THE POISSON DISTRIBUTION, ABSTRACT FRACTIONAL DIFFERENCE EQUATIONS, AND STABILITY

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Abstract. We study the initial value problem

$$(*)\begin{cases} {}_C\Delta^{\alpha}u(n) &= Au(n+1), \quad n\in\mathbb{N}_0;\\ u(0) &= u_0\in X.\end{cases}$$

when A is a closed linear operator with domain $D(A)$ defined on a Banach space X. We introduce a method based on the Poisson distribution to show existence and qualitative properties of solutions for the problem (*), using operator-theoretical conditions on A. We show how several properties for fractional differences, including their own definition, are connected with the continuous case by means of sampling using the Poisson distribution. We prove necessary conditions for stability of solutions, that are only based on the spectral properties of the operator A in case of Hilbert spaces.

1. Introduction

In the past decade, the study of existence and qualitative properties of discrete solutions for fractional difference equations has drawn a great deal of interest. To mention a few, see [1, 2, 4, 5, 23, 25, 26, 31, 40, 44].

First studies on time differences of fractional order are due to Kutter [33]. Diaz and Osler [19] introduced in 1974 a discrete fractional difference operator defined as an infinite series. Grey and Zhang [27] developed a fractional calculus for the discrete nabla (backward) difference operator. At the same time, Miller and Ross [38] defined a fractional sum via the solution of a linear difference equation. More recently, Atici and Eloe [4] introduced the Riemann-Liouville like fractional difference by using the definition of fractional sum of Miller and Ross, and developed some of its properties that allow to obtain solutions of certain fractional difference equations. Ferreira [21, 22] introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. See also Sengül [42] for related work. Holm [29, 30] further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations. See also the recent paper [12] for complementary work. Concerning qualitative properties, Goodrich [24] in a series of papers studied existence of positive solutions and geometrical properties. Applications to concrete models have been analyzed by Atici and Sengül in [7].

In spite of the significant increase of research in this area, there are still many open questions regarding fractional difference equations. In particular, the study of fractional difference equations with unbounded linear operators and their stability properties remains an open problem. These abstract fractional models, with unbounded operators, are closely connected with numerical methods for integro-differential equations [16, 41] and evolution equations with memory [35]. The theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears [7, 11]. In this paper, we propose a novel method to deal with this problem based on the sampling of fractional differential equations by means of the Poisson distribution. We will use it to prove the existence of a unique solution to the initial value problem

$$\begin{cases} {}_C\Delta^{\alpha}u(n) &= Au(n+1), \quad n\in\mathbb{N}_0;\\ u(0) &= u_0\in X.\end{cases} \tag{1.1}$$

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where A is a closed linear operator with domain $D(A)$ defined on a Banach space X. Here we use a particular choice of the notion of fractional differences (in the sense of Caputo) and we assume $0 < \alpha \leq 1$.

Mathematical understanding of the linear equation (1.1) is meant as a preliminary critical step for the subsequent analysis of full nonlinear models. The approach followed here is purely operator-theoretic and has as main ingredient the use of the Poisson distribution:

$$p_n(t) = e^{-t}\frac{t^n}{n!}, \quad n \in \mathbb{N}_0, \quad t \geq 0.$$

The method relies in to take advantage of the properties of this distribution when it is applied to continuous phenomena. More precisely, given a continuous evolution $u(t)$, $t \in [0, \infty)$ we can discretize it by means of that we will call the *Poisson transformation*

$$u(n) := \int_0^\infty p_n(t)u(t)dt, \quad n \in \mathbb{N}_0. \tag{1.2}$$

In this paper, we will show that when this procedure is applied to fractional models defined on the time scale $\mathbb{R}_+$, these transformations are well behaved and fit perfectly in the discrete fractional concepts. In other words, our approach is as follows: Suppose that a solution of the fractional Cauchy problem

$$D^\alpha u(t) = Au(t), \quad t \geq 0, \quad 0 < \alpha \leq 1,$$

exists. It happens, for instance, if A is the generator of a C_0-semigroup or A is sectorial, see [10, Sections 2.1 and 2.2], [32] and references therein. Then, by sampling each side of the above equation by means of the Poisson distribution, we obtain that $u(n)$ defined by (1.2) is a solution of

$$\Delta^\alpha u(n) = Au(n+1), \quad n \in \mathbb{N},$$

where D^α and Δ^α denote the fractional operators on $\mathbb{R}_+$ and $\mathbb{N}_0$, respectively, in the sense of Riemann-Liouville. It is remarkable that by this mechanism we recover the concept of fractional nabla sum and difference operator introduced by Atici and Eloe [6], which has been used recently and independently of the method used here by other authors in order to obtain several qualitative properties of fractional difference equations, notably concerning stability properties [13, 14]. We take advantage of this important connection to derive several sufficient conditions for stability in case of unbounded operators A. Among others, in this paper we prove the following practical criteria in Hilbert spaces: Let A be the generator of a C_0-semigroup on a Hilbert space H such that $\{\mu \in \mathbb{C} : Re(\mu) > 0\} \subset \rho(A)$ and satisfies

$$\sup_{Re(\mu)>0} \|(\mu - A)^{-1}\| < \infty,$$

then, the solution of the fractional difference equation of order $\alpha \in (0, 1)$

$${}_C\Delta^\alpha u(n) = Au(n+1), \quad n \in \mathbb{N},$$

exists and is stable for all initial conditions $u_0 \in H$.

The outline of this paper is as follows: In Section 2, we give some preliminary background in notation and definitions. The remarkable fact is that we use here a particular choice of the definition introduced by Atici and Eloe in [9] for the nabla operator. This choice, that has been used by other authors [13], [14], is proved to be the right notion in the sense that the following notable relation holds

$$\int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt = \Delta^\alpha u(n), \quad n \in \mathbb{N}_0, \tag{1.3}$$

where D_t^α denotes the Riemann-Liouville fractional derivative on $\mathbb{R}_+$ and $u(n)$ is defined by (1.2). Then, we can connect the Delta operator (i.e. the Riemann-Liouville fractional difference) in the right hand side with the Caputo-like fractional difference by means of the identity (Theorem 2.4):

$${}_C\Delta^\alpha u(n) = \Delta^\alpha u(n) - k^{1-\alpha}(n+1)u(0), \quad n \in \mathbb{N}_0.$$

In Section 3, we establish several relations between the continuous and discrete setting, including the identity (1.3). In particular, we show that the continuous and discrete convolution as well as the Laplace and Z-transform are related by means of the Poisson transformation (1.2).

In Section 4, we show how to apply the preceding results to solve the problem (1.1). After recall the notion of α-resolvent family, which is an extension of the concept of C_0-semigroup, we prove that the solution of problem (1.1) can be represented by means of the Poisson transformation of such family, whenever A is a generator (Theorem 4.4). In particular, we derive in the scalar case an explicit representation of the solution (Theorem 4.7). From a different point of view, the obtained representation can be considered as the discrete definition of the continuous Mittag-Leffler function.

In Section 5, we give some necessary conditions for stability of solutions in the sense that

$$\|u(n)\| \to 0 \text{ as } n \to \infty.$$

A practical condition that we find, in general Banach spaces, is the uniform exponential stability of a semigroup generated by A (Theorem 5.2). A more precise condition is provided in Hilbert spaces, as mentioned before (Corollary 5.3).

2. Preliminaries

For a real number a, we denote

$$\mathbb{N}_a := \{a, a+1, a+2, ...\},$$

and we write $\mathbb{N}_1 \equiv \mathbb{N}$. Let X be a complex Banach space. We denote by $s(\mathbb{N}_a; X)$ the vectorial space consisting of all vector-valued sequences $f : \mathbb{N}_a \to X$.
The forward Euler operator $\Delta_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For $m \in \mathbb{N}_2$, we define recursively $\Delta_a^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a,$$

and is called the m-th order forward difference operator. For instance, for any $f \in s(\mathbb{N}_0; X)$, we have

$$\Delta_0^m f(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(n+j), \quad n \in \mathbb{N}_0.$$

In particular, we obtain

$$(\Delta_0^1 f)(n) = f(n+1) - f(n), \quad n \in \mathbb{N}_0.$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is the identity operator, and $\Delta \equiv \Delta_0^1$. We define

$$k^\alpha(j) := \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0. \tag{2.1}$$

The following definition of fractional sum was proposed by Atici and Eloe in 2009.

Definition 2.1 ([6])**.** Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t - s + 1)^{\overline{\alpha - 1}} f(s),$$

where $t \in \mathbb{N}_a$ and $t^{\overline{\alpha}} := \dfrac{\Gamma(t+\alpha)}{\Gamma(t)}$.

In particular, in case $a = 0$ we denote

$$\Delta^{-\alpha} f(n) \equiv \nabla_0^{-\alpha} f(n) = \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)\Gamma(n-k+1)} f(k) = \sum_{k=0}^{n} k^\alpha(n-k) f(k), \quad n \in \mathbb{N}_0. \tag{2.2}$$

One of the reasons to choose this operator is because their flexibility to be handled by means of Z-transform methods. Moreover, it has a better behavior for mathematical analysis when we ask, for example, for definitions of fractional sums and differences on subspaces of $s(\mathbb{N}_0; X)$ like e.g. l_p spaces. We notice that, recently, this approach has been followed by other authors. See [13, 14].

On the other hand, we observe that for numerical treatment of fractional differential equations it is essential to have good approximations of the operator D^α of fractional differentiation. For the derivation of approximations to D^α it is convenient to make use of the discrete operators of translation: Backward and forward Euler difference operators. For applications to causal problems, backward operators are more appropriate. The Grünwald-Letnikov discretization of time-fractional difference equations is based on the backward scheme. For instance, Meerschaert et.al have published a series of papers in which they used the Grünwald-Letnikov difference scheme in approximating the fractional diffusion equation, see [36, 37]. However, in concrete applications, this scheme have some drawbacks [37]. A different numerical method was proposed by Lubich [34]. The basic idea is to combine the classical backward Euler method with a suitable quadrature rule, see [15] and references therein. We observe that both methods requires the use of the kernel (2.1). Indeed, whereas the Grünwald-Letnikov formula requires (2.1) for $\alpha < 0$ (see [37, Formula (3)]), the method of Lubich requires (2.1) for $\alpha > 0$. For a recent discussion on the matter of convergence, see [39, Section 3.3].

The next concept is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [38] and [4]. In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied.

Definition 2.2. [12] The nabla fractional difference operator of order $\alpha > 0$ is defined by

$$\nabla_a^\alpha f(t) = \Delta_a^m (\nabla_a^{-(m-\alpha)} f)(t), \quad t \in \mathbb{N}_a,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

In this paper, we use the case $a = 0$ that we denote Δ^α, that is the fractional difference operator $\Delta^\alpha : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ of order $\alpha > 0$ (in the sense of Riemann-Liouville) is defined by

$$\Delta^\alpha f(n) := \Delta_0^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m, \quad m := \lceil \alpha \rceil$.

Interchanging the order of the operators in the definition of fractional difference in the sense of Riemann-Liouville, and in analogous way as above, we can introduce the notion of fractional difference in the sense of Caputo as follows.

Definition 2.3. Let $\alpha > 0$. The *α-th fractional Caputo like difference* is defined by

$$_C\nabla_a^\alpha f(t) = \nabla_a^{-(m-\alpha)} (\Delta_a^m f)(t), \quad t \in \mathbb{N}_a,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

We use the particular case $a = 0$ i.e., the fractional difference (in the sense of Caputo) of order $\alpha > 0$ defined by

$$_C\Delta^\alpha f(n) := \Delta^{-(m-\alpha)} (\Delta_0^m f)(n), \quad n \in \mathbb{N}_0, \tag{2.3}$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Recall that the finite convolution $*$ of two sequences $f(n)$ and $g(n)$ is defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j) g(j), \quad n \in \mathbb{N}_0.$$

For further use, we note the following relation between the Caputo and Riemann-Liouville fractional differences of order $0 < \alpha < 1$.

Theorem 2.4. *For each* $0 < \alpha < 1$ *and* $f \in s(\mathbb{N}_0; X)$*, we have*

$$_C\Delta^\alpha f(n) = \Delta^\alpha f(n) - k^{1-\alpha}(n+1)f(0), \quad n \in \mathbb{N}_0,$$

where $k^{1-\alpha}$ *is defined in* (2.1).

Proof. By definition and (2.2) we have

$$\begin{aligned}
\Delta^{-(1-\alpha)}(\Delta f)(n) &= \sum_{j=0}^{n} k^{1-\alpha}(n-j)\Delta f(j) \\
&= \sum_{j=0}^{n} k^{1-\alpha}(n-j)f(j+1) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)f(j) \\
&= \sum_{j=1}^{n+1} k^{1-\alpha}(n+1-j)f(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)f(j) \\
&= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)f(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)f(j) - k^{1-\alpha}(n+1)f(0) \\
&= \Delta(\Delta^{-(1-\alpha)}f)(n) - k^{1-\alpha}(n+1)f(0),
\end{aligned}$$

and we obtain the desired result. □

3. A method based on the Poisson distribution

For each $n \in \mathbb{N}_0$, we recall that the Poisson distribution is defined by

$$p_n(t) := e^{-t}\frac{t^n}{n!}, \quad t \geq 0.$$

As expected, $p_n(t) \geq 0$ and

$$\int_0^\infty p_n(t)dt = 1, \quad n \in \mathbb{N}_0.$$

The Poisson distribution arises in connection with Poisson processes. In this section we will realize their application to abstract difference equations. The method itself uses an idea of discretization of the derivative in time used in the paper [17] (see also [16] and references therein).

First, we recall some concepts. Let $S : \mathbb{R}_+ \to \mathcal{B}(X)$ be strongly continuous, that is, for all $x \in X$ the map $t \to S(t)x$ is continuous on $\mathbb{R}_+$. We say that a family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ is exponentially bounded if there exists real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

We say that $\{S(t)\}_{t\geq 0}$ is bounded if $\omega = 0$. Note that if $\{S(t)\}_{t\geq 0}$ is exponentially bounded then, the Laplace transform

$$\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)xdt, \quad x \in X,$$

exists for all $Re(\lambda) > \omega$.

We recall that the Z-*transform* of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\widetilde{f}(z) := \sum_{j=0}^{\infty} z^{-j}f(j)$$

where z is a complex number. Note that convergence of the series is given for $|z| > R$ with R sufficiently large.

An interesting connection between the vector-valued Z-transform and the vector-valued Laplace transform can be given by means of the Poisson distribution. This is the content of the following result.

Theorem 3.1. *Let $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ be bounded. Define*

$$S(n)x := \int_0^\infty p_n(t)S(t)xdt, \quad n \in \mathbb{N}_0, \quad x \in X. \tag{3.1}$$

Then

$$\widetilde{S}(z)x = \hat{S}(1-1/z)x, \quad x \in X,$$

for all z such that $|z| > 1$.

Proof. By hypothesis, there exists $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$. Then, their Laplace transform $\hat{S}(\lambda)$ exists for all $Re(\lambda) > 0$. Moreover, we note that

$$\|S(n)x\| \leq M \int_0^\infty p_n(t)\|x\|dt = M\|x\|, \quad n \in \mathbb{N}_0,$$

and therefore, for all $x \in X$, the Z-transform $\tilde{S}(z)x$ exists for all $|z| > 1$.

Let z be such that $|z| > 1$ and define $\lambda := 1 - \dfrac{1}{z}$. Then $1 - \dfrac{1}{z}$ belongs to the circle $D(1,1) := \{w \in \mathbb{C} : |w-1| < 1\}$. In particular, $Re(1-\frac{1}{z}) > 0$. Hence,

$$\begin{aligned}\widetilde{S}(z)x &= \sum_{n=0}^\infty z^{-n}S(n)x = \sum_{n=0}^\infty z^{-n}\int_0^\infty p_n(t)S(t)xdt = \int_0^\infty e^{-t}\sum_{n=0}^\infty z^{-n}\frac{t^n}{n!}S(t)xdt \\ &= \int_0^\infty e^{-t}e^{t/z}S(t)xdt = \hat{S}(1-1/z)x\end{aligned}$$

for all $x \in X$, proving the theorem. □

Remark 3.2. An analogous result holds in case that $\{S(t)\}_{t\geq 0}$ is replaced by a continuous and bounded function $a : \mathbb{R}_+ \to \mathbb{C}$, obtaining

$$\widetilde{a}(z) = \hat{a}(1-1/z)$$

for all $|z| > 1$, where

$$a(n) := \int_0^\infty p_n(t)a(t)dt, \quad n \in \mathbb{N}_0.$$

Example 3.3. For $\alpha > 0$ define

$$g_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0 \\ 0 & t \leq 0. \end{cases} \tag{3.2}$$

Note the semigroup property:

$$g_{\alpha+\beta} = g_\alpha * g_\beta, \quad \alpha, \beta > 0.$$

We have the following interesting property of sampling

$$g_\alpha(n) := \int_0^\infty p_n(t)g_\alpha(t)dt = \int_0^\infty e^{-t}\frac{t^{n+\alpha+1}}{\Gamma(\alpha)n!}dt = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = k^\alpha(n),$$

for all $n \in \mathbb{N}_0$. By the preceding theorem (more precisely, Remark 3.2) we obtain

$$\widetilde{k}^\alpha(z) = \frac{z^\alpha}{(z-1)^\alpha}$$

for all $|z| > 1$ and

$$k^{\alpha+\beta}(n) = (k^\alpha * k^\beta)(n), \quad \alpha, \beta > 0.$$

In particular, for $\alpha, \beta > 0$ we deduce the identities

$$\Delta^{-\alpha}(\Delta^{-\beta}u)(n) = \Delta^{-(\alpha+\beta)}u(n) = \Delta^{-\beta}(\Delta^{-\alpha}u)(n), \quad \forall n \in \mathbb{N}_0. \tag{3.3}$$

Indeed,

$$\Delta^{-\alpha}(\Delta^{-\beta}u) = \Delta(k^\beta * u) = k^\alpha * (k^\beta * u) = (k^\alpha * k^\beta) * u = k^{\alpha+\beta} * u = \Delta^{\alpha+\beta}u,$$

and interchanging the role of α and β we obtain (3.3). We finally remark that, for $\alpha, \beta > 0$, we get the identity

$$\Delta^{-\alpha} k^{\beta} = k^{\alpha} * k^{\beta} = k^{\alpha+\beta}.$$

The next property connecting the continuous and discrete convolution will be very useful in the treatment of abstract difference equations.

Theorem 3.4. *Let $a : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable such that $\hat{a}(1)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1)$ exists. Then for all $x \in X$,*

$$\int_0^{\infty} p_n(t)(a * S)(t)x dt = \sum_{k=0}^{n} a(n-k)S(k)x, \quad n \in \mathbb{N}_0.$$

Proof. We first observe that by properties of the Laplace transform, for any Laplace transformable function f (scalar or vector-valued) such that $\hat{f}(1)$ exists, we have

$$f(n) := \int_0^{\infty} p_n(t) f(t) dt = \int_0^{\infty} \frac{t^n}{n!} e^{-t} f(t) dt = \frac{(-1)^n}{n!} [\hat{f}(\lambda)]^{(n)}|_{\lambda=1}.$$

See [3, Theorem 1.5.1]. Then, using the Leibniz's rule for the n-th derivative of a product (see [28, formula 0.42]), we obtain

$$\begin{aligned}
\int_0^{\infty} p_n(t)(a * S)(t)x dt &= \frac{(-1)^n}{n!}[\widehat{(a * S)}(\lambda)x]^{(n)}|_{\lambda=1} = \frac{(-1)^n}{n!}[\widehat{a}(\lambda)\widehat{S}(\lambda)x]^{(n)}|_{\lambda=1} \\
&= \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} [\widehat{a}(\lambda)]^{(n-k)} [\widehat{S}(\lambda)x]^{(k)}|_{\lambda=1} \\
&= \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(-1)^{n-k}} a(n-k) \frac{k!}{(-1)^k} S(k)x \\
&= \sum_{k=0}^{n} a(n-k)S(k)x
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}_0$, proving the theorem.

□

In order to establish our next result, we recall that the Riemann-Liouville fractional integral of order $\alpha > 0$, of a locally integrable function $u : [0, \infty) \to X$ is given by:

$$I_t^{\alpha} u(t) := (g_{\alpha} * u)(t) := \int_0^t g_{\alpha}(t-s)u(s)ds,$$

where g_{α} is defined in (3.2). The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a function u is defined by

$$D_t^{\alpha} u(t) := \frac{d}{dt} \int_0^t g_{1-\alpha}(t-s)u(s)ds.$$

The next Theorem establishes a notably and very interesting relation between the discrete and continuous fractional concepts in the sense of Riemann-Liouville, that is achieved by means of sampling with the Poisson distribution.

Theorem 3.5. *Let $u : \mathbb{R}_+ \to X$ be locally integrable and bounded. Then we have*

$$\int_0^{\infty} p_{n+1}(t) D_t^{\alpha} u(t) dt = \Delta^{\alpha} u(n), \quad n \in \mathbb{N}_0,$$

where $u(n)$ is given by (1.2).

Proof. Set $n \in \mathbb{N}_0$ and $0 < \alpha < 1$. Since u is bounded, there exists a constant $M > 0$ such that $\|u(t)\| \le M$ for all $t \ge 0$. Moreover,

$$\|g_{1-\alpha} * u(t)\| \le M \int_0^t g_{1-\alpha}(t-s)ds = M \int_0^t g_{1-\alpha}(\tau)d\tau = M g_{2-\alpha}(t),$$

for all $t \ge 0$. Therefore, $\|p_n(t)(g_{1-\alpha} * u)(t)\| \le e^{-t}\frac{1}{n!}t^{n+1-\alpha}\frac{1}{\Gamma(2-\alpha)}$ and consequently an integration by parts gives

$$\int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt = \int_0^\infty p_{n+1}(t)\frac{d}{dt}(g_{1-\alpha} * u)(t)dt = -\int_0^\infty p'_{n+1}(t)(g_{1-\alpha} * u)(t)dt,$$

for all $n \in \mathbb{N}_0$. Note that

$$-p'_{n+1}(t) = p_{n+1}(t) - p_n(t), \quad n \in \mathbb{N}_0.$$

Then

$$\int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt = \int_0^\infty p_{n+1}(t)(g_{1-\alpha} * u)(t)dt - \int_0^\infty p_n(t)(g_{1-\alpha} * u)(t)dt.$$

By Theorem 3.4, Example 3.3 and Definition we get

$$\begin{aligned}\int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt &= \sum_{k=0}^{n+1} k^{1-\alpha}(n+1-k)u(k) - \sum_{k=0}^{n} k^{1-\alpha}(n-k)u(k)\\ &= (k^{1-\alpha} * u)(n+1) - (k^{1-\alpha} * u)(n) = \Delta^{-(1-\alpha)}u(n+1) - \Delta^{-(1-\alpha)}u(n)\\ &= \Delta \circ \Delta^{-(1-\alpha)}u(n) = \Delta^\alpha u(n),\end{aligned}$$

proving the theorem. □

For further use, we give the following application.

Corollary 3.6. *Let $0 < \alpha < 1$ and $0 < \alpha < \beta$ be given. Then*

$$\Delta^\alpha k^\beta(n) = k^{\beta-\alpha}(n+1), \quad n \in \mathbb{N}_0.$$

Proof. Since $D_t^\alpha g_\beta = g_{\beta-\alpha}$ (see e.g. [10, (1.19), p.11]) we obtain by the preceding theorem

$$\Delta^\alpha k^\beta(n) = \int_0^\infty p_{n+1}(t)D_t^\alpha g_\beta(t)dt = \int_0^\infty p_{n+1}(t)g_{\beta-\alpha}(t)dt = k^{\beta-\alpha}(n+1),$$

for all $n \in \mathbb{N}_0$. □

The following result tell us that, by the method outlined here, several properties of the family $\{S(t)\}_{t\ge0}$ can be inherited by the corresponding operator valued sequence $\{S(n)\}_{n\in\mathbb{N}_0}$ defined by (3.1).

Proposition 3.7. *Let $\{S(t)\}_{t\ge0}$ be a strongly continuous and exponentially bounded family of linear operators defined on a Banach space X.*

(i) *If $\{S(t)\}_{t\ge0}$ is bounded then $\{S(n)\}_{n\in\mathbb{N}_0}$ is bounded.*

(ii) *If there are constants $M > 0$ and $\omega > 0$ such that $\|S(t)\| \le Me^{-\omega t}$ for all $t \ge 0$ then $\|S(n)\| \le \dfrac{M}{(1+\omega)^{n+1}}$ for all $n \in \mathbb{N}_0$.*

(iii) *Let X be a Banach lattice. If $S(t)x \ge 0$ for all $x \ge 0$ and $t \ge 0$ then $S(n)x \ge 0$ for all $x \ge 0$ and $n \in \mathbb{N}_0$.*

Proof. (i) Let $M > 0$ such that $\|S(t) \le M$ for all $t \ge 0$. Then

$$\|S(n)\| \le \int_0^\infty p_n(t)\|S(t)\|dt \le M\int_0^\infty p_n(t)dt = M,$$

for all $n \in \mathbb{N}_0$.
(ii) We have

$$\|S(n)\| \leq M \int_0^\infty p_n(t) e^{-\omega t} dt = M \int_0^\infty \frac{t^n}{n!} e^{-(1+\omega)t} dt = \frac{M}{(1+\omega)^{n+1}},$$

for all $n \in \mathbb{N}_0$.
(iii) Is clear from the definitions involved. □

4. Linear fractional difference equations on Banach spaces

Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. In this section we study the problem

$$\begin{cases} {}_C\Delta^\alpha u(n) &= Au(n+1), \quad n \in \mathbb{N}_0; \\ u(0) &= u_0 \in X. \end{cases} \tag{4.1}$$

First studies on the model (4.1) when A is a complex or real valued matrix, have only recently appeared [18, 8]. However, to the best of our knowledge, the unbounded case, i.e. when A is merely a closed linear operator, has not been studied in the literature.

We introduce the following notion of solution.

Definition 4.1. We say that a vector valued sequence $u \in s(\mathbb{N}_0; X)$ is a solution for (4.1) if $u(n) \in D(A)$ for all $n \in \mathbb{N}$ and $u(n)$ satisfies (4.1).

We recall the following concept (see [10], [32] and references therein).

Definition 4.2. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, the resolvent set of A, and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

By the uniqueness theorem for the Laplace transform, a 1-resolvent family is the same as a C_0-semigroup, while a 2-resolvent family corresponds to a cosine family. See e.g. [3] and the references therein for an overview on these concepts. A systematic study in the fractional case is carried out in [10].

Some properties of $(S_\alpha(t))$ are included in the following lemma. For a proof, see [32].

Lemma 4.3. *The following properties hold:*

(i) $S_\alpha(0) = I$.
(ii) $S_\alpha(t)D(A) \subset D(A)$ *and* $AS_\alpha(t)x = S_\alpha(t)Ax$ *for all* $x \in D(A)$, $t \geq 0$.
(iii) *For all* $x \in D(A) : S_\alpha(t)x = x + \int_0^t g_\alpha(t-s) A S_\alpha(s) x ds$, $t \geq 0$.
(iv) *For all* $x \in X : (g_\alpha * S_\alpha)(t)x \in D(A)$ *and*

$$S_\alpha(t)x = x + A \int_0^t g_\alpha(t-s) S_\alpha(s) x ds, \ t \geq 0.$$

Our main result in this section is the following theorem.

Theorem 4.4. *Suppose that A is the generator of a bounded α-resolvent family $S_\alpha(t)$ on X. Then the fractional difference equation of order $\alpha \in (0,1)$*

$${}_C\Delta^\alpha u(n) = Au(n+1), \quad n \in \mathbb{N}_0 \tag{4.2}$$

with initial condition $u(0) = u_0 \in D(A)$ admits the solution

$$u(n) = S_\alpha(n)(I-A)u_0 := \int_0^\infty p_n(t) S_\alpha(t)(I-A)u_0 dt, \quad n \in \mathbb{N}_0.$$

Proof. Fix $x \in X$. We first show that $S_\alpha(n)x \in D(A)$ for all $n \in \mathbb{N}_0$. Indeed, for all $n \in \mathbb{N}_0$ we have by [3, Theorem 1.5.1]

$$S_\alpha(n)x = \frac{(-1)^n}{n!}[\hat{S}_\alpha(\lambda)]^{(n)}x|_{\lambda=1},$$

where $\hat{S}_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$. Denote $a(\lambda) = \lambda^{\alpha-1}$, $b(\lambda) = \lambda^\alpha$ and $R(\lambda) = (\lambda - A)^{-1}$. Then $\hat{S}_\alpha(\lambda) = a(\lambda)R(b(\lambda))$. By the Leibniz's rule for the n-th derivative of a product, we obtain

$$[\hat{S}_\alpha(\lambda)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a(\lambda)^{(n-k)}[(R \circ b)(\lambda)]^{(k)}.$$

By the rule for the n-th derivative of a composite function (see e.g. [28, Formula 0.430]), we get

$$[(R \circ b)(\lambda)]^{(k)} = \sum_{j=1}^{k} \frac{U_j(\lambda)}{j!}[R(\lambda)]^{(j)},$$

where

$$U_j(\lambda) = [b(\lambda)^j]^{(n)} - \frac{j!}{1!}b(\lambda)[b(\lambda)^{j-1}]^{(n)} + \frac{j(j-1)}{2!}b(\lambda)^2[b(\lambda)^{j-2}]^{(n)} - ... + (-1)^{j-1}jb(\lambda)^{j-1}[b(\lambda)]^{(n)}.$$

Note that for all $m \in \mathbb{N}_0$, we have

$$[R(\lambda)]^{(m)}x = (\lambda - A)^{-(m+1)}x.$$

Therefore $[R(\lambda)]^{(j)}|_{\lambda=1}x = (I - A)^{-(j+1)}x \in D(A)$ and consequently $[(R \circ b)(\lambda)]^{(k)}|_{\lambda=1}x \in D(A)$. It follows that $[\hat{S}_\alpha(\lambda)]^{(n)}|_{\lambda=1}x \in D(A)$. In particular, this shows that $S_\alpha(n)x \in D(A)$ proving the claim.

Next, from the identity

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds, \quad t \geq 0,$$

valid for all $x \in X$, we obtain

$$p_n(t)S_\alpha(t)x = p_n(t)x + Ap_n(t)(g_\alpha * S_\alpha)(t)x, \quad t \geq 0, \quad n \in \mathbb{N}_0.$$

Therefore,

$$\begin{aligned} S_\alpha(n)x := \int_0^\infty p_n(t)S_\alpha(t)xdt &= \int_0^\infty p_n(t)xdt + \int_0^\infty p_n(t)A(g_\alpha * S_\alpha)(t)xdt \\ &= x + \sum_{k=0}^{n} k^\alpha(n-k)AS_\alpha(k)x = x + A(k^\alpha * S_\alpha)(n)x, \quad n \in \mathbb{N}_0. \end{aligned}$$

Note that $k^1(n) = 1$ for all $n \in \mathbb{N}_0$. Hence, convolving the above identity by $k^{1-\alpha}$, we obtain

$$k^{1-\alpha} * S_\alpha(n)x = k^{1-\alpha} * k^1(n)x + A(k^{1-\alpha} * k^\alpha * S_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

Using the semigroup property for the kernels k^α we have

$$k^{1-\alpha} * S_\alpha(n)x = k^{2-\alpha}(n)x + A(k^1 * S_\alpha)(n)x, \quad n \in \mathbb{N}_0.$$

This is equivalent, by definition, to the following identity

$$\Delta^{-(1-\alpha)}S_\alpha(n)x = k^{2-\alpha}(n)x + A\sum_{k=0}^{n} S_\alpha(k)x, \quad n \in \mathbb{N}_0.$$

Therefore, by Corollary 3.6 and definition of Δ we get

$$\Delta \circ \Delta^{-(1-\alpha)}S_\alpha(n)x = \Delta k^{2-\alpha}(n)x + A[\sum_{k=0}^{n+1} S_\alpha(k)x - \sum_{k=0}^{n} S_\alpha(k)x] = k^{1-\alpha}(n+1)x + AS_\alpha(n+1)x,$$

for all $n \in \mathbb{N}_0$. We note that the left hand side in the above identity corresponds to the fractional difference of order $\alpha \in (0,1)$ in the sense of Riemann-Liouville. Therefore, using Theorem 2.4, we obtain

$$_C\Delta^\alpha S_\alpha(n)x = AS_\alpha(n+1)x,$$

for all $n \in \mathbb{N}_0$. Define $u(n) := S_\alpha(n)(I-A)u_0$ for all $n \in \mathbb{N}$. It then follows that $u(n) \in D(A)$ for all $n \in \mathbb{N}$ and $u(n)$ solves (4.2). Finally, from the identity $S_\alpha(0)x = x + A(k^\alpha * S_\alpha)(0) = x + Ak^\alpha(0)S_\alpha(0) = x + AS_\alpha(0)x$ we obtain $u(0) = S_\alpha(0)(I-A)u_0 = u_0$ proving the theorem. □

In case $\alpha = 1$ we obtain the following consequence. Observe that boundedness of a C_0-semigroup automatically implies that $1 \in \rho(A)$.

Corollary 4.5. *Let A be the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Then the solution of*

$$\Delta u(n) = Au(n+1), \quad n \in \mathbb{N}$$

with initial condition $u(0) = u_0 \in X$ is given by

$$u(n) = (I-A)^{-n}u_0, \quad n \in \mathbb{N}.$$

In case $A = \lambda I$ where $\lambda \in \mathbb{C}$ we recover a well known result.

Corollary 4.6. *Let $\lambda \neq 1$. Then the solution of*

$$\Delta u(n) = \lambda u(n+1), \quad n \in \mathbb{N}$$

with initial condition $u(0) = u_0 \in X$ exists and is given by

$$u(n) = (1-\lambda)^{-n}u_0, \quad n \in \mathbb{N}.$$

We also note that in the scalar case we have explicit representation for $S_\alpha(n)$. To see this, first recall that the Mittag-Leffler function, defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

is an entire function for any $\alpha > 0$, because the series converges for all values of the argument z. It is well known that

$$\int_0^\infty e^{-\mu t} E_\alpha(\lambda t^\alpha)dt = \mu^{\alpha-1}(\mu^\alpha - \lambda)^{-1},$$

for all $Re(\mu) > |\lambda|^{1/\alpha}$ (see e.g. [32] and references therein). In particular, given $\lambda \in \mathbb{C}$, it follows from Definition 4.2 that

$$S_\alpha(t) = E_\alpha(\lambda t^\alpha), \quad t \geq 0,$$

is the α-resolvent family generated by $A = \lambda I$.

Theorem 4.7. *For all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ we have*

$$S_\alpha(n) = \sum_{k=0}^{\infty} \lambda^k \frac{\Gamma(\alpha k + n + 1)}{\Gamma(n+1)\Gamma(\alpha k + 1)}, \quad n \in \mathbb{N}_0. \tag{4.3}$$

Proof. Note that we can write

$$\frac{\Gamma(\alpha k + 1 + n)}{\Gamma(n+1)\Gamma(\alpha k + 1)} = k^{\alpha k + 1}(n),$$

where $k^{\alpha k+1}$ is defined in (2.1). Since, by [45, Vol. I, (3.11)(2)] or [28, Formula 8.328(2)], we have

$$k^\beta(n) = \frac{n^{\beta-1}}{\Gamma(\beta)}\Big(1 + O(\frac{1}{n})\Big), \quad n \in \mathbb{N}, \quad \beta > 0,$$

we conclude that the series in the right hand side of (4.3) is convergent for $|\lambda| < 1$. Finally, we obtain

$$S_\alpha(n) := \int_0^\infty e^{-t}\frac{t^n}{n!}E_\alpha(\lambda t^\alpha)dt = \int_0^\infty e^{-t}\frac{t^n}{n!}\sum_{k0}^\infty \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+1)}dt$$
$$= \sum_{k=0}^\infty \frac{\lambda^k}{\Gamma(\alpha k+1)n!}\int_0^\infty e^{-t}t^{\alpha k+n}dt = \sum_{k=0}^\infty \lambda^k \frac{\Gamma(\alpha k+n+1)}{\Gamma(\alpha k+1)\Gamma(n+1)},$$

for all $n \in \mathbb{N}_0$. □

For example, in case $\alpha = 1$ we deduce from Corollary 4.6 that for $|\lambda| < 1$,

$$\sum_{k=0}^\infty \lambda^k \frac{(k+n)!}{k!n!} = \frac{1}{(1-\lambda)^{n+1}}, \quad n \in \mathbb{N}_0.$$

In passing, we observe that such identity seems not to be well known in the literature.

Corollary 4.8. *For any $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $0 < \alpha \leq 1$ the solution of the equation*

$${}_C\Delta^\alpha u(n) = \lambda u(n+1)$$

is given by

$$u(n) = \sum_{k=0}^\infty \lambda^k \frac{\Gamma(\alpha k+n)}{\Gamma(n)\Gamma(\alpha k+1)}u(0), \quad n \in \mathbb{N}. \tag{4.4}$$

Remark 4.9. By the definition of $k^\alpha(n)$ we can rewrite (4.4) as follows

$$u(n) = \sum_{k=0}^\infty \lambda^k k^{\alpha k+1}(n-1)u(0), \quad n \in \mathbb{N}, \quad |\lambda| < 1.$$

5. Stability properties

As an application of the theorems and corollaries in the preceding section, we will give necessary conditions for stability of solutions. Recall that a vector-valued sequence $u \in S(\mathbb{N}_0, X)$ is said to be *stable* if

$$\|u(n)\| \to 0 \text{ as } n \to \infty.$$

We begin with the following criteria.

Theorem 5.1. *Let $0 < \alpha < 1$. Suppose that A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on X and that there exist constants $M > 0$ and $0 < \gamma < 1$ such that*

$$\|S_\alpha(t)\| \leq \frac{M}{t^\gamma},$$

for all $t > 0$. Then the solution of the fractional difference equation of order $\alpha \in (0,1)$

$${}_C\Delta^\alpha u(n) = Au(n+1), \quad n \in \mathbb{N}, \tag{5.1}$$

is stable for all initial condition $u(0) = u_0 \in D(A)$.

Proof. For all $n \in \mathbb{N}$, we have by Theorem 4.4, hypothesis and definition of Gamma function,

$$\|u(n)\| \leq M\|(I-A)u_0\| \int_0^\infty p_n(t)\frac{1}{t^\gamma}dt = MC\int_0^\infty e^{-t}\frac{t^{n-\gamma}}{n!}dt = MC\frac{\Gamma(n+(1-\gamma))}{n\Gamma(n)}$$
$$= MC\frac{n^{-(1-\gamma)}\Gamma(n+(1-\gamma)))}{\Gamma(n)}\frac{1}{n}\frac{1}{n^{-(1-\gamma)}},$$

and using the fact that $\lim_{k\to\infty}\dfrac{\Gamma(k+\beta)k^{-\beta}}{\Gamma(k)} = 1$ for any $\beta > 0$ (see [28, Formula 8.328(2)]) the assertion follows. □

Using the subordination principle for α-resolvent families (cf. [32]) and (the proof of) the Datko-Pazy theorem, we obtain the following theorem.

Theorem 5.2. *Suppose that A is the generator of an uniformly exponentially stable semigroup, then the solution of the fractional difference equation of order $\alpha \in (0,1]$*

$$_C\Delta^{\alpha} u(n) = Au(n+1), \quad n \in \mathbb{N}$$

is stable.

Proof. Let $T(t)$ be the uniformly stable semigroup generated by A. In case $\alpha = 1$ the result follows by Proposition 3.7(i). Suppose now that $0 < \alpha < 1$. Observing the proof of the Datko-Pazy theorem (see [20, Chapter V, Theorem 1.8, p.300]), it follows that there exist constants $M > 0$ and $p > 1$ such that

$$\|T(t)\| \leq \frac{M}{t^{1/p}}, \quad t > 0.$$

Define

$$S_\alpha(t)x = \int_0^\infty \Phi_\alpha(\tau) T(\tau t^\alpha) x dt, \quad t \geq 0,$$

where

$$\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)},$$

is a Wright type function (see [32] and references therein). Then by [10, Theorem 3.1] it follows that $S_\alpha(t)$ is an α-resolvent family generated by A. Hence

$$\|S_\alpha(t)\| \leq \frac{M}{t^{\alpha/p}} \int_0^\infty \frac{\Phi_\alpha(\tau)}{\tau^{1/p}} dt.$$

Since $p > 1$ we have by [43, Formula (W3), p.212] that

$$\int_0^\infty \frac{\Phi_\alpha(\tau)}{\tau^{1/p}} x dt = \frac{\Gamma(1-1/p)}{\Gamma(1-\alpha/p)}.$$

Hence there exists a constant $C > 0$ such that

$$\|S_\alpha(t)\| \leq \frac{C}{t^{\alpha/p}}.$$

Therefore by Theorem 5.1 with $\gamma = \alpha/p$ the assertion is proved. □

Using the Gearhart-Prüss-Greiner theorem that characterizes the uniform exponential stability of semigroups in Hilbert spaces (see [20, Chapter V, Theorem 1.11]), we obtain the following remarkable corollary.

Corollary 5.3. *Let A be the generator of a C_0-semigroup on a Hilbert space H such that $\{\mu \in \mathbb{C} : Re(\mu) > 0\} \subset \rho(A)$ and satisfy*

$$\sup_{Re(\mu)>0} \|(\mu - A)^{-1}\| < \infty.$$

Then the solution of the fractional difference equation of order $\alpha \in (0,1)$

$$_C\Delta^{\alpha} u(n) = Au(n+1), \quad n \in \mathbb{N},$$

is stable.

From Corollary 4.5 we deduce the following result in case $\alpha = 1$.

Corollary 5.4. *Let A be the generator of a bounded C_0-semigroup. Suppose that*

$$\|(I - A)^{-1}\| < 1.$$

Then the solution of the equation

$$\Delta u(n) = Au(n+1), \quad n \in \mathbb{N},$$

is stable.

Denote $\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$. In case $A = \lambda I$ with $\lambda \in \mathbb{C}$ we obtain directly from the preceding Corollary, or as application of Corollary 4.6, the following consequence.

Corollary 5.5. *Let* $\lambda \in \mathbb{C} \setminus \mathbb{D}(1,1)$. *Then the solution of*

$$\Delta u(n) = \lambda u(n+1), \quad n \in \mathbb{N}$$

is stable.

In particular, in case of $\lambda \in \mathbb{R}$, the condition on λ reads: $\lambda < 0$ and $\lambda > 2$.

Remark 5.6. Note that stability and asymptotic properties for the equation (4.1) has been studied in recent times [13, 14]. More precisely, results has been obtained in case of A is a constant matrix with real entries, see [14] or $A = \lambda I$ is an scalar. We observe that in the scalar case a characterization has been recently given, see [13, Theorem 3.1], and it coincides with ours. An spectral characterization in the general case that A is an unbounded operator seems to be a difficult problem and is left open.

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