AN OPERATOR THEORETICAL APPROACH TO A CLASS OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

CARLOS LIZAMA

Abstract. We propose a general method to obtain the representation of solutions for linear fractional order differential equations based on the theory of (a,k)-regularized families of operators. We illustrate the method in case of the fractional order differential equation

$$D_t^\alpha u'(t) + \mu D_t^\alpha u(t) = Au(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}(u'(0)+\mu u(0)) + f(t), \quad t>0, \quad 0<\alpha\le 1,$$

where A is an unbounded closed operator defined on a Banach space X and f is a X-valued function.

1. Introduction

We study in this paper existence of solutions for fractional order differential equations of the form

$$D_t^\alpha(u'(t)-u'(0)) + \mu D_t^\alpha(u(t)-u(0)) - Au(t) = f(t), \quad t>0, \quad 0<\alpha\le 1,\ \mu\ge 0 \tag{1.1}$$

where the unbounded closed linear operator $A : D(A) \subset X \to X$ is the generator of an (a,k)-regularized family defined on a complex Banach space X and f is a vector-valued function. Here D_t^α denotes the Riemann-Liouville fractional derivative. Notice that equation (1.1) can be rewritten as

$$D_t^\alpha u'(t) + \mu D_t^\alpha u(t) = Au(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}(u'(0)+\mu u(0)) + f(t), \quad t>0, \quad 0<\alpha\le 1. \tag{1.2}$$

Fractional order differential equations is a subject of increasing interest in different contexts and areas of research, see e.g. [1], [4], [9], [11], [8], [19], [21] the survey paper [7] and references therein. Our motivation to study equation (1.1) comes from recent investigations where a related class appears in connection with partial differential equations and Cauchy-time processes, a type of iterated stochastic process (see [3]). Note that when $A = 2\Delta - \epsilon^2\Delta^2$ (where Δ is the Laplace operator) $\alpha = 1$ and $\mu = 1/\epsilon^2$ the above equation was recently considered by Nane [18, Theorem 2.2]. In particular, in case $\mu = 0, \alpha = 1, A = -\Delta^2$ and $u'(0) = -\frac{2}{\pi}\Delta u_0$ with $u_0 \in D(\Delta)$, the equation

$$u''(t) = -\Delta^2 u(t) - \frac{2}{\pi t}\Delta u_0 + f(t), \quad t>0, \tag{1.3}$$

has been studied in [18, Theorem 2.1] in connection with PDE's and iterated processes. A precise interplay between entire and fractional order differential equations was investigated in reference [10].

Observe that one cannot apply semigroup theory directly to solve problem (1.1) in terms of a variation of constant formula. However, our methods based on the theory of (a,k)-regularized families allows us to construct a solution. In fact, we will show that it is possible to give an abstract operator approach to equation (1.1) by defining an ad-hoc family of strongly continuous operators. Then, we are able to show that the solution of equation (1.1) can be written in terms of a kind of variation of constants formula (cf. Theorem 3.1 below). We believe that the method indicated in this paper can be used to handle many classes of linear fractional order differential equations. Our method can be viewed as an extension of the ideas in reference [4] to state the existence of solutions for the abstract fractional order Cauchy problem.

Our plan is as follows: In section 2, we introduce some preliminaries on fractional order derivatives, the Mittag-Leffler function and the concept of (α,μ)-regularized families, which give us the necessary

2000 *Mathematics Subject Classification.* 26A33, 47D06, 45N05.

Key words and phrases. Regularized resolvent family; fractional order derivative; linear unbounded operator.

Partially supported by Laboratorio de Análisis Estocástico, Proyecto Anillo ACT-13.

framework to apply an operator theoretical approach in the analysis of solutions for the abstract fractional order differential equation (1.1). In section 3, we show a general result on existence and uniqueness of solutions for (1.1). In section 4, we relate equation (1.1) to an integral Volterra equation of convolution type to finally link the (α,μ)-regularized families with the theory of resolvent families for evolutionary integral equations developed in reference [20].

2. Preliminaries

Let $\alpha > 0$, $m = [\alpha]$ and $u : [0,\infty) \to X$, where X is a complex Banach space. The Riemann-Liouville fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s)u(s)ds, \quad t>0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t>0, \quad \beta>0,$$

and in case $\beta = 0$ we set $g_0(t) := 0$. When $\alpha = n$ is integer, we define $D_t^n := \frac{d^n}{dt^n}, n = 1,2,\ldots$ Note that $D_t^\alpha g_\beta = g_{\beta-\alpha}$ for $\beta \geq \alpha$. In particular, $D_t^\alpha 1 = g_{1-\alpha}$ for $0 < \alpha \leq 1$ and $D_t^\alpha g_\alpha = 0$.

The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

if the integral is absolutely convergent for $Re\lambda > \omega$. We have

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1}(g_{m-\alpha} * f)^{(k)}(0)\lambda^{m-1-k}. \tag{2.1}$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{iarg\lambda}$, with $-\pi < arg\lambda < \pi$. The Mittag-Leffler function (see e.g. [6]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z}d\mu, \quad \alpha,\beta>0, \quad z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counter-clockwise. It is an entire function which provides a generalization of several usual functions.
An interesting property related with the Laplace transform of the Mittag-Leffler function is the following (cf. [5, (A.27) p.267]):

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(-\rho^\alpha t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha}, \quad Re\lambda > \rho^{1/\alpha}, \quad \rho > 0. \tag{2.2}$$

In what follows, we denote $k_\mu(t) = e^{-\mu t}$. In order to give an operator theoretical approach to equation (1.1) we introduce the following definition.

Definition 2.1. *Let $\mu \geq 0$ and $0 < \alpha \leq 1$ be given. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an (α,μ)-regularized family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied*

(S1) $S_\alpha(t)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;

(S2) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$;

(S3) The following equation holds:

$$S_\alpha(t)x = k_\mu(t)x + (g_\alpha * k_\mu * AS_\alpha)(t)x, \quad x \in D(A). \tag{2.3}$$

for all $x \in D(A), t \geq 0$. In this case, $S_\alpha(t)$ is called the (α,μ)-regularized family generated by A.

Because of the uniqueness of the Laplace transform, a $(0,0)$-regularized family is the same as a C_0-semigroup whereas that a $(1,0)$-regularized family corresponds to the concept of cosine family, see [2]. We note that an (α,μ) regularized family corresponds to a (a,k)-regularized family with

$$a(t) \equiv a_\alpha(t) := \int_0^t g_\alpha(t-s)e^{-\mu s}ds$$

and

$$k(t) \equiv k_\mu(t).$$

The concept of (a,k)-regularized family was introduced in [13] and studied in a series of papers in recent years (see [12], [14], [15], [16], [17] and [22]).

Remark 2.2. As a consequence of [13]*, in the more general context of (a,k)- regularized families, we obtain that an operator A is the generator of an (α,μ)-regularized family if and only if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha(\lambda+\mu) : Re\lambda > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x := \lambda^\alpha(\lambda^\alpha(\lambda+\mu)-A)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

As in the situation of C_0-semigroups we have diverse relations of an (α,μ)-regularized family and its generator. The following result is a direct consequence of [13, Proposition 3.1 and Lemma 2.2].

Proposition 2.3. *Let $S_\alpha(t)$ be an (α,μ)-regularized family on X with generator A. Then for $\mu \geq 0$ and $0 < \alpha \leq 1$ the following holds:*

(a) For all $x \in D(A)$ we have $S_\alpha(\cdot)x \in C^1(\mathbb{R}_+;X)$.

(b) Let $x \in X$ and $t \geq 0$. Then $\int_0^t\int_0^\tau g_\alpha(s)e^{\mu(s-\tau)}AS_\alpha(t-\tau)xdsd\tau \in D(A)$ and

$$S_\alpha(t)x = e^{-\mu t}x + A\int_0^t\int_0^\tau g_\alpha(s)e^{\mu(s-\tau)}S_\alpha(t-\tau)xdsd\tau.$$

The following characterization of generators of (α,μ)-regularized families, analogous to the Hille-Yosida Theorem for C_0- semigroups, can be directly deduced from [13, Theorem 3.4].

Theorem 2.4. *Let A be a closed linear densely defined operator in a Banach space X. Then A is the generator of a (α,μ)-regularized family if and only if there exists constants $\omega \in \mathbb{R}$ and $M > 0$ such that*
(P1) $\lambda^\alpha(\lambda+\mu) \in \rho(A)$ for all $Re\lambda > \omega$ and
(P2) $H(\lambda) := \lambda^\alpha(\lambda^\alpha(\lambda+\mu)-A)^{-1}$ satisfies the estimates

$$||H^{(n)}(\lambda)|| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0,1,2...$$

We also recall from [20, Proposition 0.1] the following result, that will be useful in what follows.

Proposition 2.5. *Let Y be a Banach space. Suppose $g : \mathbb{C}_+ \to Y$ is holomorphic and satisfies $||\lambda g(\lambda)|| + ||\lambda^2 g'(\lambda)|| \leq M$ for all $Re\lambda > 0$. Then*

$$||f^{(n)}(\lambda)|| \leq \frac{Mn!}{\lambda^{n+1}}$$

for all $\lambda > 0$ and $n = 0,1,2...$

Finally, we observe that results on perturbation, approximation, asymptotic behavior, representation as well as ergodic type theorems can be also deduced from the more general context of (a,k) regularized resolvents (see [14], [15], [16], [17] and [22]).

3. Existence and Uniqueness

In this section we study the existence and uniqueness of solutions to the fractional evolution equation

$$D_t^\alpha(u'(t)-u'(0))+\mu D_t^\alpha(u(t)-u(0))-Au(t)=f(t),\quad t\geq 0,\quad 0\leq\alpha\leq 1,\ \mu\geq 0, \tag{3.1}$$

where A is the generator of an (α,μ)-regularized family. Notice that in terms of the Caputo fractional derivative, that we denote $\mathbb{D}_t^\alpha$, equation (3.1) is equivalent to

$$\mathbb{D}_t^\alpha u'(t)+\mu\mathbb{D}_t^\alpha u(t)-Au(t)=f(t),\quad 0<\alpha\leq 1. \tag{3.2}$$

We say that $u\in C^1(\mathbb{R}_+;X)$ is a *solution* of equation (3.1) if $u(t)\in D(A)$ and satisfies (3.1).

Taking, formally, the Laplace transform to (3.2) we obtain

$$(\lambda^\alpha(\lambda+\mu)-A)\hat{u}(\lambda)=\lambda^\alpha u(0)+\lambda^{\alpha-1}[\dot{u}(0)+\mu u(0)]+\hat{f}(\lambda).$$

Using the definition of $S_\alpha(t)$ by means of Laplace transform (cf. Remark 2.2), the statement of the following theorem is then easy to deduce.

Theorem 3.1. *Assume that A generates an (α,μ)-regularized family $S_\alpha(t)$ on X and that $f\in W_{loc}^{1,1}(\mathbb{R}_+,D(A))$ is given. Then the unique solution of the equation (3.1) is given by*

$$u(t)=S_\alpha(t)u(0)+(1*S_\alpha)(t)[u'(0)+\mu u(0)]+(g_\alpha*S_\alpha*f)(t),\quad 0<\alpha\leq 1,\ \mu\geq 0 \tag{3.3}$$

provided $u(0),u'(0)\in D(A)$.

Proof. We have

$$\begin{aligned} D_t^{\alpha+1}(u(t) & -u(0)-tu'(0))+\mu D_t^\alpha(u(t)-u(0))-Au(t)\\ &= D_t^{\alpha+1}u(t)-(D_t^{\alpha+1}1)u(0)-(D_t^{\alpha+1}t)u'(0)+\mu D_t^\alpha u(t)-\mu(D_t^\alpha 1)u(0)-Au(t)\\ &= D_t^{\alpha+1}u(t)+\mu D_t^\alpha u(t)-Au(t)-g_{1-\alpha}(t)u'(0)-\mu g_{1-\alpha}(t)u(0)\\ &= D_t^\alpha(u'(t)+\mu u(t))-Au(t)-g_{1-\alpha}(t)(u'(0)+\mu u(0)). \end{aligned} \tag{3.4}$$

From (2.3) and the fact that $\mu 1*k_\mu=1-k_\mu$ we deduce the identity

$$(1*S_\alpha)(t)x=x-S_\alpha(t)x+(1*g_\alpha*AS_\alpha)(t)x,\quad x\in D(A)$$

which replaced in (3.3) gives

$$u(t)=S_\alpha(t)u(0)+(u'(0)+\mu u(0))-S_\alpha(t)(u'(0)+\mu u(0))+(1*g_\alpha*AS_\alpha)(u'(0)+\mu u(0))+(g_\alpha*S_\alpha*f)(t). \tag{3.5}$$

On the other hand, from (3.3) we also have

$$u'(t)=S'_\alpha(t)u(0)+S_\alpha(t)(u'(0)+\mu u(0))+\frac{d}{dt}(g_\alpha*S_\alpha*f)(t). \tag{3.6}$$

Now we note from (2.3) the identity $S'_\alpha(t)+S_\alpha(t)=(g_\alpha*AS_\alpha)(t)$. This gives, summing up (3.5) and (3.6), the following identity

$$\begin{aligned} u'(t)+u(t) &= (g_\alpha*AS_\alpha)(t)u(0)+(u'(0)+\mu u(0))+(1*g_\alpha*AS_\alpha)(t)(u'(0)+\mu u(0))\\ &\quad +(g_\alpha*S_\alpha*f)(t)+\tfrac{d}{dt}(g_\alpha*S_\alpha*f)(t). \end{aligned} \tag{3.7}$$

Moreover, again from (3.3) we have

$$\begin{aligned} Au(t) &= AS_\alpha(t)u(0)+(1*AS_\alpha)(t)(u'(0)+\mu u(0))+(g_\alpha*AS_\alpha*f)(t)\\ &= AS_\alpha(t)u(0)+(1*AS_\alpha)(t)(u'(0)+\mu u(0))+(S'_\alpha*f)(t)+(S_\alpha*f)(t). \end{aligned} \tag{3.8}$$

Setting (3.7) and (3.8) in the last equality in (3.4) we get

$$\begin{aligned}
&D_t^{\alpha+1}(u(t)-u(0)-tu'(0))+\mu D_t^{\alpha}(u(t)-u(0))-Au(t)\\
&=D_t^{\alpha}(u'(t)+u(t))-Au(t)-g_{1-\alpha}(t)(u'(0)+\mu u(0))\\
&=-D_t^{\alpha}(g_\alpha * AS_\alpha)(t)u(0)+(D_t^{\alpha}1)(u'(0)+\mu u(0))-D_t^{\alpha}(1*g_\alpha * AS_\alpha)(t)(u'(0)+\mu u(0))\\
&+D_t^{\alpha}(g_\alpha * S_\alpha * f)(t)+D_t^{\alpha+1}(g_\alpha * S_\alpha * f)(t)+AS_\alpha(t)u(0)\\
&+(1*AS_\alpha)(t)(u'(0)+\mu u(0))-(S'_\alpha * f)(t)-(S_\alpha * f)(t)-g_{1-\alpha}(t)(u'(0)+\mu u(0))\\
&=-D_t^{\alpha}(g_\alpha * AS_\alpha)(t)u(0)+AS_\alpha(t)u(0)\\
&-D_t^{\alpha}(1*g_\alpha * AS_\alpha)(t)(u'(0)+\mu u(0))+(1*AS_\alpha)(t)(u'(0)+\mu u(0))\\
&+D_t^{\alpha}(g_\alpha * S_\alpha * f)(t)-(S_\alpha * f)(t)\\
&+D_t^{\alpha+1}(g_\alpha * S_\alpha * f)(t)-(S'_\alpha * f)(t),
\end{aligned} \tag{3.9}$$

where,

$$D_t^{\alpha}(g_\alpha * AS_\alpha)(t)=\frac{d}{dt}(g_{1-\alpha}*g_\alpha * AS_\alpha)(t)=\frac{d}{dt}(1*AS_\alpha)(t)=AS_\alpha(t),$$

$$D_t^{\alpha}(1*g_\alpha * AS_\alpha)(t)=\frac{d}{dt}(g_{1-\alpha}*1*g_\alpha * AS_\alpha)(t)=\frac{d}{dt}(1*1*AS_\alpha)(t)=1*AS_\alpha(t),$$

$$D_t^{\alpha}(g_\alpha * S_\alpha * f)(t)=\frac{d}{dt}(g_{1-\alpha}*g_\alpha * S_\alpha * f)(t)=(S_\alpha * f)(t),$$

and

$$D_t^{\alpha+1}(g_\alpha * S_\alpha * f)(t)=\frac{d^2}{dt^2}(g_{1-\alpha}*g_\alpha * S_\alpha * f)(t)=\frac{d^2}{dt^2}(1*S_\alpha * f)(t)=\frac{d}{dt}(S_\alpha * f)(t).$$

Hence, we obtain from (3.9)

$$\begin{aligned}
D_t^{\alpha+1}(u(t)-u(0)-tu'(0))+\mu D_t^{\alpha}(u(t)-u(0))-Au(t)&=\tfrac{d}{dt}(S_\alpha * f)(t)-(S'_\alpha * f)(t)\\
&=f(t).
\end{aligned} \tag{3.10}$$

□

Sufficient conditions to obtain generators of an (α,μ)-regularized family are given in the following proposition.

Proposition 3.2. *Suppose A generates a bounded analytic semigroup and $0<\alpha\le 1$, $\mu\ge 0$. Then A generates an (α,μ)-regularized family.*

Proof. Since A generates a bounded analytic semigroup, we have that $\{\tau : Re\tau \ge 0\}\subseteq \rho(A)$ and there is a constant $M>0$ such that $||\tau(\tau-A)^{-1}||\le M$ for $Re\tau>0$. Now, define $\hat{a}(\lambda)=\frac{1}{\lambda^{\alpha}(\lambda+\mu)}$ and let $\lambda=re^{i\theta}$ with $|\theta|<\pi/2$ and $r>0$. Then, since $0<\alpha\le 1$ we obtain

$$Re\hat{a}(\lambda)=\frac{\cos\alpha\theta}{r^{\alpha}((r\cos\theta+\mu)^2+(r\sin\theta)^2)}>0,$$

or equivalently $Re(\frac{1}{\hat{a}(\lambda)})>0$. We conclude that $H(\lambda):=\lambda^{\alpha}(\lambda^{\alpha}(\lambda+\mu)-A)^{-1}=\frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}-A)^{-1}$ is well defined and satisfies

$$||\lambda H(\lambda)||=||\lambda\lambda^{\alpha}\hat{a}(\lambda)\frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}-A)^{-1}||\le|\lambda^{\alpha+1}\hat{a}(\lambda)|M\le\frac{|\lambda|}{|\lambda+\mu|}M\le M_1 \text{ for all } Re\lambda>0.$$

For $H'(\lambda)$ one obtains that

$$\lambda^2 H'(\lambda) = \alpha\lambda H(\lambda) + \lambda^2 H(\lambda)^2 \frac{\hat{a}'(\lambda)}{\lambda^\alpha \hat{a}(\lambda)^2} = \alpha\lambda H(\lambda) + \frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}\lambda\frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)},$$

where

$$\lambda\frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)} = -\alpha - \frac{\lambda}{\lambda+\mu},$$

and hence we obtain that there exist a constant $C > 0$ such that for all $Re\lambda > 0$

$$||\lambda^2 H'(\lambda)|| \leq \alpha||\lambda H(\lambda)|| + ||\frac{1}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}||\ |\lambda\frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)}| \leq \alpha M_1 + MC$$

By Proposition 2.5 we obtain that (P2) in Theorem 2.4 is satisfied, proving the claim.

□

Combining the previous results, we can give the following corollary on existence of solutions of equation (3.1).

Corollary 3.3. *Let $0 \leq \alpha \leq 1$, $\mu \geq 0$. Suppose A generates a bounded analytic semigroup. Then there exist a unique solution of equation (3.1) for all $f \in W^{1,1}_{loc}(\mathbb{R}_+, D(A))$, and $u(0), u'(0) \in D(A)$.*

4. Connection with resolvent families

Given $c \in L^1_{loc}(\mathbb{R}_+)$, we recall that a closed and linear operator A is called the generator of a resolvent family (or $(c,1)$-regularized family) $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = I$;

(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A), t \geq 0$;

(R3) The following equation holds:

$$R(t)x = x + (c * AR)(t)x, \tag{4.1}$$

for all $x \in D(A), t \geq 0$.

The above is equivalent to say that there exists $\omega \geq 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{1}{\hat{c}(\lambda)} : Re\lambda > \omega\} \subset \rho(A)$ and

$$\frac{1}{\lambda\hat{c}(\lambda)}(\frac{1}{\hat{c}(\lambda)} - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, \quad x \in X.$$

The following result shows that (a_α, k_μ)-regularized families are in one to one correspondence with $(a_\alpha, 1)$- regularized families.

Theorem 4.1. *A closed and linear operator A generates an (α,μ)-regularized family $S_\alpha(t)$ if and only if A generates an $(a_\alpha, 1)$- regularized family $R(t)$. Moreover,*

$$R(t)x = S_\alpha(t)x + \mu(1 * S_\alpha)(t) \tag{4.2}$$

and

$$S_\alpha(t) = R(t)x - \mu(k_\mu * R)(t).$$

Proof. Suppose that A generates an (α,μ)-regularized family $S_\alpha(t)$. Then defining $R(t)$ as in (4.2), we obtain

$$\begin{aligned} (a_\alpha * AR)(t) &= a_\alpha * A[S_\alpha + \mu(1 * S_\alpha)](t) \\ &= (a_\alpha * AS_\alpha)(t) + \mu(a_\alpha * A(1 * S_\alpha))(t) \\ &= S_\alpha(t) - k_\mu(t) + \mu 1 * A(S_\alpha - k_\mu)(t) \\ &= S_\alpha(t) - k_\mu(t) + \mu(1 * AS_\alpha)(t) - \mu(1 * k_\mu)(t). \end{aligned}$$

Now, we can use the identity $\mu(1 * k_\mu) = 1 - k_\mu$ to get

$$(a_\alpha * AR)(t) = S_\alpha(t) + \mu(1 * S_\alpha)(t) - I = R(t) - I$$

proving that R is a resolvent family generated by A. Conversely, suppose A generates a resolvent family $R(t)$ and define $S_\alpha(t) = R(t)x - \mu(k_\mu * R)(t)$. Then, making use again of the identity $\mu(1 * k_\mu) = 1 - k_\mu$ we obtain

$$\begin{aligned}(a_\alpha * AS_\alpha)(t) &= a_\alpha * A[R(t) - \mu(k_\mu * R)(t)] \\ &= (a_\alpha * AR)(t) - \mu(a_\alpha * k_\mu * AR)(t) \\ &= R(t) - I - \mu(k_\mu * R)(t) + \mu(k_\mu * 1)(t) \\ &= R(t) - \mu(k_\mu * R)(t) - k_\mu(t) \\ &= S_\alpha(t) - k_\mu(t).\end{aligned}$$

□

According with Theorem 4.1 the solution u of equation (1.1) is linked with the solution v of the following Volterra equation of scalar type

$$v(t) = f(t) - \int_0^t a_\alpha(t-s)Av(s)ds, \tag{4.3}$$

which is given by the variation of parameters formula

$$v(t) = R(t)v(0) + \int_0^t R(t-s)\dot{f}(s)ds, \tag{4.4}$$

whenever A is the generator of a resolvent family $R(t)$.

The connection is given by means of the relation

$$u(t) = S_\alpha(t)u(0) + (1 * S_\alpha)(t)[u'(0) + \mu u(0)] + (a_\alpha * v)(t) \tag{4.5}$$

or, equivalently

$$u(t) = (k_\mu * \dot{R})(t)u(0) + e^{-\mu t}u(0) + (k_\mu * R)(t)[\dot{u}(0) + \mu u(0)] + (a_\alpha * v)(t). \tag{4.6}$$

Finally, we note that

$$(1 * S_\alpha)(t) = (k_\mu * R)(t), \quad t \geq 0. \tag{4.7}$$

References

[1] D. Araya, C. Lizama. *Almost automorphic mild solutions to fractional differential equations.* Nonlinear Analysis; Theory, Methods and Applications **69** (2008), 3692-3705.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* Monographs in Mathematics. **96**. Birkhäuser, Basel, 2001.

[3] B. Baeumer, M.M. Meerschaert, E. Nane, *Brownian subordinators and fractional Cauchy problems,* Trans. Amer. Math. Soc., **361** (7) (2009), 3915-3930.

[4] E. Bazhlekova.*Fractional Evolution Equations in Banach Spaces,* Ph.D. Thesis, Eindhoven University of Technology, 2001.

[5] R. Gorenflo, F. Mainardi. *Fractional Calculus: Integral and Differential Equations of Fractional Order*, A. Carpinteri and F. Mainardi (Editors): Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York 1997, 223-276.

[6] R. Gorenflo, F. Mainardi.*On Mittag-Leffler-type functions in fractional evolution processes.* J. Comp. Appl. Math. **118** (2000), 283-299.

[7] R. Gorenflo, F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order.* CIMS Lecture Notes. (http://arxiv.org/0805.3823).

[8] N. Heymans, I. Podlubny. *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives,* Rheologica Acta, **45** (5) (2006), 765-771.

[9] R. Hilfer. *Applications of Fractional Calculus in Physics,* World Scientific Publ. Co., Singapore, 2000.

[10] V. Keyantuo, C. Lizama. *On a connection between powers of operators and fractional Cauchy problems.* Submitted.

[11] A.A. Kilbas, H.M. Srivastava, J.J. and Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam (2006).

[12] M. Kostic, (a,k)*-regularized* C*-resolvent families: regularity and local properties.* Abstract and Applied Analysis Volume 2009 (2009), Article ID 858242, 27 pages doi:10.1155/2009/858242.

[13] C. Lizama. *Regularized solutions for abstract Volterra equations.* J. Math. Anal. Appl. **243** (2000), 278-292.

[14] C. Lizama. *On approximation and representation of k-regularized resolvent families.* Integral Equations Operator Theory **41** (2), (2001), 223-229.

[15] C. Lizama, H. Prado.*Rates of approximation and ergodic limits of regularized operator families.* J. Approximation Theory, **122** (1) (2003), 42-61.

[16] C. Lizama, J. Sánchez,*On perturbation of k-regularized resolvent families,* Taiwanese J. Math. **7** (2), (2003), 217-227.

[17] C. Lizama, H. Prado, *On duality and spectral properties of* (a,k)*-regularized resolvents.* Proceedings of the Royal Society of Edinburgh: Section A, 139 (3), (2009), 505-517.

[18] E. Nane, *Higher order PDE's and iterated processes,* Trans. Amer. Math. Soc. **360**(5) (2008), 2681-2692.

[19] I. Podlubny. *Fractional Differential Equations,* Academic Press, San Diego, 1999

[20] J. Prüss. *Evolutionary Integral Equations and Applications.* Monographs Math., **87**, Birkhäuser Verlag, 1993.

[21] S.G. Samko, A.A. Kilbas, O.I. Marichev,*Fractional Integrals and Derivatives: Theory and Applications,* Gordon and Breach, New York (1993). [Translation from the Russian edition, Nauka i Tekhnika, Minsk (1987)]

[22] S.Y.Shaw, J.C. Chen. *Asymptotic behavior of* (a,k)*-regularized families at zero.* Taiwanese J. Math. **10** (2) (2006), 531-542.

Departamento de Matemática, Universidad de Santiago de Chile , Santiago, Chile
E-mail address: carlos.lizama@usach.cl