) \leq 2.$$

(iii) Note that

$$(1-z)\mathfrak{g}_n(z) = \frac{1}{\mathfrak{f}(z)}\left(\sum_{j=0}^{n-1}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{Z}](j) - \sum_{j=0}^{n-1}[D^\alpha f](j)\; z[\Delta^{-\alpha}\mathcal{Z}](j)\right)$$

with

$$\begin{aligned}\sum_{j=0}^{n-1}[D^\alpha f](j)z[\Delta^{-\alpha}\mathcal{Z}](j) &= \sum_{j=1}^{n}[D^\alpha f](j-1)\sum_{l=1}^{j}k_\alpha(j-l)z^l \\ &= \sum_{j=1}^{n}[D^\alpha f](j-1)([\Delta^{-\alpha}\mathcal{Z}](j)-k_\alpha(j)).\end{aligned}$$

Put

$$(1-z)\mathfrak{g}_n(z) = \mathfrak{h}_n(z) - \mathfrak{r}_n(z) \tag{5.2}$$

where

$$\begin{aligned}\mathfrak{h}_n(z) &:= \frac{1}{\mathfrak{f}(z)}\Bigg([D^\alpha f](0) + \sum_{j=1}^{n-1}([D^\alpha f](j)-[D^\alpha f](j-1))[\Delta^{-\alpha}\mathcal{Z}](j) \\ &\qquad + \sum_{j=1}^{n}[D^\alpha f](j-1)k_\alpha(j)\Bigg) \\ &= \frac{1}{\mathfrak{f}(z)}\left([D^\alpha f](0) + \sum_{j=1}^{n}[D^\alpha f](j-1)k_\alpha(j) - \sum_{j=1}^{n-1}[D^{\alpha+1} f](j-1)[\Delta^{-\alpha}\mathcal{Z}](j)\right)\end{aligned}$$

and

$$\mathfrak{r}_n(z) := \frac{1}{\mathfrak{f}(z)}[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{Z}](n).$$

Let us remark that both $\mathfrak{h}_n$ and $\mathfrak{r}_n$ belong to $A^\alpha(\mathbb{D})$. We claim that

$$\lim_{n\to\infty}\|\mathfrak{h}_n-(1-z)\|_{A^\alpha(\mathbb{D})} = 0 \quad\text{and}\quad \lim_{n\to\infty}\|\mathfrak{r}_n\|_{A^\alpha(\mathbb{D})} = 0. \tag{5.3}$$

To see this, note that

$$\|\mathfrak{h}_n-(1-z)\|_{A^\alpha(\mathbb{D})} \leq M_\alpha\left\|\frac{1}{\mathfrak{f}}\right\|_{A^\alpha(\mathbb{D})}\|\mathfrak{h}_n\mathfrak{f}-(1-z)\mathfrak{f}\|_{A^\alpha(\mathbb{D})}.$$

By Lemma 5.1,

$$\begin{aligned}(1-z)\mathfrak{f}(z) =&\sum_{j=0}^{\infty}[D^{\alpha}f](j)[\Delta^{-\alpha}\mathcal{Z}](j)-\sum_{j=0}^{\infty}[D^{\alpha}f](j)\ z[\Delta^{-\alpha}\mathcal{Z}](j)\\ =&[D^{\alpha}f](0)+\sum_{j=1}^{\infty}([D^{\alpha}f](j)-[D^{\alpha}f](j-1))[\Delta^{-\alpha}\mathcal{Z}](j)\\ &+\sum_{j=1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)\\ =&[D^{\alpha}f](0)+\sum_{j=1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)-\sum_{j=1}^{\infty}[D^{\alpha+1}f](j-1)[\Delta^{-\alpha}\mathcal{Z}](j),\end{aligned}$$

where the series in the latter line converges since

$$\sum_{j=1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)\le K\sum_{j=0}^{\infty}[D^{\alpha}f](j)k_{\alpha}(j)=K\mathfrak{f}(0),$$

for some constant $K>0$. Moreover, the expansion

$$(1-z)\mathfrak{f}(z)=[D^{\alpha}f](0)+\sum_{j=1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)-\sum_{j=1}^{\infty}[D^{\alpha+1}f](j-1)[\Delta^{-\alpha}\mathcal{Z}](j)$$

shows that the series gives us the representation (4.3) for $(1-z)\mathfrak{f}$ as an element of $A^{\alpha}(\mathbb{D})$ by (4.1), (4.2) and Lemma 4.1. Hence,

$$\begin{aligned}&\|\mathfrak{h}_n\mathfrak{f}-(1-z)\mathfrak{f}\|_{A^{\alpha}(\mathbb{D})}\\ &\qquad=\Big\|-\sum_{j=n+1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)+\sum_{j=n}^{\infty}[D^{\alpha+1}f](j-1)[\Delta^{-\alpha}\mathcal{Z}](j)\Big\|_{A^{\alpha}(\mathbb{D})}\\ &\qquad=\sum_{j=n+1}^{\infty}[D^{\alpha}f](j-1)k_{\alpha}(j)+\sum_{j=n}^{\infty}|[D^{\alpha+1}f](j-1)|k_{\alpha+1}(j),\end{aligned}$$

whence $\|\mathfrak{h}_n-(1-z)\|_{A^{\alpha}(\mathbb{D})}\to 0$ as $n\to\infty$.

On the other hand,

$$\begin{aligned}\|\mathfrak{r}_n\|_{A^{\alpha}(\mathbb{D})}&\le M_{\alpha}\|1/\mathfrak{f}\|_{A^{\alpha}(\mathbb{D})}\|[D^{\alpha}f](n-1)[\Delta^{-\alpha}\mathcal{Z}](n)\|_{A^{\alpha}(\mathbb{D})}\\ &=M_{\alpha}\|1/\mathfrak{f}\|_{A^{\alpha}(\mathbb{D})}[D^{\alpha}f](n-1)k_{\alpha+1}(n)\to 0,\quad \text{as } n\to\infty,\end{aligned}$$

by the assumption in part (iii). Thus the proof is over. ∎

To end this section, we provide several examples of α-admissible functions crucial for the sequel. These examples have been inspired by [25].

General examples. Hausdorff moments. Let ν be a bounded positive Borel measure on $[0,1)$ such that

$$c_0 := \int_0^1 (1-t)^{-1} d\nu(t) < \infty.$$

Let $(c_n)_{n\geq 1}$ be the Hausdorff moment sequence associated to ν, that is,

$$c_n := \int_0^1 t^{n-1} d\nu(t), \quad n \in \mathbb{N}.$$

Proposition 5.9: *Let $a, b \geq 0$. Given ν, (c_n) as before, put $h(0) = a + b + c_0$, $h(1) = -b - c_1$ and $h(n) = -c_n$, $n \geq 2$. For $z \in \mathbb{D}$, set*

$$\mathfrak{h}(z) = \sum_{n=0}^{\infty} h(n) z^n \quad \text{and} \quad \mathfrak{f}(z) := (1-z)^{-1}\mathfrak{h}(z) = \sum_{n=0}^{\infty} f(n) z^n.$$

Then $(h(n))_{n\geq 0} \in \ell^1$ and

$$(\forall \alpha \geq 0) \quad [D^\alpha h](0) \geq 0, \ [D^\alpha h](n) \leq 0 \ (n \geq 1).$$

Also,

$$f(n) \geq 0, \quad [D^\alpha f](n) \geq 0 \ (n \geq 0) \quad \text{and} \quad \lim_{n\to\infty} n^\alpha [D^\alpha f](n) = 0 \ \text{ for all } \alpha > 0.$$

Proof. Since

$$\sum_{n=1}^{\infty} c_n = \int_0^1 \sum_{n=1}^{\infty} t^{n-1} d\nu(t) = \int_0^1 (1-t)^{-1} d\nu(t) = c_0 < \infty$$

one has $(h(n)) \in \ell^1$. Also, $h(n) \leq 0$ for every $n \geq 1$ by hypothesis. Now, note that the sequence $(t^{j-1})_{j\geq 1}$ is in ℓ^1 for every $t \in (0,1)$. Then we have, for $n \geq 2$,

$$\begin{aligned}
[D^\alpha h](n) &= -\int_0^1 \sum_{j=n}^{\infty} k_{-\alpha}(j-n) t^{j-1} d\nu(t) = -\int_0^1 [D^\alpha p_{1/t}](n) \ t^{-2} d\nu(t) \\
&= -\int_0^1 [W^\alpha p_{1/t}](n) \ t^{-2} d\nu(t) \\
&= -\int_0^1 (1-t)^\alpha t^{n-1} d\nu(t) \leq 0,
\end{aligned}$$

where the first equality is permitted because $k_{-\alpha}(j)$ has constant sign for $j \geq [\alpha]+1$, see (2.6); the second equality holds because W^α and D^α coincide on ℓ^1 by Proposition 2.3 (iv); for the last equality see Example 2.4(i). Similarly,

$$[D^\alpha h](1) = -b - \int_0^1 (1-t)^\alpha d\nu(t).$$

To prove that $[D^\alpha h](0) \geq 0$ we proceed as follows. Notice that for every $q \in \mathbb{N}$ one has $[D^q h](0) = [D^{q-1}h](0) - [D^{q-1}h](1)$. From here and the fact that $h(0) \geq 0$ and $[D^{q-1}h](1) \leq 0$ one obtains readily the inequality $[D^q h](0) \geq 0$, $q \in \mathbb{N}$, by induction. Now, for any β such that $\alpha > \beta > 0$,

$$\begin{aligned}[D^{\alpha-\beta}(D^\beta h)](0) &= \sum_{n=0}^{\infty} k_{-\alpha+\beta}(n) \sum_{j=n}^{\infty} k_{-\beta}(j-n)h(j) \\ &= \sum_{j=0}^{\infty} (k_{-\alpha+\beta} * k_{-\beta})(j)h(j) = \sum_{j=0}^{\infty} k_{-\alpha}(j) = [D^\alpha h](0).\end{aligned}$$

Thus taking $\beta = [\alpha]$,

$$\begin{aligned}[D^\alpha h](0) &= [D^{\alpha-[\alpha]}(D^{[\alpha]}h)](0) \\ &= k_{-\alpha+[\alpha]}(0)[D^{[\alpha]}h](0) + \sum_{n=1}^{\infty} k_{-\alpha+[\alpha]}(n)[D^{[\alpha]}h](n) \geq 0,\end{aligned}$$

since $k_{-\alpha+[\alpha]}(0) = 1$, $[D^{[\alpha]}h](0) \geq 0$ and $k_{-\alpha+[\alpha]}(n) \leq 0$, $[D^{[\alpha]}h](n) \leq 0$ for every $n \geq 1$.

As regards $\mathfrak{f}$ note that

$$(1-z)^{-1}\mathfrak{h}(z) = \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{n=0}^{\infty} h(n)z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} h(j)\right)z^n,$$

so that $f(0) = a + b + c_0 \geq 0$,

$$f(1) = h(0)+h(1) = a+\int_0^1 (1-t)^{-1}d\nu(t) - \int_0^1 d\nu(t) = a + \int_0^1 t(1-t)^{-1}d\nu(t) \geq 0$$

and for $n \geq 2$,

$$f(n) = \sum_{j=0}^{n} h(j) = a + c_0 - \sum_{j=1}^{n} c_j = a + \int_0^1 \left(\frac{1}{1-t} - \sum_{j=1}^{n} t^{j-1}\right)d\nu(t) \geq a \geq 0.$$

As for $D^\alpha f$, we have for all $n \geq 0$, $j \geq 0$,

$$\begin{aligned}\sum_{q=0}^{j+n} h(q) &= a + b + c_0 - b - c_1 - \sum_{q=2}^{j+n}\int_0^1 t^{q-1}d\nu(t) \\ &= a + \int_0^1 \frac{d\nu(t)}{1-t} - \sum_{q=1}^{j+n}\int_0^1 t^{q-1}d\nu(t) \\ &= a + \int_0^1 \left(\frac{1}{1-t} - \frac{1-t^{j+n}}{1-t}\right)d\nu(t) = a + \int_0^1 \frac{t^{j+n}}{1-t}d\nu(t).\end{aligned}$$

Hence, for all $n \geq 0$,

$$\begin{aligned}[D^\alpha f](n) &= \sum_{j=0}^{\infty} k_{-\alpha}(j) f(j+n) = \sum_{j=0}^{\infty} k_{-\alpha}(j) \sum_{q=0}^{j+n} h(q) \\ &= \sum_{j=0}^{\infty} k_{-\alpha}(j) a + \int_0^1 [D^\alpha p_{1/t}](0) \frac{t^{n-1}}{1-t} d\nu(t) \\ &= \int_0^1 \frac{(1-t)^\alpha t^n}{1-t} d\nu(t) > 0\end{aligned}$$

where we have used that $\sum_{j=0}^{\infty} k_{-\alpha}(j) = (1-z)^\alpha \mid_{z=1} = 0$; see (2.4).

Finally,

$$\max\{t^n(1-t)^\alpha : 0 \leq t \leq 1\} = \Big(\frac{n}{n+\alpha}\Big)^n \Big(\frac{\alpha}{n+\alpha}\Big)^\alpha$$

and therefore one can apply the dominated convergence theorem to the next integral to obtain

$$n^\alpha [D^\alpha f](n) = \int_0^1 n^\alpha t^n \frac{(1-t)^\alpha}{1-t} d\nu(t) \to 0, \quad \text{as } n \to \infty. \qquad \blacksquare$$

Remark 5.10 (Complete Bernstein functions): The above proposition is a source of examples of α-admissible functions which can be constructed from complete Bernstein functions. An analytic function $H \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is said to be a complete Bernstein function if its restricton on the half-plane $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ takes the form

$$H(\lambda) = a + b\lambda + \int_0^\infty \lambda(\lambda+s)^{-1}\, d\mu(s)$$

where $a, b \geq 0$ and μ is a positive Borel measure on $(0,\infty)$ satisfying

$$\int_0^\infty (1+s)^{-1} d\mu(s) < \infty.$$

In [25, Proposition 3.8] it is proven that complete Bernstein functions and Hausdorff moments are in close connection: Let H be as before. Then, by passing in the integral to the measure ν on $(0,1)$ given by $d\nu(t) = s(1+s)^{-2} d\mu(s)$, under the map $s = 1/(1+t)$ (which gives $\int_0^\infty (1+s)^{-1} d\mu(s) = \int_0^\infty (1-t)^{-1} d\nu(t) =: c_0$), we have

$$\int_0^\infty \frac{\lambda}{\lambda+s} d\mu(s) = \int_0^1 \frac{\lambda}{1-(1-\lambda)t} \frac{d\nu(t)}{1-t} = \int_0^1 \frac{d\nu(t)}{1-t} - \int_0^1 \frac{(1-\lambda) d\nu(t)}{1-(1-\lambda)t}, \quad \Re\lambda > 0.$$

Putting $z = 1 - \lambda$ in $\mathbb{D}$, one gets

$$H(1-z) = a+b(1-z)+c_0 - \int_0^1 \frac{z\, d\nu(t)}{1-zt} = a+b(1-z)+c_0 - \sum_{j=1}^{\infty} \left(\int_0^1 t^{j-1} d\nu(t) \right) z^j,$$

that is, $\mathfrak{h}(z) := H(1-z)$ satisfies the hypotheses of Proposition 5.9.

On the other hand, the function G defined by

$$G(\lambda) := \lambda^{-1} H(\lambda), \quad \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

is also a complete Bernstein function, see [39, Th. 6.2]. So, applying the argument conducted for H to G, and Proposition 5.9 to

$$\mathfrak{g}(z) := G(1-z) = \sum_{n=0}^{\infty} g(n) z^n,$$

we obtain $g(0) \geq 0$, $[D^\alpha g](0) \geq 0$ and $g(n) \leq 0$, $[D^\alpha g](n) \leq 0$ for all $n \geq 1$.

Assume moreover that $\mathfrak{h}(z) \neq 0$ for $z \in \mathbb{D}$ and let

$$\mathfrak{f}(z) := 1/\mathfrak{g}(z) = (1-z)^{-1} \mathfrak{h}(z).$$

Then $\mathfrak{f}(n) \geq 0$ and $[D^\alpha f](n) \geq 0$ for all $n \geq 0$, and $\lim_{n\to\infty} n^\alpha [D^\alpha f](n) = 0$ by Proposition 5.9 again. In conclusion, $\mathfrak{f}$ is an α-admissible function and Theorem 5.8 is applicable to $\mathfrak{f}$.

There are many concrete complete Bernstein functions with no zero in $\{|1-\lambda| < 1\}$; see [39, Chapter 16] for an extensive list of them. For the aims of this paper we are mainly interested in the two following examples.

Example 5.11: For $0 < r < 1$, let $H_r(\lambda) := \lambda^r$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$H_r(\lambda) = \int_0^\infty \frac{\lambda}{\lambda + t} d\mu_r(t),$$

with $d\mu_r(t) = \frac{\sin(\pi r)}{\pi} t^{r-1} dt$, is a complete Bernstein function. So, the function

$$\mathfrak{q}_s(z) = (1-z)^{-s} = (1-z)^{-1} H_{1-s}(1-z), \quad z \in \mathbb{D},$$

is an α-admissible function, for every $\alpha \geq 0$ and $0 < s < 1$, to which Theorem 5.8 applies. Recall on the other hand that $([D^\alpha k_s](n))$ can be directly computed, resulting in

$$D^\alpha k_s(n) = (\sin(\pi s)/\pi) \mathrm{B}(1 - s + \alpha, s + n)$$

for all $n \in \mathbb{N}_0$; see (2.11).

Example 5.12: Let H be given by $H(\lambda) = (\lambda - 1)^{-1}\log(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Then

$$H(\lambda) = \int_0^\infty \frac{\lambda}{\lambda + t} d\mu(t),$$

with

$$d\mu(t) = \frac{(t+1)}{t\log^2(t) + \pi^2} dt,$$

is a complete Bernstein function. Hence, the function Λ defined by

$$\Lambda(z) = \frac{-\log(1-z)}{z} = \sum_{n=0}^{\infty} L(n) z^n, \quad z \in \mathbb{D},$$

where $L(n) = \frac{1}{n+1}$ for all $n \geq 0$, is an α-admissible function for every $\alpha \geq 0$. Also, the function

$$(\mathfrak{q}_{-1}\Lambda)(z) = 1 - \sum_{n=0}^{\infty} \frac{1}{n(n+1)} z^n, \quad z \in \mathbb{D},$$

is in $A^\alpha(\mathbb{D})$. Thus Λ satisfies the conditions of Theorem 5.8. Note on the other hand that a direct calculation gives us

$$D^\alpha L(n) = B(\alpha + 1, n + 1), \quad n \in \mathbb{N}_0.$$

We wish to thank the referee for pointing out references [18, 25, 31].

6. Approximating identities with Taylor coefficients

The construction of $(1-z)$-approximate identities in $A^\alpha(\mathbb{D})$ carried out in Theorem 5.8 requires using partial sums formed with fractional differences. Such approximate identities are suitable for our objectives in Section 7 and Section 8 below, concerning general domains of operatorial functions or higher degree α. For the specific examples given at the end of Section 5 and $\alpha \in (0, 1)$ one can obtain $(1-z)$-approximate identities from partial sums of Taylor expansions.

Let $\mathfrak{f}$ be a zero-free holomorphic function on the unit disc with

$$(1/\mathfrak{f})(z) = \sum_{j=0}^{\infty} g(j) z^j.$$

Let then $\mathfrak{g}_n^0$ denote the approximating function given in [28, Lemma 4.6], that is,

$$\mathfrak{g}_n^0(z) := (1/\mathfrak{f})(z) \sum_{j=0}^{n-1} f(j) z^j, \quad |z| \leq 1.$$

Put

$$\mathfrak{f}_n^0(z) := \mathfrak{g}_n^0(z)\mathfrak{f}(z)$$

and

$$g_n^0(j) = \sum_{l=0}^{n-1} f(l) g(j-l)$$

for $j \geq n$. By [28, Lemma 4.6] we have

$$\mathfrak{g}_n^0(z) = 1 + \sum_{j=n}^{\infty} g_n^0(j) z^j. \tag{6.1}$$

In the sequel, we deal with sequences $(\mathfrak{g}_n^0)_{n\geq 1}$ as before which correspond to the α-admissible functions $\mathfrak{f} = \mathfrak{q}_s$ and $\mathfrak{f} = \Lambda$ of Example 5.11 and Example 5.12 respectively. For α in a certain range of values, Theorem 6.2 and Theorem 6.4 below prove that $(\mathfrak{g}_n^0)_{n\geq 1}$ is a bounded $(1-z)$-approximate identity in each case. The first theorem requires the following result.

Lemma 6.1: *Let $m \in \mathbb{N} \cup \{0\}$ and $v, r > 0$ be such that $v + r \geq 2$. Then*

$$\frac{\Gamma(m+r+1)}{\Gamma(m+r+v)} \sum_{j=0}^{m} \frac{\Gamma(j-\lambda)}{\Gamma(j+1)} \frac{\Gamma(r+v+m-j+\lambda)}{\Gamma(r+m+1-j)}$$
$$= \frac{\Gamma(r+v+\lambda)}{\Gamma(r)} \sum_{j=0}^{m} \frac{\Gamma(j-\lambda)}{\Gamma(j+1)} \frac{\Gamma(j+r)}{\Gamma(j+r+v)}$$

for every $\lambda \in \mathbb{C}$ such that $\Re\lambda + 2 > 0$ and $\lambda \neq -1, 0, 1, 2, \ldots$.

Proof. The equality of the statement is proven in [6, Th. 1.3] for $v, r > 0$, $m \in \mathbb{N} \cup \{0\}$ and $\lambda \in (0, \infty) \setminus \mathbb{N}$. In fact, both members of the equality are analytic functions on the connected open set $\{\Re\lambda > (-2), \lambda \neq -1, 0, 1, 2, \ldots\}$ under the condition $v + r \geq 2$. So, the lemma follows from [6, Th. 1.3] by analytic continuation. ∎

THEOREM 6.2: *Let $s \in (0,1)$. Let $\mathfrak{f} = \mathfrak{q}_s$ and let $\mathfrak{g}_n^0$ be as above, corresponding to $\mathfrak{f} = \mathfrak{q}_s$. Then for $0 < \alpha < 1 - s$ one has*

(i) $\|\mathfrak{g}_n^0\|_{A^\alpha(\mathbb{D})} \le M$ *for every n, with $M > 0$ independent of n,*
(ii) $\lim_{n\to\infty}(1-z)\mathfrak{g}_n^0(z) = 1 - z$ *in $A^\alpha(\mathbb{D})$.*

Proof. (i) Let $n \in \mathbb{N}$. First of all, note that

$$g_n^0(j) = \sum_{l=0}^{n-1} k_s(l)k_{-s}(j-l) < 0,$$

for $j \ge n$. Then by (6.1) we have

$$[D^\alpha g_n^0](0) = 1 + \sum_{l=n}^{\infty} k_{-\alpha}(l)g_n^0(l) > 0;$$

$$[D^\alpha g_n^0](j) = \sum_{l=n}^{\infty} k_{-\alpha}(l-j)g_n^0(l) > 0 \quad (1 \le j \le n-1).$$

Also, if $j \ge n$ by (6.1) and the α-admissibility of $\mathfrak{q}_s \equiv (k_{-s}(n))$ we obtain

$$\begin{aligned}[D^\alpha g_n^0](j) &= \sum_{l=j}^{\infty} k_{-\alpha}(l-j)\sum_{u=0}^{n-1} k_s(u)k_{-s}(l-u)\\ &= \sum_{u=0}^{n-1} k_s(u)\sum_{l=j}^{\infty} k_{-\alpha}(l-j)k_{-s}(l-u)\\ &= \sum_{u=0}^{n-1} k_s(u)[D^\alpha k_{-s}](j-u) < 0.\end{aligned}$$

Secondly, by (4.3),

$$\sum_{j=0}^{\infty} D^\alpha g_n^0(j)k_{\alpha+1}(j) = \mathfrak{g}_n^0(1) = \frac{1}{\mathfrak{q}_{\mathfrak{s}}(1)}\sum_{j=0}^{n-1} k_s(j) = 0$$

since $\mathfrak{q}_s(1) = 0$.

Finally, note that, for $l \ge n$,

$$0 = \delta_0(l) = \sum_{u=0}^{l} k_s(u)k_{-s}(l-u)$$

and so

$$\sum_{u=0}^{n-1} k_s(u)k_{-s}(l-u) = -\sum_{u=n}^{l} k_s(u)k_{-s}(l-u).$$

Therefore,

$$\begin{aligned}
\|\mathfrak{g}_n^0\|_{A^\alpha(\mathbb{D})} &= \sum_{j=0}^{\infty} |[D^\alpha g_n^0](j)| k_{\alpha+1}(j) \\
&= \sum_{j=0}^{n-1} [D^\alpha g_n^0(j)] k_{\alpha+1}(j) - \sum_{j=n}^{\infty} [D^\alpha g_n^0](j) k_{\alpha+1}(j) \\
&= 2\sum_{j=0}^{n-1} [D^\alpha g_n^0](j) k_{\alpha+1}(j) \\
&\overset{(6.1)}{=} 2\sum_{j=0}^{n-1} k_{\alpha+1}(j) \sum_{l=n}^{\infty} k_{-\alpha}(l-j) g_n^0(l) + 2 \\
&= 2 - 2\sum_{j=0}^{n-1} k_{\alpha+1}(j) \sum_{l=n}^{\infty} k_{-\alpha}(l-j) \sum_{q=n}^{l} k_s(q) k_{-s}(l-q) \\
&= 2 - 2\sum_{j=0}^{n-1} k_{\alpha+1}(j) \sum_{q=n}^{\infty} k_s(q) [D^s k_{-\alpha}](q-j) \\
&= 2 + K_{\alpha,s} \sum_{q=n}^{\infty} k_s(q) \sum_{j=0}^{n-1} k_{\alpha+1}(j) \frac{\Gamma(-\alpha+q-j)}{\Gamma(s+1+q-j)},
\end{aligned}$$

with

$$K_{\alpha,s} = -\frac{2\Gamma(1+\alpha+s)}{\Gamma(1+\alpha)\Gamma(-\alpha)} > 0,$$

where we have used (2.11) in the last equality. Let us have a closer look at the general term of the latter series. For $q \geq n$, put

$$Q_{s,\alpha,n}(q) := \frac{\Gamma(s+q+1)}{\Gamma(q+1)} \sum_{j=0}^{n-1} \frac{\Gamma(\alpha+1+j)}{\Gamma(j+1)} \frac{\Gamma(-\alpha+q-j)}{\Gamma(s+1+q-j)},$$

so that

$$k_s(q) \sum_{j=0}^{n-1} k_{\alpha+1}(j) \frac{\Gamma(-\alpha+q-j)}{\Gamma(s+1+q-j)} = \frac{Q_{s,\alpha,n}(q)}{(q+s)\Gamma(s)\Gamma(\alpha+1)}.$$

Putting $m = n-1$, $v = 1-s$, $r = q-n+s+1$, $\lambda = -\alpha-1$, rewriting $Q_{s,\alpha,n}(q)$ accordingly, and using Lemma 6.1, one obtains, once having come back to n, s, q, α,

$$\frac{Q_{s,\alpha,n}(q)}{(q+s)\Gamma(s)\Gamma(\alpha+1)} = \frac{\Gamma(q-n+1-\alpha)}{(q+s)\Gamma(q-n+s+1)}\sum_{j=0}^{n-1} k_{\alpha+1}(j)k_s(j+q-n+1).$$

Hence, using that

$$k_s(r+1) \leq k_s(r)$$

for $0 < s < 1$ and $r \geq 0$, and (2.7) one has

$$\begin{aligned}
\|\mathfrak{g}_n^0\|_{A^\alpha(\mathbb{D})} &= 2 + K_{\alpha,s}\sum_{j=0}^{n-1} k_{\alpha+1}(j)\sum_{p=0}^{\infty}\frac{\Gamma(p+1-\alpha)}{(p+n+s)\Gamma(p+1+s)}k_s(j+p+1)\\
&\leq 2 + K_{\alpha,s}\sum_{j=0}^{n-1} k_{\alpha+1}(j)k_s(j+1)\sum_{p=0}^{\infty}\frac{\Gamma(p+1-\alpha)}{(p+n+s)\Gamma(p+1+s)}\\
&< 2 + K'_{\alpha,s}\sum_{p=0}^{\infty}\frac{\sum_{j=0}^{n-1}(j+1)^{\alpha}(j+1)^{s-1}}{p+n+s}\,\frac{\Gamma(p+s+1-\alpha-s)}{\Gamma(p+s+1)}\\
&\leq 2 + K''_{\alpha,s}\sum_{p=0}^{\infty}\frac{n^{\alpha+s}}{(p+n+s)}(p+s)^{-(\alpha+s)}\\
&= 2 + K''_{\alpha,s}n^{\alpha+s}\left(\frac{1}{s^{\alpha+s}(s+n)} + \sum_{p=1}^{\infty}\frac{1}{(p+s+n)(p+s)^{\alpha+s}}\right)\\
&\leq 2 + K''_{\alpha,s}n^{\alpha+s}\left(\frac{1}{s^{\alpha+s}(s+n)} + \sum_{p=1}^{\infty}\int_{p+s-1}^{p+s}\frac{du}{(u+n)u^{\alpha+s}}\right)\\
&\leq 2 + K''_{\alpha,s}n^{\alpha+s}\left(\frac{1}{s^{\alpha+s}(s+n)} + \int_{0}^{\infty}\frac{du}{(u+n)u^{\alpha+s}}\right)\\
&= 2 + K''_{\alpha,s}n^{\alpha+s}\left(\frac{1}{s^{\alpha+s}(s+n)} + \frac{\Gamma(1-\alpha-s)\Gamma(\alpha+s)}{n^{\alpha+s}}\right)\\
&\leq M_{\alpha,s}
\end{aligned}$$

for some constants $K'_{\alpha,s} > 0$, $K''_{\alpha,s} > 0$, $M_{\alpha,s} > 0$.

(ii) Let $n \in \mathbb{N}$ and $|z| < 1$. Recall the notation

$$\mathfrak{g}_n(z) = \frac{1}{\mathfrak{f}(z)} \sum_{l=0}^{n-1} [D^\alpha f](l)\ [\Delta^{-\alpha} \mathcal{Z}](l)$$

in Section 5 and put

$$\mathfrak{s}_n(z) = \frac{(1-z)}{\mathfrak{f}(z)} \sum_{l=0}^{n-1} [\Delta^{-\alpha} \mathcal{Z}](l) \sum_{j=n}^{\infty} k_{-\alpha}(j-l) f(j).$$

Then we can write

$$\begin{aligned}
(1-z)\mathfrak{g}_n^0(z) &= \frac{(1-z)}{\mathfrak{f}(z)} \sum_{j=0}^{n-1} f(j)(k_{-\alpha} * \Delta^{-\alpha} \mathcal{Z})(j) \\
&= \frac{(1-z)}{\mathfrak{f}(z)} \sum_{l=0}^{n-1} [\Delta^{-\alpha} \mathcal{Z}](l) \sum_{j=l}^{n-1} k_{-\alpha}(j-l) f(j) \\
&= (1-z)\mathfrak{g}_n(z) - \frac{(1-z)}{\mathfrak{f}(z)} \sum_{l=0}^{n-1} [\Delta^{-\alpha} \mathcal{Z}](l) \sum_{j=n}^{\infty} k_{-\alpha}(j-l) f(j) \\
&= (1-z)\mathfrak{g}_n(z) - \mathfrak{s}_n(z),
\end{aligned}$$

and one has

$$\lim_{n\to\infty} (1-z)\mathfrak{g}_n^0 = 1 - z$$

in $A^\alpha(\mathbb{D})$ if and only if

$$\lim_{n\to\infty} \|\mathfrak{s}_n\|_{A^\alpha(\mathbb{D})} = 0,$$

by Theorem 5.8 (iii). To show that the latter limit is zero we proceed as follows.

Since

$$z[\Delta^{-\alpha} \mathcal{Z}](l) = [\Delta^{-\alpha} \mathcal{Z}](l+1) - k_\alpha(l+1)$$

and

$$[Dk_{-\alpha}](j) = -k_{-\alpha-1}(j+1)$$

one gets

$$\begin{aligned}
\mathfrak{f}(z)\mathfrak{s}_n(z) =& \sum_{l=0}^{n-1}[\Delta^{-\alpha}\mathcal{Z}](l)\sum_{j=n}^{\infty}k_{-\alpha}(j-l)f(j)\\
&- \sum_{l=1}^{n}[\Delta^{-\alpha}\mathcal{Z}](l)\sum_{j=n}^{\infty}k_{-\alpha}(j-l+1)f(j)\\
&+ \sum_{l=0}^{n-1}k_{\alpha}(l+1)\sum_{j=n}^{\infty}k_{-\alpha}(j-l)f(j)\\
=&\left(\sum_{j=n}^{\infty}k_{-\alpha}(j)f(j)+\sum_{l=1}^{n}k_{\alpha}(l)\sum_{j=n}^{\infty}k_{-\alpha}(j-l+1)f(j)\right)\\
&- \sum_{l=1}^{n}[\Delta^{-\alpha}\mathcal{Z}](l)\sum_{j=n}^{\infty}k_{-\alpha-1}(j-l+1)f(j)\\
&- [\Delta^{-\alpha}\mathcal{Z}](n)\sum_{j=n}^{\infty}k_{-\alpha}(j-n+1)f(j)\\
=&\mathfrak{s}_n^1(z)+\mathfrak{s}_n^2(z)+\mathfrak{s}_n^3(z),
\end{aligned}$$

where the meaning of $\mathfrak{s}_n^1(z), \mathfrak{s}_n^2(z), \mathfrak{s}_n^3(z)$ is clear.

By (2.4) and Lemma 4.1 we have

$$\begin{aligned}
\|\mathfrak{s}_n^1\|_{A^\alpha(\mathbb{D})} \leq& \left|\sum_{j=n}^{\infty}k_{-\alpha}(j)f(j)\right| + \left|\sum_{l=1}^{n}k_{\alpha}(l)\sum_{j=n}^{\infty}k_{-\alpha}(j-l+1)f(j)\right|\\
\leq& \left(-\sum_{j=n}^{\infty}k_{-\alpha}(j)\right)f(n)+\sum_{l=1}^{n}k_{\alpha}(l)\left(-\sum_{j=n}^{\infty}k_{-\alpha}(j-l+1)\right)f(n)\\
=& \left(\sum_{j=0}^{n-1}k_{-\alpha}(j)\right)k_s(n)+\sum_{l=1}^{n}k_{\alpha}(l)\left(\sum_{u=0}^{n-l}k_{-\alpha}(u)\right)k_s(n)\\
=& k_s(n)\left[k_{1-\alpha}(n-1)+\sum_{l=1}^{n}k_{\alpha}(l)k_{1-\alpha}(n-l)\right]\\
=& k_s(n)[k_{1-\alpha}(n-1)+(k_\alpha * k_{1-\alpha})(n)-k_{1-\alpha}(n)]\\
=& k_s(n)[k_{1-\alpha}(n-1)+(1-k_{1-\alpha}(n)] \to 0, \qquad \text{as } n\to\infty,
\end{aligned}$$

since, within the above sums, $f(j)>0$, $k_\alpha(l)>0$ and $k_{-\alpha}(j), k_{-\alpha}(j-l+1)<0$ for $j\geq n$ and $1\leq l\leq n$, and the sequence $(f(j))$ is decreasing.

By a similar argument as above, using this time (2.4), (4.2), (4.3) and Lemma 4.1, and that $k_{-(\alpha+1)}(q) \geq 0$ if $q \geq 2$, $k_{-\alpha}(1) = -\alpha$, we get

$$\begin{aligned}
\|\mathfrak{s}_n^2\|_{A^\alpha(\mathbb{D})} =& \sum_{l=1}^{n-1} k_{\alpha+1}(l) \sum_{j=n}^{\infty} k_{-(\alpha+1)}(j-l+1) f(j) \\
&+ k_{\alpha+1}(n) \bigg| \sum_{j=n}^{\infty} k_{-(\alpha+1)}(j-n+1) f(j) \bigg| \\
\leq& \sum_{l=1}^{n-1} k_{\alpha+1}(l) \sum_{j=n}^{\infty} k_{-(\alpha+1)}(j-l+1) f(j) \\
&+ k_{\alpha+1}(n) \bigg[|k_{-(\alpha+1)}(1)| f(n) + \sum_{j=n+1}^{\infty} k_{-(\alpha+1)}(j-n+1) f(j) \bigg] \\
\leq& f(n) \sum_{l=1}^{n-1} k_{\alpha+1}(l) \sum_{j=n}^{\infty} k_{-(\alpha+1)}(j-l+1) \\
&+ (\alpha+1) k_{\alpha+1}(n) f(n) + k_{\alpha+1}(n) \bigg(\sum_{j=n+1}^{\infty} k_{-(\alpha+1)}(j-n+1) \bigg) f(n) \\
=& - f(n) \sum_{l=1}^{n-1} k_{\alpha+1}(l) \sum_{u=0}^{n-l} k_{-(\alpha+1)}(u) + (\alpha+1) k_{\alpha+1}(n) f(n) \\
&- k_{\alpha+1}(n) \bigg(\sum_{u=0}^{1} k_{-(\alpha+1)}(u) \bigg) f(n) \\
=& - f(n) \sum_{l=1}^{n-1} k_{\alpha+1}(l) (k_{-(\alpha+1)} * k_1)(n-l) + (\alpha+1) k_{\alpha+1}(n) f(n) \\
&- k_{\alpha+1}(n) (k_{-(\alpha+1)} * k_1)(1) f(n) \\
=& \bigg(- \sum_{l=1}^{n-1} k_{\alpha+1}(l) k_{-\alpha}(n-l) + (\alpha+1) k_{\alpha+1}(n) + \alpha k_{\alpha+1}(n) \bigg) f(n) \\
=& [-(k_{\alpha+1} * k_\alpha)(n) + k_{\alpha+1}(n) k_\alpha(0) + (\alpha+1) k_{\alpha+1}(n) + \alpha k_{\alpha+1}(n)] f(n) \\
=& [-1 + (1 + (\alpha+1) + \alpha) k_{\alpha+1}(n)] f(n) \\
=& [-1 + 2(\alpha+1) k_{\alpha+1}(n)] f(n) \\
=& [-1 + 2(\alpha+1) k_{\alpha+1}(n)] k_s(n) \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

since $\alpha + s - 1 < 0$.

Finally,

$$\begin{aligned}\|\mathfrak{s}_n^3\|_{A^\alpha(\mathbb{D})} =& k_{\alpha+1}(n)\left|\sum_{j=n}^{\infty} k_{-\alpha}(j-n+1)f(j)\right| \\ =& k_{\alpha+1}(n)|[D^\alpha f](n-1)-f(n-1)| \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

since $\mathfrak{f}=\mathfrak{q}_s$ satisfies Proposition 5.9 (so its last part in particular) and $\alpha+s-1<0$.

All in all, we have shown that $\|\mathfrak{f}\mathfrak{s}_n\|_{A^\alpha(\mathbb{D})} \to 0$ as $n\to\infty$. It follows that $\lim_{n\to\infty}\|\mathfrak{s}_n\|_{A^\alpha(\mathbb{D})}=0$ since $1/\mathfrak{f}=\mathfrak{q}_{-s}\in A^\alpha(\mathbb{D})$. ∎

Remark 6.3: Let $0<s<1$. If we suppose that $\alpha\geq 1-s$, then the sequence of functions $(\mathfrak{g}_n^0)_{n\geq 1}$ is not a bounded $(1-z)$-approximate identity. Indeed, proceeding as in the proof of Theorem 6.2 (i), we get

$$\|\mathfrak{g}_n^0\|_{A^{1-s}(\mathbb{D})}=2-\sum_{j=0}^{n-1}\frac{2k_{2-s}(j)}{\Gamma(2-s)\Gamma(s-1)}\sum_{v=0}^{\infty}\frac{k_s(j+v+1)}{(v+n+s)(v+s)}.$$

Since the sequence $(k_s(n))$ is decreasing and

$$\sum_{j=0}^{n-1}k_{2-s}(j)=k_{3-s}(n-1)\geq c_s n^{2-s},$$

with $c_s>0$, one has by [21, Eq. (1)]

$$\|\mathfrak{g}_n^0\|_{A^{1-s}(\mathbb{D})}\geq 2+C_s\sum_{v=0}^{\infty}H_n(v),$$

where

$$H_n(v)=\frac{n^{2-s}}{(v+n+s)^{2-s}(v+s)}$$

increases to $(v+s)^{-1}$ as $n\to\infty$. Therefore $\|\mathfrak{g}_n^0\|_{A^{1-s}(\mathbb{D})}$ is not uniformly bounded in n and so, by (2.10), $\|\mathfrak{g}_n^0\|_{A^\alpha(\mathbb{D})}$ is not uniformly bounded in n for every $\alpha\geq 1-s$.

The second theorem in this section gives an analog of Theorem 6.2 for the function Λ defined (in Example 5.12) by

$$\Lambda(z)=-z^{-1}\log(1-z), \quad z\in\mathbb{D}.$$

Theorem 6.4: *Let $\mathfrak{f}=\Lambda$ and let $\mathfrak{g}_n^L$ denote the function $\mathfrak{g}_n^0$ corresponding to Λ as above. Then*

(i) $\|\mathfrak{g}_n^L\|_{A^\alpha(\mathbb{D})}\leq M$ *for every n and $0\leq\alpha\leq 1$, with $M>0$ independent of n,*
(ii) $\lim_{n\to\infty}(1-z)\mathfrak{g}_n^L(z)=1-z$ *in $A^\alpha(\mathbb{D})$ for $0\leq\alpha<1$.*

Proof. (i) Let $n \geq 1$. Note that, by the equality prior to (6.1),

$$g_n^L(n) = (f * g)(n) - f(n)g(0) = -(n+1)^{-1}.$$

Thus we have $[Dg_n^L](0) = 1$, $[Dg_n^L](j) = 0$ for $1 \leq j \leq n-2$, by (6.1), and

$$[Dg_n^L](n-1) = -g_n^L(n) = -(n+1)^{-1}.$$

Moreover, for $j \geq n$ we have

$$g_n^L(j) = \sum_{l=0}^{n-1} f(l)g(j-l) < 0$$

and the 1-admisibility of $\mathfrak{f}$ implies

$$[Dg_n^L](j) = g_n^L(j) - g_n^L(j+1) = \sum_{l=0}^{n-1} f(j)[Dg](j-l) < 0.$$

Since $\mathfrak{g}_n^L \in A^1(\mathbb{D})$, by (4.3) one gets

$$1 + \sum_{j=n-1}^{\infty} [Dg_n^L](j)k_2(j) = \sum_{j=0}^{\infty}[Dg_n^L](j)k_2(j) = \mathfrak{g}_n^L(1) = \frac{1}{\mathfrak{f}(1)} \sum_{j=0}^{n-1} \frac{1}{j+1} = 0$$

and then

$$\|\mathfrak{g}_n^L\|_{A^1(\mathbb{D})} = \sum_{j=0}^{\infty} |[Dg_n^L](j)|k_2(j) = 1 - \sum_{j=n-1}^{\infty} [Dg_n^L](j)k_2(j) = 2.$$

Finally, the inclusions given in (2.10) imply the result.

(ii) Write $(1-z)\mathfrak{g}_n^L(z) = (1-z)\mathfrak{g}_n(z) - \mathfrak{s}_n(z)$ where $\mathfrak{g}_n$ and $\mathfrak{s}_n = \frac{1}{\mathfrak{f}}(\mathfrak{s}_n^1 + \mathfrak{s}_n^2 + \mathfrak{s}_n^3)$ have the same meaning as in the proof of Theorem 6.2. Then, as in that theorem, it is enough to prove that $\|\mathfrak{s}_n\|_{A^\alpha(\mathbb{D})} \to 0$ as $n \to \infty$ to arrive at the conclusion. To show this, note that

$$\begin{aligned}
\|\mathfrak{s}_n^1\|_{A^\alpha(\mathbb{D})} &\leq -f(n)[W^{-1}k_{-\alpha}](n) - f(n) \sum_{l=1}^{n} k_\alpha(l)[W^{-1}k_{-\alpha}](n-l+1) \\
&= \frac{k_{1-\alpha}(n-1)}{n+1} + \frac{1}{n+1} \sum_{l=1}^{n} k_\alpha(l)k_{1-\alpha}(n-l) \\
&= \frac{k_{1-\alpha}(n-1)}{n+1} + \frac{1}{n+1}(k_1(n) - k_{1-\alpha}(n)) \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where the first equality is obtained from (2.4); also,

$$\begin{aligned}\|\mathfrak{s}_n^2\|_{A^\alpha(\mathbb{D})} &\le -f(n)\sum_{l=1}^{n} k_{\alpha+1}(l)k_{-\alpha}(n-l)\\ &= -\frac{1}{n+1}(k_1(n)-k_{\alpha+1}(n)-k_{-\alpha}(n)) \to 0, \quad \text{as } n\to\infty,\end{aligned}$$

and by (2.6) and (2.4) once again,

$$\begin{aligned}\|\mathfrak{s}_n^3\|_{A^\alpha(\mathbb{D})} &= k_{\alpha+1}(n)\sum_{j=n}^{\infty}(-k_{-\alpha}(j-n+1))f(j)\\ &\le \frac{k_{\alpha+1}(n)}{n+1}\Big(-\sum_{u=1}^{\infty}k_{-\alpha}(u)\Big)\\ &= \frac{k_{\alpha+1}(n)}{n+1} \to 0, \quad \text{as } n\to\infty. \qquad \blacksquare\end{aligned}$$

Remark 6.5: Let $\alpha = 1$ and write

$$(1-z)\mathfrak{g}_n^L(z) = 1 - z + g_n^L(n) - \sum_{j=n+1}^{\infty}[Dg_n^L](j-1)z^j$$

so that

$$\|(1-z)\mathfrak{g}_n^L(z) - (1-z)\|_{A^1(\mathbb{D})} \ge |g_n^L(n)|k_2(n-1) = \frac{n}{n+1}.$$

This implies that the sequence $(\mathfrak{g}_n^L)_{n\ge1}$ is not a $(1-z)$-approximate identity.

7. Domains of operator functions in terms of Taylor coefficients

In this section, we characterize the domain of operators $\mathfrak{f}(T)$ given by the functional calculus associated with a (C,α)-bounded operator T and α-admissible functions $\mathfrak{f}$. To do this, we transfer the results of Section 5 to operators. Our results extend the results obtained in [28] from the case $\alpha = 0$ to the case of arbitrary $\alpha > 0$. Part of the proofs mimic those of [28], but even in these cases we include them for the convenience of readers.

Proposition 7.1: *Let $\alpha > 0$. Let $\mathfrak{f}$ be an α-admissible function, and $(\mathfrak{g}_n)_{n\ge1}$ be defined as in Section 6. Let T be a (C,α)-bounded operator on X with* $\mathrm{Ker}(I-T) = \{0\}$.

(i) *If $\mathfrak{f}(1) < \infty$ then $\lim_{n\to\infty}\mathfrak{g}_n(T) = I$ in the operator norm.*
(ii) *If $\mathfrak{f}(1) = \infty$ then $\|\mathfrak{g}_n(T)\| \le 2K_\alpha(T)$, $n \ge 1$.*

(iii) *If* $\mathfrak{f}(1)=\infty$, $(1-z)\mathfrak{f}(z)\in A^\alpha(\mathbb{D})$ *and* $[D^\alpha f](j)j^\alpha\to 0$ *as* $j\to\infty$, *then*

$$\lim_{n\to\infty}(I-T)\mathfrak{g}_n(T)=I-T \quad \textit{in the operator norm.}$$

(iv) *If* $\mathfrak{f}(1)=\infty$, $(1-z)\mathfrak{f}(z)\in A^\alpha(\mathbb{D})$ *and* $x\in X$ *is such that*

$$[D^\alpha f](j-1)[\Delta^{-\alpha}\mathcal{T}](j)\mathfrak{f}(T)^{-1}x\to 0 \quad \textit{as } j\to\infty,$$

then

$$\lim_{n\to\infty}(I-T)\mathfrak{g}_n(T)x=(I-T)x \quad \textit{in norm.}$$

Proof. Assertions (i), (ii) and (iii) are straightforward consequences of Theorem 5.8 and the estimate (4.4) involving the functional calculus set up in Section 4 for T. As regards part (iv), applying the functional calculus to (5.2) we have

$$(I-T)\mathfrak{g}_n(T)=\mathfrak{h}_n(T)-[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n)\mathfrak{f}(T)^{-1}, \tag{7.1}$$

and then, applying the continuity of the calculus to the limit involving $\mathfrak{h}$ in (5.3), one obtains $\lim_{n\to\infty}\mathfrak{h}_n(T)=I-T$ in the operator norm. Then the result follows by the hypothesis on the second term in (7.1). ∎

Corollary 7.2: *Let* $\alpha>0$ *and let* $\mathfrak{f}$ *be an* α*-admissible function such that* $(1-z)\mathfrak{f}(z)\in A^\alpha(\mathbb{D})$, *and let* T *be a* (C,α)*-bounded operator on* X *with* $\mathrm{Ker}(I-T)=\{0\}$. *Then*

$$\lim_{n\to\infty}\mathfrak{g}_n(T)x=x,\quad x\in\overline{\mathrm{Ran}}(I-T),$$

if and only if

$$\lim_{n\to\infty}[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n)w=0,\quad w\in\overline{\mathrm{Ran}}(I-T).$$

Proof. First we prove the "if" part. Let $y\in X$. Take $s\in(0,1)$. By Proposition 4.8 we have $x=(I-T)^s y\in\overline{(I-T)X}$ and then we get

$$\lim_{n\to\infty}[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n)\mathfrak{f}(T)^{-1}x=0$$

from the hypothesis. Therefore it follows by Proposition 7.1 (iv) that

$$\lim_{n\to\infty}\mathfrak{g}_n(T)(I-T)^{1+s}y=(I-T)^{1+s}y \quad \text{in norm.} \tag{7.2}$$

Then the uniform boundedness of $\{\mathfrak{g}_n(T)\}_{n\in\mathbb{N}}$ given in Proposition 7.1 (i) and (ii), together with (7.2) and (4.8), imply that

$$\begin{aligned}\|\mathfrak{g}_n(T)(I-T)y-(I-T)y\| \leq &\|\mathfrak{g}_n(T)(I-T)y-\mathfrak{g}_n(T)(I-T)^{1+s}y\|\\ &+\|\mathfrak{g}_n(T)(I-T)^{1+s}y-(I-T)^{1+s}y\|\\ &+\|(I-T)^{1+s}y-(I-T)y\|\to 0,\end{aligned}$$

as $n\to\infty$, $s\to 0^+$. Hence, we conclude that

$$\lim_{n\to\infty}\mathfrak{g}_n(T)x=x,\quad x\in\overline{\mathrm{Ran}}(I-T).$$

Conversely, assume $\lim_{n\to\infty}\mathfrak{g}_n(T)x=x$ for all $x\in\overline{\mathrm{Ran}}(I-T)$. Then

$$\lim_{n\to\infty}(I-T)\mathfrak{g}_n(T)=I-T$$

in the operator norm, and by (7.1) one gets

$$\lim_{n\to\infty}[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n)\mathfrak{f}(T)^{-1}=0$$

strongly. Since $\mathfrak{f}(T)(I-T)\in\mathcal{B}(X)$ we have $\mathrm{Ran}(I-T)\subset\mathrm{Dom}\,\mathfrak{f}(T)$, hence

$$\lim_{n\to\infty}D^\alpha f(n-1)\Delta^{-\alpha}\mathcal{T}(n)w=0,$$

for all $w\in\overline{\mathrm{Ran}}(I-T)$. ∎

In view of the above proposition and corollary and to simplify our formulation below, we assume without loss of generality that $X=\overline{\mathrm{Ran}}(I-T)$. Note that in this case if T is (C,α)-bounded then it is (C,β)-ergodic for all $\beta>\alpha$ (Theorem 3.3).

Theorem 7.3: *Let $\alpha>0$ and let $\mathfrak{f}$ be an α-admissible function such that $(1-z)\mathfrak{f}(z)\in A^\alpha(\mathbb{D})$ and $[D^\alpha f](j)j^\alpha\to 0$ as $j\to\infty$. If T is a (C,α)-bounded operator on X with $\overline{\mathrm{Ran}}(I-T)=X$, the following assertions are equivalent for a given x in X:*

(i) $x\in\mathrm{Dom}\,\mathfrak{f}(T)$.
(ii) *The series $\sum_{j\geq 0}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x$ converges in norm.*
(iii) *The series $\sum_{j\geq 0}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x$ converges weakly.*

Furthermore, if one of the equivalent conditions (i)–(iii) holds, then

$$\mathfrak{f}(T)x=\sum_{j\geq 0}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x.$$

Proof. Since $\overline{\mathrm{Ran}}(I-T)=X$ we have that $\mathfrak{g}_n(T)$ converges to the identity operator in the strong topology, by Corollary 7.2. From the hypothesis, $\mathfrak{f}$ is α-regularizable by $1-z$ so that $\mathfrak{f}(T)$ is well-defined (as closed operator).

(i)⇒(ii) Let $x\in \mathrm{Dom}\,\mathfrak{f}(\mathrm{T})$. Since $\mathfrak{g}_n(T)\mathfrak{f}(T)\subset(\mathfrak{g}_n\mathfrak{f})(T)$, one gets

$$\sum_{j=0}^{n-1}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x=(\mathfrak{g}_n\mathfrak{f})(T)x=\mathfrak{g}_n(T)\mathfrak{f}(T)x\to\mathfrak{f}(T)x,\quad n\to\infty.$$

(ii)⇒(iii) This is obvious.

(iii)⇒(i) Assume $y:=\lim_{n\to\infty}\sum_{j=0}^{n-1}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x$ weakly, for some $x\in X$. Let $\mathfrak{f}_n$ be as prior to Theorem 5.7. Then,

$$\begin{aligned}[(1-z)\mathfrak{f}](T)x&=\lim_{n\to\infty}(I-T)\mathfrak{f}_n(T)x\\&=(I-T)\sum_{j\ge0}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x=(I-T)y,\end{aligned}$$

with convergence in the weak topology of X. Hence $x\in\mathrm{Dom}\,\mathfrak{f}(\mathrm{T})$ and

$$\mathfrak{f}(T)x=(I-T)^{-1}[(1-z)\mathfrak{f}](T)x=y. \qquad \blacksquare$$

Remark 7.4: Note that from (5.2) we have

$$(I-T)\mathfrak{f}_n(T)=(\mathfrak{f}\mathfrak{h}_n)(T)-[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n).$$

In addition,

$$[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n)=T[D^\alpha f](n-1)[\Delta^{-\alpha}\mathcal{T}](n-1)+[D^\alpha f](n-1)k_\alpha(n).$$

Then, following ideas of [28, Th. 5.6], the previous identities imply that the assertions (i)–(iii) of Theorem 7.3 are equivalent to the Cesàro convergence and/or the Cesàro weak convergence of

$$\sum_{j\ge0}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x.$$

Also, similarly to [28, Th. 5.6], if X is a reflexive Banach space then assertions (i)–(iii) of Theorem 7.3 are equivalent to

$$\sup_N\left\|\sum_{j=0}^{N}[D^\alpha f](j)[\Delta^{-\alpha}\mathcal{T}](j)x\right\|<\infty.$$

The details of all of these claims are rather straightforward and left for verification to the reader.

8. Fractional Poisson equation and the logarithm

As before, we assume that T is a (C,α)-bounded operator on X with $(I-T)X$ dense in X. Then it is readily seen by (4.8) that $((I-T)^s)_{s\geq 0}$ is a C_0-semigroup on X, where $(I-T)^s$ is defined by the holomorphic functional calculus. The semigroup extends to a holomorphic C_0-semigroup $((I-T)^s)_{\Re(s)\geq 0}$, but we will not need this fact in the sequel. Let $\log(I-T)$ denote the infinitesimal generator of $((I-T)^s)_{\mathfrak{Re}\, s>0}$. Next, we discuss the solvability of the fractional Poisson equation for T, as well as the domain of the generator $\log(I-T)$. To do this, we apply the results on domains of operatorial functions of Section 7.

Fractional Poisson equation. By (abstract) fractional Poisson equation we mean the equation $(I-T)^s u = x$ where x is given and u is the unknown. If $s=1$, then $(I-T)u = x$ is just a standard abstract Poisson equation, according to the established terminology; see [16].

By hypothesis, $I-T$ is injective and so

$$(I-T)^{-s} := ((I-T)^s)^{-1}$$

is such that

$$\operatorname{Ran}(I-T)^s = \operatorname{Dom}(I-T)^{-s}.$$

In fact, $(I-T)^{-s} = \mathfrak{q}_s(T)$ where $\mathfrak{q}_s$ is α-admissible; see Example 5.11. Obviously, the equation has a solution if and only if x lies in $\operatorname{Ran}(I-T)^s$ and the solution u is $u=(I-T)^{-s}x$. First, we characterize the property $x\in \operatorname{Ran}(I-T)^s$ through convergence of series involving Cesàro sums of T.

Theorem 8.1: *Let $\alpha>0$ and let T be a (C,α)-bounded operator on X with $\overline{(I-T)X} = X$ and let $0<s<1$. For $x\in X$ the following assertions are equivalent:*

(i) $x \in \operatorname{Ran}(I-T)^s$.
(ii) *The series* $\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha-s}}[\Delta^{-\alpha}\mathcal{T}](n)x$ *converges in norm.*
(iii) *The series* $\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha-s}}[\Delta^{-\alpha}\mathcal{T}](n)x$ *converges weakly in X.*

Furthermore, if one of the equivalent conditions (i)–(iii) holds then

$$(I-T)^{-s}x = \frac{\sin(\pi s)\Gamma(1-s+\alpha)}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(s+n)}{\Gamma(n+\alpha+1)}[\Delta^{-\alpha}\mathcal{T}](n)x.$$

Proof. The equivalence between (i) and either weak or norm convergence of the series

$$\frac{\sin(\pi s)\Gamma(1-s+\alpha)}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(s+n)}{\Gamma(n+\alpha+1)}[\Delta^{-\alpha}\mathcal{T}](n)x$$

follows by Theorem 7.3 and (2.11). To finish the proof it is then enough to apply (2.12). ∎

The mean ergodic theorem for power-bounded operators T says that

$$\sup_n \|M_T^0(n)\| < \infty$$

jointly with

$$\overline{\mathrm{Ran}}(I-T) = X$$

imply that $M_T^1(n)x$ converges (to 0), as $n \to \infty$, for every $x \in X$. Further, if $x \in \mathrm{Ran}(I-T)^s$ with $0 < s < 1$ then the convergence takes place with the polynomial rate $\|M_T^1(n)x\| = o(n^{-s})$ as $n \to \infty$, [16, 28]. It appears that a similar result holds for (C,α)-bounded operators as the next statement shows. Recall, $M_T^\beta(n) = k_{\beta+1}(n)^{-1}[\Delta^{-\beta}\mathcal{T}](n)$ for $n \geq 0$, $\beta > 0$.

COROLLARY 8.2: *Let $\alpha > 0$ and let T be a (C,α)-bounded operator on X where $X = \overline{\mathrm{Ran}}(I-T)$. Let $\beta > \alpha$. Then $\lim_{n\to\infty} M_T^\beta(n)x = 0$ for every $x \in X$.*

Moreover, if $x \in \mathrm{Ran}(I-T)^s$ with $0 < s < 1$ then

$$\|M_T^\beta(n)x\| = \begin{cases} o(1), & \alpha < \beta < \alpha+1-s, \\ o(n^{-s-\beta+\alpha+1}), & \alpha+1-s \leq \beta < \alpha+1, \\ o(n^{-s}), & \beta \geq \alpha+1. \end{cases}$$

Proof. The ergodicity of T for $\beta > \alpha$ is a direct consequence of Theorem 3.3. Regarding the rate of convergence for $x \in \mathrm{Ran}(I-T)^s$, the case $\alpha < \beta \leq \alpha+1-s$ is redundant, following from the ergodicity of T. For $\beta = \alpha+1$, the series $\sum_{n=1}^{\infty} n^{s-\alpha-1}[\Delta^{-\alpha}\mathcal{T}](n)x$ is convergent in norm by Theorem 8.1. Then, by (vector-valued) Kronecker's Lemma [16, p. 103], we have

$$\left\|n^{s-\alpha-1}\sum_{j=0}^{n}[\Delta^{-\alpha}\mathcal{T}](j)x\right\| = n^{s-\alpha-1}+\left\|n^{s-\alpha-1}\sum_{j=1}^{n}[\Delta^{-\alpha}\mathcal{T}](j)x\right\| \to 0, \quad n\to\infty.$$

Hence

$$\|M_T^{\alpha+1}(n)x\| = \left\|\frac{(k_1 * k_\alpha * \mathcal{T})(n)}{k_{\alpha+2}(n)}x\right\| \sim \left\|\frac{1}{n^{\alpha+1}}\sum_{j=0}^{n}[\Delta^{-\alpha}\mathcal{T}](j)x\right\| = o(n^{-s}),$$

as $n \to \infty$.

Now, assume $\alpha+1-s<\beta<\alpha+1$. Let $\chi_{[0,n]}$ stand for the characteristic function of the interval of integers $[0,n]$. Then, for some constant $K>0$,

$$\begin{aligned}(n+1)^{\beta-\alpha+s-1}&\|M_T^{\beta}(n)x\|\\ &\le \frac{K}{(n+1)^{\alpha+1-s}}\Big\|\sum_{j=0}^{n}k_{\beta-\alpha-1}(j)[\Delta^{-(\alpha+1)}\mathcal{T}](n-j)x\Big\|\\ &\le K\sum_{j=0}^{\infty}|k_{\beta-\alpha-1}(j)|\chi_{[0,n]}(j)\frac{\|[\Delta^{-(\alpha+1)}\mathcal{T}](n-j)x\|}{(n-j+1)^{\alpha+1-s}}\to 0, \quad \text{as } n\to\infty,\end{aligned}$$

since $(q+1)^{-(\alpha+1-s)}\|[\Delta^{-(\alpha+1)}\mathcal{T}](q)x\|\to 0$ as $q\to\infty$ (case $\beta=\alpha+1$) and $k_{\alpha-\beta-1}$ lies in ℓ^1.

Finally, for $\beta>\alpha+1$, in particular T is a $(C,\beta-1)$-bounded operator, and it follows by previous arguments that $\|M_T^{\beta}(n)x\|=o(n^{-s})$, as $n\to\infty$, for every $x\in \operatorname{Ran}(I-T)^s$. ∎

Remark 8.3: Assume for a moment that $s=1$ and then that $x\in\operatorname{Ran}(I-T)$. In the second part of the proof of Lemma 3.1, where assumptions $\beta\ge\alpha+1$ and $\alpha<\beta<\alpha+1$ are explicitly and separately considered, it is shown that

$$M_T^{\beta}(n)x=\mathrm{O}(n^{-1}) \text{ if } \beta\ge\alpha+1 \quad\text{and}\quad M_T^{\beta}(n)x=\mathrm{O}(n^{\alpha-\beta}),\ \text{if } \alpha<\beta<\alpha+1,$$

as $n\to\infty$, for every (C,α) bounded operator T (for some comments in the case $\alpha=0$, that is, for power-bounded operators, and $\beta=1$, see [24, Prop. 1.1]). Corollary 8.2 is an (improved) extension of such estimates for $x\in\operatorname{Ran}(I-T)^s$, $0<s<1$.

The fact that the rate of convergence of the series in the above theorem and corollary are given in terms of (fractional) Cesàro sums seems to be appropriate, on account of the general character of (C,α)-bounded operators. However, if one restricts the range of α to values between 0 and $1-s$, it is then possible to express convergence only involving the Taylor series of $(I-T)^{-s}$. The next theorem extends the corresponding results obtained for power-bounded operators in [16, 28].

Theorem 8.4: *Let s be such that $0<s<1$ and let $\alpha\in(0,1-s)$. Let T be a (C,α)-bounded operator on X with $\overline{(I-T)X}=X$. Then the following assertions are equivalent:*

(i) $x\in\operatorname{Ran}(I-T)^s$.

(ii) *The series* $\sum_{n=1}^{\infty} \frac{1}{n^{1-s}} T^n x$ *converges in norm (or weakly).*

If one of the equivalent conditions (i) or (ii) holds, then

$$(I-T)^{-s}x = \sum_{n=0}^{\infty} k_s(n) T^n x.$$

Proof. The argument follows similar lines to those of the previous Theorem 8.1 and Theorem 7.3, by using the bounded $(1-z)$-approximate identity $\mathfrak{g}_n^0$ of Section 6 instead $\mathfrak{g}_n$. Suppose first that x belongs to $\mathrm{Ran}(I-T)^s$. Then $\mathfrak{g}_n^0(T)\mathfrak{q}_s(T) \subset (\mathfrak{g}_n^0\mathfrak{q}_s)(T)$ and hence we have

$$\begin{aligned}\sum_{n=0}^{N-1} k_s(n)T^n x &= \left(\sum_{n=0}^{N-1} k_s(n)T^n\right)\mathfrak{q}_{-s}(T)\mathfrak{q}_s(T)x \\ &= \mathfrak{g}_N^0(T)(I-T)^{-s}x \longrightarrow (I-T)^{-s}x, \quad N\to\infty,\end{aligned}$$

since $\lim_{N\to\infty}\mathfrak{g}_N^0(T) = I$ strongly on X by Theorem 6.2. Conversely, suppose now that there exists $y := \lim_{N\to\infty}\sum_{n=0}^{N-1} k_s(n)T^n x$ for some $x \in X$ (weakly or in norm). Then

$$\begin{aligned}[(1-z)\mathfrak{q}_s](T)x &= \lim_{N\to\infty}(I-T)\mathfrak{q}_s(T)\mathfrak{g}_N^0(T)x \\ &= \lim_{N\to\infty}(I-T)\left(\sum_{n=0}^{N-1} k_s(n)T^n x\right) = (I-T)y,\end{aligned}$$

so that $x \in \mathrm{Dom}(I-T)^{-s}$ and $(I-T)^{-s}x = y$. The equivalence between (i) and (ii) follows now from (2.7). ∎

Corollary 8.5: *Let s be such that $0<s<1$ and let $\alpha\in(0,1-s)$. Assume that T is a (C,α)-bounded operator on X with $X=\overline{\mathrm{Ran}}(I-T)$. Let $\beta>\alpha$. Then $\lim_{n\to\infty} M_T^\beta x = 0$ for every $x\in X$.*

Moreover, if $x\in\mathrm{Ran}(I-T)^s$ then

$$\|M_T^\beta(n)x\| = \begin{cases} o(1), & \alpha<\beta<1-s, \\ o(\frac{1}{n^{\beta+s-1}}), & 1-s\le\beta\le 1.\end{cases}$$

Proof. The first part of the corollary is a consequence of Theorem 3.3. As for the rates of convergence, the proof runs parallel to that of Corollary 8.2, using Kronecker's Lemma to first show that $\|M_T^1(n)x\| = o(n^{-s})$, as $n\to\infty$, and then convolution for $1-s\le\beta<1$. ∎

Remark 8.6: Under assumptions $0 < s < 1$ and $0 < \alpha < 1 - s$ considered in the above corollary, the convergence rates obtained for $M_T^\beta(n)x$, with $x \in \mathrm{Ran}(I-T)^s$, are better than those given in Corollary 8.2. Also, note that Corollary 8.5 gives us

$$M_T^1(n)x = o(n^{-s}) \quad \text{as } n \to \infty,$$

for $\alpha \in (0, 1-s)$, which is an extension to (C,α)-bounded operators of the corresponding result for power-bounded operators proven in [16] and [28].

The operator logarithmic function and the discrete Hilbert transform. For $z \in \mathbb{D}$, there is the decomposition

$$\log(1-z) = -\sum_{n\geq 1} \frac{z^j}{n} = \mathfrak{h}(z) - \Lambda(z)$$

with

$$\mathfrak{h}(z) := 1 - \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \quad \text{and} \quad \Lambda(z) := \sum_{n=0}^{\infty} \frac{z^n}{n+1}. \tag{8.1}$$

The function Λ, with Taylor coefficients $L(n) := (n+1)^{-1}$, has been studied in Example 5.12, where it has been shown that it is an α-admissible function with $D^\alpha L(n) n^\alpha \to 0$ as $n \to \infty$. Also, $(1-z)\Lambda = \mathfrak{h} \in A^\alpha(\mathbb{D})$ for all $\alpha \geq 0$.

The functional calculus of Section 4 enables us to define the closed operator

$$\log(I-T) := [\log(I-z)](T)$$

by regularization. It is readily seen that $\log(I-T)$ is the infinitesimal generator of the holomorphic semigroup $((I-T)^s)_{\Re s>0}$.

Theorem 8.7: *Let $\alpha > 0$. Let T be a (C,α)-bounded operator on X with $\overline{\mathrm{Ran}}(I-T) = X$. Given $x \in X$ the following are equivalent:*

(i) *$x \in \mathrm{Dom}(\log(I-T))$.*
(ii) *The series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}[\Delta^{-\alpha}\mathcal{T}](n)x$ converges in norm (or weakly).*

Furthermore, if either (i) or (ii) holds true then

$$\log(I-T)x = (\psi(\alpha+1) - \psi(1))x - \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)}[\Delta^{-\alpha}\mathcal{T}](n)x,$$

where $\psi(x) = \frac{d}{dx}\ln(\Gamma(x))$ is the digamma function.

Proof. Let $\mathfrak{h}$ and Λ be defined by (8.1). Then we have

$$(1-z)\log(1-z) = (1-z)\mathfrak{h}(z) - (1-z)\Lambda(z) = (1-z)\mathfrak{h}(z) - \mathfrak{h}(z)$$

in $A^\alpha(\mathbb{D})$ and therefore

$$(I-T)\log(I-T) = (I-T)\mathfrak{h}(T) - \mathfrak{h}(T)$$

in $\mathcal{B}(X)$, from which one obtains

$$\log(I-T) = (I-T)^{-1}[(I-T)\mathfrak{h}(T)] - (I-T)^{-1}\mathfrak{h}(T) = \mathfrak{h}(T) - \Lambda(T)$$

as closed operators on X. Moreover, $\mathfrak{h}(T)$ is bounded and so the domains of $\log(I-T)$ and $\Lambda(T)$ coincide. Hence $x \in \mathrm{Dom}(\log(I-T))$ if and only

$$\sum_{n\geq 0} \frac{n!}{\Gamma(n+\alpha+2)}[\Delta^{-\alpha}\mathcal{T}](n)x$$

converges (in norm or weakly), according to Theorem 7.3 and (2.13). To get the equivalence between (i) and (ii) it is now enough to use (2.14).

As regards the range of $\log(I-T)$, note that on $\mathrm{Dom}(\log(I-T))$ we have

$$\log(I-T) = \mathfrak{h}(T) - \Lambda(T) = (1-T)\Lambda(T) - \Lambda(T) = -T\Lambda(T)$$

and so, by Theorem 7.3 and (2.13),

$$\begin{aligned}
\log(I-T)x &= -\sum_{n=0}^{\infty}[D^\alpha L](n)T[\Delta^{-\alpha}\mathcal{T}](n)x \\
&= -\sum_{n=0}^{\infty}[D^\alpha L](n)[\Delta^{-\alpha}\mathcal{T}](n+1)x + \sum_{n=0}^{\infty}[D^\alpha L](n)k_\alpha(n+1)x \\
&= -\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)}[\Delta^{-\alpha}\mathcal{T}](n)x + \sum_{n=0}^{\infty}\frac{\alpha}{(n+1)(n+\alpha+1)}x \\
&= -\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)}[\Delta^{-\alpha}\mathcal{T}](n)x + (\psi(\alpha+1) - (\psi(\alpha))x
\end{aligned}$$

for all $x \in \mathrm{Dom}(\log(I-T))$, where in the latter equality we have applied that

$$\psi(\alpha+1) - \psi(1) = \sum_{n=0}^{\infty}\frac{\alpha}{(n+1)(n+\alpha+1)}$$

(see [7]). We have completed the proof. ∎

Remark 8.8: Notice that one also has $\psi(\alpha+1)-\psi(1)=\int_0^1 \frac{1-u^\alpha}{1-u}du$ since

$$\sum_{n=0}^{\infty}\frac{\alpha}{(n+1)(n+\alpha+1)} = \int_0^\infty \sum_{n=0}^{\infty}(e^{-(n+1)t}-e^{-(n+\alpha+1)t})dt$$
$$= \int_0^\infty \frac{1-e^{-\alpha t}}{1-e^{-t}}e^{-t}\, dt = \int_0^1 \frac{1-u^\alpha}{1-u}\, du.$$

Remark 8.9: Suppose for a moment that the operator T is power-bounded on the Banach space X. Then the formal expression

$$H_T := \sum_{n=1}^{\infty}\frac{1}{n}T^n \tag{8.2}$$

defines a closed operator on X which is called the one-sided ergodic Hilbert transform, see [16] and [28]. In fact, $H_T=-\log(I-T)$ with

$$\mathrm{Dom}(H_T)=\mathrm{Dom}(\log(I-T))$$

in particular [12, 28].

Let us assume again that T is a (C,α)-bounded operator on X, as usually in the present paper. Then the representation (8.2) of H_T does not look suitable for arbitrary $\alpha>0$ since $\|T^n\|=\mathrm{O}(n^\alpha)$ as $n\to\infty$. Instead, one could define in this case the one-side Cesàro-Hilbert transform of order α (or α-ergodic Hilbert transform, for short) as the operator given by

$$H_T^{(\alpha)}x=\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)}\Delta^{-\alpha}\mathcal{T}(n)x+c_\alpha x,$$

with

$$c_\alpha=-\int_0^1(1-u^\alpha)(1-u)^{-1}du.$$

Then, in view of Theorem 8.7, we have

$$\log(I-T)x=-H_T^{(\alpha)}x$$

for all $x\in X$ such that

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}[\Delta^{-\alpha}\mathcal{T}](n)x$$

converges in X. Nevertheless, we show below that if $\alpha\in(0,1)$, then $-\log(I-T)$ admits the representation (8.2) as in the case of power bounded operators. This result extends [12, Prop. 3.3] and [28, Th. 6.2].

Theorem 8.10: *Let α be such that $0 < \alpha < 1$. Let T be a (C,α)-bounded operator on X with $\overline{\mathrm{Ran}}(I-T) = X$. For a given $x \in X$ the following are equivalent:*

(i) $x \in \mathrm{Dom}(\log(I-T))$.

(ii) *The series $\sum_{n=1}^{\infty} \frac{1}{n} T^n x$ converges (in norm or weakly).*

If one of the equivalent conditions (i) and (ii) holds then

$$\log(I-T)x = -\sum_{n=1}^{\infty} \frac{1}{n} T^n x.$$

Proof. As in the proof of Theorem 8.7,

$$\log(I-T) = \mathfrak{h}(T) - \Lambda(T)$$

so that $x \in \mathrm{Dom}(\log(I-T))$ if and only if $x \in \mathrm{Dom}(\Lambda(T))$. On the other hand, one has that $x \in \mathrm{Dom}(\Lambda(T))$ if and only if $\sum_{n=0}^{\infty} \frac{1}{n+1} T^n x$ converges, and in this case

$$\Lambda(T)x = \sum_{n=0}^{\infty} \frac{1}{n+1} T^n x.$$

The latter assertion can be proved by the same argument as in the proof of Theorem 8.4, by replacing there $(I-T)^s$ with $\Lambda(T)$ now, and $\mathfrak{g}_n^0(T)$ with $\mathfrak{g}_n^L(T)$ (recall that $\mathfrak{g}_n^L$ is a bounded $(1-z)$-approximate identity in $A^\alpha(\mathbb{D})$ for $0 < \alpha < 1$; see Theorem 6.4). The identity

$$\log(I-T)x = -\sum_{n=1}^{\infty} \frac{1}{n} T^n x,$$

for $x \in \mathrm{Dom}(\log(I-T)) = \mathrm{Dom}(\Lambda(T))$, follows then from the identity

$$\log(I-T) = \mathfrak{h}(T) - \Lambda(T)$$

by taking n-partial sums and letting $n \to \infty$. ∎

Remark 8.11: As pointed out in Remark 6.5, $(\mathfrak{g}_n^L)_{n\geq 1}$ is not an approximate identity in $A^\alpha(\mathbb{D})$ when $\alpha = 1$ and therefore a type of argument as above is not enough to prove the analog to the equivalence established in Theorem 8.10, in this case. Also, as regards part (ii) of that theorem, the convergence rate of $M_T^1(n)$ to 0 is not a simple matter which requires special treatment. For every power-bounded T, it has been elucidated in [24].

Recall that by standard properties of holomorphic semigroups,

$$\bigcup_{s>0} \mathrm{Ran}(I-T)^s \subset \mathrm{Dom}(\log(I-T)).$$

Another interesting task arising in the study of operator logarithms is to determine whether one may have the equality $\mathrm{Dom}(\log(I-T)) = \bigcup_{s>0} \mathrm{Ran}(I-T)^s$ so that the domain of $\log(I-T)$ can be described in terms of more accesible spaces $\mathrm{Ran}(I-T)^s$. In [28, Th. 6.3], it was shown that the equality fails for every power-bounded operator T with $1 \notin \sigma(T)$. The argument to prove this result relies on the sectorial functional calculus (Th. 4.5) and can be mimicked in the case of (C,α)-bounded operators. So we have the following.

Proposition 8.12: *Let $\alpha > 0$ and let T be a (C,α)-bounded operator on a Banach space X such that $\overline{\mathrm{Ran}}(I-T) = X$ and $1 \notin \sigma(T)$. Then we have the inclusion*

$$\bigcup_{s>0} \mathrm{Ran}(I-T)^s \subset \mathrm{Dom}(\log(I-T))$$

and it is strict.

9. Application to concrete operators

In this section we show two examples to illustrate results of the paper.

Example 9.1: Let $1 \le p < \infty$. With $\|T\|_{op,p}$ we denote the operator norm of operators T in $\mathcal{B}(L^p(0,1))$. Let V be the Volterra integral operator on $L^p(0,1)$ given by

$$Vf(t) := \int_0^t f(s)ds, \quad t \in [0,1],\ f \in L^p(0,1).$$

Define

$$T_V := I - V.$$

Estimates involving powers of the operator T_V were given in [29, Th. 11] for $p=1$. Long after that, such estimates were extended to arbitrary $p \in [1,\infty)$. Namely, there exist $A>0$, $B>0$ such that

$$A\, n^{|(1/4)-(1/2p)|} \le \|T_V^n\|_{op,p} \le B\, n^{|(1/4)-(1/2p)|}, \quad n \in \mathbb{N};$$

see [37, Th. 2.2]. Thus T_V is power-bounded on $L^p(0,1)$ exclusively in the Hilbertian case $p=2$.

In fact, powers T_V^n, $n \in \mathbb{N}$, and means $M_{T_V}^\alpha$ of T_V can be expressed in the integral form

$$M_{T_V}^\alpha(n)f(t) = f(t) - \frac{1}{k_{\alpha+1}(n)} \int_0^t L_{n-1}^{(\alpha+1)}(t-u)f(u)du, \qquad t \in [0,1],\, n \in \mathbb{N},\, f \in L^p(0,1),$$

where $\alpha \geq 0$ and $L_{n-1}^{(\alpha+1)}$ is the generalized Laguerre polynomial of degree $n-1$, see [29, (5.3) and (6.14)]. In other words, we have

$$M_{T_V}^\alpha(n)f = (\delta_0 - [k_{\alpha+1}(n)]^{-1} L_{n-1}^{(\alpha+1)}) \star f, \quad n \in \mathbb{N},\, f \in L^p(0,1), \tag{9.1}$$

where δ_0 is the Dirac mass at $\{0\}$ and "$\star$" is the convolution product in the Banach algebra $L^1(0,1)$. Moreover, the sequence

$$\Phi_{n-1}^{(\alpha+1)} := [k_{\alpha+1}(n)]^{-1} L_{n-1}^{(\alpha+1)}, \quad n \in \mathbb{N},$$

is a bounded approximate identity in $L^1(0,1)$ for $\alpha > 1/2$, which is to say

$$\Phi_{n-1}^{(\alpha+1)} \star f \to f \quad \text{as } n \to \infty,$$

for all $f \in L^1(0,1)$, and $\sup_n \|\Phi_{n-1}^{(\alpha+1)}\|_{L^1} < \infty$ [29, Lemma 1 and (6.14)]. Using these properties, it can be shown that T_V is (C,α)-bounded on $L^1(0,1)$ if and only if T_V is (C,α)-ergodic on $L^1(0,1)$ if and only if $\alpha > 1/2$ [29, Th. 11].

The next result characterizes the solutions of the fractional Volterra equation

$$(I - T_V)^s g = V^s g = f,$$

and the domain of $\log(V)$.

Proposition 9.2: *Let $p \in [1,\infty)$, let V be the Volterra operator on $L^p(0,1)$ and set $T_V = I - V$ as above. Then the following properties hold*

(i) *$(I - T_V)(L^p(0,1))$ is dense in $L^p(0,1)$.*

(ii) *For every $\alpha > 1/2$ the operator T_V is (C,α)-ergodic on $L^p(0,1)$.*

(iii) *Let $\alpha > 1/2$ and $0 < s < 1$. If $f \in L^p(0,1)$, then the Volterra integral equation*

$$\frac{1}{\Gamma(s)} \int_0^t (t-u)^{s-1} g(u)\, du = f(t), \quad 0 \leq t \leq 1, \tag{9.2}$$

has a (unique) solution $g \in L^p(0,1)$ if and only if

$$\sum_{n=1}^\infty n^{s-1} (\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f \quad \textit{is norm-convergent.}$$

In this case,

$$g = \frac{\sin(\pi s)}{\pi} \frac{\Gamma(1-s+\alpha)}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} (\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f.$$

Also,

$$\|(\delta_0 - \Lambda_{n-1}^{(\alpha+2)}) \star f\|_p = o(n^{-s}), \quad \text{as } n \to \infty.$$

If, moreover, $0 < s < 1/2$ *and* $\alpha \in (1/2, 1-s)$, *the equation* (9.2) *has a (unique) solution* $g \in L^p(0,1)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{s-1} (\delta_0 - L_{n-1}^{(1)}) \star f \quad \text{is norm-convergent,}$$

with

$$g = \sum_{n=0}^{\infty} k_s(n) (\delta_0 - L_{n-1}^{(1)}) \star f.$$

Also,

$$\frac{1}{n} \left\| \sum_{j=1}^{n} (\delta_0 - \Lambda_{j-1}^{(1)}) \star f \right\|_p = o(n^{-s}), \quad \text{as } n \to \infty.$$

(iv) *Let* $\alpha > 1/2$. *Then* $f \in \mathrm{Dom}(\log V)$ *if and only if the series*

$$\sum_{n=1}^{\infty} n^{-1} (\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f \quad \text{is norm-convergent.}$$

In this case,

$$(\log V) f = (\psi(\alpha+1) - \psi(1)) f - \sum_{n=1}^{\infty} \frac{1}{n} (\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f.$$

If moreover $\alpha \in (1/2, 1)$, *then*

$$f \in \mathrm{Dom}(\log V) \subset L^p(0,1) \Longleftrightarrow \sum_{n=1}^{\infty} n^{-1} (\delta_0 - L_{n-1}^{(1)}) \star f \text{ is norm-convergent.}$$

In this case, $(\log V) f = -\sum_{n=1}^{\infty} \frac{1}{n} (\delta_0 - L_{n-1}^{(1)}) \star f$.

Proof. (i) This is standard.

(ii) Let $\Phi_{n-1}^{(\alpha+1)}$ be as prior to the proposition. Clearly, $L^1(0,1) \star L^p(0,1)$ is dense in $L^p(0,1)$. Then the uniform (in n) L^1-boundedness of $\Phi_{n-1}^{(\alpha+1)}$ and the fact that

$$\|h \star g\|_p \le \|h\|_1 \|g\|_p, \quad (h \in L^1(0,1), g \in L^p(0,1))$$

readily imply that $\Phi_{n-1}^{(\alpha+1)} \star f \to f$ as $n \to \infty$, for all $f \in L^p(0,1)$. Hence,

$$M_{T_V}^{\alpha}(n)f \to 0 \quad \text{as } n \to \infty,$$

for all $f \in L^p(0,1)$.

(iii) The Volterra equation is $(I - T_V)^s g = V^s g = f$ and so the assertions of this part (iii) follow from (i), (9.1), Theorem 8.1, Corollary 8.2, Theorem 8.4 and Corollary 8.5, respectively.

(iv) This is as in part (iii), this time applying Theorem 8.7 and Theorem 8.10. ∎

Remark 9.3: It is natural to ask whether the condition $\alpha > |(1/2) - (1/p)|$ in the above Proposition is sufficient (or even necessary) to get the α-ergodicity of T_V on the space $L^p(0,1)$, but this is not part of our aims here and so we do not address this (involved) problem in this paper. Note also that for $p = \infty$ all the results in Proposition 9.2 hold true by replacing the space $L^\infty([0,1])$ with $C([0,1])$.

Regarding convergence, the norm-convergence can be replaced with weak convergence in the statement of Proposition 9.2.

Example 9.4: Let $0 < \beta < 1$ and let $\ell^2_\beta(\mathbb{N}_0)$ denote the Hilbert space of sequences f such that $\|f\|_{2,\beta}^2 := \sum_{j=0}^{\infty} |f(j)|^2 k_\beta(j) < \infty$. Let T_S be the backward shift operator on $\ell^2_\beta(\mathbb{N}_0)$ given by

$$(T_S f)(j) = f(j+1), \quad f \in \ell^2_\beta(\mathbb{N}_0),\, j \in \mathbb{N}_0.$$

Then $\|T_S^n\|^2 \sim (n+1)^{1-\beta}$, so T_S is not power-bounded on $\ell^2_\beta(\mathbb{N}_0)$, but T_S is (C,α)-bounded for $\alpha > (1-\beta)/2$; see [2].

We have that $I - T_S$ is the first order finite difference operator $W = D$ (recall Section 2), that is,

$$(I - T_S)f(n) = f(n) - f(n+1), \quad n \in \mathbb{N}_0.$$

It is very simple to show that the space $c_{00}(\mathbb{N}_0)$ of eventually null sequences satisfies

$$c_{00}(\mathbb{N}_0) \subset (I - T_S)(c_{00}(\mathbb{N}_0)),$$

whence one gets the density of $(I - T_S)(\ell^2_\beta(\mathbb{N}_0))$ in $\ell^2_\beta(\mathbb{N}_0)$. As a consequence, T_S is (C,α)-ergodic for $\alpha > (1-\beta)/2$ by Theorem 3.3. Thus we can apply the results of Section 8 to T_S in a similar way we have done in the above example for T_V.

Proposition 9.5: *Let T_S be the backward shift acting on $\ell^2_\beta(\mathbb{N}_0)$, $0 < \beta < 1$, as above and assume $\alpha > (1-\beta)/2$.*

(i) *Let $0 < s < 1$. Take $f \in \ell^2_\beta(\mathbb{N}_0)$. Then the problem in differences*

$$D^s u = f \tag{9.3}$$

has a (unique) solution $u \in \ell^2_\beta(\mathbb{N}_0)$ if and only if

$$\sum_{n=1}^{\infty} n^{s-1-\alpha} \sum_{j=0}^{n} k_\alpha(n-j) f(j+\cdot)$$

is norm (or weak) convergent in $\ell^2_\beta(\mathbb{N}_0)$, and in this case

$$u = \frac{\sin(\pi s)\Gamma(1-s+\alpha)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(n+\alpha+1)} \sum_{j=0}^{n} k_\alpha(n-j) f(j+\cdot).$$

Also,

$$\left\| \sum_{j=0}^{n} (n-j)^{\alpha-1} f(j+\cdot) \right\|_{2,\beta} = o(n^{\alpha+1-s}), \text{ as } n \to \infty.$$

If, moreover, $0 < s < (1+\beta)/2$ and $(1-\beta)/2 < \alpha < 1-s$, then the equation (9.3) has a (unique) solution $u \in \ell^2_\beta(\mathbb{N}_0)$ if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{1-s}} f(n+\cdot)$ is convergent in $\ell^2_\beta(\mathbb{N}_0)$ and then the solution u is given by

$$u = \sum_{n=0}^{\infty} k_s(n) f(n+\cdot) = W^{-s} f.$$

(ii) *One has $f \in \mathrm{Dom}(\log D)$ if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \sum_{j=0}^{n} k_\alpha(n-j) f(j+\cdot)$ is convergent in $\ell^2_\beta(\mathbb{N}_0)$. In this case,*

$$(\log D) f = (\psi(\alpha+1) - \psi(1)) f - \sum_{n=1}^{\infty} B(\alpha+1, n) \sum_{j=0}^{n} k_\alpha(n-j) f(j+\cdot),$$

where ψ is the digamma function.

If, moreover, $\alpha \in ((1-\beta)/2, 1)$, then $f \in \mathrm{Dom}(\log D) \subset \ell^2_\beta(\mathbb{N}_0)$ if and only if $\sum_{n=1}^{\infty} \frac{f(n+\cdot)}{n}$ is convergent in $\ell^2_\beta(\mathbb{N}_0)$ and then

$$(\log D) f = -\sum_{n=1}^{\infty} \frac{1}{n} f(n+\cdot).$$

Remark 9.6: The space $\ell^2_\beta(\mathbb{N}_0)$ coincides, up to equivalent norms, with the weighted Bergman space $\mathcal{B}_\nu$ for $\nu = -\beta$ formed by the holomorphic functions $\mathfrak{f}$ on the unit disc such that

$$\|\mathfrak{f}\|_{\nu,2} := \left(\int_{\mathbb{D}} (\nu+1)|f(z)|^2(1-|z|^2)^\nu dA(z) \right)^{1/2} < \infty,$$

where $dA(z)$ is the normalized area measure $\frac{dx\,dy}{\pi}$ on $\mathbb{D}$ (usually, ν takes the form $\nu = \mu - 2$ with $\mu > 1$). Naturally, the operator T_S, transferred on $\mathcal{B}_\nu$, reads

$$T_S\mathfrak{f}(z) = \frac{\mathfrak{f}(z) - \mathfrak{f}(0)}{z}, \quad |z| < 1,$$

whence one gets

$$T_S^n\mathfrak{f}(z) = \frac{1}{z^n}(\mathfrak{f}(z) - \sum_{j=0}^{n-1} \frac{\mathfrak{f}^{(j)}(0)}{(n-1)!} z^{n-1}), \quad |z| < 1, n \in \mathbb{N}$$

and so

$$[(I - T_S)\mathfrak{f}](z) = \frac{(z-1)\mathfrak{f}(z) - \mathfrak{f}(0)}{z}, \quad |z| < 1,$$

which are quite more manageable using Taylor coefficients, that is, $\ell^2_\beta(\mathbb{N}_0)$.

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References

[1] L. Abadias, *A Katznelson–Tzafriri theorem for Cesàro bounded operators*, Studia Mathematica **234** (2016), 59–82.

[2] L. Abadias, G. Bello-Burguet and D. Yakubovich, *Operator inequalities, functional models and ergodicity*, Journal of Mathematical Analysis and Applications **498** (2021), Article no. 124984.

[3] L. Abadias and A. Bonilla, *Growth orders and ergodicity for absolutely Cesàro bounded operators*, Linear Algebra and its Applications **561** (2019), 253–267.

[4] L. Abadias, M. De León and J. L. Torrea, *Non-local fractional derivatives. Discrete and continuous*, Journal of Mathematical Analysis and Applications **449** (2017), 734–755.

[5] L. Abadias, C. Lizama, P. J. Miana and M. P. Velasco, *Cesàro sums and algebra homorphisms of bounded operators*, Israel Journal of Mathematics **216** (2016), 471–505.

[6] L. Abadias and P. J. Miana, *Generalized Cesàro operators, fractional finite differences and gamma functions*, Journal of Functional Analysis **274** (2018), 1424–1465.

[7] B. N. Al-Saqabi, S. L. Kalla and H. M. Srivastava, *A certain family of infinite series associated with digamma functions*, Journal of Mathematical Analysis and Applications **159** (1991), 361–372.

[8] S. Asmussen and M. Bladt, *Poisson equation for queues driven by a Markovian marked point process*, Queueing Systems **17** (1994), 235–274.

[9] I. Assani and M. Lin, *On the one-sided ergodic Hilbert transform*, in *Ergodic Theory and Related Fields*, Contemporary Mathematics, Vol. 430, American Mathematical Society, Providence, RI, 2007, pp. 221–39.

[10] F. M. Atici and P. W. Eloe, *Initial value problems in discrete fractional calculus*, Proceedings of the American Mathematical Society **137** (2009), 981–989.

[11] L. W. Cohen, *On the mean ergodic theorem*, Annals of Mathematics **41** (1940), 505–509.

[12] G. Cohen, C. Cuny and M. Lin, *The one-sided ergodic Hilbert transform in Banach spaces*, Studia Mathematica **196** (2010), 251–263.

[13] G. Cohen and M. Lin, *The one-sided ergodic Hilbert transform of normal contractions. Characteristic Functions, Scattering Functions and Transfer Functions*, Operator Theory: Advances and Applications, Vol. 197, Birkhäuser, Basel, 2010, pp. 77–98.

[14] C. Cuny and M. Lin, *Pointwise ergodic theorems with rate and application to the CLT for Markov chains*, Annales de l'Institut Henri Poincaré Probabilités et Statistiques **45** (2009), 710–733.

[15] Y. Derriennic, *On the mean ergodic theorem for Cesàro bounded operators*, Colloquium Mathematicum **84/85** (2000), 443–455.

[16] Y. Derriennic and M. Lin, *Fractional Poisson equations and ergodic theorems for fractional coboundaries*, Israel Journal of Mathematics **123** (2001), 93–130.

[17] N. Dunford and J. T. Schwartz, *Linear operators. Part I: General theory*, Wiley Classics Library, John Wiley & Sons, New York, 1988.

[18] N. Dungey, *Subordinated discrete semigroups of operators*, Transactions of the American Mathematical Society **363** (2011), 1721–1741.

[19] E. Ed-Dari, *On the* (C, α) *Cesáro bounded operators*, Studia Mathematica **161** (2004), 163–175.

[20] R. Emilion, *Mean-Bounded operators and mean ergodic theorems*, Journal of Functional Analysis **61** (1985), 1–14.

[21] A. Erdélyi and F. G. Tricomi, *The aymptotic expansion of a ratio of Gamma functions*, Pacific Journal of Mathematics **1** (1951), 133–142.

[22] J. E. Galé and A. Wawrzyńczyk, *Standard ideals in weighted algebras of Korenblyum and Wiener types*, Mathematica Scandinavica **108** (2011), 291–319.

[23] P. W. Glynn, *Poissons equation for the recurrent M/G/1 queue*, Advances in Applied Probability **26** (1994), 1044–1062.

[24] A. Gomilko, M. Haase and Y. Tomilov, *On rates in mean ergodic theorems*, Mathematical Research Letters **18** (2011), 201–213.

[25] A. Gomilko and Y. Tomilov, *On discrete subordination of power bounded and Ritt operators*, Indiana University Mathematics Journal **67** (2018), 781–829.

[26] C. S. Goodrich and C. Lizama, *A transference principle for nonlocal operators using a convolutional approach: Fractional monotonocity and convexity*, Israel Journal of Mathematics **236** (2020), 533–589

[27] M. Haase, *The Functional Calculus for Sectorial Operators*, Operator Theory: Advances and Applications, Vol. 169, Birkhäuser, Basel, 2006.

[28] M. Haase and Y. Tomilov, *Domain characterizations of certain functions of power-bounded operators*, Studia Mathematica **196** (2010), 265–288.

[29] E. Hille, *Remarks on ergodic theorems*, Transactions of the American Mathematical Society **57** (1945), 246–269.

[30] S. Jiang, Y. Liu and S. Yao, *Poisson's equation for discrete-time single-birth processes*, Statistics & Probability Letters **85** (2014), 78–83.

[31] W. Jurkat, *Questions of signs in power series*, Proceedings of the American Mathematical Society **5** (1954), 964–970.

[32] Th. Kaluza, *Über die Koeffizienten reziproker Potenzreihen*, Mathematische Zeitschrift **28** (1928), 161–170.

[33] U. Krengel, *Ergodic Theorems*, De Gruyter Studies in Mathematics, Vol. 6, W. de Gruyter, Berlin 1985

[34] Y.-C. Li, R. Sato and S.-Y. Shaw, *Boundednes and growth orders of means of discrete and continuous semigroups of operators*, Studia Mathematica **187** (2008), 1–35.

[35] C. Lizama, *lp-maximal regularity for fractional difference equations on UMD spaces*, Mathematische Nachrichten **288** (2015), 2079–2092.

[36] C. Lizama, *The Poisson distribution, abstract fractional difference equations, and stability*, Proceedings of the American Mathematical Society **145** (2017), 3809–3827.

[37] A. Montes-Rodríguez, J. Sánchez-Álvarez and J. Zemánek, *Uniform Abel–Kreiss boundedness and the extremal behavior of the Volterra operator*, Proceedings of the London Mathematical Society **91** (2005), 761–788.

[38] R. Sato, *Growth orders of means of discrete semigroups of operators in Banach spaces*, Taiwanese Journal of Mathematics **14** (2010), 1111–1116.

[39] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions*, De Gruyter Studies in Mathematics, Vol. 37, Walter de Gruyter, Berlin, 2012.

[40] L. Suciu and J. Zemánek, *Growth conditions on Cesàro means of higher order*, Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum **79** (2013), 545–581.

[41] Y. Tomilov and J. Zemánek, *A new way of constructing examples in operator ergodic theory*, Mathematical Proceedings of the Cambridge Philosophical Society **137** (2004), 209–225.

[42] W. Whitt, *Asymptotic formulas for Markov processes with applicatons to simulations*, Operations Research **40** (1992), 279–291.

[43] T. Yoshimoto, *Uniform and strong ergodic theorems in Banach spaces*, Illinois Journal of Mathematics **42** (1998), 525–543; *Correction*, ibid. **43** (1999), 800–801.

[44] A. Zygmund, *Trigonometric Series. Vols. I, II*, Cambridge University Press, New York, 1959.