A CHARACTERIZATION OF WELL-POSEDNESS FOR ABSTRACT CAUCHY PROBLEMS WITH FINITE DELAY

CARLOS LIZAMA AND FELIPE POBLETE

Abstract. Let A be a closed operator defined on a Banach space X and F be a bounded operator defined on a appropriate space. In this paper, we characterize the mildly well-posedness of the first order abstract Cauchy problem with finite delay,

$$\begin{cases} u'(t) = Au(t) + Fu_t, & t > 0; \\ u(0) = x; & \\ u(t) = \phi(t), & -r \le t < 0 \end{cases},$$

solely in terms of a strongly continuous one-parameter family $\{G(t)\}_{t\geq 0}$ of bounded linear operators that satisfy the functional equation

$$G(t+s)x = G(t)G(s)x + \int_{-r}^{0} G(t+m)[SG(s+\cdot)x](m)dm$$

for all $t, s \geq 0,\ x \in X$. In case $F \equiv 0$ this property reduces to the characterization of well-posedness for the first order abstract Cauchy problem in terms of the functional equation that satisfy the C_0-semigroup generated by A.

1. Introduction

Let X be a complex Banach space. In this paper, we study the first order abstract Cauchy problem with finite delay

$$\begin{cases} u'(t) = Au(t) + Fu_t, & t > 0; \\ u(0) = x; & \\ u(t) = \phi(t), & -r \le t < 0 \end{cases}, \tag{1.1}$$

where A is a closed linear operator with domain $D(A)$, $F : L^p([-r, 0], X) \to X$ is a bounded linear map, r is a positive number and ϕ is a given initial function.

The field of linear (and nonlinear) delay differential equations has undergone a significant development for several decades. In addition, its interaction with other scientific fields has also increasing interest, in particular, in the study of biological models.

In case $F \equiv 0$ it is well known that (1.1) is well-posed (in a strong or mild sense) if and only if A is the generator of a C_0-semigroup, that is, a strongly continuous family of bounded and linear operators $\{T(t)\}_{t\geq 0}$ satisfying $T(0) = I$ and the Cauchy's functional equation

$$T(t+s)x = T(t)T(s)x, \quad t, s \geq 0, \quad x \in X. \tag{1.2}$$

The theory of C_0-semigroups is a well-established and developed theory, that starts from the original monograph of Hille and Phillips [9]. For an up to date reference and historical remarks, see e.g. Engel and Nagel [6].

In case $F \neq 0$ there is an important amount of literature on the subject. For instance, Hale [8] and Webb [20] began an abstract analysis, i.e. in the setting of Banach spaces, applying

2000 *Mathematics Subject Classification.* 39B72; 34K37; 34K06; 35R10.

Key words and phrases. C_0-semigroups; finite delay; Cauchy problem; Functional equations; well posedness.

The second author is partially supported by FONDECYT #1170466 and DID S-2017-43.

methods coming from semigroup theory. After that, Travis and Webb [18, Section 4] studied existence and stability of solutions when A is the generator of a compact semigroup, or an analytic semigroup [19]. Fitzgibbon [7] was among the first to treat the nonautonomous case i.e. $A = A(t)$. Jiang, Guo and Huang [10] studied the well-posedness of the linear abstract problem with unbounded delay operators. More recently, Ashyralyev and Agirseven [4] analyzed the well-posedness of (1.1) when the delay admits the form of a nonautonomous and unbounded operator.

After the method of semigroups, most of the approaches consists into associate to a given delay equation an expanded space E (phase space) and a lifted unbounded operator $(B, D(B))$ and to demonstrate that the solutions of the abstract Cauchy problem associated to $(B, D(B))$ in E naturally correspond to those of the delay equation. Then, the task is to show that the lifted operator $(B, D(B))$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on E, thus implying the Cauchy problem is well-posed. See e.g. [5] and the monograph of Bátkai and Piazzera [4].

However, this last method produces significant mathematical difficulties when we deal with e.g. the regularity problem. For instance, suppose that the operator A in (1.1) generates an analytic semigroup, a condition which is frequently assumed in the investigation of the regularity problem. Then the lifted generator $(B, D(B))$ of the system does not generate an analytic semigroup any more on the expanded spaces (cf. [11]).

First attempts to treat directly (1.1), that is without any assumption on the operator A and neither appealing to some phase space, were made by Petzeltová [16], [17]. By replacing X with a suitable interpolation space, she proves the existence of a family of bounded and linear operators $R(t)$ satisfying $R'(t) = AR(t) + FR_t, R(0) = I, R_0 = 0$.

In a recent paper, Liu [12] employed a direct method to deal with the regularity problem. Liu developed a theory of retarded type operators $\{G(t)\}$, or fundamental solutions, for (1.1) defined in a Hilbert space H. Among other interesting things, Liu proves that the following functional equation is satisfied:

$$G(t+s)x = G(t)G(s)x + \int_{-r}^{0} G(t+m)[SG(s+\cdot)x](m)dm, \quad t, s \geq 0, \ x \in H, \tag{1.3}$$

where S is the so-called structure operator, which depends of F. We mention the remarkable fact that in case $F \equiv 0$ the functional equation (1.3) coincides with (1.2).

Then it is natural to ask: Could the functional equation (1.3) completely characterize the well-posedness of the abstract Cauchy problem with delay (1.1), in some sense?.

In this paper, we answer this question in the affirmative. We prove that a characterization of (1.1) by means of (1.3) is true not only in Hilbert spaces but also in any general Banach space. Specifically, we show that the existence of a strongly continuous family $\{G(t)\}_{t\geq 0}$ of linear and bounded operators, extended over $-r \leq t < 0$ as the null operator, satisfying $G(0) = I$ and the functional equation (1.3), is equivalent to the well-posedness of the following integrated (or mild) version of the problem (1.1):

$$\begin{cases} u(t) = x + A\displaystyle\int_0^t u(s)ds + \int_0^t Fu_s ds, & t \geq 0; \\ u(t) = \phi(t), & -r \leq t < 0 \end{cases}, \tag{1.4}$$

where $x \in X$ and ϕ is a measurable function on $L^p([-r, 0], X)$. A very remarkable fact is that we not need the operator A as a generator of a C_0-semigroup.

In consequence, the main novelty of this paper is that we are able to present a theory of fundamental solutions for equations with bounded delay operators and then a kind of variation of constants formula. As a result, stability and stationary solutions could be deduced, as done for instance in the recent reference [14] where F is considered also unbounded, and also extensions and algebraic properties for solution families of vector-valued differential equations with delay could be studied, following for instance the reference [1]. Several concrete examples are analyzed to illustrate the theory.

The paper is organized as follows: In the second section we will present a proper definition of a resolvent family with delay F in terms of the functional equation (1.3) and then we introduce the concept of mildly well-posedness. We will exhibit different properties of this type of resolvent families, being the most important that they are exponentially bounded, which allow us to define a kind of generator of the family by making use of Laplace transform tools. We finish the section showing that the mildly well-posedness of (1.1) implies the existence of a resolvent family with delay F generated by the operator A. In the third section, we study sufficient conditions on the resolvent family with delay F in order to ensure that the problem (I_{st}) is mildly well-posed. A notable result is the following: If A is the generator of a C_0-semigroup defined on a Banach space X and and the delay operator $F : L^p([-r, 0], X) \to X$ is defined by

$$F\phi = \int_{-r}^{0} H(\theta)\phi(\theta)d\theta,$$

where H is an $\mathcal{B}(X)$-valued and q-integrable function on $[-r, 0]$ with $\frac{1}{q} + \frac{1}{p} = 1$, then the problem

$$\begin{cases} u'(t) = Au(t) + Fu_t, & t \geq 0; \\ u(0) = x; & \\ u(t) = \phi(t), & -r \leq t < 0 \end{cases},$$

is mildly well-posed. Also, we will identify explicitly the generator of the resolvent family in terms of $\{G(t)\}_{t\geq 0}$. Finally, we show some applications, concrete examples and further properties of a resolvent family with delay F generated by A.

2. Preliminaries

Most of the notation used throughout this paper is standard. Hence, we will denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the sets of natural, integer, real and complex numbers respectively. For the rest of the paper, X and Y always are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscripts will be dropped when there is no danger of confusion. We denote by $\mathcal{B}(X, Y)$ to the space of all bounded linear operators from X to Y. In the case $X = Y$, we will write briefly $\mathcal{B}(X)$. Let $A\colon D(A) \subset X \to X$ be a closed linear operator with domain $D(A)$. The domain of A endowed with the graph norm will be denoted by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Further, the resolvent operator $(\lambda - A)^{-1}x$ will be denoted by $R(\lambda, A)x$ for $\lambda \in \rho(A)$ and $x \in X$.

Let $1 \leqslant p < \infty$ be given, $J \subseteq \mathbb{R}$ an interval of real numbers and X a Banach space. By $L^p(J, X)$, we denote the Banach space of all p-integrable functions (in the sense of Bochner) endowed with the norm

$$\|f\|_p = \left(\int_J \|f(t)\|_X^p dt \right)^{1/p}.$$

Analogously, for $n \in \mathbb{N}$ and $1 \leqslant p < \infty$ define the Sobolev spaces:

$$W^{n,p}(J, X) = \{f \in L^p(J, X) \ : \ f^{(k)} \in L^p(J, X) \text{ for } \ k \in \{1, \ldots n\}\}.$$

They are Banach spaces when endowed with the norm $\|f\|_{W^{n,p}} = \|f\|_p + \|f'\|_p + \cdots + \|f^{(n)}\|_p$.

Throughout the article we adopt the following notations: Given $u \in L^p_{loc}([-r,\infty),X)$, we denote, for any $t \geq 0$, the history function $u_t \in L^p([-r,0],X)$ described by the formula $u_t(\theta) = u(\theta+t)$ for $\theta \in [-r,0]$, and $\mathfrak{F}u : [0,\infty) \to X$ by

$$\mathfrak{F}u(t) = Fu_t.$$

In relation to the above, we infer from [6, pag. 35] that $\mathbb{R}_+ \ni t \mapsto u_t \in L^p([-r,0],X)$, is a continuous function, in particular $\mathfrak{F}u$ so is.

For two strongly continuous families of operators $T, G : \mathbb{R}_+ \to \mathcal{B}(X)$, we denote the convolution operator $T * G : \mathbb{R}_+ \times X \to X$ by

$$(T * G)(t)x = \int_0^t T(t-s)G(s)xds.$$

In addition, the product space $X \times L^p([-r,0],X)$ with norm $\|\Phi\|_{\widetilde{X}} = (\|x\|^p + \|\phi\|^p_{L^p})^{\frac{1}{p}}$, for all $\Phi = (x,\phi) \in X \times L^p([-r,0],X)$, will be denoted by $\widetilde{X}$.

For a function $f \in L^1_{loc}(\mathbb{R},X)$ we consider the Laplace transform

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad \lambda \in \mathbb{C}. \tag{2.1}$$

The *abscissa of convergence of* $\hat{f}$ is given by

$$abs(f) := \inf\{\Re(\lambda) : \hat{f}(\lambda) \ exists\}.$$

Is well known that the set of those $\lambda \in \mathbb{C}$ for which the Laplace integral (2.1) converges is either empty or a right half-plane. A function f is called Laplace transformable if $abs(f) < \infty$. We observe that if f is exponentially bounded i.e. there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|f(t)\| \leq Me^{\omega t}$, then $abs(f) < \infty$.

3. Well-posedness

We start this section defining what we understand by a mild solution and the mildly well-posedness of the problem (1.1).

Definition 3.1. A function $u : [-r,\infty) \to X$ is called a mild solution of the problem (1.1) associated to $(x,\phi) \in \widetilde{X}$ if $u|_{[0,\infty)} \in C([0,\infty),X)$, $\int_0^t u(s)ds \in D(A)$ for all $t \geq 0$ and u satisfies (1.4).

Definition 3.2. We say that the problem (1.1) is mildly well-posed if, for every $\Phi \in \widetilde{X}$, there is a unique mild solution u_Φ of the problem (1.1) and if $\Phi_n \in \widetilde{X}$ is such that $\Phi_n \to 0$ then $u_{\Phi_n}(t) \to 0$ uniformly for t on compact subintervals of $\mathbb{R}_+$.

The next lemma will be useful throughout this paper.

Lemma 3.3. Let $I \subseteq \mathbb{R}$ be a measurable set and $1 \leqslant p < \infty$. Assume that the function $f \in L^1(I, L^p([-r,0],X))$. If $F : L^p([-r,0],X) \to X$ is a bounded linear operator, then $t \mapsto Ff(t)$ is integrable and

$$F\int_I f(t)dt = \int_I Ff(t)dt.$$

Furthermore, the function $\zeta\colon [-r,0] \to X$ defined by $\zeta(\theta) = \displaystyle\int_I f(t)(\theta)dt$ belongs to $L^p([-r,0],X)$ and

$$\left(\int_I f(t)dt\right)(\theta) = \int_I f(t)(\theta)dt, \text{ for almost all } \theta \in [-r,0].$$

Proof. To abbreviate the text of this proof, for $p \geqslant 1$ we will write L^p_τ instead $L^p([-\tau,0],X)$. Moreover, to differentiate the integration in sense of Bochner of functions in the Banach spaces $L^1(I,L^p([-\tau,0],X))$ and $L^1(I,L^1([-\tau,0],X))$ we will use the symbols ${}_{L^p_\tau}\!\int_I$ and ${}_{L^1_\tau}\!\int_I$ respectively.

On the one hand, since $f \in L^1(I,L^p([-\tau,0],X))$ and $F : L^p([-\tau,0],X) \to X$ is bounded by [2, Proposition 1.1.6] we obtain that $t \mapsto Ff(t)$ is integrable and

$$F\left({}_{L^p_\tau}\!\int_I f(t)dt\right) = \int_I Ff(t)dt.$$

We note that there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(I,L^p_\tau)$ such that $f_n(t) \to f(t)$ for almost all $t \in I$ and

$$\lim_{n\to\infty}\int_I \|f_n(t)-f(t)\|_{L^p_\tau}dt = 0.$$

Moreover, we note that $\{f_n\}_{n\in\mathbb{N}} \subseteq L^1(I,L^1_\tau)$ and applying the Hölder inequality we obtain

$$\begin{aligned}\int_I \|f_n(t)-f(t)\|_{L^1_\tau}dt &= \int_I\left(\int_{-r}^0 \|f_n(t)(\theta)-f(t)(\theta)\|d\theta\right)dt\\ &\leqslant \tau^{1/q}\int_I\left(\int_{-\tau}^0 \|f_n(t)(\theta)-f(t)(\theta)\|^p d\theta\right)^{1/p}dt.\end{aligned}$$

Since the right side of the last inequality converges to 0 as $n\to\infty$, we obtain that $f \in L^1(I,L^1_\tau)$. Moreover, we observe that ${}_{L^p_\tau}\!\int_I f_n(t)dt = {}_{L^1_\tau}\!\int_I f_n(t)dt$, for all $n\in\mathbb{N}$. This implies

$$\begin{aligned}\left\|{}_{L^p_\tau}\!\int_I f(t)dt - {}_{L^1_\tau}\!\int_I f(t)dt\right\|_{L^1_\tau} &\leqslant \left\|{}_{L^p_\tau}\!\int_I (f(t)-f_n(t))dt + {}_{L^1_\tau}\!\int_I f_n(t)-f(t)dt\right\|_{L^1_\tau}\\ &\leqslant \int_I \|f(t)-f_n(t)\|_{L^p_\tau}dt + \int_I \|f_n(t)-f(t)\|_{L^1_\tau}dt.\end{aligned}$$

Thus, since the right side converges to 0 as $n\to\infty$, we conclude that,

$$\left({}_{L^p_\tau}\!\int_I f_n(t)dt\right)(\theta) = \left({}_{L^1_\tau}\!\int_I f_n(t)dt\right)(\theta) \text{ for almost all } \theta\in[-\tau,0]. \tag{3.1}$$

Define the mapping $\mathcal{I} : L^1(I,L^1_\tau) \to L^1(I\times[-\tau,0],X)$ described by $(\mathcal{I}f)(t,\theta) = f(t)(\theta)$. This operator is an isomorphism between $L^1(I,L^1_\tau)$ and $L^1(I\times[-\tau,0],X)$. Hence, $\mathcal{I}f$ is integrable in $L^1(I\times[-r,0],X)$. Applying Fubini's Theorem, we conclude that the function $\theta \mapsto \int_I \mathcal{I}(t,\theta)dt$ is L^1_τ-integrable.

We claim that $\left({}_{L^p_\tau}\!\int_I f(t)dt\right)(\theta) = \int_I (\mathcal{I}f)(t,\theta)dt$ for almost all $\theta\in[-\tau,0]$. In fact, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of simple functions on $L^1(I,L^p_\tau)$ such that $f_n(t)\to f(t)$ for almost all $t\in I$ and $\lim_{n\to\infty}\int_I \|f_n(t)-f(t)\|_{L^1_\tau}dt = 0$. This is equivalent to

$$\lim_{n\to\infty}\int_{-\tau}^0\int_I \|(\mathcal{I}f_n)(t,\theta)-(\mathcal{I}f)(t,\theta)\|dtd\theta = 0.$$

It follows from the equality (3.1) that

$$\left\| {}^{L^1_\tau}\!\!\int_I f(t)dt - \int_I (\mathcal{I}f)(t,\cdot)dt \right\|_{L^1_\tau} = \left\| {}^{L^1_\tau}\!\!\int_I (f(t)-f_n(t))dt + \int_I (\mathcal{I}f_n)(t,\cdot) - (\mathcal{I}f)(t,\cdot)dt \right\|_{L^1_\tau}$$
$$\leqslant \int_I \|f(t)-f_n(t)\|_{L^p_\tau} dt + \int_{-\tau}^0 \int_I \|(\mathcal{I}f_n)(t,\theta) - (\mathcal{I}f)(t,\theta)\| dt d\theta.$$

The right side of the preceding inequality converges to 0 as $n \to \infty$. Using the inequality (3.1) we obtain that $\left({}^{L^p_\tau}\!\!\int_I f(t)dt \right)(\theta) = \int_I (\mathcal{I}f)(t,\theta)dt = \int_I f(t)(\theta)dt$, for almost all $\theta \in [-\tau, 0]$. ☐

The following lemma is a direct consequence of the previous result.

Lemma 3.4. Let $u \in L^p_{loc}([-r,\infty), X)$ be exponentially bounded on $\mathbb{R}_+$ with constants $L, \omega > 0$. Then, for any $\lambda > \omega$, the function $\int_0^\infty e^{-\lambda t} u_t dt \in L^p([-r,0], X)$ satisfies

$$F \int_0^\infty e^{-\lambda t} u_t dt = \int_0^\infty e^{-\lambda t} F u_t dt. \tag{3.2}$$

Proof. Let $\lambda > \omega$ be fixed and $Q_\lambda : [0,\infty) \to L^p([-r,0], X)$ defined by $Q_\lambda(t) = e^{-\lambda t} u_t$. Since u is exponentially bounded and $u|_{[-r,r]} \in L^p([-r,r], X)$, we have that

$$\begin{aligned}\int_0^\infty \|Q_\lambda(t)\|_{L^p} dt =& \int_0^\infty \left(\int_{-r}^0 e^{-\lambda t p} \|u(t+\theta)\|^p d\theta \right)^{1/p} dt \\ =& \int_r^\infty \left(\int_{-r}^0 e^{-\lambda t p} \|u(t+\theta)\|^p d\theta \right)^{1/p} dt + \int_0^r \left(\int_{-r}^0 e^{-\lambda t p} \|u(t+\theta)\|^p d\theta \right)^{1/p} dt \\ \leq& L \int_r^\infty e^{(\omega-\lambda)t} \left(\int_{-r}^0 e^{\omega \theta p} d\theta \right)^{1/p} dt + \int_0^r e^{-\lambda t} \left(\int_{-r}^r \|u(\theta)\|^p d\theta \right)^{1/p} dt < \infty.\end{aligned}$$

Thus Q_λ is integrable and the function $\int_0^\infty e^{-\lambda t} u_t dt \in L^p([-r,0], X)$ is well defined. Since $F : L^p([-r,0], X) \to X$ is bounded, by Lemma 3.3 we conclude that (3.2) is satisfied. ☐

In what follows, for each $\varphi \in L^p([-r,0], X)$ we denote:

$$\phi_\uparrow(t) = \begin{cases} \varphi(t), & -r \leq t \leq 0; \\ 0, & t > 0 \end{cases}.$$

Observe that $\phi_\uparrow \in L^p_{loc}([-r,\infty), X)$. Also, if $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is extended over $-r \leq t < 0$ as the null operator, for any $s \geq 0$ and $x \in X$ we denote $G_s x(\theta) = G(s+\theta)x$, where $-r \leq \theta < 0$. Note that $G_s x \in L^p([-r,0], X)$.

Definition 3.5. A strongly continuous family $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$, extended over $-r \leq t < 0$ as the null operator, is called a resolvent family with delay F if the following equation

$$\begin{aligned} G(t+s)x &= G(t)G(s)x + \int_0^t G(t-m)\mathfrak{F}(G_s x)_\uparrow(m) dm, \\ G(0) &= I, \end{aligned} \tag{3.3}$$

is satisfied for all $s, t \geq 0$ and $x \in X$.

The functional equation (3.3) corresponds to a rigorous representation of (1.3) in functional analytical terms. See also [13, eq. (2.15)]. In the case $F \equiv 0$ the equation (3.3) coincides, and will play the same role, with the well-known Cauchy functional equation

$$T(t+s) = T(t)T(s), \quad t, s \geq 0,$$

associated to the abstract Cauchy problem of first order.

Concerning properties, our first result is the following.

Proposition 3.6. *Suppose that $\{G(t)\}_{t\geq 0}$ is a resolvent family with delay F. Then, $\{G(t)\}_{t\geq 0}$ is an exponentially bounded family.*

Proof. Let $t > 1$, $x \in X$ and set $s = t-1$. Using the Hölder inequality, the functional equation (3.3) and the fact that $M_0 = \sup_{0\leq t\leq 1} \|G(t)\| < \infty$, we obtain

$$\begin{aligned} \|G(t)x\| &\leq \|G(1)\|\|G(s)\|\|x\| + \int_0^1 \|G(1-m)\|\|\mathfrak{F}(G_s x)_{\uparrow}(m)\| dm \\ &\leq \|G(1)\|\|G(s)\|\|x\| + M_0 \left(\int_0^1 \|\mathfrak{F}(G_s x)_{\uparrow}(m)\|^p dm\right)^{1/p} \\ &\leq \|G(1)\|\|G(s)\|\|x\| + M_0\|F\| \left(\int_0^1 \|(G_s x)_{\uparrow m}\|^p_{L^p([-r,0],X)} dm\right)^{1/p}. \end{aligned} \tag{3.4}$$

Furthermore, it follows from the fact that $\{G(t)\}_{t\geq 0}$ is extended over $-r \leq t < 0$ as the null operator, that

$$\begin{aligned} \int_0^1 \|(G_s x)_{\uparrow m}\|^p_{L^p([-r,0],X)} dm &= \int_0^1 \int_{-r}^0 \|(G_s x)_{\uparrow}(m+\theta)\|^p d\theta dm \\ &= \int_0^1 \int_{-r+m}^m \|(G_s x)_{\uparrow}(\theta)\|^p d\theta dm \\ &\leq \int_0^1 \int_{-r}^1 \|(G_s x)_{\uparrow}(\theta)\|^p d\theta dm = \int_{-r}^0 \|(G_s x)(\theta)\|^p d\theta \\ &= \int_{-r}^0 \|G(s+\theta)x\|^p d\theta = \int_{-r+s}^s \|G(\theta)x\|^p d\theta \\ &\leq r\|x\|^p \sup_{0\leq\theta\leq s} \|G(\theta)\|^p. \end{aligned} \tag{3.5}$$

Thus, combining the inequalities (3.4) and (3.5), we obtain for all $t > 1$:

$$\begin{aligned} \|G(t)x\| &\leq \|G(1)\|\|G(t-1)\|\|x\| + M_0\|F\|r^{1/p}\|x\| \sup_{0\leq\theta\leq t-1} \|G(\theta)\| \\ &\leq (\|G(1)\| + M_0\|F\|r^{1/p}) \sup_{0\leq\theta\leq t-1} \|G(\theta)\|\|x\|. \end{aligned}$$

Since the last inequality is valid for all $x \in X$, we conclude

$$\|G(t)\| \leq C_0(M_0 + \sup_{1<\theta\leq t-1} \|G(\theta)\|), \quad t > 2,$$

where $C_0 = \|G(1)\| + M_0\|F\|r^{1/p}$. Thus, for all $t > 2$:

$$\sup_{1<\theta\leq t} \|G(\theta)\| \leq C_0(M_0 + \sup_{1<\theta\leq t-1} \|G(\theta)\|). \tag{3.6}$$

Let $f : (1,\infty) \to \mathbb{R}_+$ be defined by $f(t) = \sup_{1<\theta\le t} \|G(\theta)\|$, for $t > 1$ the integer n_t such that $1 \le t - n_t \le 2$ and $M_1 = \sup_{1\le t\le 2} f(t)$. It follows from the inequality (3.6) that for $t > 2$

$$\begin{aligned} f(t) &\le M_0(C_0 + C_0^2 + ... + C_0^{n_t} f(t-n_t)) \\ &\le M_0(1+M_1)\left(\left(C_0 + \frac{1}{C_0}\right) + \left(C_0 + \frac{1}{C_0}\right)^2 + ... + \left(C_0 + \frac{1}{C_0}\right)^{n_t}\right) \\ &\le M_0(1+M_1)\left(C_0 + \frac{1}{C_0}\right)^{n_t} n_t \\ &\le M_0(1+M_1)\left(C_0 + \frac{1}{C_0}\right)^{t-1} (t-1) \le M_0(1+M_1)e^{\ln(C_0+\frac{1}{C_0})(t-1)}e^t \\ &\le Me^{\omega t} \end{aligned}$$

where $M = M_0(1+M_1)e^{-\ln(C_0+\frac{1}{C_0})}$ and $\omega = \ln(C_0 + \frac{1}{C_0}) + 1$. Thus, f is exponentially bounded on $(2,\infty)$, which implies that the family $\{G(t)\}_{t\ge 0}$ is exponentially bounded. $\square$

Given $\lambda \in \mathbb{C}$, e_λ denotes the function on $L^p([-r,0],\mathbb{C})$ described by $e_\lambda(\theta) = e^{\lambda\theta}$ for $\theta \in [-r,0]$. Observe that for each $x \in X$, $e_\lambda x \in L^p([-r,0],X)$. Furthermore, B_λ denotes the linear bounded operator described by $B_\lambda x = F(e_\lambda x)$. Let A be a closed linear operator. The set $\rho(A,F)$ is defined as the set of all values $\lambda \in \mathbb{C}$ for which the operator $\lambda I - A - B_\lambda : D(A) \to X$ has a bounded inverse, denoted by $R(\lambda,A,F)$, on the Banach space X.

The following theorem show that the resolvent families with delay F are exactly those strongly continuous operator-valued functions whose Laplace transforms are resolvents.

Theorem 3.7. *Assume that $\{G(t)\}_{t\ge 0}$ be a resolvent family with delay F. Then, G is Laplace transformable and there exists a linear operator $A : D(A) \subseteq X \to X$ such that $\lambda \in \rho(A,F)$ and*

$$R(\lambda,A,F)x = \hat{G}(\lambda)x$$

for all $x \in X$, $\lambda > \omega := abs(G)$.

Proof. By Proposition 3.6 the family $\{G(t)\}_{t\ge 0}$ is exponentially bounded with constants $M,\omega > 0$, i.e. $\|G(t)\| \le Me^{\omega t}$ for all $t \in \mathbb{R}_+$. Observe that, for each $t,s \in \mathbb{R}_+$ and $x \in X$

$$\begin{aligned} \|(G * \mathfrak{F}(G_s x)_\uparrow)(t)\| &\le \|F\| \int_0^t \|G(t-m)\| \|(G_s x)_{\uparrow m}\|_{L^p([-r,0],X)} dm \\ &\le \int_0^t e^{\omega(t-m)} \left(\int_{-r}^0 \|(G_s x)_\uparrow(m+\theta)\|^p d\theta\right)^{1/p} dm \\ &= \int_0^t e^{\omega(t-m)} \left(\int_{-r+m}^m \|(G_s x)_\uparrow(\theta)\|^p d\theta\right)^{1/p} dm \\ &\le \int_0^t e^{\omega(t-m)} \left(\int_{-r}^t \|(G_s x)_\uparrow(\theta)\|^p d\theta\right)^{1/p} dm \\ &= \int_0^t e^{\omega(t-m)} \left(\int_{-r}^0 \|(G_s x)(\theta)\|^p d\theta\right)^{1/p} dm \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t e^{\omega(t-m)} \left(\int_{-r+s}^s \|G(\theta)x\|^p d\theta \right)^{1/p} dm \\
&\le M \int_0^t e^{\omega(t-m)} \left(\int_0^s e^{\omega\theta p} \|x\|^p d\theta \right)^{1/p} dm \\
&= M \left(\frac{e^{\omega sp} - 1}{\omega} \right) \left(\frac{e^{\omega t} - 1}{\omega} \right) \|x\|. \qquad (3.7)
\end{aligned}
$$

Also, for $\lambda, \mu > \omega$

$$
\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} G(t+s)x dt ds = \frac{\hat{G}(\lambda)x - \hat{G}(\mu)x}{\mu - \lambda}.
$$

Let $\lambda, \mu > \omega$ and $x \in X$. We infer from (3.7) and Lemmas 3.4 and 3.3

$$
\begin{aligned}
\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} (G * \mathfrak{F}(G_s x)_\uparrow)(t) dt ds &= \int_0^\infty e^{-\mu s} \left(\int_0^\infty e^{-\lambda t} (G * \mathfrak{F}(G_s x)_\uparrow)(t) dt \right) ds \\
&= \int_0^\infty e^{-\mu s} \hat{G}(\lambda) \left(\int_0^\infty e^{-\lambda t} \mathfrak{F}(G_s x)_\uparrow(t) dt \right) ds \\
&= \int_0^\infty e^{-\mu s} \hat{G}(\lambda) F \left[\int_0^\infty e^{-\lambda t} (G_s x)_{\uparrow_t} dt \right] ds \\
&= \hat{G}(\lambda) F \int_0^\infty \int_0^\infty e^{-\mu s} e^{-\lambda t} (G_s x)_{\uparrow_t} dt ds. \qquad (3.8)
\end{aligned}
$$

Observe that for almost all $\theta \in [-r, 0]$

$$
\begin{aligned}
\left(\int_0^\infty \int_0^\infty e^{-\mu s} e^{-\lambda t} (G_s x)_{\uparrow_t} dt ds \right)(\theta) &= \int_0^\infty \int_0^{-\theta} e^{-\mu s} e^{-\lambda t} G(s+t+\theta) x dt ds \\
&= \int_0^{-\theta} e^{-\lambda t} \left(\int_0^\infty e^{-\mu s} G(s+t+\theta) x ds \right) dt \\
&= \int_0^{-\theta} e^{-\lambda t} \left(\int_{t+\theta}^\infty e^{-\mu(s-t-\theta)} G(s) x ds \right) dt \\
&= \int_0^{-\theta} e^{(\mu-\lambda)t} e^{\mu\theta} \left(\int_0^\infty e^{-\mu s} G(s) x ds \right) dt \\
&= e^{\mu\theta} \left(\frac{e^{-(\mu-\lambda)\theta} - 1}{\mu - \lambda} \right) \hat{G}(\mu) x \\
&= \left(\frac{e^{\lambda\theta} - e^{\mu\theta}}{\mu - \lambda} \right) \hat{G}(\mu) x. \qquad (3.9)
\end{aligned}
$$

Thus, combining (3.8) with (3.9), we conclude that

$$
\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} (G * \mathfrak{F}(G_s x)_\uparrow)(t) dt ds = \hat{G}(\lambda) \left(\frac{B_\lambda - B_\mu}{\mu - \lambda} \right) \hat{G}(\mu) x.
$$

Applying the double Laplace transform to the equation (3.3) we obtain,

$$
\frac{\hat{G}(\lambda)x - \hat{G}(\mu)x}{\mu - \lambda} = \hat{G}(\lambda)\hat{G}(\mu)x + \hat{G}(\lambda) \left(\frac{B_\lambda - B_\mu}{\mu - \lambda} \right) \hat{G}(\mu) x, \qquad (3.10)
$$

which is equivalent to

$$
\hat{G}(\lambda)x - \hat{G}(\mu)x = (\mu - \lambda)\hat{G}(\lambda)\hat{G}(\mu)x + \hat{G}(\lambda)(B_\lambda - B_\mu)\hat{G}(\mu)x. \qquad (3.11)
$$

Let $x \in \ker \hat{G}(\lambda)$ be given. Since $G(0)x = x$ and $\hat{G}(\lambda)x = 0$ for all $\lambda > \omega$ we have that $x = 0$, and thus $\ker \hat{G}(\lambda) = \{0\}$ and $\hat{G}(\lambda) : X \to \mathrm{Im}(\hat{G}(\lambda))$ is invertible. Let $\lambda, \mu > \omega$ be fixed. We define the mapping

$$\begin{aligned} A : \mathrm{Im}(\hat{G}(\lambda)) &\to X \\ x \to Ax &= \lambda x - B_\lambda x - \hat{G}^{-1}(\lambda)x. \end{aligned} \tag{3.12}$$

Since (3.11) is satisfied, we obtain that $\mathrm{Im}(\hat{G}(\lambda)) = \mathrm{Im}(\hat{G}(\mu))$ and for $x \in \mathrm{Im}(\hat{G}(\lambda))$ the identity

$$\lambda x - B_\lambda x - \hat{G}^{-1}(\lambda)x = \mu x - B_\mu x - \hat{G}^{-1}(\mu)x,$$

holds. Thus, the operator A does not depend of the selection of $\lambda > \omega$. Hence A is well defined. Is not difficult to see, by the definition of A that $\hat{G}(\lambda) = (\lambda - A - B_\lambda)^{-1}$. Since $\hat{G}(\lambda)$ is bounded we conclude that $\lambda \in \rho(A, F)$ for all $\lambda > \omega$, proving the theorem. □

We say that the operator A defined by (3.12) is the generator of the resolvent family with delay F. Before to state the main theorem of this section, we need the following lemma.

Lemma 3.8. Suppose that $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a strongly continuous family, extended as the null operator as $-r \leq t < 0$, and $h \in L^1_{loc}([-r,\infty), X)$. Then,

$$\mathfrak{F}(G * h)(t) = (\mathfrak{F}G * h)(t),$$

for all $t \geq 0$.

Proof. Let $t \geq 0$ be given and $Q : [0,t] \to L^p([-r,0], X)$ defined by $Q(s) = G_{t-s}h(s)$. Observe that

$$\begin{aligned} \int_0^t \|Q(s)\|_{L^p([-r,0],X)} ds &\leq \int_0^t \left(\int_{-r}^0 \|G(t-s+\theta)h(s)\|^p d\theta \right)^{1/p} ds \\ &\leq \sup_{-r\leq\omega\leq t} \|G(\omega)\| \int_0^t \left(\int_{-r}^0 \|h(s)\|^p d\theta \right)^{1/p} ds \\ &\leq \sup_{-r\leq\omega\leq t} \|G(\omega)\| r^{1/p} \int_0^t \|h(s)\| ds < \infty. \end{aligned}$$

Thus Q is integrable. Then by Lemma 3.3 we have

$$F \int_0^t Q(s)ds = \int_0^t FQ(s)ds = \int_0^t FG_{t-s}h(s)ds = (\mathfrak{F}G * h)(t). \tag{3.13}$$

Now, for almost all $\theta \in [-r, 0]$, and because $G(t)$ is the null operator for $t \in [-r, 0)$, we obtain

$$\int_0^t Q(s)ds(\theta) = \int_0^t G(t-s+\theta)h(s)ds = \int_0^{t+\theta} G(t-s+\theta)h(s)ds = (G * h)_t(\theta).$$

Thus, by (3.13) we obtain that

$$\mathfrak{F}(G * h)(t) = (\mathfrak{F}G * h)(t),$$

for all $t \geq 0$. Hence the proof is complete. □

The following is the main result of this section and one of the main results of this paper.

Theorem 3.9. *Let A be a closed operator. Suppose that the problem* (1.1) *associated to A is mildly well-posed. Then, the family $\{G(t)\}_{t\geq 0}$ of operators from X into itself, defined by $G(t)x = u_\Phi(t)$, $\Phi = (x, 0) \in \widetilde{X}$, and extended as the null operator for $-r \leq t < 0$, is a resolvent family with delay F generated by the operator A.*

Proof. Is not difficult to see, by the uniqueness, that $\{G(t)\}_{t\geq 0}$ is well defined and is also a strongly continuous family of linear operators. Further, let $t \geq 0$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be such that $x_n \to 0$. Then $\Phi_n = (x_n, 0) \to (0,0)$ on $\widetilde{X}$ and we have that $G(t)x_n = u(t, \Phi_n) \to 0$ as $n \to \infty$. Therefore, $G(t)$ is a bounded linear operator on X. Also

$$G(t)x = x + A(1 * G)(t)x + (1 * \mathfrak{F}G)(t)x, \tag{3.14}$$

for all $t \geq 0$ and $x \in X$.

Let $s \geq 0$, $x \in X$ be given. Consider $\widetilde{G} : [-r, \infty) \to \mathcal{B}(X)$ defined by

$$\widetilde{G}(t)x = \begin{cases} G(s+t)x, & t \geq 0; \\ 0, & -r \leq t < 0 \end{cases},$$

and $v : [-r, \infty) \to X$ defined by $v(t) = \widetilde{G}(t)x - (G * \mathfrak{F}(G_s x)_{\uparrow})(t)$. We observe that $v(t) = 0$ for all $-r \leq t < 0$ and $(1 * v)(t) \in D(A)$ for all $t \geq 0$, since $(1 * G)(t)x \in D(A)$ for all $t \geq 0$ $x \in X$, A is closed operator and the equation (3.14) is satisfied. From equation (3.14) we have

$$\begin{aligned} A(1 * v)(t) &= A(1 * \widetilde{G})(t)x - A((1 * G) * \mathfrak{F}(G_s x)_{\uparrow})(t) \\ &= A(1 * \widetilde{G})(t)x - (G * \mathfrak{F}(G_s x)_{\uparrow})(t) + (1 * \mathfrak{F}(G_s x)_{\uparrow})(t) \\ &\quad + ((1 * \mathfrak{F}G) * \mathfrak{F}(G_s x)_{\uparrow})(t). \end{aligned} \tag{3.15}$$

On the one hand, for $t \geq 0$, using Lemma 3.3 we obtain that

$$\begin{aligned} A(1 * \widetilde{G})(t)x &= A\int_0^t G(s+r)xdr = A\int_s^{t+s} G(r)xdr = A\int_0^{t+s} G(r)xdr - A\int_0^s G(r)xdr \\ &= G(t+s)x - x - \int_0^{t+s} FG_r xdr - G(s)x + x + \int_0^s FG_r xdr \\ &= G(t+s)x - G(s)x - \int_s^{t+s} FG_r xdr = \widetilde{G}(t)x - G(s)x - \int_s^{t+s} FG_r xdr \\ &= \widetilde{G}(t)x - G(s)x - F\int_s^{t+s} G_r xdr, \end{aligned}$$

and for almost all $\theta \in [-r, 0]$

$$\begin{aligned} \left(\int_s^{t+s} G_r xdr\right)(\theta) &= \int_s^{t+s} G(r+\theta)xdr = \int_0^t G(s+r+\theta)xdr \\ &= \int_{-\theta}^t G(s+r+\theta)xdr + \int_0^{-\theta} G(s+r+\theta)xdr \\ &= \int_0^t \widetilde{G}(r+\theta)xdr + \int_0^t (G_s x)_{\uparrow}(r+\theta)dr \\ &= \left(\int_0^t \widetilde{G}_r xdr\right)(\theta) + \left(\int_0^t (G_s x)_{\uparrow_r} dr\right)(\theta). \end{aligned}$$

Thus,

$$A(1 * \widetilde{G}x)(t) = \widetilde{G}(t)x - G(s)x - (1 * \mathfrak{F}\widetilde{G})(t)x - (1 * \mathfrak{F}(G_s x)_{\uparrow})(t)x. \tag{3.16}$$

On the other hand, using Lemma 3.8, we have that

$$((1 * \mathfrak{F}G) * \mathfrak{F}(G_s x)_{\uparrow})(t) = (1 * \mathfrak{F}(G * \mathfrak{F}(G_s x)_{\uparrow}))(t). \tag{3.17}$$

It follows from the equations (3.15), (3.16) and (3.17) that

$$\begin{aligned} A(1*v)(t) &= \widetilde{G}(t)x - (G*\mathfrak{F}(G_s x)_{\uparrow})(t) - G(s)x - (1*\mathfrak{F}\widetilde{G})(t)x + (1*\mathfrak{F}(G*\mathfrak{F}(G_s x)_{\uparrow}))(t) \\ &= v(t) - G(s)x - (1*\mathfrak{F}v)(t). \end{aligned}$$

Hence, we have obtained that v is a mild solution of the problem (1.1) associated to $(G(s)x, 0) \in \widetilde{X}$. By the uniqueness we obtain

$$v(t) = G(t)G(s)x, \quad t \geq 0.$$

Since, $s \geq 0$ and $x \in X$ was arbitrarily selected, we conclude

$$G(t+s)x - \int_0^t G(t-m)\mathfrak{F}(G_s x)_{\uparrow}(m)dm = G(t)G(s)x,$$

for all $t, s \geq 0$, $x \in X$. Hence $\{G(t)\}_{t\geq 0}$ is a resolvent family with delay F. In particular G is Laplace transformable by Proposition 3.6.

Now we will show that A is the generator of $\{G(t)\}_{t\geq 0}$. Let A_0 be the generator operator of G defined in (3.12). By the closedness of the operator A we obtain that $\hat{G}(\lambda)x \in D(A)$ for all $\lambda > \omega := abs(G)$ and $x \in X$. Thus, if $x \in D(A_0)$ then we have $x = \hat{G}(\lambda)(\lambda - A_0 - B_\lambda)x \in D(A)$ which implies $D(A_0) \subseteq D(A)$. On the other hand, applying the Laplace transform to (3.14) we obtain

$$x = (\lambda - A - B_\lambda)\hat{G}(\lambda)x = (\lambda - A - B_\lambda)R(\lambda, A_0, F)x.$$

If $(\lambda - A - B_\lambda)$ is an injective operator for all $\lambda > \omega$, then we can conclude that

$$R(\lambda, A_0, F)x = R(\lambda, A, F)x,$$

for all $x \in X$. Consequently $D(A) \subseteq D(A_0)$ and $A = A_0$.

In order to show that $(\lambda - A - B_\lambda)$ is injective, let $\lambda > \omega$ and $x \in \ker(\lambda - A - B_\lambda)$ be fixed. On the one hand, let $v : [-r, \infty) \to X$ be defined by $v(t) = e^{\lambda t}x$. We note, for $t \geq 0$, that

$$x + A(1*v(t)) + (1*\mathfrak{F}v)(t) = x + A\left(\frac{e^{\lambda t}}{\lambda}x - \frac{x}{\lambda}\right) + B_\lambda\left(\frac{e^{\lambda t}}{\lambda}x - \frac{x}{\lambda}\right) = e^{\lambda t}x = v(t).$$

Thus v is a mild solution of the problem (1.1) associated to $(x, \phi) \in \widetilde{X}$, where $\phi(\theta) = e^{\lambda\theta}x$, $\theta \in [-r, 0]$. On the other hand, since $(1*G)(t)x \in D(A)$ is satisfied for all $x \in X$ and $t \geq 0$ we conclude, by the closedness of A, that $(1*(G*\mathfrak{F}\phi_{\uparrow}))(t)x \in D(A)$. We infer from (3.14) and Lemma 3.8 that

$$\begin{aligned} (G*\mathfrak{F}\phi_{\uparrow})(t)x &= (1*\mathfrak{F}\phi_{\uparrow})(t)x + A(1*G*\mathfrak{F}\phi_{\uparrow})(t)x + (1*\mathfrak{F}G*\mathfrak{F}\phi_{\uparrow})(t)x \\ &= (1*\mathfrak{F}\phi_{\uparrow})(t)x + A(1*G*\mathfrak{F}\phi_{\uparrow})(t)x + (1*\mathfrak{F}(G*\mathfrak{F}\phi_{\uparrow}))(t)x \\ &= A(1*G*\mathfrak{F}\phi_{\uparrow})(t)x + (1*\mathfrak{F}(G*\mathfrak{F}\phi_{\uparrow} + \phi_{\uparrow}))(t)x. \end{aligned}$$

Thus, since $\phi_{\uparrow}(t) = 0$ for all $t \geq 0$, we have

$$(G*\mathfrak{F}\phi_{\uparrow} + \phi_{\uparrow})(t)x = A(1*(G*\mathfrak{F}\phi_{\uparrow} + \phi_{\uparrow}))(t)x + (1*\mathfrak{F}(G*\mathfrak{F}\phi_{\uparrow} + \phi_{\uparrow}))(t)x,$$

for all $x \in X$, $t \geq 0$. Hence, $G*\mathfrak{F}\phi_{\uparrow} + \phi_{\uparrow}$ is a mild solution of the problem (1.1) associated to $(\phi_{\uparrow}, 0) \in \widetilde{X}$. Consequently, $v - G*\mathfrak{F}\phi_{\uparrow} - \phi_{\uparrow}$ is a mild solution of the problem (1.1) associated to $(0, x) \in \widetilde{X}$ and by the uniqueness we have $v(t) = G(t)x + G*\mathfrak{F}\phi_{\uparrow}(t) + \phi_{\uparrow}(t)$ for all $t \in [-r, \infty)$. Since $\hat{G}(\lambda)x$ exists for all $\lambda > \omega$ we conclude that $\hat{v}(\lambda)$ so is. The only way this will happen is that $x = 0$. Therefore, $(\lambda - A - B_\lambda)$ is an injective operator. Hence the proof is complete. □

4. A characterization

In this section we characterize the mildly well-posedness of problem (1.1) in terms of a resolvent family $\{G(t)\}_{t\geq 0}$ generated by a closed linear operator A. To make this possible, will be useful to consider the following strongly continuous family of operators $\{\mathbb{1}(t)\}_{t\geq -r}$ defined by

$$\mathbb{1}(t)x = \begin{cases} x, & t \geq 0; \\ 0, & -r \leq t < 0 \end{cases}.$$

It follows from Lemma 3.3 that $\widehat{\mathfrak{F}\mathbb{1}}(\lambda)x = \frac{1}{\lambda}B_\lambda x$ for all $\lambda \geq 0$, $x \in X$.

Proposition 4.1. *Suppose that $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a resolvent family with delay F generated by the operator A. Then, the following assertions holds:*

(i) $(1 * G)(t)x \in D(A)$ *for all* $x \in X$, $t \geq 0$.
(ii) $G(t)x = x + A(1 * G)(t)x + (1 * \mathfrak{F}G)(t)x$ *for all* $x \in X$, $t \geq 0$.
(iii) $G(t)x = x + (G * A)(t)x + (G * \mathfrak{F}\mathbb{1})(t)x$ *for all* $x \in D(A)$, $t \geq 0$.
(iv) *The operator A is closed with dense domain on X.*

Proof. Let $x \in X$ and $\lambda > abs(G)$ be given. Since $\hat{G}(\lambda)x = R(\lambda, A, F)x$ by Theorem 3.7, we obtain that $\hat{G}(\lambda)x \in D(A)$ and

$$A\left(\frac{1}{\lambda}\hat{G}(\lambda)x\right) = \hat{G}(\lambda)x - \frac{1}{\lambda}x - \frac{1}{\lambda}B_\lambda \hat{G}(\lambda)x.$$

Then it follows from [2, Proposition 1.7.6] that $(1 * G)(t)x \in D(A)$ and

$$G(t)x = x + A(1 * G)(t)x + (1 * \mathfrak{F}G)(t)x,$$

for all $t \geq 0$, $x \in X$, showing the items (i) and (ii). Since

$$x = \hat{G}(\lambda)(\lambda - A - B_\lambda)x, \tag{4.1}$$

holds for all $x \in D(A)$, we obtain, by the inversion of Laplace transform, that

$$x = G(t)x - (G * A)(t)x - (G * \mathfrak{F}\mathbb{1})(t)x,$$

for all $t \geq 0$ and $x \in D(A)$. Hence (iii) is satisfied. Let $\{x_n\} \subseteq D(A)$ be a sequence such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. It follows from (4.1) that

$$x = \hat{G}(\lambda)(\lambda x - y - B_\lambda x).$$

Thus $x \in D(A)$ and $y = \lambda x - B_\lambda x - G(\lambda)^{-1}x = Ax$, by the definition of A in (3.12). Hence A is closed. Finally observe that

$$\left\| \frac{1 * G(s)x}{s} - x \right\| \leq \frac{1}{s}\int_0^s \|G(\mu)x_n - x\|\, d\mu \leq \sup_{\mu\in[0,s]} \|G(\mu)x - x\|.$$

Then, taking into account the strong continuity at $t = 0$, we obtain

$$\lim_{s\to 0^+} \frac{(1 * G)(s)x}{s} = x. \tag{4.2}$$

Since it was proved that $(1 * G)(t)x \in D(A)$ for all $t \geq 0$, defining

$$x_n = \frac{(1 * G)(\frac{1}{n})}{\frac{1}{n}}x, \quad n \in \mathbb{N},$$

it follows from (4.2) that $x_n \in D(A)$ and $\lim_{n\to\infty} x_n = x$, proving the density of $D(A)$. $\square$

Remark 4.2. We notice that, when $F \equiv 0$, A is the generator of a C_0-semigroup see ([6, Definition 1.2]) and the denseness of $D(A)$ is always present under the conditions of the above theorem, which is a well known result in the theory of C_0-semigroups.

The next result allows to represent the generator of a resolvent family with delay F directly in terms of $\{G(t)\}_{t\geq 0}$ without the help of the Laplace transform.

Proposition 4.3. *Suppose that $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a resolvent family with delay F generated by the operator A. Then $A = B$ where*

$$D(B) := \left\{ x \in X : \lim_{t\to 0^+} \frac{G(t)x - x}{t} \text{ exists} \right\}$$

and

$$Bx := \lim_{t\to 0^+} \frac{G(t)x - x}{t}, \quad x \in D(B). \tag{4.3}$$

Proof. Consider the linear operator B defined in (4.3). We observe, for $0 < t \leq r$, that

$$\begin{aligned} \frac{1}{t}\|(G * \mathfrak{F}\mathbb{1})(t)x\| &\leq \frac{1}{t}\int_0^t \|G(t-s)\|\|F\|\|\mathbb{1}_s x\|_{L^p([-r,0],X)} ds \\ &\leq \sup_{0\leq s\leq t} \|G(s)\|\|F\| \frac{1}{t}\int_0^t \left(\int_{-r}^0 \|\mathbb{1}(s+\theta)x\|^p d\theta\right)^{1/p} ds \\ &= \sup_{0\leq s\leq t} \|G(s)\|\|F\| \frac{1}{t}\int_0^t \left(\int_{-r+s}^s \|\mathbb{1}(\theta)x\|^p d\theta\right)^{1/p} ds \\ &= \sup_{0\leq s\leq t} \|G(s)\|\|F\| \frac{1}{t}\int_0^t \left(\int_0^s \|x\|^p d\theta\right)^{1/p} ds \\ &\leq \sup_{0\leq s\leq t} \|G(s)\|\|F\|\|x\| t^{1/p}. \end{aligned} \tag{4.4}$$

Let $x \in D(A)$ be given. It follows from (iii) of Proposition 4.1 that

$$\frac{G(t) - x}{t} = \frac{(G * 1)(t)Ax}{t} + \frac{(G * \mathfrak{F}\mathbb{1})(t)x}{t}, \quad t \geq 0.$$

By (4.2) and (4.4), the right side of the above equation converges to Ax as $t \to 0^+$. Hence $x \in D(B)$ and $Bx = Ax$.

On the other hand, we observe, for $0 < t \leq r$, that

$$\begin{aligned} \frac{1}{t}\|(1 * \mathfrak{F}G)(t)x\| &\leq \frac{1}{t}\int_0^t \|F\|\|G_s x\|_{L^p([-r,0],X)} ds \\ &= \|F\| \frac{1}{t}\int_0^t \left(\int_{-r}^0 \|G(s+\theta)x\|^p d\theta\right)^{1/p} ds \\ &= \sup_{0\leq s\leq t} \|G(s)\|\|F\| \frac{1}{t}\int_0^t \left(\int_{-r+s}^s \|G(\theta)x\|^p d\theta\right)^{1/p} ds \\ &= \sup_{0\leq s\leq t} \|G(s)\|\|F\| \frac{1}{t}\int_0^t \left(\int_0^s \|G(\theta)x\|^p d\theta\right)^{1/p} ds \\ &\leq \sup_{0\leq s\leq t} \|G(s)\|\|F\|\|x\| t^{1/p}. \end{aligned} \tag{4.5}$$

Let $x \in D(B)$ be given. It follows from items $(i), (ii)$ and (iii) of Proposition 4.1 that $x_t := \frac{(1*G)(t)}{t} x \in D(A)$ and

$$Ax_t = A\frac{(1*G)(t)x}{t} = \frac{G(t)x - x}{t} - \frac{(1*\mathfrak{F}G)(t)x}{t},$$

for all $t \geq 0$. By (4.5) the right side of the last equation converges to Bx and x_t converges to x as $t \to 0$. By the closedness of operator A we can conclude that $x \in D(A)$ and $Ax = Bx$. This finishes de proof. □

Our next Theorem, which is one of the main results in this section, gives sufficient conditions for the mildly well-posedness of the problem (1.1) in terms of the functional equation (3.3).

Theorem 4.4. *Assume that $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a resolvent family with delay F generated by A. Then the problem (1.1) is mildly well-posed.*

Proof. (Existence) Let $(x, \phi) \in \widetilde{X}$ be given and define $v(t) = G(t)x + (G * \mathfrak{F}\phi_\uparrow)(t) + \phi_\uparrow(t)$, $t \geq -r$. Since G is a strongly continuous family and $\phi_\uparrow(t) = 0$ for all $t \geq 0$, we have that v is continuous on $[0, \infty)$. Let $t \geq 0$ be fixed. It follows from (i) and (iv) of Proposition 4.1 that $(1 * G)(t) \in D(A)$ and A is a closed operator. Thus, we obtain $((1 * G) * \mathfrak{F}\phi_\uparrow)(t) \in D(A)$. Further $(1 * v)(t) = (1 * G)(t)x + ((1 * G) * \mathfrak{F}\phi_\uparrow)(t)$ which implies $(1 * v)(t) \in D(A)$. On the one hand, it follows from item (ii) of Proposition 4.1 and Lemma 3.8 that

$$\begin{aligned}
A(1*v)(t) &= A(1*G)(t)x + A((1*G)*\mathfrak{F}\phi_\uparrow)(t) \\
&= G(t)x - x - (1*\mathfrak{F}G)(t)x + (G*\mathfrak{F}\phi_\uparrow)(t) - (1*\mathfrak{F}\phi_\uparrow)(t) - ((1*\mathfrak{F}G)*\mathfrak{F}\phi_\uparrow)(t) \\
&= v(t) - x - (1*\mathfrak{F}\phi_\uparrow)(t) - (1*\mathfrak{F}G)(t)x - ((1*\mathfrak{F}G)*\mathfrak{F}\phi_\uparrow)(t) \\
&= v(t) - x - (1*\mathfrak{F}\phi_\uparrow)(t) - (1*\mathfrak{F}G)(t) - (1*(\mathfrak{F}G*\mathfrak{F}\phi_\uparrow))(t) \\
&= v(t) - x - (1*\mathfrak{F}\phi_\uparrow)(t) - (1*\mathfrak{F}G)(t) - (1*\mathfrak{F}(G*\mathfrak{F}\phi_\uparrow))(t).
\end{aligned}$$

On the other hand, we have

$$(1*\mathfrak{F}v)(t) = (1*\mathfrak{F}G)(t)x + (1*\mathfrak{F}(G*\mathfrak{F}\phi_\uparrow))(t) + (1*\mathfrak{F}\phi_\uparrow)(t).$$

Taking into account the above two identities, we conclude that

$$A(1*v)(t) = v(t) - x - (1*\mathfrak{F}v)(t).$$

It proves that v is a mild solution of problem (1.1) associated to $(x, \phi) \in \widetilde{X}$.

(Uniqueness) Suppose that u_1, u_2 are two mild solutions of the problem (1.1) associated to the initial condition $\Phi = (x, \phi) \in \widetilde{X}$. Then, $v : [-r, \infty) \to X$ defined by $v(t) = (u_1 - u_2)(t)$ is a mild solution of the problem (1.1) associated to $(0, 0) \in \widetilde{X}$. It follows from item (iii) of Proposition 4.1 that

$$x = G(t)x - (G*A)(t)x - (G*\mathfrak{F}\mathbb{1})(t)x,$$

for all $t \geq 0$ and $x \in D(A)$. The last equality and the fact $(1*v)(t) \in D(A)$ for all $t \geq 0$, allows us to conclude

$$\begin{aligned}
(1*(1*v))(t) &= (G*(1*v))(t) - ((G*A)*(1*v))(t) - ((G*\mathfrak{F}\mathbb{1})*(1*v))(t) \\
&= (G*(1*v))(t) - (G*(1*A(1*v)))(t) - (G*(1*(\mathfrak{F}\mathbb{1}*v)))(t).
\end{aligned} \tag{4.6}$$

Note that $\mathfrak{F}\mathbb{1} * v = 1 * \mathfrak{F}v$. Indeed, let $t \geq 0$ be given, then $f^t : [0, t] \to L^p([-r, 0], X)$ defined by $f^t(s) = \mathbb{1}_{t-s}v(s)$ is integrable. By Lemma 3.3

$$F\int_0^t f^t(s)ds = \int_0^t Ff^t(s)ds = \int_0^t F\mathbb{1}_{t-s}v(s)ds = (\mathfrak{F}\mathbb{1}*v)(t), \tag{4.7}$$

and for almost all $\theta \in [-r, 0]$ we have

$$\begin{aligned}\left(\int_0^t f^t(s)ds\right)(\theta) &= \int_0^t \mathbb{1}(t+\theta-s)v(s)ds = \int_{-\theta}^{t-\theta} \mathbb{1}(t-s)v(s+\theta)ds \\ &= \int_0^t \mathbb{1}(t-s)v(s+\theta)ds - \int_0^{-\theta} \mathbb{1}(t-s)v(s+\theta)ds \\ &+ \int_t^{t-\theta} \mathbb{1}(t-s)v(s+\theta)ds \\ &= \int_0^t v(s+\theta)ds = \int_0^t v_s(\theta)ds.\end{aligned} \tag{4.8}$$

Then, using Lemma 3.3 and the equations (4.7), (4.8) we can conclude that, for each $t \geq 0$

$$(1 * \mathfrak{F}v)(t) = (\mathfrak{F}\mathbb{1} * v)(t).$$

Thus, combining the above equality with (4.6) and the fact that v is a mild solution associated to $(0,0) \in \widetilde{X}$ we obtain

$$\begin{aligned}(1 * (1 * v))(t) &= (G * (1 * v - 1 * A(1 * v) - 1 * (1 * \mathfrak{F}v)))(t) \\ &= (G * (1 * (v - A(1 * v) - (1 * \mathfrak{F}v))))(t) = 0,\end{aligned}$$

for all $t \geq 0$. Thus, by Titchmarsh's theorem, we conclude that $v(t) = 0$ for all $t \geq 0$, proving the uniqueness.

(Continuity) Suppose that $\Phi_n = (x_n, \phi_n) \in \widetilde{X}$ is such that $\Phi_n \to 0$. Let $t \geq 0$ and $p > 1$ be given. By the Hölder inequality and the uniform boundedness principle we have

$$\begin{aligned}\|u(t,\Phi_n)\| &\leq \|G(t)x_n\| + \|(G * \mathfrak{F}\phi_{n\uparrow})(t)\| \\ &\leq \|G(t)x_n\| + \left(\int_0^t \|G(s)\|^q ds\right)^{\frac{1}{q}} \left(\int_0^t \|\mathfrak{F}\phi_{n\uparrow}(s)\|^p ds\right)^{\frac{1}{p}} \\ &\leq \|G(t)x_n\| + \left(t \sup_{s\in[0,t]} \|G(s)\|^q\right)^{\frac{1}{q}} \left(t\|F\|^p \int_{-r}^t \|\phi_{n\uparrow}(s)\|^p ds\right)^{\frac{1}{p}} \\ &= \|G(t)x_n\| + t\|F\| \sup_{s\in[0,t]} \|G(s)\| \left(\int_{-r}^0 \|\phi_{n\uparrow}(s)\|^p ds\right)^{\frac{1}{p}}.\end{aligned}$$

Using that $\Phi_n \to 0$ as $n \to \infty$ we conclude that the right hand side of the above equation converges to zero uniformly for t in compact intervals. The case for $p = 1$ is proven similarly. Hence, the problem (1.1) is mildly well-posed.

□

As a directly consequence of the proof of the preceding theorem we obtain the following kind of variation of constants formula.

Proposition 4.5. *Suppose that $\{G(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is a resolvent family with delay F generated by A. Then, for all $\phi \in L^p([-r,0],X)$ the function $v : [-r,\infty) \to X$ defined by*

$$v(t) = G(t)x + (G * \mathfrak{F}\phi_{\uparrow})(t) + \phi_{\uparrow}(t),$$

is the unique mild solution of problem (1.1) *associated to* $(x,\phi) \in \widetilde{X}$.

Now, we are ready to state the main result of this paper.

Theorem 4.6. *Suppose that $A : D(A) \subseteq X \to X$ a closed linear operator. Then the problem* (1.1) *is midly well-posed if and only if A is the generator of a resolvent family with delay F.*

Proof. The result is implied by Theorems 4.4 and 3.9. □

Remark 4.7. With similar arguments, the previous result can be extended to include equations of the form:

$$\begin{cases} u(t) = x + A\int_0^t u(s)ds + \int_0^t Fu_s ds + \int_0^t \sum_{i=1}^m C_i x(t-r_i)ds, & t \geq 0; \\ u(t) = \phi(t), & -r \leq t < 0 \end{cases},$$

where $C_i \in \mathcal{B}(X)$ and $0 < r_i \leq r$.

5. Applications and examples

In this section we search for practical criteria in order to verify that a strongly continuous family of bounded and linear operators satisfy the functional equation (3.3). For this, we consider the case where the operator A generates a C_0-semigroup $T(t) = e^{At}$ and the delay operator $F : L^p([-r,0],X) \to X$ is described by

$$F\phi = \int_{-r}^0 H(\theta)\phi(\theta)d\theta, \tag{5.1}$$

where H is an $\mathcal{B}(X)$-valued q-integrable function on $[-r,0]$ with $\frac{1}{q} + \frac{1}{p} = 1$. In particular, we will show that in such case the operator A is the generator of a resolvent family with delay F. We complete this section exhibiting two explicit examples of resolvent families with delay F.

Note that since e^{At} is a C_0-semigroup, there exist constants $L \geq 1$ and $\omega \in \mathbb{R}$ such that $\|e^{At}\| \leq Le^{\omega t}$, $t \geq 0$.

It is clear that $F : L^p([-r,0],X) \to X$ is bounded. Also, for $T > 0$ and $u \in L^p_{loc}([-r,T],X)$, using Hölder inequality and Fubini's theorem we obtain

$$\begin{aligned} \int_0^T \|Fu_t\|^p dt &\leq \|H\|^p_{L^q} \int_0^T \left(\int_{-r}^0 \|u(t+\theta)\|^p d\theta\right) dt \\ &= \|H\|^p_{L^q} \int_0^T \left(\int_{-r+t}^t \|u(\theta)\|^p d\theta\right) dt \\ &\leq \|H\|^p_{L^q} \int_0^T \left(\int_{-r}^T \|u(\theta)\|^p d\theta\right) dt \\ &\leq \|H\|^p_{L^q} T \int_{-r}^T \|u(\theta)\|^p d\theta. \end{aligned} \tag{5.2}$$

In what follows we consider the operator F defined in (5.1). We begin with the following result on the existence and uniqueness of solutions.

Proposition 5.1. *The integral problem,*

$$u(t) = e^{At}x + \int_0^t e^{A(t-s)}Fu_s ds, \quad x \in X, \quad u(t) = 0, \quad t \in [-r,0), \tag{5.3}$$

admits a unique solution $u \in C([0,\infty),X)$ exponentially bounded.

Proof. We define the operator $K : L^p([0,T],X) \to C([0,T],X)$ by $Ku(t) = e^{At}x + \int_0^t e^{A(t-s)}Fu_s ds$. Here u is extended to $t \in [-r,0)$ by $u(t) = 0$. It is easy to see that K maps $L^p([0,T],X)$ into itself. Also, using the bound (5.2) and Jensen inequality we obtain

$$\begin{aligned}
\int_0^T \|(Ku)(t) - (Ku)(t)\|^p \, dt &= \int_0^T \left\| \int_0^t e^{(t-s)A} F(u_s - v_s) ds \right\|^p dt \\
&\le \int_0^T \sup_{0 \le s \le T} \|e^{(t-s)A}\|^p t^{p-1} \int_0^t \|F(u_s - v_s)\|^p ds dt \\
&\le L e^{p\omega T} \|H\|_{L^q}^p \int_0^T t^p \int_0^t \|(u(s) - v(s))\|^p ds dt \\
&\le \left(L e^{p\omega T} \|H\|_{L^q}^p \int_0^T t^p dt \right) \int_0^T \|(u(s) - v(s))\|^p ds.
\end{aligned} \tag{5.4}$$

Let $T > 0$ be fixed such that $Le^{\omega T}\|H\|_{L^q} \left(\int_0^T t^p dt\right)^{\frac{1}{p}} < 1$. We infer from (5.4) that K is a contraction and therefore there exists a unique solution $u \in L^p([0,T],X)$ of (5.3). In particular $u \in C([0,T],X)$.

To continuously extend the function u to the interval $[T,2T]$ satisfying (5.3), we consider the operator $K_2 : L^p([T,2T],X) \to L^p([T,2T],X)$ defined by $K_2u(t) = e^{A(t-T)}u(T^-) + \int_T^t e^{A(t-s)}Fu_s ds$, where $u(T^-) = \lim_{t \to T^-} u(t)$. Similarly as in (5.4) we obtain that K_2 is a contraction with constant $Le^{\omega T}\|H\|_{L^q} \left(\int_0^T t^p dt\right)^{\frac{1}{p}}$. Hence, the function u can be continuously extended to the interval $[T,2T]$. Now we observe that, for $t \in (T,2T]$

$$\begin{aligned}
u(t) &= e^{A(t-T)}u(T^-) + \int_T^t e^{A(t-s)} F u_s ds \\
&= e^{A(t-T)} \left(e^{AT}x + \int_0^T e^{A(T-s)} F u_s ds \right) + \int_T^t e^{A(t-s)} F u_s ds \\
&= e^{At}x + \int_0^t e^{A(t-s)} F u_s ds,
\end{aligned}$$

showing that (5.3) is satisfied. Inductively, we can continuously extend the function u defined on $[nT,(n+1)T]$, $n \in \mathbb{N}$ to the interval $[(n+1)T,(n+2)T]$ satisfying (5.3). Is not difficult to see that such solution u is unique.

Now we will show that u is exponentially bounded. By (5.3) we have that

$$\begin{aligned}
\|u(t)\| &\le e^{t\omega}\|x\| + \int_0^t e^{(t-s)\omega} \int_{-r}^0 \|H(\theta)\| \|u(s+\theta)\| d\theta ds \\
&\le e^{t\omega}\|x\| + \left(\int_{-r}^0 \|H(\theta)\| d\theta \right) \int_0^t e^{(t-s)\omega} \sup_{-r+s \le \tau \le s} \|u(\tau)\| ds \\
&\le e^{t\omega}\|x\| + M_r \int_0^t e^{(t-s)\omega} \sup_{0 \le \tau \le s} \|u(\tau)\| ds,
\end{aligned}$$

where $M_r = \int_{-r}^0 \|H(\theta)\| d\theta$, which immediately implies

$$e^{-t\omega} \sup_{0 \le \tau \le t} \|u(\tau)\| \le \|x\| + M_r \int_0^t e^{-s\omega} \sup_{0 \le \tau \le s} \|u(\tau)\| ds.$$

We now apply Gronwall's inequality to obtain $e^{-t\omega}\sup_{0\le\tau\le t}\|u(\tau)\| \le \|x\|e^{tM_r}$ which implies

$$\|u(t)\| \le \sup_{0\le\tau\le t}\|u(\tau)\| \le e^{t(\omega+M_r)}\|x\|. \tag{5.5}$$

□

In view of the above, we can define the family $\{G(t)\}_{t\ge0}$ by $G(t)x = u_x(t)$ where u_x is the unique continuous solution of (5.3). We note that $\{G(t)\}_{t\ge0}$ is a strongly continuous family of bounded linear operators and, by (5.5), the family is exponentially bounded. The family $\{G(t)\}_{t\ge0}$ is defined by the null operator for all $t\in[-r,0)$ and satisfies that

$$G(t)x = e^{tA}x + \int_0^t e^{(t-s)A}FG_s x ds, \quad t\ge 0,\ x\in X. \tag{5.6}$$

The next result show that $\{G(t)\}_{t\ge0}$ satisfies the functional equation (3.3).

Theorem 5.2. *The family $\{G(t)\}_{t\ge0}$, defined above, is a resolvent family with delay F.*

Proof. Let $x\in X$ and $\lambda > \omega+M_r$ be fixed. Applying the Laplace transform in both sides of the equation (5.6) we have

$$\hat G(\lambda)x = (\lambda-A)^{-1}x + (\lambda-A)^{-1}B_\lambda\hat G(\lambda)x,$$

which is equivalent to

$$(\lambda-A)(I-(\lambda-A)^{-1}B_\lambda)\hat G(\lambda)x = x.$$

We observe that

$$\|(\lambda-A)^{-1}B_\lambda x\| \le \frac{\|B_\lambda x\|}{\Re(\lambda)-\omega} \le \frac{M_r\|x\|}{\Re(\lambda)-\omega},$$

thus $\|(\lambda-A)^{-1}B_\lambda\| < 1$ for all $\Re(\lambda) > \omega+M_r$. With this, for $\Re(\lambda) > \omega+M_r$ we conclude that the operator $(\lambda-A)(I-(\lambda-A)^{-1}B_\lambda) = (\lambda-A-B_\lambda)$ has bounded inverse and

$$\hat G(\lambda)x = R(\lambda,A,F)x.$$

Since $\hat G(\mu)x\in D(A)$, for all $x\in X$, we have

$$\begin{aligned}\hat G(\lambda)x - \hat G(\mu)x &= \hat G(\lambda)(\mu-A-B_\mu)\hat G(\mu)x - \hat G(\lambda)(\lambda-A-B_\lambda)\hat G(\mu)x\\ &= (\mu-\lambda)\hat G(\lambda)\hat G(\mu)x + \hat G(\lambda)(B_\lambda-B_\mu)\hat G(\mu)x,\end{aligned}$$

which is equivalent to

$$\frac{\hat G(\lambda)x-\hat G(\mu)x}{\mu-\lambda} = \hat G(\lambda)\hat G(\mu)x + \hat G(\lambda)\left(\frac{B_\lambda-B_\mu}{\mu-\lambda}\right)\hat G(\mu)x.$$

In view of the equation (3.10), and by the uniqueness of the inversion of the double Laplace transform, we obtain that the functional equation (3.3) is satisfied and thus $\{G(t)\}_{t\ge0}$ is a resolvent family with delay F. □

The following result provides a remarkable sufficient condition.

Theorem 5.3. *If A is the generator of a C_0-semigroup defined on a Banach space X and F is defined as in* (5.1), *then the problem*

$$\begin{cases} u'(t) = Au(t) + Fu_t, & t\ge 0;\\ u(0) = x; & \\ u(t) = \phi(t), & -r\le t<0\end{cases},$$

is mildly well-posed.

Proof. By Theorem 4.4 and Proposition 4.3 the problem

$$\begin{cases} u'(t) = Bu(t) + Fu_t, & t \geq 0; \\ u(0) = x; \\ u(t) = \phi(t), & -r \leq t < 0 \end{cases},$$

is mildly well-posed, where

$$D(B) := \left\{ x \in X : \lim_{t \to 0^+} \frac{G(t)x - x}{t} \; exists \right\}$$

and

$$Bx := \lim_{t \to 0^+} \frac{G(t)x - x}{t}, \quad x \in D(B).$$

It follows from (5.6) that $\lim_{t \to 0^+} \frac{e^{tA}x - x}{t}$ exists if and only if $\lim_{t \to 0^+} \frac{G(t)x - x}{t}$ exists. Thus $D(B) = D(A)$ and $Ax = Bx$ for all $x \in D(A)$. This proves the theorem. □

Example 5.4. We set $X = \mathbb{R}$ and let $a \in \mathbb{R}$ be given. For $\phi \in L^p([-1, 0], \mathbb{R})$ and $x_0 \in \mathbb{R}$ we consider the scalar problem

$$\begin{cases} u'(t) = au(t) + u(t-1), & t > 0; \\ u(0) = x; \\ u(t) = \phi(t), & -1 \leq t < 0 \end{cases}. \tag{5.7}$$

We observe that the mild solution of the above problem, for the initial function $\phi(t) \equiv 0$, is represented by the integral equation

$$G(t)x = \begin{cases} e^{at}x + \displaystyle\int_0^t e^{(t-s)a} G(s-1)x ds, & t \geq 0; \\ 0, & -1 \leq t < 0 \end{cases},$$

whose solution is the continuous function

$$G(t)x = \sum_{k=0}^{\infty} \frac{(t-k)^k}{k!} e^{(t-k)a} \mathbb{1}_{[k,\infty)}(t)x, \quad x \in X.$$

It is not difficult to see that $\|G(t)\| \leq e^{t(a+1)}$ for all $t \geq 0$ and identifying Fu_t, as in Remark 4.7, by $Fu_t = u(t-1)$ for all $t \geq 0$, we can conclude the following two assertions: The first is that $\hat{G}(\lambda) = (\lambda - a - e^{-\lambda})^{-1} = (\lambda - A - B_\lambda)^{-1}$ for all $\lambda > a + 1$, which implies that

$$\frac{\hat{G}(\lambda)x - \hat{G}(\mu)x}{\mu - \lambda} = \hat{G}(\lambda)\hat{G}(\mu)x + \hat{G}(\lambda) \left(\frac{B_\lambda - B_\mu}{\mu - \lambda} \right) \hat{G}(\mu)x, \quad x \in X.$$

The second is that

$$Bx := \lim_{t \to 0^+} \frac{G(t)x - x}{t} = ax, \text{ for all } x \in D(B) = \mathbb{R}.$$

In view of the equation (3.10), and by the inversion of the double Laplace transform, we obtain that the functional equation (3.3) is satisfied and thus $\{G(t)\}_{t \geq 0}$ is a resolvent family with delay F. Also by Proposition 4.3 we obtain that B is the corresponding generator.

It follows from Theorem 4.4 that under the above described conditions the problem (5.7) is mildly well-posed and it follows from Proposition 4.5 that the solution of (5.7) is given by $u(t) = G(t)x + (G * \mathfrak{F}\phi_{\uparrow})(t)$, $t \geq 0$.

Example 5.5. We consider the one dimensional diffusion equation with finite delay

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + u(t-1,x), & t \geq 0, \quad x \in [0,\pi]; \\ u(t,0) = u(t,\pi) = 0, & t \geq 0; \\ u(0,x) = u_0(x), & x \in [0,\pi]; \\ u(t,x) = \phi(t,x), & -1 \leq t < 0, \quad x \in [0,\pi] \end{cases} . \tag{5.8}$$

where, $u_0 \in L^2([0,\pi],\mathbb{R})$ and $\phi \in L^p([-1,0], L^2([0,\pi],\mathbb{R}))$. To study this system in an abstract setting as (1.1), we choose the space $X = L^2([0,\pi],\mathbb{R})$, $x = u_0(\cdot) \in X$, $\phi(t) = \phi(t,\cdot)$ for all $-r \leq t < 0$ and the operator $A : D(A) \subseteq X \to X$ given by $Ax = x''$ with domain

$$D(A) = \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}.$$

It is well known that A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on X.

Similarly to Example 5.4 the family $\{G(t)\}_{t\geq 0}$ defined on X by

$$G(t)x = \sum_{k=0}^{\infty} T(t-k)\frac{(t-k)^k}{k!}\mathbb{1}_{[k,\infty)}(t)x, \text{ for all } t \geq 0,$$

is a continuous solution of the integral equation

$$G(t)x_0 = \begin{cases} e^{at}x + \displaystyle\int_0^t e^{(t-s)a}G(s-1)xds, & t \geq 0; \\ 0, & -1 \leq t < 0 \end{cases} , \tag{5.9}$$

Let $L, \omega > 0$ be constants such that $\|T(t)\| \leq Le^{t\omega}$ for all $t \geq 0$. We observe that $\|G(t)\| \leq Le^{t(\omega+1)}$. If $x \in X$ and $\lambda > \omega + 1$. Then, applying the Laplace transform in both sides of equation (5.9) and noting that $B_\lambda x = e^{-\lambda}x$ for all $x \in X$, we obtain

$$\hat{G}(\lambda)x = (\lambda - A)^{-1}x + (\lambda - A)^{-1}B_\lambda \hat{G}(\lambda)x,$$

which is equivalent to

$$(\lambda - A)(I - (\lambda - A)^{-1}B_\lambda)\hat{G}(\lambda)x = x.$$

We observe that

$$\|(\lambda - A)^{-1}B_\lambda x\| \leq \frac{\|B_\lambda x\|}{\Re(\lambda) - \omega} \leq \frac{\|x\|}{\Re(\lambda) - \omega},$$

thus $\|(\lambda - A)^{-1}B_\lambda\| < 1$ for all $\Re(\lambda) > \omega + 1$. With this, for $\Re(\lambda) > \omega + 1$ we conclude that the operator $(\lambda - A)(I - (\lambda - A)^{-1}B_\lambda) = (\lambda - A - B_\lambda)$ has bounded inverse and

$$\hat{G}(\lambda)x = R(\lambda, A, F)x.$$

Since $\hat{G}(\mu)x \in D(A)$, for all $x \in X$, we have

$$\begin{aligned} \hat{G}(\lambda)x - \hat{G}(\mu)x &= \hat{G}(\lambda)(\mu - A - B_\mu)\hat{G}(\mu)x - \hat{G}(\lambda)(\lambda - A - B_\lambda)\hat{G}(\mu)x \\ &= (\mu - \lambda)\hat{G}(\lambda)\hat{G}(\mu)x + \hat{G}(\lambda)(B_\lambda - B_\mu)\hat{G}(\mu)x, \end{aligned}$$

which is equivalent

$$\frac{\hat{G}(\lambda)x - \hat{G}(\mu)x}{\mu - \lambda} = \hat{G}(\lambda)\hat{G}(\mu)x + \hat{G}(\lambda)\left(\frac{B_\lambda - B_\mu}{\mu - \lambda}\right)\hat{G}(\mu)x.$$

In view of the equation (3.10), and again by the inversion of the double Laplace transform, we obtain that the functional equation (3.3) is satisfied and thus $\{G(t)\}_{t\geq 0}$ is a resolvent family with delay F. Furthermore, we observe that

$$Bx := \lim_{t\to 0^+} \frac{G(t)x - x}{t} = \lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax, \quad \text{for all } x \in D(B) = D(A).$$

Thus, A is the generator of the resolvent family $\{G(t)\}_{t\geq 0}$ with delay F.

It follows from Theorem 4.4 and Proposition 4.5 that the problem (5.8) is mildly well-posed and that the solution of (5.7) is given by $u(t) = G(t)x + (G * \mathfrak{F}\phi_{\uparrow})(t)$, where $x = u_0$ and $\phi(t) = \phi(t, \cdot)$ for all $-r \leq t < 0$.

References

[1] L. Abadias, C. Lizama and P.J. Miana. *Sharp extensions and algebraic properties for solution families of vector-valued differential equations.* Banach J. Math. Anal. 10 (1) (2016), 169–208.

[2] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* Second edition. Monographs in Mathematics, 96. Birkhuser/Springer Basel AG, Basel, 2011.

[3] A. Ashyralyev and D. Agirseven. *Well-posedness of delay parabolic equations with unbounded operators acting on delay terms.* Bound. Value Probl. 126 (2014) 1–15.

[4] A. Bátkai and S. Piazzera. *Semigroups for Delay Equations.* Research Notes in Mathematics, 10. A K Peters, Ltd., Wellesley, MA, 2005. xii+259 pp.

[5] O. Diekmann and M. Gyllenberg. *Equations with infinite delay: blending the abstract and the concrete.* J. Differential Equations. 252 (2) (2012), 819–851.

[6] K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate texts in Mathematics, 194, Springer, New York, 2000.

[7] W.F. Fitzgibbon. *Stability for abstract nonlinear Volterra equations involving finite delay.* J. Math. Anal. Appl. 60 (2) (1977), 429–434.

[8] J.K. Hale. *Functional Differential Equations*, Springer-Verlag, New York, 1971.

[9] E. Hille and R. S. Phillips. *Functional Analysis and Semi-groups.* Amer. Math. Soc. Coll. Publ., vol. 31. Amer. Math. Soc., Providence, R. I., 1957.

[10] W. Jiang, F. Guo and F. Huang. *Well-posedness of linear partial differential equations with unbounded delay operators.* J. Math. Anal. Appl. 293 (1) (2004), 310–328.

[11] K. Liu, *Retarded stationary Ornstein–Uhlenbeck processes driven by Lévy noise and operator selfdecomposability.* Potential Anal. 33 (3) (2010), 291–312.

[12] K. Liu. *On regularity property of retarded Ornstein-Uhlenbeck processes in Hilbert spaces.* J. Theoret. Probab. 25 (2) (2012), 565–593.

[13] K. Liu, L. Hu, J. Luo. *Stability property and essential spectrum of linear retarded functional differential equations.* J. Comput. Appl. Math. 244 (2013), 19–35.

[14] K. Liu, *On stationarity of stochastic retarded linear equations with unbounded drift operators.* Stoch. Anal. Appl. 34 (4) (2016), 547–572.

[15] C. Lizama and F. Poblete. *On a functional equation associated with (a, k)-regularized resolvent families.* Abstr. Appl. Anal., Art. ID 495487, 23, 2012.

[16] H. Petzeltová. *Solution semigroup and invariant manifolds for functional equations with infinite delay.* Math. Bohem. 118 (2) (1993), 175-193.

[17] H. Petzeltová and J. Milota. *Resolvent operator for abstract functional-differential equations with infinite delay.* Numer. Funct. Anal. Optim. 9 (7-8) (1987), 779–807.

[18] C.C. Travis and G.F. Webb. *Existence and stability for partial functional differential equations.* Trans. Amer. Math. Soc. 200 (1974), 395–418.

[19] C.C. Travis and G.F. Webb. *Existence, stability, and compactness in the α-norm for partial functional differential equations.* Trans. Amer. Math. Soc. 240 (1978), 129–143.

[20] G. Webb. *Functional differential equations and nonlinear semigroups in Lp-spaces,* J. Differential Equations 29 (1976), 71–89.

Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, CHILE
E-mail address: carlos.lizama@usach.cl

Universidad Austral de Chile, Facultad de Ciencias, Instituto de Ciencias Físicas y Matemáticas, Valdivia, CHILE.
E-mail address: felipe.poblete@uach.cl