l_p- MAXIMAL REGULARITY FOR FRACTIONAL DIFFERENCE EQUATIONS ON UMD SPACES

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ABSTRACT. Let T be a bounded linear operator defined on a UMD Banach space X. We introduce an operator theoretical method for linear fractional difference equations based on the notion of α-resolvent sequence of bounded and linear operators. Then, we define and characterize the l_p- maximal regularity of solutions for the problem

$$(*)\begin{cases} \Delta^\alpha u(n) & = Tu(n) + f(n), \quad n \in \mathbb{Z}_+, \quad 0 < \alpha \leq 1; \\ u(0) & = 0, \end{cases}$$

solely in terms of the R-boundedness of the set

$$\left\{ z^{1-\alpha}(z-1)^\alpha \Big(z^{1-\alpha}(z-1)^\alpha - T \Big)^{-1} : \ |z| = 1, \ z \neq 1 \right\}.$$

1. INTRODUCTION

In the past decade, the study of existence and qualitative properties of discrete solutions for fractional difference equations has drawn a great deal of interest. To mention a few references, see [1, 2, 8, 9, 28, 30, 31, 35, 40] and [43].

First studies on time differences of fractional order are due to Kutter [38]. Diaz and Osler [25] introduced in 1974 a discrete fractional difference operator defined as an infinite series. Grey and Zhang [32] developed a fractional calculus for the discrete nabla (backward) difference operator. At the same time, Miller and Ross [39] defined a fractional sum via the solution of a linear difference equation. More recently, Atici and Eloe [8] introduced the Riemann-Liouville like fractional difference by using the definition of fractional sum of Miller and Ross, and developed some of its properties that allow to obtain solutions of certain fractional difference equations. Ferreira [26, 27] introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. Holm [33, 34] further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations. See also the recent paper [18] for related work. Concerning qualitative properties, Goodrich [29] in a series of papers studied existence of positive solutions and geometrical properties. On the other hand, the theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears [11, 12]. For instance, applications to concrete models have been analyzed recently by Atici and Sengül in [11].

In spite of the significant increase of research in this area, there are still many open questions regarding fractional difference equations. In particular, the study of regularity properties on vector-valued Lebesgue spaces l_p remains an open problem. The maximal regularity property is key to handle vector-valued nonlinear difference equations by operator theoretical methods, because it is a prior and essential step to the use of fixed point

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arguments, see the monograph [3]. However, the literature on the subject is scarce, and no attempt to study maximal regularity of fractional difference equations has been done.

In this paper, we want to give an step further to the development of the theory of vector-valued fractional difference equations. We are successful to completely characterize the maximal regularity of solutions for the problem

$$\left\{ \begin{array}{ll} \Delta^{\alpha}u(n) & = Tu(n) + f(n), \quad n \in \mathbb{Z}_+, \quad 0 < \alpha \leq 1; \\ u(0) & = 0, \end{array} \right. \tag{1.1}$$

in Lebesgue vector-valued spaces defined on the set $\mathbb{Z}_+$. Our approach use as main ingredients Blunck's operator-valued multiplier theorem [13], and the introduction of a special sequence of bounded operators, that we called α-resolvent families, which will play a central role in the representation of the solution of the problem (1.1) by means of a kind of discrete variation of parameters formula. We observe that maximal regularity for evolution equations using methods of operator valued Fourier multiplier theorems has been studied intensively in the last two decades; see [23], [36], [3] and references therein. For instance, on the real line, it has been treated by means of the Amann-Weiss multiplier theorem [42], [4] and on compact intervals by use of the Arendt-Bu multiplier theorem [6], [7]. In contrast, the study of maximal regularity for evolution equations on the time scale $\mathbb{Z}_+$ is limited only to the case of first and second order difference equations [22], [3]. We remark that Fourier multipliers has been used to characterize the maximal regularity of fractional evolution equations in compact intervals by S. Bu in [15] and [16]. See also [37] for a related result using a different concept of fractional derivative. For the time scale $\mathbb{R}$, Fourier multipliers has been employed by Ponce in [41] in order to characterize maximal regularity of fractional differential equations on vector-valued Hölder spaces.

The outline of this paper is as follows: Section 2 is devoted to preliminaries, recalling the notions of UMD spaces, R-boundedness and operator-valued Fourier multipliers. This concepts are needed for the application of Blunck‘s theorem to fractional difference equations analyzed here in the context of Banach spaces. We also establish the definition of the fractional difference operator that we will use and that seems to be more convenient for our purposes. In this line of ideas, we note the recent paper [18], where it was proved that many concepts of fractional differences currently used in the literature are simply related by traslation. We remark that our definition is at the basis of this equivalence. Section 3 introduces a new concept that we called α-resolvent sequences, denoted by $S_\alpha(n)$, as a necessary tool for the study of l_p-maximal regularity. We show how this tool help us to prove an explicit representation of the solution for the fractional difference equation (1.1) with initial value $u(0) = x$, namely (see Theorem 3.7):

$$u(n) = S_\alpha(n)x + (S_\alpha * f)(n-1), \quad n \in \mathbb{N}.$$

On the other hand, the notion of α-resolvent families has own interest because it should correspond to the vector-valued concept of Mittag-Leffler operator sequence. In this context, Theorem 3.4 gives a interesting characterization, showing that α-resolvent families must have the form

$$S_\alpha(n) = \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma(j\alpha+\alpha)} T^j, \quad n \in \mathbb{Z}_+$$

Section 3 is devoted to the main result of this paper, that shows a characterization of l_p-maximal regularity of the equation (1.1) solely in terms of the data of the problem (see Theorem 4.2). In other words, we prove that if a vector valued sequence $f \in l_p$ is defined on a UMD-space X then the solution u of (1.1) exists and is such that $u,\ \Delta^\alpha u \in l_p$ if and only if the set

$$\left\{ z^{1-\alpha}(z-1)^{\alpha}\Big(z^{1-\alpha}(z-1)^{\alpha}-T\Big)^{-1} : |z|=1, \ z\neq 1 \right\}$$

is R-bounded. Finally, a simpler criteria in case of Hilbert spaces is given (Corollary 4.5).

2. Preliminaries

In this section, we provide the necessary preliminaries on UMD spaces, R-boundedness, fractional differences and operator-valued Fourier multipliers, needed in the forthcoming sections. Additional information on this topics can be found in the monographs [24] and [3].

2.1. UMD spaces.

There are many important statements in vector-valued harmonic analysis and probability theory that are equivalent to the UMD property [17]. The significance of the UMD property for vector-valued multiplier theorems was recognized in [14].

Definition 2.1. A Banach space X is said to has the Unconditional Martingale Difference property (UMD) if for each $p\in(1,\infty)$ there is a constant $C_p>0$ such that for any martingale $(f_n)_{n\geq 0}\subset L^p(\Omega,\Sigma,\mu;X)$ and any choice of signs $(\xi_n)_{n\geq 0}\subset\{-1,1\}$ and any $N\in\mathbb{Z}_+$ the following estimate holds

$$\Big|\Big|f_0+\sum_{n=1}^{N}\xi_n(f_n-f_{n-1})\Big|\Big|_{L^p(\Omega,\Sigma,\mu;X)}\leq C_p||f_N||_{L^p(\Omega,\Sigma,\mu;X)}.$$

Remark 2.2.

(i) A Banach space X is said to be $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R},X)$ for some (and then all) $p\in(1,\infty)$. Here, the Hilbert transform H of a function $f\in\mathcal{S}(\mathbb{R},X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf(t):=PV-\int\frac{1}{t-s}f(s)ds.$$

It is a well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. For more information and details on the Hilbert transform and UMD Banach spaces we refer to Amann's book [5, Section III.4.3-III.4.5].

(ii) The UMD spaces include Hilbert spaces, Sobolev spaces $W^s_p(\Omega)$, $1<p<\infty$, Lebesgue spaces $L^p(\Omega,\mu)$, $L^p(\Omega,\mu;X)$, l_p, $1<p<\infty$, where X is a UMD space, the reflexive noncommutative L^p spaces, Hardy spaces, Lorentz and Orlicz spaces, any von Neumann algebra and the Schatten-von Neumann classes $C_p(H)$, $1<p<\infty$, of operators on Hilbert spaces. On the other hand, the space of continuous functions $C(K)$ do not have the UMD property.

(iii) If $1<p<\infty$, and the Lebesgue-Bochner space $L^p((0,1),X)$ has an unconditional basis, then X is UMD.

Theorem 2.3. ([24]) *The following properties hold:*

(i) *If X is UMD and Y is a closed linear subspace of X, then Y and X/Y are UMD spaces.*

(ii) *A Banach space X is UMD if and only if its dual X^* is UMD.*

(iii) *A UMD space is always uniformly convex, but not conversely.*

(iv) *If X and Y are UMD spaces, then $X\oplus Y$ is a UMD space.*

(v) *Every UMD space is super-reflexive but not conversely. In particular, L^1 and L^∞ are not UMD spaces.*

2.2. **R-boundedness.** Let X be a Banach space. We define the means

$$\|(x_1,...,x_n)\|_R := \frac{1}{2^n}\sum_{\epsilon_j\in\{-1,1\}^n}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|,$$

for $x_1,...,x_n \in X$.

Definition 2.4. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there is a constant $c \geq 0$ such that

$$\|(T_1x_1,...,T_nx_n)\|_R \leq c\|(x_1,...,x_n)\|_R, \tag{2.1}$$

for all $T_1,...,T_n \in \mathcal{T},\ x_1,...,x_n \in X,\ n \in \mathbb{N}$. The least c such that (2.1) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

Denote by r_j the j-th Rademacher function, that is $r_k(t) := sign(\sin(2^k\pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector-valued function $t \to r_j(t)x$. An equivalent definition using the Rademacher functions replaces (2.1) by

$$\Big\|\sum_{k=1}^n r_k\otimes T_kx_k\Big\|_{L^2(0,1;Y)} \leq c\Big\|\sum_{k=1}^n r_k\otimes x_k\Big\|_{L^2(0,1;X)}. \tag{2.2}$$

The Rademacher functions are an ortonormal sequence in $L^2[0,1]$. For more information, see [3, Section 2.2].

2.3. **Fractional differences.** For a real number a, we denote

$$\mathbb{N}_a := \{a, a+1, a+2, ...\},$$

and we write $\mathbb{N}_1 \equiv \mathbb{N}$. Let X be a complex Banach space. We denote by $s(\mathbb{N}_a;X)$ the vectorial space consisting of all vector-valued sequences $f : \mathbb{N}_a \to X$.
The forward Euler operator $\Delta_a : s(\mathbb{N}_a;X) \to s(\mathbb{N}_a;X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For $m \in \mathbb{N}_2$, we define recursively $\Delta_a^m : s(\mathbb{N}_a;X) \to s(\mathbb{N}_a;X)$ by

$$\Delta_a^m := \Delta_a^{m-1}\circ\Delta_a,$$

and is called the m-th order forward difference operator. For instance, for any $f \in s(\mathbb{N}_0;X)$, we have

$$\Delta_0^m f(n) = \sum_{j=0}^m \binom{m}{j}(-1)^{m-j}f(n+j), \quad n \in \mathbb{N}_0.$$

In particular, we obtain

$$(\Delta_0^1 f)(n) = f(n+1) - f(n), \quad n \in \mathbb{N}_0,$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a;X) \to s(\mathbb{N}_a;X)$ is the identity operator, and $\Delta \equiv \Delta_0^1$.

Definition 2.5. Let $\alpha > 0$ be given and $f : \mathbb{N}_0 \to X$. We define the fractional sum of order α as follows

$$\Delta^{-\alpha}f(n) = \sum_{k=0}^n k^\alpha(n-k)f(k), \quad n \in \mathbb{N}_0, \tag{2.3}$$

where

$$k^\alpha(j) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0.$$

Remark 2.6. The following definition of fractional sum was proposed by Atici and Eloe [10] in 2009 : Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s+1)^{\overline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_a$ and $t^{\overline{\alpha}} := \dfrac{\Gamma(t+\alpha)}{\Gamma(t)}$. Our definition corresponds to the particular case $a = 0$, in other words $\Delta^{-\alpha} \equiv \nabla_0^{-\alpha}$. One of the reasons to choose this operator is because their flexibility to be handled by means of Z-transform methods. Moreover, it has a better behavior for mathematical analysis when we ask, for example, for definitions of fractional sums and differences on subspaces of $s(\mathbb{N}_0; X)$ like e.g. l_p spaces. We notice that, recently, this approach has been followed by other authors, see [20, 21]. It is also interesting to note that it was recently proved that different notions of fractional sum, existing in the current literature, are equivalent with Definition 2.5 modulo translation, see [18].

The next concept is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [39] and [8]. In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied.

Definition 2.7. The fractional difference operator of order $\alpha > 0$ (in the sense of Riemann-Liouville) is defined by

$$\Delta^{\alpha} f(n) := \Delta_0^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Interchanging the order of the operators in the definition of fractional difference in the sense of Riemann-Liouville, and in analogous way as above, we can introduce the notion of fractional difference in the sense of Caputo as follows.

Definition 2.8. Let $\alpha > 0$. The *α-th fractional Caputo like difference* is defined by

$$(2.4) \qquad {}_C\Delta^{\alpha} f(n) := \Delta^{-(m-\alpha)} (\Delta_0^m f)(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Recall that the finite convolution $*$ of two sequences $f(n)$ and $g(n)$ is defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j) g(j), \quad n \in \mathbb{N}_0.$$

For further use, we note the following relation between the Caputo and Riemann-Liouville fractional differences of order $0 < \alpha < 1$.

Theorem 2.9. *(*[18]*) For each $0 < \alpha < 1$ and $u \in s(\mathbb{N}_0; X)$, we have*

$$ {}_C\Delta^{\alpha} u(n) = \Delta^{\alpha} u(n) - k^{1-\alpha}(n+1) u(0), \quad n \in \mathbb{N}_0.$$

2.4. **Fourier multipliers.** We recall that the *z-transform* of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\widetilde{f}(z) := \sum_{j=0}^{\infty} z^{-j} f(j)$$

where z is a complex number. Note that convergence of the series is given for $|z| > R$ with R sufficiently large. The discrete time Fourier transform of a vector valued sequence $f : \mathbb{Z} \to X$ is defined by

$$\widehat{f}(z) := \sum_{j=-\infty}^{\infty} z^{-j} f(j), \text{ where } z = e^{it}, \quad t \in (-\pi, \pi),$$

whenever it exists.

Remark 2.10. Note that the z-transform around the unit circle in the complex plane corresponds to the discrete time Fourier transform, whenever both exists.

In what follows we denote $\mathbb{T} := (-\pi, \pi) \setminus \{0\}$. The following Fourier multiplier theorem for operator valued symbols is due to S. Blunck [13].

Theorem 2.11. [13, Theorem 1.3] *Let $p \in (1, \infty)$ and let X be a UMD space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be differentiable and such that the set*

$$\left\{M(t), (z-1)(z+1)M'(t) : z = e^{it},\ t \in \mathbb{T}\right\}$$

is R-bounded. Then there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that

$$\widehat{(T_M f)}(z) = M(t)\widehat{f}(z), \text{ for all } z = e^{it},\ t \in \mathbb{T}. \tag{2.5}$$

The converse of Blunck's theorem holds without restriction on the Banach space X, in the following sense:

Theorem 2.12. *([13]) Let $p \in (1, \infty)$ and let X be a Banach space. Let $M : \mathbb{T} \to \mathcal{B}(X)$ be an operator valued function. Suppose that there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that* (2.5) *holds. Then the set*

$$\{M(t) : t \in \mathbb{T}\}$$

is R-bounded.

3. α-resolvent sequences

In this section, we introduce an operator theoretical method to study the linear fractional difference equation

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}, \tag{3.1}$$

with initial condition $u(0) = x \in X$ and where $T \in \mathcal{B}(X)$. We begin with the following definition.

Definition 3.1. Let T be bounded operator defined on a Banach space X and $\alpha > 0$. We call T the generator of an α-resolvent sequence if there exists a sequence of bounded and linear operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ that satisfies the following properties

(i) $S_\alpha(0) = I$

(ii) $S_\alpha(n+1) = k^\alpha(n+1)I + T\sum_{j=0}^{n} k^\alpha(n-j)S_\alpha(j)$ for all $n \in \mathbb{N}_0$.

In this case, $S_\alpha(n)$ is called the α-resolvent sequence generated by T.

The sequence of operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ will help us to obtain an explicit representation for the solution of equation (3.1) by means of a kind of variation of parameters formula. The following Lemma follows easily from the definition.

Lemma 3.2. *If T generates an α-resolvent family, then it is unique.*

Proof. Let $Q_\alpha(n)$ and $S_\alpha(n)$ two α-resolvent families generated by T. Define $P_\alpha(n) = S_\alpha(n) - Q_\alpha(n)$. Then $P_\alpha(0) = 0$ and

$$P_\alpha(n+1) = T\sum_{j=0}^{n} k^\alpha(n-j)P_\alpha(j),$$

which implies $P_\alpha(n) = 0$ for all $n \in \mathbb{N}$.

□

Example 3.3. In case $\alpha = 1$ we have the recurrence relation

$$S_1(n) = I + T\sum_{j=0}^{n-1} S_1(j)$$

where $S_1(0) = I$. Then, a simple calculation gives $S_1(n) = (I+T)^n$ for all $n \in \mathbb{N}_0$.

Other cases are not simple to obtain explicitly and we need to use transform methods. In order to do that we recall that given two sequences $\{a(n)\}_{n\in\mathbb{N}_0}$ and $\{S(n)\}_{n\in\mathbb{N}_0}$, the convolution of a and S can be defined by

$$(a * S)(n) := \sum_{k=0}^{n} a(n-k)S(k),$$

whenever the right hand side makes sense. For example, with this notation we observe that we can rewrite the definition of fractional sum as follows

$$\Delta^{-\alpha}u(n) = (k^\alpha * u)(n), \quad n \in \mathbb{N}_0.$$

and (ii) in Definition 3.1 as

$$S_\alpha(n+1) = k^\alpha(n+1)I + T(k^\alpha * S_\alpha)(n), \quad n \in \mathbb{N}_0.$$

Using the property of the z-transform on the convolution, we obtain formally

$$\widetilde{S_\alpha}(z) = z\left(\frac{z}{\widetilde{k^\alpha}(z)} - T\right)^{-1}$$

whenever $\dfrac{z}{\widetilde{k^\alpha}(z)} \in \rho(T)$, where $\rho(T)$ denotes the resolvent set of T. Since

$$\widetilde{k^\alpha}(z) = \frac{z^\alpha}{(z-1)^\alpha},$$

for $|z| > 1$, we have

$$\widetilde{S_\alpha}(z) = z((z-1)^\alpha z^{1-\alpha} - T)^{-1},$$

whenever the right hand side exists. Then, applying the inverse z-transform, we obtain

$$S_\alpha(n) = \frac{1}{2\pi i}\int_C z^n((z-1)^\alpha z^{1-\alpha} - T)^{-1}dz,$$

where C is a circle, centered at the origin of the complex plane, that encloses all spectral values of $(z-1)^\alpha z^{1-\alpha} - T$.

In order to have an equivalent and more useful representation for our purposes, we prove the following interesting characterization that completely describes α-resolvent families.

Theorem 3.4. *Let $\alpha > 0$ and T be a bounded operator defined on a Banach space X. The following properties are equivalent*

(i) *T is the generator of an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$.*

(ii) $S_\alpha(n) = \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma(j\alpha+\alpha)} T^j.$

Proof. Assume (i) and define

$$\delta_j(n) = \begin{cases} 1 & \text{if} \quad n = j; \\ 0 & \text{if} \quad n \neq j. \end{cases}$$

Note that $(\delta_j * \delta_k)(n) = \delta_{j+k}(n)$ and $(\delta_j * k^\beta)(n) = k^\beta(n-j)$ for $j,k \in \{0,1,2...,n\}$ and $\beta > 0$. Then, by definition

$$\begin{aligned} Q_\alpha(n) &:= \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma(j\alpha+\alpha)} T^j = \sum_{j=0}^{n} k^{(j+1)\alpha}(n-j) T^j \\ &= \sum_{j=0}^{n} \delta_j * k^{(j+1)\alpha}(n) T^j = k^\alpha(n) I + \sum_{j=1}^{n} \delta_j * k^{(j+1)\alpha}(n) T^j, \end{aligned}$$

and $Q_\alpha(0) = I$. Since,

$$T(k^\alpha * Q_\alpha)(n) = T k^\alpha * k^\alpha(n) + \sum_{j=1}^{n} \delta_j * k^{(j+2)\alpha}(n) T^{j+1} = \sum_{j=0}^{n} \delta_j * k^{(j+2)\alpha}(n) T^{j+1}, \tag{3.2}$$

and

$$\begin{aligned} Q_\alpha(n+1) - k^\alpha(n+1) &= \sum_{j=1}^{n+1} \delta_j * k^{(j+1)\alpha}(n+1) T^j \\ &= \sum_{j=1}^{n+1} \delta_1 * \delta_{j-1} * k^{(j+1)\alpha}(n+1) T^j \\ &= \sum_{j=1}^{n+1} \delta_{j-1} * k^{(j+1)\alpha}(n) T^j \\ &= \sum_{j=0}^{n} \delta_j * k^{(j+2)\alpha}(n) T^{j+1}, \end{aligned}$$

it follows from (3.2) that

$$Q_\alpha(n+1) - k^\alpha(n+1) = T(k^\alpha * Q_\alpha)(n).$$

Therefore, by the uniqueness Lemma 3.2 we obtain (ii). Conversely, define $S_\alpha(n)$ by (ii). Proceeding as previously, we obtain that $S_\alpha(n)$ satisfies Definition 3.1, proving the theorem. □

Remark 3.5. It is interesting to note that in case $\alpha = 1$ we have from the above Theorem

$$S_1(n) = \sum_{j=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n-j+1)\Gamma(j+1)} T^j = \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} T^j = \sum_{j=0}^{n} \binom{n}{j} T^j = (I+T)^n,$$

which recover Example 3.3.

We will need the following Lemma that have own interest.

Lemma 3.6. *Let $0 < \alpha < 1$, $a : \mathbb{N}_0 \to \mathbb{C}$ and $S : \mathbb{N}_0 \to X$ be given. Then*

$$\Delta^\alpha(a * S)(n) = (a * \Delta^\alpha S)(n) + S(0)a(n+1), \quad n \in \mathbb{N}_0, \tag{3.3}$$

holds.

Proof. By definition, we have

$$\begin{aligned}\Delta^\alpha(a*S)(n) &= \Delta\circ\Delta^{-(1-\alpha)}(a*S)(n) = \Delta^{-(1-\alpha)}(a*S)(n+1) - \Delta^{-(1-\alpha)}(a*S)(n)\\ &= (k^{1-\alpha}*a*S)(n+1) - (k^{1-\alpha}*a*S)(n)\\ &= \sum_{j=0}^{n+1}(k^{1-\alpha}*S)(n+1-j)a(j) - \sum_{j=0}^{n}(k^{1-\alpha}*S)(n-j)a(j)\\ &= \sum_{j=0}^{n}[(k^{1-\alpha}*S)(n+1-j) - (k^{1-\alpha}*S)(n-j)]a(j) + (k^{1-\alpha}*S)(0)a(n+1)\\ &= \sum_{j=0}^{n}\Delta(k^{1-\alpha}*S)(n-j)a(j) + k^{1-\alpha}(0)S(0)a(n+1)\\ &= \sum_{j=0}^{n}\Delta^\alpha S(n-j)a(j) + S(0)a(n+1),\end{aligned}$$

proving the Lemma. □

Now we are ready to prove the second main result of this section.

Theorem 3.7. *Let* $0<\alpha<1$ *and* $f:\mathbb{N}\to X$ *be given. The unique solution of* (3.1) *with initial condition* $u(0)=x$ *can be represented by*

$$u(n) = S_\alpha(n)u(0) + (S_\alpha * f)(n-1), \quad n\in\mathbb{N}.$$

Proof. Let $n\in\mathbb{N}$ be given. We have

$$\Delta^\alpha u(n) = \Delta^\alpha S_\alpha(n)u(0) + \Delta^\alpha(S_\alpha*f)(n-1), \tag{3.4}$$

where, by definition

$$\Delta^\alpha S_\alpha(n) = \Delta^\alpha k^\alpha(n) + T\Delta^\alpha(k^\alpha*S_\alpha)(n-1).$$

Note that $\Delta^\alpha k^\alpha(n) = \Delta\circ\Delta^{-(1-\alpha)}k^\alpha(n) = \Delta(k^{1-\alpha}*k^\alpha)(n) = \Delta k^1(n) = 0$, therefore using Lemma 3.6 we obtain

$$\Delta^\alpha S_\alpha(j+1) = T\Delta^\alpha(k^\alpha*S_\alpha)(j) = T(\Delta^\alpha k^\alpha*S_\alpha)(j) + k^\alpha(0)TS_\alpha(j+1) = TS_\alpha(j+1),$$

for all $j\in\mathbb{N}_0$. Hence $\Delta^\alpha S_\alpha(n) = TS_\alpha(n)$ and from (3.4) we have

$$\Delta^\alpha u(n) = TS_\alpha(n)u(0) + \Delta^\alpha(S_\alpha*f)(n-1). \tag{3.5}$$

Now, again by Lemma 3.6 we get

$$\Delta^\alpha(S_\alpha*f)(j) = (\Delta^\alpha S_\alpha*f)(j) + S_\alpha(0)f(j+1),$$

for all $j\in\mathbb{N}_0$. Therefore from (3.5) we conclude that

$$\begin{aligned}\Delta^\alpha u(n) &= TS_\alpha(n)u(0) + (\Delta^\alpha S_\alpha*f)(n-1) + f(n)\\ &= TS_\alpha(n)u(0) + T(S_\alpha*f)(n-1) + f(n)\\ &= T[S_\alpha(n)u(0) + (S_\alpha*f)(n-1)] + f(n)\\ &= Tu(n) + f(n).\end{aligned}$$

□

Remark 3.8. In the border case $\alpha=1$ we obtain the representation proved in [3, Proposition 1.3.1].

4. A characterization of maximal l_p- regularity

Let $T \in \mathcal{B}(X)$ be given and $f : \mathbb{Z}_+ \to X$ be a vector valued sequence. In this section we consider the discrete time evolution equation of fractional order

$$\begin{cases} \Delta^\alpha u(n) = Tu(n) + f(n), & n \in \mathbb{N}, \\ u(0) = 0. \end{cases} \tag{4.1}$$

where $0 < \alpha < 1$. By Theorem 3.7 the solution of equation (4.1) can be represented by

$$u(n) = (S_\alpha * f)(n-1), \quad n \in \mathbb{N},$$

where $S_\alpha(n)$ is explicitly given in Theorem 3.4(ii). Note that, by Lemma 3.6

$$\Delta^\alpha u(n) = T(S_\alpha * f)(n-1) + f(n). \tag{4.2}$$

In analogy to the border case $\alpha = 1$, we introduce the following definition

Definition 4.1. Let $1 < p < \infty$, $0 < \alpha \le 1$ and $T \in \mathcal{B}(X)$ be given. We say that equation (4.1) has maximal l_p-regularity if

$$(\mathcal{K}_\alpha f)(n) := T \sum_{j=0}^{n} S_\alpha(n-j) f(j)$$

defines a bounded operator $\mathcal{K}_\alpha \in \mathcal{B}(l_p(\mathbb{Z}_+; X))$ for some $p \in (1, \infty)$.

In other words, and in view of the relation (4.2), the question is if $f \in l_p(\mathbb{N}_0, X)$ implies $u,\ \Delta^\alpha u \in l_p(\mathbb{N}_0, X)$.

In what follows we will need the following hypothesis:

$(H)_\alpha$ The operator $(z-1)^\alpha z^{1-\alpha} I - T$ is invertible for all $|z| = 1,\ z \neq 1$.

Denote $\mathbb{D} : \{z \in \mathbb{C} \,:\, |z| \le 1\}$ and define the set

$$\Omega_\alpha := \{z \in \mathbb{C} \,:\, z = (w-1)^\alpha w^{1-\alpha},\ w \in \partial\mathbb{D},\ w \neq 1\}.$$

For example, in case $\alpha = 1$ we have $\Omega_1 = \{z \in \mathbb{C} : |z+1| = 1, \quad z \neq 0\}$. Other cases are illustrated in the figures below.

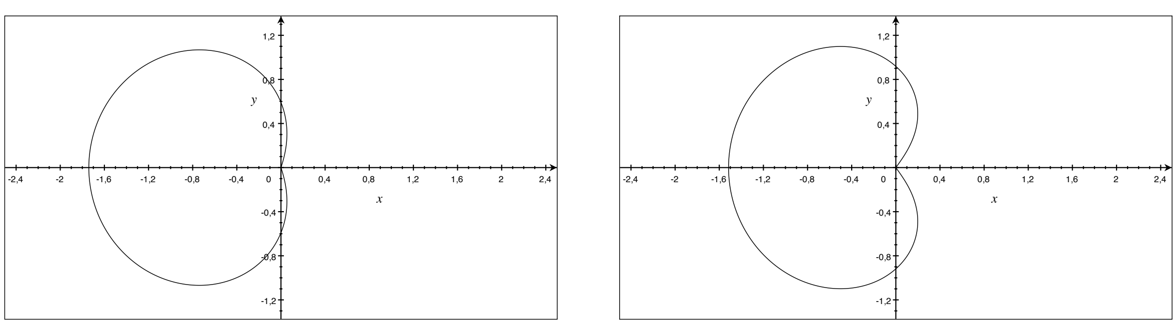

Figure 1. left: $\alpha = 0.8$; right: $\alpha = 0.6$

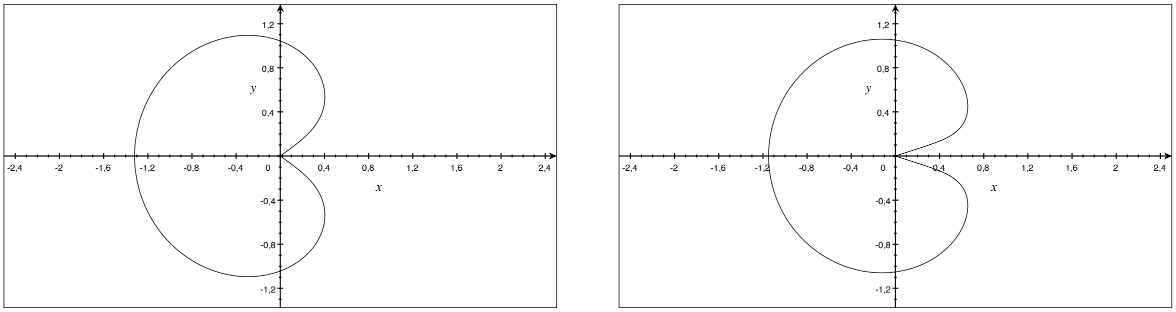

FIGURE 2. left: $\alpha = 0.4$; right: $\alpha = 0.2$

We note that if $(I+T)$ is power bounded and analytic (see definitions below) then $(H)_1$ holds. See [3, Theorem 1.3.9]. Our main result in this paper is the following theorem.

Theorem 4.2. *Let $0 < \alpha \le 1$, $p > 1$ and X be a UMD space. Suppose there is $T \in \mathcal{B}(X)$ such that $(H)_\alpha$ holds. Then the following assertions are equivalent.*

(i) *Equation (4.1) has maximal l_p-regularity.*

(ii) *The set $\left\{ z^{1-\alpha}(z-1)^\alpha \Big(z^{1-\alpha}(z-1)^\alpha - T \Big)^{-1} : |z| = 1,\ z \neq 1 \right\}$ is R-bounded.*

Proof. (ii) $\Longrightarrow$ (i). Define

$$M(t) = z^{1-\alpha}(z-1)^\alpha \Big(z^{1-\alpha}(z-1)^\alpha - T \Big)^{-1}, \quad z = e^{it}, \quad t \in (-\pi, \pi).$$

Denote $f_\alpha(t) = e^{it}(1-e^{-it})^\alpha$. Then we can rewrite $M(t) = f_\alpha(t)(f_\alpha(t) - T)^{-1}$ and a calculation gives

$$M'(t) = \frac{f_\alpha'(t)}{f_\alpha(t)} M(t) - \frac{f_\alpha'(t)}{f_\alpha(t)} M(t)^2,$$

where

$$f_\alpha'(t) = i f_\alpha(t) + \frac{i\alpha}{e^{i\alpha} - 1} f_\alpha(t) = i f_\alpha(t) + \frac{i\alpha}{z-1} f_\alpha(t), \quad z = e^{it}.$$

Therefore,

$$(z-1)(z+1)M'(t) = [i(z-1)(z+1) + i\alpha(z+1)]M(t) - [i(z-1)(z+1) + i\alpha(z+1)]M(t)^2,$$

where $a_\alpha(t) = i(z-1)(z+1) + i\alpha(z+1)$ is bounded for $z = e^{it}$, $t \in (-\pi, \pi)$. We conclude from [3, Proposition 2.2.5] that the set

$$\left\{ (z-1)(z+1)M'(t) : z = e^{it},\ t \in \mathbb{T} \right\},$$

is R-bounded. Therefore, by Theorem 2.11 we obtain that there is an operator $T_\alpha \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that

$$\widehat{(T_\alpha f)}(z) = M(t)\widehat{f}(z), \text{ for all } z = e^{it},\ t \in \mathbb{T}, \tag{4.3}$$

and for all $f \in l^p(\mathbb{Z}; X)$. From the identity

$$T(z^{1-\alpha}(z-1)^\alpha - T)^{-1} = z^{1-\alpha}(z-1)^\alpha (z^{1-\alpha}(z-1)^\alpha - T)^{-1} - I$$

and (4.3) we obtain that the left hand side in the identity

$$T(z^{1-\alpha}(z-1)^\alpha - T)^{-1}\hat{f}(z) = z^{1-\alpha}(z-1)^\alpha (z^{1-\alpha}(z-1)^\alpha - T)^{-1}\widehat{f}(z) - I\widehat{f}(z), \tag{4.4}$$

defines a bounded operator on $l^p(\mathbb{Z}; X)$ given by $R_\alpha f(n) = T_\alpha f(n) - f(n)$.

For $f \in l^p(\mathbb{Z}; X)$ we define the operator

$$K_\alpha f(n) = \begin{cases} T(S_\alpha * f)(n) & n \in \mathbb{N}, \\ \\ 0 & otherwise. \end{cases} \tag{4.5}$$

Note that the z-transform of $S_\alpha(n)$ satisfy

$$((z-1)^\alpha z^{1-\alpha} - T)\widetilde{S}_\alpha(z) = zI, \tag{4.6}$$

for all $|z| > R$ where $R > 0$ is sufficiently large. By hypothesis (H_α), the left hand side is invertible for $|z| = 1,\ z \neq 1$ and therefore, en view of (4.4), the discrete Fourier transform of $K_\alpha f(n-1)$ coincides with $R_\alpha f(n)$ for all $n \in \mathbb{N}$, by uniqueness. It proves (i). Conversely, suppose (i). Let $f \in l^p(\mathbb{Z}; X)$ and define

$$K_\alpha f(n) = \begin{cases} \mathcal{K}_\alpha f(n) & n \in \mathbb{N}_0, \\ \\ 0 & otherwise. \end{cases} \tag{4.7}$$

and

$$T_\alpha f(n) := K_\alpha f(n-1) + f(n), \quad n \in \mathbb{Z}.$$

Then, for $z = e^{it}$, $t \in (-\pi, \pi)$ we have

$$\begin{aligned} \widehat{T_\alpha f}(z) &= \sum_{j\in\mathbb{Z}} z^{-j} T_\alpha f(j) = \sum_{j=1}^{\infty} z^{-j} K_\alpha f(j-1) + \sum_{j\in\mathbb{Z}} z^{-j} f(j) \\ &= z^{-1} \sum_{j=0}^{\infty} z^{-j} K_\alpha f(j) + \sum_{j\in\mathbb{Z}} z^{-j} f(j) \\ &= z^{-1} \sum_{j=0}^{\infty} z^{-j} \mathcal{K}_\alpha f(j) z^{-j} + \widehat{f}(z). \end{aligned}$$

By hypothesis (H_α) and (4.6) we have that the z-transform $\widetilde{S}_\alpha(z)$ exists for $z = e^{it}$ and therefore

$$\widehat{T_\alpha f}(z) = z^{-1} T \widehat{\mathbf{S}}_\alpha(z) \widehat{f}(z) + \widehat{f}(z),$$

where

$$\mathbf{S}_\alpha(n) = \begin{cases} S_\alpha(n) & n \in \mathbb{N}_0, \\ \\ 0 & otherwise. \end{cases} \tag{4.8}$$

Hence, (4.6) and (4.4) implies

$$\begin{aligned} \widehat{T_\alpha f}(z) &= T(z^{1-\alpha}(z-1)^\alpha - T)^{-1} \widehat{f}(z) + \widehat{f}(z) \\ &= z^{1-\alpha}(z-1)^\alpha (z^{1-\alpha}(z-1)^\alpha - T)^{-1} \widehat{f}(z) - \widehat{f}(z) + \widehat{f}(z) \\ &= M(t)\widehat{f}(z). \end{aligned}$$

Then, the result is consequence of Theorem 2.12. □

Remark 4.3. Under the hypothesis that equation (4.1) has l_p-maximal regularity, we deduce that the operator $(z^{1-\alpha}(z-1)^\alpha - T)$ in $(H)_\alpha$ is always surjective. Indeed, given $x \in X$ we define

$$f(n) = \begin{cases} x & n = 0, \\ \\ 0 & otherwise. \end{cases}$$

Hence, by hypothesis we obtain that there exists $u_x \in l_p(\mathbb{Z}, X)$ such that $(z^{1-\alpha}(z-1)^\alpha - T)\widehat{u}_x(z) = \widehat{f}(z) = x$ where $z = e^{it}, t \in (-\pi, \pi)$.

Remark 4.4. In case $\alpha = 1$ we recover and extend Theorem 1.1 (equivalence of (a) and (b)) in Blunck's paper [13], where is assumed that T is power bounded and analytic. See also [3, Theorem 3.1.4] and [3, Section 4] for applications of discrete maximal regularity to nonlinear problems.

For Hilbert spaces, the hypothesis of R-boundedness can be replaced by boundedness. In such case we can show an alternative condition on the operator T in order to obtain l_p-maximal regularity. Recall that an operator $T \in \mathcal{B}(X)$ is called power bounded if the set $\{T^n : n \in \mathbb{N}\}$ is bounded, and is called analytic if the set $\{n(T-I)T^n : n \in \mathbb{N}\}$ is bounded. A discussion on power bounded and analytic operators can be found in [3, Section 1.3] and [19].

Corollary 4.5. *Let $T \in \mathcal{B}(H)$ be a normal operator defined on a Hilbert space H and assume that $T - 2I$ is power bounded and analytic, then for each $f \in l_p(\mathbb{Z}_+, X), p > 1$, there is a unique $u \in l_p(\mathbb{Z}_+, X)$ such that*

$$\left\{ \begin{array}{rcll} \Delta^\alpha u(n) &=& Tu(n) + f(n), & n \in \mathbb{N}, \\ \\ u(0) &=& 0, & \end{array} \right. \tag{4.9}$$

for any $0 < \alpha < 1$.

Proof. Since $T - 2I$ is power bounded and analytic, it follows from [3, Theorem 1.3.9] that

$$\sigma(T) \subset \{z \in \mathbb{C} : |z-2| < 1 \} \cup \{3\}.$$

Define $f_\alpha(z) = z^{1-\alpha}(z-1)^\alpha$. Then $f_\alpha(z) \in \Omega_\alpha$ for all $z \in \partial\mathbb{D}$, $z \neq 1$ and condition (H_α) is satisfied. Hence there exists $\epsilon > 0$ such that $d(f_\alpha(z), \sigma(T)) > \epsilon > 0$. Since T is normal, it follows that

$$\|(f_\alpha(z) - T)^{-1}\| \leq \frac{1}{d(f_\alpha(z), \sigma(T))} < \frac{1}{\epsilon}$$

for all $|z| = 1, z \neq 1$. It follows that condition (ii) in Theorem 4.2 is satisfied and the corollary is proved. ☐

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