A SEMIGROUP APPROACH TO FRACTIONAL POISSON PROCESSES

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Abstract. It is well-known that Fractional Poisson processes (FPP) constitute an important example of a Non-Markovian structure. That is, the FPP has no Markov semigroup associated via the customary Chapman-Kolmogorov equation. This is physically interpreted as the existence of a memory effect. Here, solving a difference-differential equation, we construct a family of contraction semigroups $(T_\alpha)_{\alpha\in]0,1]}$, $T_\alpha = (T_\alpha(t))_{t\geq 0}$. If $C([0,\infty[, B(X))$ denotes the Banach space of continuous maps from $[0,\infty[$ into the Banach space of endomorphisms of a Banach space X, it holds that $T_\alpha \in C([0,\infty[, B(X))$ and $\alpha \mapsto T_\alpha$ is a continuous map from $]0,1]$ into $C([0,\infty[, B(X))$. Moreover, T_1 becomes the Markov semigroup of a Poisson process.

1. Introduction

Many phenomena in nature may be described mathematically by functions of a small number of independent variables and parameters. In particular, if such a phenomenon is given by a function of spatial position and time, its description gives rise to a wealth of models, which often result in equations, usually containing a large variety of derivatives with respect to these variables. Apart from the spatial variable(s), which are essential for the problems to be considered, the time variable play a special role.

For practical purposes, the time variable is measured at discrete events. An important class of linear, discrete time systems consists of systems represented by linear constant-coefficient difference- partial differential equations. Discrete linear time-invariant systems that satisfy difference-partial differential equations are very common; they include data filtering, time series analysis, and digital filtering systems and algorithms.

An important distinction between linear constant-coefficient differential equations associated with continuous-time systems and linear constant-coefficient difference equations associated with discrete-time systems is that for causal systems the difference equation can be reformulated as an explicit relationship that states how successive values of the output can be computed from previously computed output values and the input. This recursive procedure for calculating the response of a difference equation is extremely useful in implementing causal systems.

In this paper, we study the following difference-differential equation

$$\lambda p(n+1,t) - \lambda p(n,t) = -\, {}_{RL}D_t^{\alpha} p(n+1,t) \tag{1.1}$$

where $0 < \alpha \leq 1$. With appropriate boundary and initial values, we prove the remarkable property that the solution of (1.1) coincides with the probability distribution

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function of the fractional Poisson process. Our idea is based on the use of tools of operator semigroup theory [1, 16].

The fractional Poisson process was introduced and studied by Repin and Saichev [19], Jumarie [8], Laskin [10, 11], Mainardi et al. [13, 14], Uchaikin et al. [21], Beghin and Orsingher [3, 4], and Meerschaert et.al. [15]. Fractional Poisson process is a stochastic process that captures the long-memory effect which results in the non-exponential waiting time probability distribution function empirically observed in complex classical and quantum systems. The fractional Poisson probability distribution found physical and mathematical applications in the field of quantum optics and combinatorial numbers [11]. Thus, the Fractional Poisson process (FPP) is an important physical example of a non Markovian evolution. In our case, we prove in Theorem 2.6 that the Markov semigroup of the Poisson process can be obtained as a suitable limit of non Markov FPP-semigroups.

2. Preliminaries and main results

Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. For a vector-valued sequence $f:\mathbb{N}_0\to X$ we denote the forward Euler operator $\Delta f(n) := f(n+1)-f(n)$. We consider the difference-abstract Cauchy problem

$$\begin{cases} \Delta u(n) &= Au(n+1), \quad n\in\mathbb{Z}_+ \\ u(0) &= u_0\in X. \end{cases} \tag{2.1}$$

Equations in the form of time-difference space-differential equations that can be modeled by the linearized system (2.1) appear in several theoretical and applied branches of mathematics. However, a systematic study in the context of abstract Cauchy problems like (2.1) seems to be missing in the literature. We introduce the following notion of solution (see [12]).

Definition 2.1. We say that a vector valued sequence $u\in s(\mathbb{N}_0;X)$ is a solution for (2.1) if $u(n)\in D(A^n)$ for all $n\in\mathbb{N}$ and $u(n)$ satisfies (2.1).

The following result follows easily from the definitions and therefore their proof is omitted (see [12] for an extension to abstract fractional difference equations).

Lemma 2.2. Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Then the solution of $\Delta u(n) = Au(n+1)$, $n\in\mathbb{N}$ with initial condition $u_0\in X$ is given by $u(n) = (I+A)^{-n}u_0$, $n\in\mathbb{N}$.

Example 2.3. Consider a locally compact space E and $\mathcal{E}$ its Borel σ-algebra. Let denote $b\mathcal{E}$ the Banach space of all complex bounded measurable functions endowed with the uniform norm. Given a Markov transition kernel $Q: E\times\mathcal{E}\to[0,1]$, define $Qf(x) = \int_E Q(x,dy)f(y)$ for all $f\in b\mathcal{E}$, $x\in E$, and $u(n,f) = Q^n f$, where Q^n is the nth-iterated kernel. Therefore, one has the difference equation

$$\begin{cases} \Delta u(n,f) = Au(n+1,f), \quad n\in\mathbb{Z}_+ \\ u(0,f) = f, \end{cases}$$

where $A = Q-I$. Consider the space $\Omega = E^{\mathbb{N}}$, the process $X_n(\omega) = \omega_n\in E$, for all $\omega = (\omega_k)_{k\in\mathbb{N}}\in\Omega$, the σ-algebras $\mathcal{F}_n = \sigma(X_k,\ k\leq n)$, $\mathcal{F} = \sigma(X_n;\ n\in\mathbb{N})$. It is well-known in Markov Theory that given $x\in E$, there exists a unique probability $\mathbb{P}_x$

on $(\Omega, \mathcal{F})$ such that $(X_n)_{n\in\mathbb{N}}$ becomes an homogeneous Markov chain with respect to the family $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with transition Q and initial probability δ_x. So that,

$$u(n.f)(x) = \mathbb{E}_x\left(f(X_n)\right) = \int_\Omega f(X_n(\omega))\mathbb{P}_x(d\omega).$$

Example 2.4. Consider $E = \mathbb{R}$ and call $C_0(E)$ the Banach space of continuous functions which vanish at infinity. Suppose that a linear operator A on $C_0(E)$ satisfies the following properties:

- (M1) $D(A)$ is dense on $C_0(E)$
- (M2) A satisfies the *positive maximum principle*, that is, if whenever $f \in D(A)$, $x_0 \in E$, and $\sup_{x\in E} f(x) = f(x_0) \geq 0$, we have $Af(x_0) \leq 0$.
- (M3) $R(\lambda - A)$ is dense in $C_0(E)$.

It is proved in [5], Theorem 2.2, that all the above are necessary and sufficient conditions for the closure of A to be the generator of a strongly continuous, positive, contraction semigroup $(T_t)_{t\geq 0}$ on $C_0(E)$. One says that A is conservative, whenever $f_n \in D(A)$ and $\sup_n \|f_n\|_\infty < \infty$ such that $f_n(x) \to 1$, for all $x \in E$, then $Af_n(x) \to 0$. Thus, if A is conservative and satisfies (M1), (M2), (M3), its closure (denoted again A) generates a Markov semigroup.

In that case, Chapman-Kolmogorov equations are written as

$$\frac{d}{dt}T_t f = AT_t f = T_t Af, \quad T_0 f = f, \quad (f \in C_0(E)), \tag{2.2}$$

or, in integral form

$$T_t f = f + \int_0^t AT_s f ds, \quad (t \geq 0).$$

Consider a discretization of the above equation in the following form. Take times $t_n = nh$, where $h > 0$ is small and $n \in \mathbb{N}$. Call $u_h(n, f) = T_{nh}f$, $A_h = Ah$. One can approach equation (2.2) by

$$\Delta u_h(n, f) = A_h u_h(n + 1, f).$$

Define the Sobolev space

$$W^{1,1}(\mathbb{R}_+, e^{-t}) := \{f \in L^1(\mathbb{R}_+, e^{-t}) \ / \ \exists\varphi \in L^1(\mathbb{R}_+, e^{-t}),\ f(t) = c_0 + \int_0^t \varphi(s)ds,\ t \in \mathbb{R}_+\}$$

and define an operator $(A_\alpha, D(A_\alpha))$ on $W^{1,1}(\mathbb{R}_+, e^{-t})$ by

$$A_\alpha f(t) := D_t^\alpha f(t), \quad t \in \mathbb{R}_+,$$

where D_t^α denotes the Riemann-Liouville fractional derivative of order $\alpha > 0$ and

$$D(A_\alpha) := \{f \in W^{1,1}(\mathbb{R}_+, e^{-t}) \ : \ (g_{1-\alpha} * f)(0) = 0\},$$

where we denote $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $\beta > 0$ and $g_0 := \delta_0$, the Dirac measure concentrated at zero. In particular, note that $D(A_1) = \{f \in W^{1,1}(\mathbb{R}_+, e^{-t}) \ : \ f(0) = 0\}$. We observe that the solution of the scalar equation

$$D_t^\alpha f(t) = -\lambda f(t) + h(t)$$

with initial condition $(g_{1-\alpha} * f)(0) = 0$ can be computed explicitly in terms of the generalized two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C},$$

and is given by

$$f(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) h(s) ds. \tag{2.3}$$

An interesting property related with the Laplace transform of the Mittag-Leffler function is the following [6, (A.27) p.267] and [17]:
(2.4)

$$\int_0^{\infty} e^{-\lambda t} t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm \omega t^{\alpha}) dt = \frac{k! \lambda^{\alpha-\beta}}{(\lambda^{\alpha} \mp \omega)^{k+1}}, \quad Re(\lambda) > |\omega|^{1/\alpha}, \quad \alpha, \beta > 0,\ k \in \mathbb{Z}_+.$$

For a general presentation of fractional calculus and applications, we refer to [9], [20] and [17]. Our next result proves that the operators A_α are generators of C_0-semigroups on the space of integrable weighted functions and also gives an explicit representation for the resolvent operator.

Theorem 2.5. For each $0 < \alpha \leq 1$ the operator $(A_\alpha, D(A_\alpha))$ is the generator of a C_0-semigroup of contractions on $L^1(\mathbb{R}_+, e^{-t})$. Moreover

$$(\lambda I + A_\alpha)^{-1} h(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) h(s) ds, \quad t \geq 0, \lambda > 0$$

for all $h \in L^1(\mathbb{R}_+, e^{-t})$.

Proof. Let $f \in D(A_\alpha)$ be given and suppose $(\lambda I + A_\alpha) f = 0$. Then, by (2.3), $f \equiv 0$. It shows that $\lambda I + A_\alpha$ is injective. Let $h \in L^1(\mathbb{R}_+, e^{-t})$ be given and define

$$g_{\alpha,\lambda}(t) := t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad \lambda > 0, \quad t \geq 0. \tag{2.5}$$

Observe that $g_{\alpha,\lambda}(t) \geq 0$. Now, let f be defined as in (2.3), i.e. $f := g_{\alpha,\lambda} * h$. Then f satisfies, after a calculation using Fubini's theorem, the following estimate

$$\begin{aligned} \|f\|_{L^1(\mathbb{R}_+, e^{-t})} &= \|g_{\alpha,\lambda} * h\|_{L^1(\mathbb{R}_+, e^{-t})} = \int_0^{\infty} e^{-t} \Big| \int_0^t g_{\alpha,\lambda}(t-s) h(s) ds \Big| dt \\ &\leq \int_0^{\infty} \int_s^{\infty} e^{-t} \Big| g_{\alpha,\lambda}(t-s) h(s) \Big| dt ds \\ &= \int_0^{\infty} \int_0^{\infty} e^{-(\tau+s)} \Big| g_{\alpha,\lambda}(\tau) h(s) \Big| d\tau ds \\ &= \int_0^{\infty} e^{-s} \int_0^{\infty} e^{-\tau} g_{\alpha,\lambda}(\tau) d\tau \ |h(s)| ds = \frac{1}{1+\lambda} \|h\|_{L^1(\mathbb{R}_+, e^{-t})}, \end{aligned} \tag{2.6}$$

where we have used the identity $\int_0^{\infty} e^{-t} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) dt = \frac{1}{1+\lambda}$, $\lambda > 0$, which follows from (2.4). In particular, (2.6) shows that $f \in L^1(\mathbb{R}_+, e^{-t})$. Moreover, using the Laplace transform, is easy to see that

$$(g_{1-\alpha} * g_{\alpha,\lambda})(t) = E_{\alpha,1}(-\lambda t^{\alpha}).$$

Since $E_{\alpha,1}(0) = 1$, it shows that $f \in D(A_\alpha)$. A simple calculation gives $(\lambda I + A_\alpha)f = h$. It proves that $\lambda I + A_\alpha$ is surjective and therefore invertible and, in view of the estimate (2.6), satisfies

$$\|(\lambda I + A_\alpha)^{-1}\| \le \frac{1}{1+\lambda}.$$

It shows that $\|\lambda(\lambda I + A_\alpha)^{-1}\| \le 1$, for all $\lambda > 0$. The conclusion follows by application of the Hille-Yosida theorem (see e.g. [16, Theorem 3.1]). □

Using the Trotter-Kato theorem on approximation of C_0-semigroups, we can prove the following interesting result.

Theorem 2.6. For each $0 < \alpha \le 1$ the C_0-semigroup $T_\alpha(t)$ generated by A_α on $L^1(\mathbb{R}_+, e^{-t})$ satisfies

$$T_\alpha(t)f \to T_1(t)f \quad \text{as} \quad \alpha \to 1,$$

for all $f \in L^1(\mathbb{R}_+, e^{-t})$, where the convergence is uniform in t on compact subsets of $\mathbb{R}_+$.

Proof. It is clear that $\|T_\alpha(t)\| \le e^t$ for all $0 < \alpha \le 1$ and $t \ge 0$. By the preceding theorem we have $(\lambda I + A_\alpha)^{-1} f = g_{\alpha,\lambda} * f$ for all $f \in L^1(\mathbb{R}_+, e^{-t})$, where $g_{\alpha,\lambda}$ is defined in (2.5). Note that by (2.4):

$$\int_0^\infty e^{-t} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) dt = \frac{1}{1+\lambda} = \int_0^\infty e^{-t} e^{-\lambda t} dt. \tag{2.7}$$

To prove that $T_\alpha(t)f \to T_1(t)f$ it suffices to show that given any sequence $(\alpha_n)_{n\in\mathbb{N}}$, such that $\alpha_n \to 1$ as $n \to \infty$ it holds that $T_{\alpha_n}(t)f \to T_1(t)f$. Indeed, denote $h_n(t) = t^{\alpha_n - 1} E_{\alpha_n,\alpha_n}(-\lambda t^{\alpha_n})$ and $h(t) = e^{-\lambda t}$, and observe that all these functions are non negative. Since $h_n(t) \to h(t)$ a.e. and (2.7) holds, Lemma 3.1 implies that

$$\int_0^\infty e^{-t} |t^{\alpha_n - 1} E_{\alpha_n,\alpha_n}(-\lambda t^{\alpha_n}) - e^{-\lambda t}| dt \to 0. \tag{2.8}$$

Therefore,

$$\begin{aligned} \|(\lambda I + A_{\alpha_n})^{-1} f - (\lambda I + A_1)^{-1} f\|_{L^1} &= \|(g_{\alpha_n,\lambda} - g_{1,\lambda}) * f\|_{L^1} \\ &\le \|g_{\alpha_n,\lambda} - g_{1,\lambda}\|_{L^1} \|f\|_{L^1} \to 0 \end{aligned}$$

as $\alpha_n \to 1$. The claim follows from [16, Theorem 4.2]. □

Remark 1. Recall that a C_0-semigroup $T(t)$ on a Banach space X is hypercyclic if there are $x \in X$ whose orbit $\{T(t)x; t \ge 0\}$ under $T(t)$ is dense in X. The C_0-semigroup $T(t)$ is said to be Devaney chaotic if it is hypercyclic and the set of periodic points $Per(T) := \{x \in X; T(t)x = x \text{ for some } t > 0\}$ is dense in X. The interested reader is referred to the recent book [7] for more information about hypercyclicity and linear chaos. Observe that the semigroup generated by A_1 is chaotic in the sense of Devaney, because it corresponds to the translation semigroup defined on $L^1(\mathbb{R}_+, \rho)$ where the weight $\rho(t) = e^{-t}$ satisfies the criteria established in [2, Theorem 2.5].

For each $\alpha > 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$ define

$$f_{\alpha,\lambda}(t) := E_{\alpha,1}(-\lambda t^\alpha), \quad t \in \mathbb{R}_+.$$

Note that $f_{\alpha,\lambda} \in L^1(\mathbb{R}_+, e^{-t})$ for each $\lambda > 0$ and $0 < \alpha \le 1$. This follows from (2.4). After this preliminaries, we can give the main result of this paper.

Theorem 2.7. Let $0 < \alpha \leq 1$ and $\lambda > 0$ be given. The unique solution of the differential-difference equation

$$\begin{cases} \Delta u(n,t) &= -\dfrac{1}{\lambda} A_\alpha u(n+1,t), \quad n \in \mathbb{Z}_+; \\ u(0,t) &= f_{\alpha,\lambda}(t), \quad t > 0, \end{cases}$$

is the probability distribution function of the fractional Poisson process

$$u(n,t) = \frac{\lambda^n t^{\alpha n}}{n!} E^{(n)}_{\alpha,1}(-\lambda t^\alpha), \quad n \in \mathbb{Z}_+, \quad t > 0. \tag{2.9}$$

Proof. From the proof of the previous theorem, we note that $(\lambda I + A_\alpha)^{-1} g = g_{\alpha,\lambda} * g$ for all $\lambda > 0$, and therefore

$$(\lambda I + A_\alpha)^{-n} g = g_{\alpha,\lambda} * g_{\alpha,\lambda} * \ldots * g_{\alpha,\lambda} * g =: g^{*n}_{\alpha,\lambda} * g,$$

for all $g \in L^1(\mathbb{R}_+, e^{-t})$. By Theorem 2.5, we have that the operator $A := -\frac{1}{\lambda} A_\alpha$ is the generator of a bounded C_0-semigroup. It follows from Lemma 2.2 that

$$\begin{aligned} u(n,t) &= (I + \frac{1}{\lambda} A_\alpha)^{-n} u(0,t) = \lambda^n (\lambda + A_\alpha)^{-n} u(0,t) \\ &= \lambda^n (\lambda + A_\alpha)^{-n} f_{\alpha,\lambda} = \lambda^n g^{*n}_{\alpha,\lambda} * f_{\alpha,\lambda}. \end{aligned}$$

Note that $\widehat{g^{*n}_{\alpha,\lambda} * f_{\alpha,\lambda}}(\mu) = [\widehat{g_{\alpha,\lambda}}(\mu)]^n \widehat{f_{\alpha,\lambda}}(\mu)$ where the hat indicates Laplace transform. Since $\widehat{g_{\alpha,\lambda}}(\mu) = \frac{1}{\mu^\alpha + \lambda}$ and $\widehat{f_{\alpha,\lambda}}(\mu) = \frac{\mu^{\alpha-1}}{\mu^\alpha + \lambda}$. we obtain $\widehat{g^{*n}_{\alpha,\lambda} * f_{\alpha,\lambda}}(\mu) = \frac{\mu^{\alpha-1}}{(\mu^\alpha + \lambda)^{n+1}}$. Therefore formula (2.4) shows that $u(n,t) = \lambda^n g^{*n}_{\alpha,\lambda} * f_{\alpha,\lambda}(t) = \lambda^n \frac{t^{\alpha n}}{n!} E^{(n)}_{\alpha,1}(-\lambda t^\alpha)$, which proves the theorem. □

In case $\alpha = 1$ we recover the well-known Poisson distribution.

Corollary 2.8. Let $\lambda > 0$ be given. The unique solution of the partial differential-difference equation

$$\begin{cases} \Delta u(n,t) &= -\dfrac{1}{\lambda} \dfrac{\partial}{\partial t} u(n+1,t), \quad n \in \mathbb{Z}_+; \\ u(0,t) &= e^{-\lambda t} \quad t \geq 0; \\ u(n,0) &= 0, \end{cases}$$

is the probability distribution of the Poisson process $u(n,t) = \dfrac{\lambda^n t^n}{n!} e^{-\lambda t}$, $n \in \mathbb{Z}_+$, $t \geq 0$.

3. Concluding remarks

In this paper we systematically work with one-dimensional space-time variables. This framework allows us to give a new interpretation of the fractional Poisson process as a Cauchy problem discretized in time, thus concluding that the fractional Riemann-Liouville operator is the generator of a family of C_0-semigroups depending on the fractional order, on the Lebesgue space $L^1(\mathbb{R}_+.e^{-t})$. Moreover, using the Trotter-Kato theorem, we show that this family of semigroups converges uniformly to the Markovian semigroup associated to the Poisson process.

This paper provides a new insight into a class of non Markovian evolutions, since it is well-known that the Fractional Poisson Process is a paradigmatic example of evolution with large memory. It teach us that suitable discretizations in space-time variables may lead to a representation of memory effects via C_0-semigroups of non Markovian type.

Appendix

Lemma 3.1. Consider the probability measure $\mu(dt) = e^{-t}dt$ on the positive real line. Let be given a sequence $(h_n)_{n\in\mathbb{N}}$ of positive integrable functions such that

(H1) $h_n(t) \to h(t)$ almost everywhere as $n \to \infty$, h integrable, and
(H2) $\int_0^\infty h_n(t)\mu(dt) \to \int_0^\infty h(t)\mu(dt)$.

Then $(h_n)_{n\in\mathbb{N}}$ is uniformly integrable and $h_n \to h$ in $L^1(\mu)$.

Proof. Take $c > 0$. Then $h_n 1_{\{h_n \le c\}} \to h 1_{\{h\le c\}}$, a.e. by (H1) and $h_n 1_{\{h_n\le c\}} \le c$. Thus, Lebesgue's Dominated Convergence Theorem implies that

$$\int_0^\infty h_n(t)1_{\{h_n\le c\}}(t)\mu(dt) \to \int_0^\infty h(t)1_{\{h\le c\}}(t)\mu(dt).$$

And (H2) yields

$$\int_{\{h_n>c\}} h_n(t)\mu(dt) \to \int_{\{h>c\}} h(t)\mu(dt).$$

So that $\limsup_n \int_{\{h_n>c\}} h_n(t)\mu(dt) = \int_{\{h>c\}} h(t)\mu(dt)$. Given $\epsilon > 0$, choose $c_0 > 0$ to have

$$\int_{\{h>c_0\}} h(t)\mu(dt) < \epsilon/2$$

and there exists n_0 such that $\sup_{n\ge n_0} \int_{\{h_n>c_0\}} h_n(t)\mu(dt) < \epsilon$. Since any finite family of integrable functions is uniformly integrable, so is $(h_1, \dots, h_{n_0-1})$, and one obtains the same property for the whole sequence $(h_n)_{n\in\mathbb{N}}$. Finally, the family $(h_n - h)_{n\in\mathbb{N}}$ is uniformly integrable as well, and this sequence converges to 0 a.e. Therefore, $\|h_n - h\|_1 \to 0$. □

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