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Abstract Linear Fractional Evolution Equations

Abstract: We review the fundamental theory of solution operators associated to abstract linear fractional evolution equations. We provide their basic results concerning generation, analyticity and inversion. We show the subordination principle as well as perturbation, approximation, ergodicity and compactness properties. The immediate norm continuity is also analyzed. We show how this functional-operator machinery can be applied to solve the fractional order Cauchy problem. We finish this review with some comments and open problems.

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1 Fractional resolvents

1.1 The theory of α-resolvents

Let Y be a Banach space. For a vector-valued function $f : \mathbb{R}_+ \to Y$ we recall that the Riemann-Liouville fractional integral of order $\beta \geq 0$ is defined by

$$J_t^\beta f(t) = (g_\beta * f)(t) := \int_0^t g_\beta(t-s)f(s)ds,$$

where $g_0(t) := \delta(t)$, the Dirac delta, and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $t > 0$. We begin with a purely algebraic notion of the theory of α-resolvents of bounded and linear operators essentially due to Chen and Li [14, Definition 3.1].

Definition 1.1. *Let X be a Banach space and $\alpha > 0$. A one parameter family $\{S_\alpha(t)\}_{t\geq 0}$ of bounded linear operators from X to X is called an α-resolvent family if the following conditions are satisfied:*
(a) $S_\alpha(0) = I$;

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(b) $S_\alpha(s)S_\alpha(t) = S_\alpha(t)S_\alpha(s)$ *for all* $s,t \geq 0$;
(c) The functional equation

$$S_\alpha(s)J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s)S_\alpha(t) = J_t^\alpha S_\alpha(t) - J_s^\alpha S_\alpha(s),$$

holds for all $t,s \geq 0$.

The integrals in (c) are understood strongly in the sense of Bochner. We observe the remarkable fact that in the scalar case, i.e. $X = \mathbb{C}$, we have that the Mittag-Leffler function $E_\alpha(zt^\alpha), z \in \mathbb{C}$, satisfies the functional equation (c). See [53, Example 3.10]. In particular, it shows that the functional equation (c) is a proper generalization of Cauchy's functional equation (corresponding to the case $\alpha = 1$) and D'Alembert functional equation (corresponding to the case $\alpha = 2$) because for $\alpha = 1$ we have $E_1(zt) = e^{zt}$ and for $\alpha = 2$ we have $E_2(zt^2) = \cosh(\sqrt{z}\,t)$, $z \in \mathbb{C}$.

Remark 1.2. *For* $0 < \alpha < 1$ *an equivalent functional equation to (c) was proposed by Peng and Li [70, 69]. See [53, Remark 3.11]. Other equivalent representation involving the sum* $S_\alpha(t+s)$ *appears in [63]. A better comprehension of this functional equations in the context of the double Laplace transform and the connection with the problem of extension from local to global can be found in the reference [1].*

Definition 1.3. *An* α*-resolvent family* $\{S_\alpha(t)\}_{t\geq 0}$ *is called uniformly continuous if*

$$\lim_{t\to s} \|S_\alpha(t) - S_\alpha(s)\| = 0, \tag{1}$$

for every $s \geq 0$.

The linear operator A defined by

$$D(A) := \{x \in X \;:\; \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} \text{ exists } \} \tag{2}$$

and

$$Ax := \Gamma(\alpha+1) \lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} \text{ for } x \in D(A) \tag{3}$$

is called the *generator* of the α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, $D(A)$ is the domain of A. This formula was proposed by the first time in [10, Proposition 3.1].

Theorem 1.4. *[55, Lemma 3.1 and Lemma 3.2] Let* $\alpha > 0$. *A linear operator* A *is the generator of a uniformly continuous* α*-resolvent family if and only if* A *is a bounded operator.*

We remark that in certain Banach spaces, like L^∞, the uniform continuity of an α-resolvent family is automatic [47, Theorem 3.2]. Observe that if A is a bounded operator, then

$$S_\alpha(t) := \sum_{n=0}^{\infty} g_{\alpha n+1}(t)A^n = \sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{\Gamma(\alpha n+1)} = E_\alpha(At^\alpha), \quad t \geq 0, \tag{4}$$

defines a uniformly continuous α-resolvent family. Therefore α-resolvent families are, in some sense, abstract versions of Mittag-Leffler functions.

Definition 1.5. *An α-resolvent family of bounded linear operators $\{S_\alpha(t)\}_{t\geq 0}$ on X is called a strongly continuous α-resolvent family if $\lim_{t\to s} \|S_\alpha(t)x - S_\alpha(s)x\| = 0$ for all $s \geq 0$ and every $x \in X$.*

The following characterization is sometimes used as starting point for their definition.

Theorem 1.6. *[14, Theorem 3.4], [53, Theorem 3.1 and Theorem 4.1] Let $\alpha > 0$. Let A be a linear operator in X with domain $D(A)$. A strongly continuous family $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators in X is an α-resolvent family generated by A if and only if the following conditions are satisfied*
(i) $S_\alpha(0) = I$;
(ii) $S_\alpha(t)x \in D(A)$ *and* $S_\alpha(t)Ax = AS_\alpha(t)x$ *for all* $x \in D(A)$ *and* $t \geq 0$;
(iii) $S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)AS_\alpha(s)x ds, \quad t \geq 0, \quad x \in D(A).$

This characterization shows that α-resolvent families are particular cases of the theory of resolvent families associated to Volterra integral equations, introduced by J. Prüss [72, Chapter 1, Section 1]. The following properties can be found in [50, Lemma 2.2], [53, Theorem 4.1] and [14, Proposition 3.3], for instance.

Proposition 1.7. *Let $\{S_\alpha(t)\}_{t\geq 0}$ be an strongly continuous α-resolvent family and let A be its generator. Then*

(i) For all $x \in X$, $\int_0^t g_\alpha(t-s)S_\alpha(s)x ds \in D(A)$ *and*

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(s)x ds, \quad t \geq 0, \quad x \in X. \tag{5}$$

(ii) $D(A)$*, the domain of* A*, is dense in* X *and* A *is a closed operator.*

A simple application is the following generalization of the Kallmann-Rota inequality [52, Theorem 3.2].

Corollary 1.8. *Let A be the generator of a bounded and strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, i.e. satisfying $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$. If $x \in D(A^2)$ then*

$$\|Ax\|^2 \leq 8M^2 \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \|x\|\|A^2 x\|.$$

1.2 Integrated α-resolvents

The concept of n-times integrated resolvents ($n \in \mathbb{N}_0$) was introduced by Arendt and Kellermann [6, Definition 1.1], after previous work of W. Arendt and M. Hieber on β-times integrated semigroups ($\beta > 0$). Following [53] (see also [14, Definition 3.7]), the definition of β-times integrated α-resolvent families can be introduced as follows:

Definition 1.9. *Let X be a Banach space and $\alpha > 0, \beta > 0$. A one parameter family $\{R_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called an (α,β)-resolvent family if the following conditions are satisfied:*

(a) $\lim_{t\to 0} t^{1-\beta} R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)} I$ *if* $0 < \beta < 1$, $R_{\alpha,1}(0) = I$ *and* $R_{\alpha,\beta}(0) = 0$ *if* $\beta > 1$.

(b) $R_{\alpha,\beta}(s)R_{\alpha,\beta}(t) = R_{\alpha,\beta}(t)R_{\alpha,\beta}(s)$ *for all* $s, t > 0$;

(c) The functional equation

$$R_{\alpha,\beta}(s)J_t^\alpha R_{\alpha,\beta}(t) - J_s^\alpha R_{\alpha,\beta}(s)R_{\alpha,\beta}(t) = g_\beta(s)J_t^\alpha R_{\alpha,\beta}(t) - g_\beta(t)J^\alpha R_{\alpha,\beta}(s),$$

holds for all $t, s > 0$.

For the development of many properties, it is important to observe that the notion of (α,β)-resolvent families is included into the theory of (a,k)-regularized families [50] with $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$.

Of course, $(\alpha, 1)$-resolvent families are α-resolvent families. For $0 < \alpha = \beta < 1$ the above definition was studied by Li and Peng [39]. We note that this concept was introduced earlier [5], but without reference to the condition near to zero given in (a).

Remark 1.10. *An equivalent identity to (c) in the spirit of Remark 1.2 has been proved in [44, Theorem 5].*

Remark 1.11. *(Notation) In what follows, an (α,α)-resolvent family $\{R_{\alpha,\alpha}(t)\}_{t>0}$ will be simply denoted by $\{R_\alpha(t)\}_{t>0}$ and an $(\alpha,1)$-resolvent family by $\{S_\alpha(t)\}_{t\geq 0}$.*

The linear operator A defined by

$$D(A) := \{x \in X \;:\; \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ exists } \} \tag{6}$$

and

$$Ax := \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ for } x \in D(A) \tag{7}$$

is called the *generator* of the (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$.

If A is a bounded operator, then

$$R_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} g_{\alpha n+\beta}(t)A^n = t^{\beta-1}\sum_{n=0}^{\infty}\frac{A^n t^{\alpha n}}{\Gamma(\alpha n+\beta)} = t^{\beta-1}E_{\alpha,\beta}(At^\alpha), \quad t>0, \tag{8}$$

defines a uniformly continuous (α,β)-resolvent family. Given $\beta > 1$, observe that the family $\{R_{\alpha,\beta}(t)\}_{t>0}$ is $(\beta-1)$-times integrated with respect to $\{R_{\alpha,1}(t)\}_{t\geq 0}$ because the identity

$$R_{\alpha,\beta}(t) = g_{\beta-1} * R_{\alpha,1}(t) = J_t^{\beta-1}R_{\alpha,1}(t), \quad t>0,$$

holds. More generally, if A is the generator of an (α,β)-resolvent family, then for all $\gamma \geq 0$ we have that A is the generator of an $(\alpha,\beta+\gamma)$-resolvent family [50, Remark 2.4 (4)]. The following characterization is often used as definition.

Theorem 1.12. *[53, Theorem 3.1 and Theorem 4.3] Let $\alpha > 0$ and $\beta > 0$ be given. A strongly continuous family $\{R_{\alpha,\beta}(t)\}_{t>0} \subset \mathcal{B}(X)$ of bounded linear operators in X is an (α,β)-resolvent family generated by A if and only if the following conditions are satisfied*

(i) $\lim_{t\to 0} t^{1-\beta}R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)}I$ *if* $0<\beta<1$, $R_{\alpha,1}(0) = I$ *and* $R_{\alpha,\beta}(0) = 0$ *if* $\beta > 1$.

(ii) $R_{\alpha,\beta}(t)x \in D(A)$ *and* $R_{\alpha,\beta}(t)Ax = AR_{\alpha,\beta}(t)x$ *for all* $x \in D(A)$ *and* $t \geq 0$;

(iii) $R_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AR_{\alpha,\beta}(s)x\,ds, \quad t\geq 0, \quad x \in D(A).$

For $\beta > 1$, we have that $D(A)$ is closed, but not necessarily densely defined [14, Proposition 3.10]. In the diagonal case $\alpha = \beta$ this notion appears by the first time in [5, Definition 2.3]. If $0 < \alpha = \beta < 1$ then A must be densely defined [39, Theorem 3.1]. We have the following criteria.

Theorem 1.13. *[5, Theorem 2.6] Let A be the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\geq 0}$, then for any $\alpha \in [1,2)$, A generates an (α,α)-resolvent family.*

Roughly speaking, the notion of $(\alpha,1)$-resolvent families is associated with the Caputo fractional derivative, whereas the notion of (α,α)-resolvent family is linked with the Riemann-Liouville fractional derivative. Other relevant cases are $(\alpha,\gamma+(1-\gamma)\alpha)$-resolvent families with $0<\alpha<1,\ 0\leq\gamma\leq 1$, see [27], and $(\alpha,\alpha+\gamma(2-\alpha))$-resolvent families with $1<\alpha<2,\ 0\leq\gamma\leq 1$, see [64], because they are related with the notion of Hilfer fractional derivative that interpolates between the Caputo and Riemann-Liouville fractional derivative (For $0<\alpha<1$ take $\gamma=1$ and $\gamma=0$, respectively).

Assuming that A is the generator of an γ-times integrated semigroup ($\gamma\geq 0$), i.e. an $(1,\gamma+1)$-resolvent family, then $(\alpha,\alpha\gamma+1)$-resolvent families and $(\alpha,\alpha(\gamma+1))$-resolvent families are important for $0<\alpha<1$ because these are the key for the treatment of existence, regularity and representation of fractional diffusion equations, see [32]. The same happens with $(\alpha,\frac{\alpha\gamma}{2}+1)$-resolvent families and $(\alpha,\alpha(\frac{\gamma}{2}+1))$-resolvent families for $1\leq\alpha\leq 2$ because these are present in the theoretical analysis of fractional wave equations, see [31].

Remark 1.14**.** *There are weaker concepts of fractional resolvent operator functions in the literature. For instance, Chen and Li [14, Definition 3.6] introduced the notion of C-regularized resolvent functions. See also the section of comments.*

1.3 Generation

A family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ is called exponentially bounded (or of type (M,ω)) if there are constants $M>0$ and $\omega\in\mathbb{R}$ such that

$$\|S(t)\|\leq Me^{\omega t},\quad t\geq 0. \tag{9}$$

If A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ satisfying (9), we write $A\in\mathcal{C}^\alpha(M,\omega)$. Also, set $\mathcal{C}^\alpha(\omega):=\cup_{M\geq 0}\mathcal{C}^\alpha(M,\omega)$ and $\mathcal{C}^\alpha:=\cup_{\omega\geq 0}\mathcal{C}^\alpha(\omega)$.

We observe that in the cases $\alpha=1$ and $\alpha=2$ the exponential boundedness is a consequence of the corresponding functional equation.

In the exponentially bounded case we have the following remarkable result due to Bazhlekova.

Theorem 1.15. *[8, Theorem 2.1] Assume that $A\in\mathcal{C}^\alpha$ for some $\alpha>2$, then $A\in\mathcal{B}(X)$.*

Another important consequence of exponential boundedness is the following useful characterization of α-resolvent families via the Laplace transform.

Theorem 1.16. *[50, Proposition 3.1] Let $\alpha > 0$. Let $\{S_\alpha(t)\}_{t\geq 0}$ be a strongly continuous α-resolvent family of type (M, ω). Then $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent family with generator A if and only if for every $\lambda > \omega$, $(\lambda^\alpha - A)^{-1}$ exists in $\mathcal{B}(X)$ and the identity*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\, dt, \text{ for all } x \in X, \tag{10}$$

holds.

More generally, the same characterization is also true in the case of exponentially bounded (α, β)-resolvent families.

Theorem 1.17. *[14, Theorem 3.11] Let $\alpha > 0$ and $\beta \geq 1$. Let $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ be a strongly continuous (α, β)-resolvent family in $\mathcal{B}(X)$ of type (M, ω). Then $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (α, β)-resolvent family with generator A if and only if for every $\lambda > \omega$, $(\lambda^\alpha - A)^{-1}$ exists in $\mathcal{B}(X)$ and the identity*

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} R_{\alpha,\beta}(t)x dt, \text{ for all } x \in X. \tag{11}$$

holds.

For a generalization in the case $0 < \beta < 1$, see [2, Theorem 10]. The following result extends the Hille-Yosida theorem for C_0-semigroups to the context of α-resolvent families.

Theorem 1.18. *[50, Theorem 3.4] and [10, Theorem 3.1] A closed linear and densely defined operator A with domain $D(A)$ is the generator of a strongly continuous α-resolvent family of type (M, ω) if and only if the following conditions hold:*
(H1) $\lambda^\alpha \in \rho(A)$ for all $Re\lambda > \omega$.
(H2) $H(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}$ satisfies the estimates

$$\|\frac{d^n}{d\lambda^n} H(\lambda)\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad Re\lambda > \omega, \quad n \in \mathbb{N}_0.$$

For recent generation theorems for fractional resolvent families, see the paper [66] by Mu and Li. We also refer to the paper [38] for related results.

An analogous result holds in the case of (α,β)-resolvent families. It can be extracted from [50].

An important consequence is the following: If A is the generator of an (α,α)-resolvent family of type (M,ω), say $\{R_\alpha(t)\}_{t>0}$ for some $0<\alpha<1$, then A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ given by

$$S_\alpha(t) = (g_{1-\alpha} * R_\alpha)(t), \quad t \geq 0.$$

1.4 Analyticity

Given $\theta \in [0,\pi]$ and $\omega \in \mathbb{R}$, we denote $\Sigma_\theta(\omega) := \{z \in \mathbb{C}\backslash\{0\} \ : \ |arg(z-\omega)| < \theta\}$. In case $\omega = 0$ we simply denote $\Sigma_\theta = \Sigma_\theta(0)$.

Definition 1.19. *Let $0 < \theta_0 \leq \pi/2$. An α-resolvent family (resp. an (α,β)-resolvent family) is called an analytic α-resolvent family of angle θ_0 (resp. an analytic (α,β)-resolvent family of angle θ_0) if it admits an analytic extension to the sector Σ_{θ_0} and the analytic extension is strongly continuous on Σ_{θ_0} for every $\theta \in (0,\theta_0)$.*

Definition 1.20. *An analytic α-resolvent family $S_\alpha(z)$ (resp. an analytic (α,β)-resolvent family $R_{\alpha,\beta}(z)$) is said to be of analyticity type (θ_0,ω_0) if for each $\theta \in (0,\theta_0)$ and each $\omega > \omega_0$ there exists a constant $M = M(\theta,\omega)$ such that*

$$\|S_\alpha(z)\| \leq M e^{\omega Re(z)}, \quad z \in \Sigma_\theta.$$

(resp. $\|R_{\alpha,\beta}(z)\| \leq M e^{\omega Re(z)}$, $z \in \Sigma_\theta$.) If $\omega_0 = 0$, an analytic α-resolvent family (resp. an analytic (α,β)-resolvent family) is called a bounded analytic α-resolvent family (resp. bounded analytic (α,β)-resolvent family).

If A generates an analytic α-resolvent family (resp. (α,β)-resolvent family) of type (θ_0,ω_0) we write $\mathcal{A}^\alpha(\theta_0,\omega_0)$ (resp. $\mathcal{A}^{\alpha,\beta}(\theta_0,\omega_0)$). In addition, we denote $\mathcal{A}^\alpha(\theta_0) := \bigcup_{\omega_0\in\mathbb{R}_+} \mathcal{A}^\alpha(\theta_0,\omega_0)$ and $\mathcal{A}^\alpha := \bigcup_{\theta_0\in(0,\pi/2)} \mathcal{A}^\alpha(\theta_0)$ (resp. $\mathcal{A}^{\alpha,\beta}(\theta_0) := \bigcup_{\omega_0\in\mathbb{R}_+} \mathcal{A}^{\alpha,\beta}(\theta_0,\omega_0)$ and $\mathcal{A}^{\alpha,\beta} := \bigcup_{\theta_0\in(0,\pi/2)} \mathcal{A}^{\alpha,\beta}(\theta_0)$). For $\alpha = 1$ we obtain the set of all generators of analytic semigroups.

Theorem 1.21. *[7, Theorem 2.14] Let $0 < \alpha < 2$, $0 < \theta_0 < \min\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\}$ and $\omega_0 \geq 0$. An operator A belongs to $\mathcal{A}^\alpha(\theta_0,\omega_0)$ if and only if*
(i) $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \Sigma_{\theta_0+\pi/2}(\omega_0)$,
(ii) For any $\omega > \omega_0$, $\theta < \theta_0$, there is a constant $C = C(\theta,\omega)$ such that

$$\|\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\| \leq \frac{C}{|\lambda-\omega|}; \quad \lambda \in \Sigma_{\theta+\pi/2}(\omega).$$

For generators of analytic (α,β)-resolvent families we have the following characterization.

Theorem 1.22. *[14, Theorem 4.6] Let* $0<\alpha<2$, $\beta>1, \theta_0\in(0,\pi/2]$ *and* $\omega_0\geq 0$. *An operator* A *belongs to* $\mathcal{A}^{\alpha,\beta}(\theta_0,\omega_0)$ *if and only if*
(i) $\lim_{\lambda\to\infty}\lambda^{\alpha-\beta+1}R(\lambda^\alpha,A)x=0$ *for any* $x\in X$,
(ii) $\lambda^\alpha\in\rho(A)$ *for each* $\lambda\in\Sigma_{\theta_0+\pi/2}(\omega_0)$,
(iii)For any $\omega>\omega_0$, $\theta<\theta_0$, *there is a constant* $C=C(\theta,\omega)$ *such that*

$$\|\lambda^{\alpha-\beta}(\lambda^\alpha-A)^{-1}\|\leq\frac{C}{|\lambda-\omega|};\quad \lambda\in\Sigma_{\theta+\pi/2}(\omega).$$

If the generator A is densely defined, then the condition (i) is automatically fulfilled. Other equivalent conditions can be found in [14, Section 4].

Recall that a closed linear operator A densely defined on a Banach space X is called *sectorial of angle* $\omega\in[0,\pi)$ ($A\in Sect(\omega)$, in short) if
(a) $\sigma(A)\subset\overline{\Sigma_\omega}$ for $\omega>0$.
(b) For every $\omega'\in(\omega,\pi)$, $\sup\{\|z(z-A)^{-1}\| \,:\, z\in\mathbb{C}\setminus\overline{\Sigma_{\omega'}}\}<\infty$.

For bounded analytic α-resolvent families, we have the following important criteria.

Theorem 1.23. *[41, Lemma 2.7] Let* $\alpha\in(0,2)$ *and* $0<\theta_0<\min\{\frac{\pi}{2},\frac{\pi}{\alpha}-\frac{\pi}{2}\}$. *The following assertions are equivalent:*
(i) $A\in\mathcal{A}^\alpha(\theta_0,0)$
(ii) $\Sigma_{\alpha(\pi/2+\theta_0)}\subset\rho(A)$ *and for all* $0<\theta<\theta_0$ *there exists a constant* M_θ *such that*

$$\|\lambda(\lambda-A)^{-1}\|\leq M_\theta,\quad \lambda\in\Sigma_{\alpha(\pi/2+\theta)}.$$

(iii)$-A\in Sect(\pi-\alpha(\frac{\pi}{2}+\theta_0))$.

A practical criteria is the following.

Theorem 1.24. *[10, Proposition 4.1] If* $\{\lambda:\Re(\lambda)>0\}\subset\rho(A)$ *and for some constant* $C>0$:

$$\|(\lambda-A)^{-1}\|\leq\frac{C}{Re(\lambda)},\quad Re(\lambda)>0,$$

then for any $\alpha\in(0,1)$, $A\in\mathcal{A}^\alpha(\min\{(\frac{1}{\alpha}-1)\frac{\pi}{2},\frac{\pi}{2}\},0)$.

1.5 Inversion

One of the fundamental problems in the theory of α-resolvent families, from the point of view of applications to fractional partial differential equations, is the relation between the α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ and its generator. The reason is that for $x \in D(A)$, $S_\alpha(t)x$ is the solution of the fractional Cauchy problem

$$D_t^\alpha u(t) = Au(t),$$

with initial conditions $u(0) = x, u^{(k)}(0) = 0, k = 1, ..., n-1; n-1 < \alpha \leq n.$

The following result corresponds to the vector-valued version of the complex inversion formula for the Laplace transform. The main difference with the finite dimensional case is that this is not true except in the case of the class of UMD-spaces. Recall that a Banach space X is UMD (or belongs to the class $\mathcal{HT}$) if the Hilbert transform is bounded on $L^2(\mathbb{R}, X)$. If X is not UMD then we have either handle integrated versions of α-resolvent families or restrict the validity of the formula for vectors belonging to the domain of powers of the generator A. See e.g. [68, Corollary 7.5] in case $\alpha = 1$.

Theorem 1.25. *[21, Theorem 3.2] Let X be a UMD space. Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M,ω) and let $\gamma > max(0,\omega)$. For every $x \in X$ we have*

$$S_\alpha(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x d\lambda$$

Remark 1.26. *Checking the conditions in [21, Theorem 3.2], we note that the above theorem also holds for (α,β)- resolvent families, whenever $\alpha > 0$ and* $0 < \beta \leq 1$.

Remark 1.27. *If $\{S_\alpha(z)\}_{z\in\Sigma_\theta}$ is a bounded analytic α-resolvent family with generator A defined in a Banach space X, then for $t > 0$,*

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda,$$

where $\frac{\pi}{2} < \theta < \theta_0 + \frac{\pi}{2}$, with θ_0 the analytic angle of $S_\alpha(t)$, and the contour

$$\Gamma_\theta = \{\rho e^{i\theta} : \rho_0 < \rho < \infty\} \cup \{\rho e^{-i\theta} : \rho_0 < \rho < \infty\} \cup \{\rho_0 e^{i\varphi} : -\theta < \varphi < \theta\}$$

is oriented counterclockwise with $\rho_0 > 0$ any given constant. See [7].

An alternative representation of bounded analytic fractional resolvent families can be given by functional calculus via the Mittag–Leffler function $E_\alpha(z)$:

$$S_\alpha(t) = \frac{1}{2\pi i} \int\limits_{\Gamma_\theta} E_\alpha(\lambda t^\alpha)(\lambda - A)^{-1} d\lambda, \quad t > 0,$$

where $\frac{\pi}{2}\alpha < \theta < (\theta_0 + \frac{\pi}{2})\alpha$, *with* θ_0 *the analytic angle of* $S_\alpha(t)$, *see [16, p.183].*

The next result is implicitly contained in [48, Theorem 2.1] and is a consequence of the Post-Widder inversion formula of the Laplace transform and [9, Lemma 4.1]. A formula for (α, β)-resolvent families have a more complicated writing and can be found in [48, Theorem 2.1]

Theorem 1.28. *Let* X *be a Banach space and* A *the generator of a strongly continuous* α*-resolvent family* $\{S_\alpha(t)\}_{t\geq 0}$ *of type* (M, ω)*. Then,*

$$S_\alpha(t)x = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{n+1} b^\alpha_{k,n+1} \Big(I - \frac{t^\alpha}{n^\alpha} A\Big)^{-k} x, \quad x \in X,$$

where $b^\alpha_{k,n}$ *are given by the recurrence relations*

$b^\alpha_{1,1} = 1, b^\alpha_{k,n} = (n - 1 - k\alpha)b^\alpha_{k,n-1} + \alpha(k - 1)b^\alpha_{k-1,n-1}, 1 \leq k \leq n, n = 2, 3, ...$

and $b^\alpha_{k,n} = 0, k > n, n = 1, 2, ...$

Some immediate consequences are characterizations of positivity of α-resolvent families in terms of positivity of the resolvent operator [48, Theorem 2.1].

In the case $\alpha = 1$ i.e. that A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$, the formula reads:

$$S_1(t)x = \lim_{n\to\infty} \Big(I - \frac{t}{n} A\Big)^{-n} x, \quad x \in X, \quad t \geq 0,$$

where the convergence is uniform in bounded t-intervals for each fixed $x \in X$. This formula has important implications for the numerical approximation of the trajectories of $\{S_1(t)\}_{t\geq 0}$, especially for implicit approximation schemes.

2 Properties of fractional resolvents

2.1 Spectral properties

Let $a \in \mathbb{C}$, $\alpha > 0$ be given, and define $s_\alpha(t) := t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)$ and $r_\alpha(t) := E_\alpha(at^\alpha)$. In order to study spectral mapping theorems, the following result due to Li and Zheng is fundamental:

Theorem 2.1. *[42, Lemma 3.1]. Let A be a closed linear and densely defined operator A with domain $D(A)$ the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$, then*

$$(a-A)\int\limits_0^t s_\alpha(t-s)S_\alpha(s)xds = r_\alpha(t)x - S_\alpha(t)x, \quad x \in X.$$

and

$$\int\limits_0^t s_\alpha(t-s)S_\alpha(s)(a-A)xds = r_\alpha(t)x - S_\alpha(t)x, \quad x \in D(A).$$

We denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$, $\sigma_a(A)$ the spectrum, point spectrum, residual spectrum and approximate point spectrum of A, respectively. We have the following result.

Theorem 2.2. *[42, Theorem 3.2] Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$, then*
(i) $E_\alpha(t^\alpha \sigma(A)) \subset \sigma(S_\alpha(t))$;
(ii) $E_\alpha(t^\alpha \sigma_p(A)) \subset \sigma_p(S_\alpha(t))$;
(iii)$E_\alpha(t^\alpha \sigma_a(A)) \subset \sigma_a(S_\alpha(t))$;
(iv) $E_\alpha(t^\alpha \sigma_r(A)) \subset \sigma_r(S_\alpha(t))$.

When $\alpha = 1$, this is the well known spectral inclusions for C_0-semigroups [68, Section 2.2]. If $\alpha = 2$, since $E_2(z) = \cosh(z)$, so it gives the spectral inclusions for cosine operator functions [67].

A generalization of this spectral inclusions appear in [58] in the context of (a,k)-regularized families. The case of (α,β)-resolvent families, read as follows:

Theorem 2.3. *[58] Let A be the generator of a strongly continuous (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ where $\alpha \in (0,2]$ and $\beta > 0$. Then*
(i) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma(A)) \subset \sigma(R_{\alpha,\beta}(t))$;
(ii) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_p(A)) \subset \sigma_p(R_{\alpha,\beta}(t))$;
(iii)$t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_a(A)) \subset \sigma_a(R_{\alpha,\beta}(t))$;
(iv) $t^{\beta-1}E_{\alpha,\beta}(t^\alpha \sigma_r(A)) \subset \sigma_r(R_{\alpha,\beta}(t))$.

It is proved in [20] that contrary to the case of semigroups, where the spectral mapping theorem does not hold without further assumptions, such a theorem does hold for $(1,\beta)$-resolvent families ($\beta > 1$). Namely, if A is the generator of an $(1,\beta)$-resolvent family $R_{1,\beta}(t)$ then

$$\sigma_*(R_{1,\beta}(t)) \cup \{0\} = \Big\{ \int_0^t g_\beta(t-s)e^{\lambda s} \,:\, \lambda \in \sigma_*(A) \Big\} \cup \{0\}, \quad t>0, \tag{1}$$

where $*$ stands for "p", "a" or "r".

2.2 Subordination principle

The subordination principle, presented earlier by Pruss [72] for general resolvent families, was studied in detail by Bajlekova for α-resolvent families in her Ph.D. Thesis [7]. It is very important in this theory. We need the Wright type function

$$\begin{aligned}\Phi_\gamma(z) &:= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n+1-\gamma)} \\ &= \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-z)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha) \\ &= \frac{1}{2\pi i}\int_\Gamma \mu^{\gamma-1} exp(\mu - z\mu^\gamma)d\mu,\end{aligned} \tag{2}$$

where $0<\gamma<1$ and Γ is a contour which starts and ends at $-\infty$ and encircles the origin counterclockwise. This has sometimes also been called the Mainardi function. We note that the expressions above holds for all $z \in \mathbb{C}$ with the exception of (2) that holds only for all $z \in \mathbb{R}_+$. The Wright function connects to the Mittag-Leffler function through the Laplace transform:

$$E_\gamma(z) = \int_0^\infty \Phi_\gamma(t)e^{zt}dt, \quad z\in\mathbb{C}, \quad 0<\gamma<1. \tag{3}$$

Theorem 2.4. *[7, 8, Theorem 3.1] Let $0<\alpha<\beta\le 2$, $\gamma=\alpha/\beta$, $\omega\ge 0$. If $A\in\mathcal{C}^\beta(\omega)$ then $A\in\mathcal{C}^\alpha(\omega^{|/\gamma})$ and the following representation holds*

$$S_\alpha(t) = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma})S_\beta(s)ds = \int_0^\infty \Phi_\gamma(\tau)S_\beta(t^\gamma\tau)d\tau, \quad t>0. \tag{4}$$

Moreover, $\{S_\alpha(t)\}_{t\ge 0}$ can be analytically extended to the sector Σ_θ with $\theta = \min\{(\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2}\}$.

Since $\Phi_{1/2}(z) = \pi^{-1/2} e^{-z^2/4}$, the formula (4) coincides with the abstract Weierstrass formula

$$S_1(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} S_2(s) ds, \quad t > 0, \tag{5}$$

relating the C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ with the cosine operator function $\{S_2(t)\}_{t\geq 0}$. Other particular case, frequently used in applications, is $\beta = 1$. That is, assuming that A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ we have

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t > 0, \quad 0 < \alpha < 1. \tag{6}$$

Concerning (α, β)-resolvent families, an interesting and useful extension of the above theorem was proven by Abadías and Miana.

Theorem 2.5. *[2, Theorem 12] Let $0 < \eta_1 \leq 2$, $0 < \eta_2$. If A generates an exponentially bounded (η_1, η_2)-resolvent family $\{S_{\eta_1,\eta_2}(t)\}_{t>0}$, then A generates an exponentially bounded $(\alpha\eta_1, \alpha\eta_2 + \beta)$- resolvent family given by*

$$S_{\alpha\eta_1,\alpha\eta_2+\beta}(t) = \int_0^\infty \psi_{\alpha,\beta}(t,s) S_{\eta_1,\eta_2}(s) ds, \quad t > 0,\ 0 < \alpha < 1,\ \beta \geq 0. \tag{7}$$

Here,

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} \sum_{n=0}^\infty \frac{(-st^{-\alpha})^n}{n!\Gamma(-\alpha n + \beta)}, \quad 0 < \alpha < 1,\ \beta \geq 0,$$

is called scaled Wright function [2]. Note that if $0 < \eta_1 \leq 2, \eta_2 = 1, \beta = 1 - \alpha$ with $0 < \alpha < 1$ we retrieve Theorem 2.4. Also, choosing $\eta_1 = \eta_2 = 1$ and $\beta = 0$ we obtain the following useful consequence for applications: If A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ then A is the generator of an (α, α)-resolvent family given by:

$$R_\alpha(t) = t^{\alpha-1} \int_0^\infty \alpha s \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t > 0. \tag{8}$$

Concerning powers of the generator, the following generalized subordination principle was given in [14, Theorem 3.1]

Theorem 2.6. *Let $-A$ generate a bounded α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on X for some $\alpha \in (0,2]$ and let $0 < \gamma < 2$. Then $-A^\beta$ generates a bounded analytic*

γ-resolvent family $\{S_\gamma^\beta(t)\}_{t\geq 0}$ of angle $\varphi = \min\{\frac{\pi}{2}, \frac{\pi}{\gamma}(1-\beta) + \frac{\pi}{2}(\frac{\alpha}{\gamma}\beta - 1)\}$ on X for each $\beta \in (0, \frac{2-\gamma}{2-\alpha})$, and the following generalized subordination principle

$$S_\gamma^\beta(t) = \int_0^\infty f_{\alpha,\gamma}^\beta(t,s) S_\alpha(s) ds, \quad t > 0,$$

holds in the strong sense, where

$$f_{\alpha,\gamma}^\beta(t,s) = \frac{-1}{2\pi i} \int_{\partial\Sigma_\omega} E_\gamma(-\mu^\beta t^\gamma)(-\mu)^{\frac{1}{\alpha}-1} e^{-(-\mu)^{1/\alpha} s} d\mu$$

with the path $\partial\Sigma_\omega$ oriented in the positive sense (from $\infty e^{i\omega}$ to $\infty e^{-i\omega}$) where $\omega \in (\pi - \frac{\pi}{2}\alpha, \min\{\pi, \frac{1}{\beta}(\pi - \frac{\pi}{2}\gamma)\})$ and $(-\rho e^{\pm i\omega})^{1/\alpha} = \rho^{1/\alpha} e^{\mp i(\pi-\omega)/\alpha}$.

We observe that $f_{1,1}^\beta$ is just the Bochner subordination kernel which is a probability density function [78, Chapter IX].

Let $0 < \gamma < \alpha \leq 2$. Then we can take $\beta = 1$ and obtain

$$\int_0^\infty e^{-\mu t} f_{\gamma,\alpha}^1(t,s) dt = \mu^{\gamma/\alpha - 1} e^{-\mu^{\gamma/\alpha} s}, \quad Re(\mu) > 0.$$

This yields $f_{\gamma,\alpha}^1(t,s) = t^{-\gamma/\alpha}\Phi_{\gamma/\alpha}(st^{-\gamma/\alpha})$. For other properties on this kernel, we refer the reader to [15, Section 3].

2.3 Immediate norm continuity

A family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ on X is called immediately norm continuous (or equicontinuous) if the function $t \to S(t)$ is norm continuous form $(0,\infty)$ into $\mathcal{B}(X)$. In case of semigroups, this is a very important class that includes analytic semigroups, among others [23, Definition 4.17, Chapter II]. We have the following extension of this property for α-resolvent families.

Theorem 2.7. *[24, Lemma 3.8] Let $0 < \alpha < 1$. Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is an analytic α-resolvent family of analyticity type (ω,θ), then $\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous.*

A characterization in known only in case of Hilbert spaces, and is a consequence of a more general result in the context of resolvent operators due to Lizama [49].

Theorem 2.8. *[49] Let $0 < \alpha \leq 2$. Let A be the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ defined in a Hilbert space H and satisfying $\|S_\alpha(t)\| \leq Me^{\omega t}$ for some $M > 0$ and $\omega > 0$. The following assertions are equivalent:*
(i) $\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous;
(ii) $\lim_{|\mu|\to\infty} \|(\mu_0 + i\mu)^{\alpha-1}\Big((\mu_0 + i\mu)^\alpha - A\Big)^{-1}\| = 0$ for some $\mu_0 > \omega$.

We observe that an extension of the above result to (α,β)-resolvent families can be found in [56, Theorem 5.7]. In case $\alpha = 1$ this result reduces to a characterization obtained by You [79]. Let us recall the following definition.

Definition 2.9. *Let $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. By $\Theta^\gamma_\omega(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy*
(i) $\sigma(A) \subset \overline{\Sigma_\omega} = \{z \in \mathbb{C} \setminus \{0\} : |arg(z)| \leq \omega\} \cup \{0\}$ and
(ii) for every $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\|R(z; A)\| \leq C_\mu |z|^\gamma \text{ for all } z \in \mathbb{C} \setminus \overline{\Sigma_\mu}.$$

A linear operator A will be called almost sectorial operator on X if $A \in \Theta^\gamma_\omega(X)$.

Observe that the limit case $\gamma = -1$ corresponds to the notion of sectorial operator. Let $0 < \alpha \leq 1$ and $A \in \Theta^\gamma_\omega(X)$, $-1 < \gamma < 0 < \omega < \pi/2$. We define the operator families

$$S_\alpha(t) := \frac{1}{2\pi i} \int\limits_{\Gamma_\theta} E_\alpha(-zt^\alpha) R(z; A)dz, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}$$

and

$$P_\alpha(t) := \frac{1}{2\pi i} \int\limits_{\Gamma_\theta} E_{\alpha,\alpha}(-zt^\alpha) R(z; A)dz, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |arg(t)|$. By [77, Theorem 3.1] we have that for each $t \in \Sigma_{\frac{\pi}{2}-\omega}$, the operators $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded in X. In case $\alpha = 1$ we have that $S_1(t)$ defines an analytic semigroup for $t > 0$ that satisfies

$$\|S_1(t)\| \leq Ct^{-\gamma-1}, \qquad \|A^\beta S_1(t)\| \leq Ct^{-\gamma-Re\beta-1}, \quad t > 0, \quad \Re(\beta) > 0.$$

From the subordination principle (Theorem 2.4), or the proof of [77, Theorem 3.1, formulae (3.2) and (3.3)], we have that the formula

$$S_\alpha(t) = \int\limits_0^\infty \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t \in \Sigma_{\frac{\pi}{2}-\omega} \tag{9}$$

defines an α-resolvent family for $t > 0$, and

$$P_\alpha(t) = \int_0^\infty \alpha s \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t \in \Sigma_{\frac{\pi}{2}-\omega}, \tag{10}$$

defines an (α, α)-resolvent family given by $R_\alpha(t) := t^{\alpha-1} P_\alpha(t)$. Moreover, there exist constants $C_s = C(\alpha, \gamma) > 0$ and $C_p = C(\alpha, \gamma) > 0$ such that for all $t > 0$

$$\|S_\alpha(t)\| \leq C_s t^{-\alpha(1+\gamma)}, \quad \|P_\alpha(t)\| \leq C_p t^{-\alpha(1+\gamma)}.$$

See e.g. [81, Proposition 2.1 p.42]. In particular,

$$\|R_\alpha(t)\| \leq C_p t^{-\alpha\gamma-1}.$$

The limit case $\gamma = -1$ was proved in [45, Theorem 3.1] in the range $1 < \alpha < 2$. Also in [45, Theorem 3.2] it was proved that for $t > 0$ and $x \in X$, we have $R_\alpha(t)x \in D(A)$ and

$$\|AR_\alpha(t)\| \leq \frac{C}{t}, \quad 1 < \alpha < 2,$$

for some positive constant C. By Theorem 2.4, [77, Theorem 3.2]) and [30, Theorem 5] we obtain

Theorem 2.10. *Let* $0 < \alpha < 1$ *and* $A \in \Theta_\omega^\gamma(X)$, $-1 < \gamma < 0 < \omega < \pi/2$ *be given. Then A generates an immediately norm continuous α-resolvent family $S_\alpha(t)$ given by* (9), *and an (α, α)-resolvent family $R_\alpha(t) := t^{\alpha-1}P_\alpha(t)$ where $P_\alpha(t)$ given by* (10) *is immediately norm continuous.*

Concerning (α, β)-resolvent families, Ponce [71] proved the following result

Theorem 2.11. *[71, Proposition 11] Let* $\alpha > 0$ *and* $1 < \beta \leq 2$. *Assume that A is the generator of an exponentially bounded (α, β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t \geq 0}$. Then $\{R_{\alpha.\beta}(t)\}_{t \geq 0}$ is immediately norm continuous.*

2.4 Perturbation

A classical result for C_0-semigroups is the following: If A is the generator of a C_0-semigroup and $B \in \mathcal{B}(X)$, then $A + B$ is again the generator of a C_0-semigroup. This is not true in general for α-resolvent families with $0 < \alpha < 1$. See [9] for an example. However, in the case $1 \leq \alpha \leq 2$ perturbations by bounded operators are always possible. In the next theorem, proved by Bazhlekova, we show this even in the case of bounded time-dependent perturbations. For $\alpha = 2$ an analogous theorem was presented by Lutz in [60].

Theorem 2.12. *[9] Let $1 < \alpha < 2$ and A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ of type (M,ω) and for every $t > 0$, $B(t) \in \mathcal{B}(X)$. If the function $t \to B(t)$ is continuous in the uniform operator topology then $A + B(t)$ generates a strongly continuous α-resolvent family $\{Q_\alpha(t)\}_{t\geq 0}$ given by the formula*

$$Q_\alpha(t) = \sum_{n=0}^{\infty} S_{\alpha,n}(t), \tag{11}$$

where

$$S_{\alpha,0}(t) := S_\alpha(t), \quad S_{\alpha,n}(t) := \int_0^t K_\alpha(t-s)B(s)S_{\alpha,n-1}(s)ds, \quad n \in \mathbb{N},$$

with

$$K_\alpha(t) := \int_0^t g_{\alpha-1}(t-s)S_\alpha(s)ds.$$

Moreover, if $K_T = \max_{t\in[0,T]} \|B(t)\|$, we have for all $t \in [0,T]$ the bounds

$$\|Q_\alpha(t)\| \leq Me^{\omega t}E_\alpha(MK_T t^\alpha)$$

and

$$\|Q_\alpha(t) - S_\alpha(t)\| \leq Me^{\omega t}(E_\alpha(MK_T t^\alpha) - 1).$$

A treatment of the perturbation problem from the point of view of the subjacent fractional abstract Cauchy problem in case $0 < \alpha \leq 1$ was given by El-Borai [22].

In the case of (α,β)-resolvent families we have the following alternative result which corresponds to a generalization of a perturbation result due to Miyadera and Voigt for C_0-semigroups.

Theorem 2.13. *[59, Theorem 3.1] Let A be the generator of an (α,β)-resolvent family $R_{\alpha,\beta}(t)$ of type (M,ω), where $\alpha \geq \beta$ and $\overline{D(A)} = X$. Let $B : D(B) \subseteq X \to X$ be a linear operator such that $D(A) \subseteq D(B)$. Suppose that there exists constants $\mu > \omega$ and $\gamma \in [0,1)$ such that*

$$\int_0^\infty e^{-\mu r}\|(g_{\alpha-\beta} * BR_{\alpha,\beta})(r)x\|dr \leq \gamma\|x\|, \quad x \in D(A),$$

then $A+B$ generates an (α,β)-resolvent family $\{R^B_{\alpha,\beta}(t)\}_{t\geq 0}$ on X of type $(\frac{M}{1-\gamma},\mu)$ that satisfies

$$R^B_{\alpha,\beta}(t)x = R_{\alpha,\beta}(t)x + \int_0^t R^B_{\alpha,\beta}(t-r)(g_{\alpha-\beta} * BR_{\alpha,\beta})(r)xdr, \quad x \in D(A).$$

2.5 Approximation

We will consider two kinds of approximation. The first is for fixed $\alpha \in (0,2]$ the approximation of the generators, i.e., the relations between the strong convergence of a sequence of α-resolvent families and that of the resolvents of their generators, as in the Trotter-Kato theorem for C_0-semigroups (cf. [68, Section 3.4]). The second is the approximation of the orders. As we know by the subordination principle, if $A \in C^\alpha$, then $A \in C^\beta$ for $\beta < \alpha$, so it is also natural to ask whether $S_\beta(t) \to S_\alpha(t)$ strongly as $\beta \to \alpha$, where $\{S_\beta(t)\}_{t\geq 0}$ is the β-resolvent family generated for A.

Recall that $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent operator of A whenever it exists.

Theorem 2.14. *[42, Theorem 4.2] Let $\alpha \in (0,2]$ and let $\{S^0_\alpha(t)\}_{t\geq 0}$ and $\{S^n_\alpha(t)\}_{t\geq 0}$ be strongly continuous α-resolvent families generated by A_0 and A, respectively. Assume there are constants $M > 0$ and $\omega > 0$ such that $\|S^n_\alpha(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, $n \in \mathbb{N}_0$. The following assertions are equivalent:*

(i) $S^n_\alpha(t)x \to S^0_\alpha(t)x$ as $n \to \infty$ for all $x \in X$, uniformly for t on every bounded interval.

(ii) $R(\lambda, A_n)x \to R(\lambda, A_0)x$ as $n \to \infty$ for all $x \in X$ and $\lambda > \omega^\alpha$.

Remark 2.15. *Using [51, Theorem 2.5] we observe that an analogous result holds for (α,β)-regularized families.*

For the second kind of approximation we have the following result.

Theorem 2.16. *[42, Theorem 4.5] Let A be the generator of a strongly continuous α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. Let $\{S_\beta(t)\}_{t\geq 0}$ be the β-resolvent family generated by A for $\beta \leq \alpha$. Then $S_\beta(t)x \to S_\alpha(t)x$ as $\beta \to \alpha$ for $t \geq 0$.*

2.6 Ergodicity

Let A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$. We analyze the behavior, as $t\to\infty$, of the following family of bounded and linear operators

$$A_t^{\alpha,\beta}x := \frac{1}{g_{\alpha+\beta}(t)}\int_0^t g(t-s)R_{\alpha,\beta}(s)x\,ds, \quad t>0, \quad x\in X.$$

Note that $A_t^{1,1}$ corresponds to the Cesáro mean of the semigroup $R_{1,1}(t)$. The family of operators $A_t^{1,\beta}$, $\beta>0$ was studied by S. Y. Shaw [74, Theorem 5]. The family $A_t^{2,\beta}$ was considered by Lizama and Prado [58, Example 6]. The following result corresponds to a strong ergodic theorem with rates.

Theorem 2.17. *[58, Theorem 1] Let $\alpha,\beta>0$ and A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$ such that*

$$\|R_{\alpha,\beta}(t)\|\le M_\beta t^{\beta-1}, \quad t>0.$$

The following assertions hold:
(i) The mapping $Px := \lim_{t\to\infty} A_t^{\alpha,\beta}x$ is a bounded linear projection with $Ran(P)=Ker(A)$, $Ker(P)=\overline{Ran(A)}$ and

$$D(P)=Ker(A)\oplus\overline{Ran(A)}.$$

(ii) For $0<\gamma\le 1$ and $x\in Ker(A)\oplus\overline{Ran(A)}$ we have

$$\|A_t^{\alpha,\beta}x-Px\| = O(t^{-\alpha\gamma}).$$

Other properties are given in [58]. The next result is the corresponding uniform ergodic theorem.

Theorem 2.18. *[58, Theorem 2] Let $\alpha,\beta>0$ and A be the generator of an (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t>0}$ such that*

$$\|R_{\alpha,\beta}(t)\|\le M_\beta t^{\beta-1}, \quad t>0.$$

The following assertions are equivalent
(i) $D(P)=X$ and $\|A_t^{\alpha,\beta}-P\|\to 0$ as $t\to\infty$.
(ii) $Ran(A)$ is closed.
(iii) $Ran(A^2)$ is closed.
(iv) $X=Ker(A)\oplus Ran(A)$.
Moreover, the convergence of the limit has order $O(t^{-\alpha})$.

Other properties in terms of the companion family $B_t x$ (see [58, Formula (11)]) are given in the reference [58]. We also observe, that there are analogous abelian ergodic theorems with rates of approximation for the nets:

$$A_\lambda^\alpha = \lambda^\alpha(\lambda^\alpha - A)^{-1} \text{ and } B_\lambda^\alpha = (\lambda^\alpha - A)^{-1}$$

where $\lambda^\alpha \in \rho(A)$ for $\lambda > 0$. See [58, Theorem 3 and Theorem 4] for details.

2.7 Compactness

A family $\{S(t)\}_{t\geq 0}$ of bounded and linear operators is called compact for $t > 0$ if for every $t > 0$, $S(t)$ is a compact operator.

The following theorem extends the compactness criteria for C_0-semigroups, see e.g. [23, Chapter II, Theorem 4.29].

Theorem 2.19. *[56, Theorem 3.4] [24, Theorem 3.6] [71, Proposition 16] Let $0 < \alpha < 2$ and A be the generator of an exponentially bounded α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is immediately norm continuous. Then the following assertions are equivalent:*
(i) $S_\alpha(t)$ is a compact operator for all $t > 0$.
(ii) $(\mu - A)^{-1}$ is a compact operator for all (some) $\mu > \omega^{1/\alpha}$.

Let A be the generator of an analytic semigroup $S_1(t)$ for $t > 0$ and $0 < \alpha < 1$. We consider the subordinated α-resolvent family generated by A

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s) S_1(st^\alpha)ds, \quad t > 0.$$

We know that $S_\alpha(t)$ is analytic. We have the following result.

Theorem 2.20. *[24, Theorem 3.11] Let A be the generator of an analytic semigroup $S_1(t)$ for $t > 0$. If $(\mu - A)^{-1}$ is compact for every (some) $\mu > 0$, then for any $\alpha \in (0, 1)$, $S_\alpha(t)$ is a compact analytic α-resolvent family for every $t > 0$.*

We now consider the (α, α)-resolvent family $R_\alpha(t) = t^{\alpha-1}P_\alpha(t)$, where

$$P_\alpha(t) = \int_0^\infty \alpha s \Phi_\alpha(s) S_1(st^\alpha)ds, \quad t > 0.$$

Theorem 2.21. *[77, Lemma 3.1 and Theorem 3.5] Let $0 < \alpha \leq 1$ and $A \in \Theta_\omega^\gamma(X)$, $-1 < \gamma < 0 < \omega < \pi/2$ be given. If $(\mu - A)^{-1}$ is compact for every (some) $\mu > 0$, then $S_\alpha(t)$ and $P_\alpha(t)$ are compact for every $t > 0$.*

For the case of (α,β)-resolvent families we have the following result.

Theorem 2.22. *[71, Theorem 14] Let A be the generator of an exponentially bounded (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ for some $\alpha > 0$ and $1 < \beta \leq 2$. Then $R_{\alpha,\beta}(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Two distinguished cases are the following

Theorem 2.23. *[71, Proposition 17] Let $3/2 < \alpha < 2$ and A be the generator of an exponentially bounded and immediately norm continuous $(\alpha,\alpha-1)$-resolvent family $\{R_{\alpha,\alpha-1}(t)\}_{t\geq 0}$. Then $R_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Theorem 2.24. *[71, Proposition 18] Let $1/2 < \alpha < 1$ and A be the generator of an exponentially bounded and immediately norm continuous (α,α)-resolvent family $\{R_{\alpha}(t)\}_{t\geq 0}$. Then $R_{\alpha}(t)$ is compact for all $t > 0$ if and only if $(\lambda - A)^{-1}$ is compact for every (some) $\lambda \in \rho(A)$.*

Concerning compactness of the generator, we have the following criteria.

Theorem 2.25. *[47, Theorem 4.2] [4, Theorem 1] Let A be the generator of an α-resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$. The following assertions are equivalent*
(i) A is a compact operator.
(ii) $S_{\alpha}(t) - I$ is a compact operator for all $t > 0$.
(iii)$\lambda R(\lambda, A) - I$ is compact a compact operator for all (some) $\lambda \in \rho(A)$.

2.8 Fractional powers of generators

If $0 \in \rho(A)$ for a sectorial operator A, then we can define its fractional powers as follows: For $b > 0$, define A^{-b} by

$$A^{-b} := \frac{1}{2\pi i} \int\limits_{\Gamma(\xi)} \lambda^{-b}(\lambda - A)^{-1} d\lambda$$

where the path $\Gamma(\xi)$ runs in the resolvent set of A from $\infty e^{-i\xi}$ to $\infty e^{i\xi}$, while avoiding the negative real axis and the origin, and λ^{b} is taken as the principal branch. Noticing that $A^{-b} \in \mathcal{B}(X)$ is injective for all $b > 0$, we can define $A^{b} := (A^{-b})^{-1}$ and $A^{0} := I$.

On the other hand, for a sectorial operator A without the assumption that $0 \in \rho(A)$, since $A + \epsilon$ is sectorial and $0 \in \rho(A + \epsilon)$ it makes sense to consider the operator $(A+\epsilon)^b$ and define the fractional powers of A by

$$A^b := s - \lim_{\epsilon\to 0^+}(A+\epsilon)^b,$$

for $b > 0$. For further information we refer the reader to the monographs [28] and [61].

Theorem 2.26. *[41, Theorem 3.1] Let* $\alpha \in (0,2], \gamma \in (0,2)$ *and* $A \in Sect(\pi - \frac{\alpha}{2}\pi)$.

(i) For each $\beta \in \left(0, \frac{2\pi-\pi\gamma}{2\pi-\pi\alpha}\right)$ *we have* $-A^\beta \in \mathcal{A}^\gamma(\varphi_0)$, *with* $\varphi_0 := \min\{\frac{\pi}{2}, -\frac{\beta}{\gamma}(\pi - \frac{\pi}{2}\alpha) + \frac{\pi}{\gamma} - \frac{\pi}{2}\}$.

(ii) If $0 \in \rho(A)$, *then the* γ*-resolvent family generated by* $-A^\beta$, $S_\gamma^\beta(t)$, *can be represented by*

$$S_\gamma^\beta(t) = \frac{1}{2\pi i}\int_{\Gamma_\omega} E_\gamma(-\mu^\beta t^\gamma)(A-\mu)^{-1}d\mu, \quad t>0,$$

where Γ_ω *is a smooth path in the resolvent set of* A *from* $\infty e^{-i\omega}$ *to* $\infty e^{i\omega}$, *avoiding the negative axis and zero, with* $\omega \in (\pi - \alpha\frac{\pi}{2}, \frac{1}{\beta}(\pi - \gamma\frac{\pi}{2})$.

The purpose of the following result is to characterize the domains $D(A^\gamma)$, $0 < \gamma < 1$, of fractional powers of sectorial operators. Recall that $-A$ generates a bounded analytic α-resolvent family for some $0 < \alpha < 2$ if and only if A is sectorial. See Theorem 1.23.

Theorem 2.27. *[16, Theorem 4.8] Let* $-A$ *be the generator of a bounded analytic* α*-resolvent family* $\{S_\alpha(t)\}_{t\geq 0}$ *for some* $0<\alpha<1$. *Let* $0<\gamma<1$ *and* $x \in X$. *The following assertions are equivalent:*

(i) $x \in D(A^\gamma)$;

(ii) $\int_0^1 g_{\beta-\gamma}(t)S_\alpha(t^{1/\alpha})x\,dt \in D(A^\beta)$ *for all (some)* $\gamma<\beta<1$;

(iii) $\lim_{\epsilon\to 0}\int_\epsilon^\infty g_{\beta-\gamma}(t)A^\beta S_\alpha(t^{1/\alpha})x\,dt$ *exists for all (some)* $\gamma<\beta<1$.

Moreover, for all $x \in D(A^\gamma)$ *and* $0<\gamma<\beta<1$:

$$A^\gamma x = s - \lim_{\epsilon\to 0}\frac{\Gamma(1-\alpha(\beta-\gamma))}{\Gamma(1-\beta+\gamma)}\int_\epsilon^\infty g_{\beta-\gamma}(t)A^\beta S_\alpha(t^{1/\alpha})x\,dt.$$

An analogous result holds in case $1 \le \alpha < 2$. See [16, Theorem 4.9]. Concerning integrated α-resolvent families, see [16, Theorem 4.15]. For a related result, see Zacher [80, Theorem 3.2, Remark 3.2 (ii) and Theorem 3.3].

Concerning square root reductions we recall that if $-A$ generate a bounded 2-resolvent family, i.e. a bounded cosine operator function $\{C(t)\}_{t\in\mathbb{R}}$, on a UMD space X, then $iA^{1/2}$ generates a bounded C_0-group $\{U(t)\}_{t\in\mathbb{R}}$ on X and

$$C(t) = \frac{U(t)+U(-t)}{2}, \quad t \in \mathbb{R}.$$

It is surprising that for a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ with some $\alpha \in (0,2)$, the square root reduction of $S_\alpha(t)$ always exists on a Banach space (not necessarily to be UMD).

Theorem 2.28. *[15, Proposition 5.6] Let $\alpha \in (0,2)$. The operator A generates a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ on X if and only if $\pm i[(-A)]^{1/2}$ generate bounded analytic $\alpha/2$-resolvent families $\{S^{\pm}_{\alpha/2}(t)\}_{t\ge 0}$ on X. Moreover,*

$$S_\alpha(t) = \frac{S^{+}_{\alpha/2}(t)+S^{-}_{\alpha/2}(t)}{2}, \quad t \ge 0.$$

A complementary result is the following: As we know, when A generates a C_0-group, then A^2 also generates a cosine operator function. The extension of this result to α-resolvent families read as follows

Theorem 2.29. *[42, Proposition 2.8] Suppose that A and $-A$ both generate an α-resolvent family for some $\alpha \in (0,1]$. Then A^2 generates a 2α-resolvent family.*

2.9 Spatial regularity

In this section we state some results on spatial regularity for α-resolvent families $\{S_\alpha(t)\}_{t\ge 0}$. See also [72, Theorem 2.2 p.57].

Theorem 2.30. *[7, Proposition 2.15] Let $\alpha \in (0,2)$ and assume $A \in \mathcal{A}^\alpha(\theta_0,\omega_0)$ then for any $x \in X$ and $t > 0$ we have $S_\alpha(t)x \in D(A)$ and*

$$\|AS_\alpha(t)\| \le Ce^{\omega t}(1+t^{-\alpha}), \quad t>0, \quad \omega>\omega_0.$$

Theorem 2.31. *[16, Lemma 4.1] Let $-A$ be the generator of a bounded analytic α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ for some $0<\alpha<2$. Then*

(i) If $1 \le \alpha < 2$, then $S_\alpha(t)x \in D(A^\infty) := \bigcap_{k=1}^{\infty} D(A^k)$ for all $x \in X$, $t>0$, and

$$\|A^\beta S_\alpha(t)\| \le Ct^{-\alpha\beta},$$

for each $\beta > 0$. *Moreover,* $D(A^\infty)$ *is dense in* X.

(ii) If $0 < \alpha < \frac{1}{m}$ *for some* $m \in \mathbb{N}$, *then for each* $x \in D(A^{m-1})$ *we have* $S_\alpha(t)x \in D(A^m)$ *and*

$$\|A^m S_\alpha(t)x\| \le Ct^{-m\alpha}\|x\| + \sum_{k=1}^{m} g_{1-k\alpha}(t)\|A^{m-k}x\|, \quad t > 0.$$

Theorem 2.32. *[77, Theorem 3.3] Let* $0 < \alpha \le 1$ *and* $A \in \Theta_\omega^\gamma(X)$, $-1 < \gamma < 0 < \omega < \pi/2$. *Let* $0 < \beta < 1 - \gamma$. *For all* $x \in D(A)$ *and* $t > 0$

$$\|AS_\alpha(t)x\| \le Ct^{-\alpha(1+\gamma)}\|Ax\|,$$

where C *is a constant depending on* γ, α.

For the next results, recall that $\{R_\alpha(t)\}_{t>0}$ denotes an (α,α)-resolvent family.

Theorem 2.33. *[76, Theorem 2.3] Let* $0 < \alpha < 1$ *and* $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$. *Then there exists a constant* $C > 0$ *such that*

$$\|R_\alpha(t)\| \le Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0. \tag{12}$$

Theorem 2.34. *[81, Section 2.2.2 Lemma 2.10 (iv)] Let* $0 < \alpha < 1$. *Let* A *be the generator of an exponentially stable semigroup* $\{T(t)\}_{t\ge 0}$, *i.e. there exist constants* $\delta > 0$ *and* $M > 0$ *such that* $\|T(t)\| \le Me^{-\delta t}$. *Then*

$$\|R_\alpha(t)\| \le Mt^{\alpha-1}E_{\alpha,\alpha}(-\delta t^\alpha), \quad t > 0.$$

The following important result is due to Cuesta [19]. However, it should be noted that his notion of sectorial operator differs from ours.

Theorem 2.35. *[19, Theorem 1] Suppose that* A *is sectorial of negative type and angle* $\theta \in [0, \pi(1 - \alpha/2))$ *then there exists* $C > 0$ *depending solely on* θ *and* α *such that*

$$\|S_\alpha(t)\| \le \frac{CM}{1 + |\omega|t^\alpha}.$$

For additional results, see the end of the subsection 2.3.

3 The fractional abstract Cauchy problem

Recall that the Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^C D_t^\alpha f(t) := J_t^{m-\alpha}\frac{d^m}{dt^m} f(t)$$

where m is the smallest integer greater than or equal to α. We consider the solutions of fractional Cauchy problems. First we give the definition of solutions to the inhomogeneous initial value problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \in (0,\tau) \\ u^{(k)}(0) = x_k, & k = 0,1,...,m-1 \end{cases} \tag{1}$$

where $\tau \in (0,\infty]$, $f \in L^1_{loc}([0,\tau), X)$ and A is a closed densely defined operator in a Banach space X. Recall that the connection between ${}^C D_t^\alpha$ and the Riemann-Liouville fractional derivative ${}^{RL} D_t^\alpha$ is given by

$${}^C D_t^\alpha u(t) \;=\; {}^{RL} D_t^\alpha \Big(u(t) - \sum_{k=0}^{m-1} g_{k+1}(t) x_k \Big).$$

Definition 3.1. *A function $u \in C([0,\tau), X)$ is called a strong (or classical) solution of* (1) *if $u(t)$ satisfies:*
(a) $u \in C([0,\tau), D(A)) \cap C^{m-1}([0,\tau), X)$.
(b) $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} g_{k+1} x_k) \in C^m([0,\tau), X)$.
(c) $u(t)$ satisfies (1)*.*

Definition 3.2. *A function $u \in C([0,\tau), X)$ is called a mild solution of* (1) *if $(g_\alpha * u)(t) \in D(A)$ and*

$$u(t) = \sum_{k=0}^{m-1} g_{k+1}(t) x_k + A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \in [0,\tau).$$

If A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ then the mild solution of (1) on $\mathbb{R}_+$ can be represented by

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t) x_k + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \geq 0, \tag{2}$$

whenever $x_k \in X$ for all $k = 0,...,m-1$.

3.1 The fractional order homogeneous problem

We briefly analyze the homogeneous fractional Cauchy problem:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), & t \in (0,\tau) \\ u^{(k)}(0) = x_k, & k = 0,1,...,m-1. \end{cases} \tag{3}$$

In case $\tau = \infty$ and analogously to the cases $\alpha = 1$ and $\alpha = 2$ we have the following concept of well-posedness in the sense of Hadamard [72].

Definition 3.3. *The problem* (3) *is called well-posed if for any* $x_k \in D(A), k = 0, 1, ..., m-1$, *there is a unique strong solution* $u_k(t)$ *of* (3), *and* $x_{k,n} \in D(A)$, $x_{k,n} \to 0$ *as* $n \to \infty$, *imply* $u_{k,n}(t) \to 0$ *as* $n \to \infty$ *in* X, *uniformly on compact intervals.*

By [72, Proposition 1.1] the problem (3) is well-posed if and only if A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$, and the unique strong solution is given by

$$u(t) = \sum_{k=0}^{m-1} (g_k * S_\alpha)(t)x_k, \quad t \geq 0,$$

whenever $x_k \in D(A)$ for all $k = 0, 1, ..., m-1$. In particular,

$$u(t) = S_\alpha(t)x$$

is the unique strong solution of the problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), & t > 0 \\ u(0) = x, \quad u^{(k)}(0) = 0, & k = 1, ..., m-1, \end{cases} \tag{4}$$

for each $x \in D(A)$ (recall that g_0 is the Dirac delta). See e.g. [14, Proposition 3.3]. According to Theorem 1.15, if A is the generator of an α-resolvent family, then the study of well-posedness for the problem (4) should be resctricted to the case $0 < \alpha \leq 2$. However, note that the concept of well-posedness may vary.

3.2 The fractional order inhomogeneous problem

We now turn to the following problem

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \in (0, \tau) \\ u^{(k)}(0) = 0, & k = 0, 1, ..., m-1. \end{cases} \tag{5}$$

For the existence of strong solutions of (5) we have

Theorem 3.4. *[41, Proposition 4.3] Let* $\alpha \in (0, 2]$. *Suppose that* A *is the generator of a strongly continuous* α*-resolvent family* $\{S_\alpha(t)\}_{t \geq 0}$ *and* $f \in C([0, \tau), X)$. *Then the following statements are equivalent:*

(a) (5) *has a strong solution on* $[0, \tau)$.

(b) $S_\alpha * f$ *is differentiable on* $[0, \tau)$.

(c) $\frac{d}{dt}(g_\alpha * S_\alpha * f)(t) \in D(A)$ *for* $t \in [0, \tau)$ *and* $A(\frac{d}{dt}(g_\alpha * S_\alpha * f)(t))$ *is continuous on* $[0, \tau)$.

In the case $\alpha \in [1,2]$, *the condition (c) can be replaced by*
(c)' $(g_{\alpha-1} * S_\alpha * f)(t) \in D(A)$ *for* $t \in [0,\tau)$ *and* $A(g_{\alpha-1} * S_\alpha * f)(t)$ *is continuous on* $[0,\tau)$.

As consequence we have

Corollary 3.5. *[41, Corollary 4.4], [46, Corollary 3.4] Let* $\alpha \in (0,2]$. *Suppose that* A *is the generator of a strongly continuous* α*-resolvent family. Then* (5) *has a strong solution on* $[0,\tau)$ *if one of the following conditions is satisfied:*
(a) f *is continuously differentiable on* $[0,\tau)$.
(b) $\alpha \in [1,2]$, $f(t) \in D(A)$ *for* $t \in [0,\tau)$ *and* $Af \in L^1_{loc}([0,\tau), X)$.
(c) $\alpha \in (0,1)$, $f(t) \in D(A)$ *for* $t \in [0,\tau)$ *and* $g_\alpha * f$ *is continuously differentiable on* $[0,\tau)$.
(d) $\alpha \in (1,2)$, $(g_{\alpha-1} * f) \in L^1((0,\tau), D(A))$.

Maximal regularity results for (5) has been studied by a number of authors under different methods and in several classes of spaces of vector valued functions. We refer the interested reader to [11, 12, 13, 29, 54] and references therein. A characterization of existence of mild solutions for (5) with boundary condition $u(0) = u(\tau)$, in terms of α-resolvent families, has been proved in [30].

If $-A$ is an m-accretive operator defined on a Hilbert space H, and $f \in L^2([0,\tau]; H)$ maximal regularity conditions for (5) with Riemann-Liouville fractional derivative of order $\alpha \in (1,2]$, was studied by Bazhlekova [7]. She rewrite the problem as a system of equations of order $\alpha/2$ in $\mathcal{H} := D((-A)^{1/2}) \times H$. If $-A$ is a positive operator, the survey [17] review some regularity results for (5). See also [18] for more on regularity properties of solutions of fractional evolution equations.

The solvability and maximal L^p regularity of the linear nonautonomous problem i.e. $A = A(t)$ where $A(t)$ is a family of linear closed densely defined operators defined on a Banach space X, such that the domain of $A(t)$ does not depend on t, with the Riemann-Liouville fractional derivative, was considered by Bazhlekova [7, Section 6.1].

3.3 The fractional order Cauchy problem: $0 < \alpha \leq 2$.

We consider the fractional order linear Cauchy problem:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t), & t \geq 0 \\ u(0) = u_0, \quad \max\{0, \alpha - 1\}(u'(0) - u_1) = 0, & 0 < \alpha \leq 2. \end{cases} \tag{6}$$

According to the previous subsections, if A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ and $0 < \alpha \leq 1$ then the mild solution of (6) can be represented by

$$u(t) = S_\alpha(t)u_0 + \frac{d}{dt}(g_\alpha * S_\alpha * f)(t), \quad t \geq 0. \tag{7}$$

In the case that A is the generator of an (α,α)-resolvent family $\{R_\alpha(t)\}_{t>0}$ we obtain that $S_\alpha(t) := (g_{1-\alpha} * R_\alpha)(t)$ is an α-resolvent family generated by A and the representation

$$u(t) = S_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s)ds, \quad t \geq 0. \tag{8}$$

Therefore, starting with A as the generator of an (α,α)-resolvent family is more natural and appropriate. See the remark at the end of Subsection 1.3. These is the more general hypothesis to be used in concrete applications.

It should be observed that if the problem (6) is considered with Riemann-Liouville fractional derivative, then we have

$$u(t) = R_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s)ds, \quad t > 0. \tag{9}$$

See also the paper of Fan [25] for additional information and results. Using the subordination principle, one can give explicit descriptions of (α,β)-resolvent families in some cases. This fact has been widely used in the literature. For example, if A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ and $0 < \alpha < 1$ then we have that $R_\alpha(t) = t^{\alpha-1}P_\alpha(t)$, where

$$P_\alpha(t) = \alpha \int_0^\infty s\Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0,$$

is the (α,α)-resolvent family generated by A, and

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0,$$

is the α-resolvent family generated by A. For up to date information, applications and historical remarks concerning this formulation, see the monograph of Gal and Warma [26, Section 2].

In the case that $1 < \alpha \leq 2$ we have the representation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + (g_{\alpha-1} * S_\alpha * f)(t), \quad t \geq 0. \tag{10}$$

Assuming that A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ we obtain that $R_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$ is an (α,α)-resolvent family generated by A and

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(s)u_1 ds + \int_0^t R_\alpha(t-s)f(s)ds, \quad t \geq 0. \tag{11}$$

Using again the subordination principle, we can give an explicit description: If A is the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\geq 0}$ and $1 < \alpha < 2$ then we have that

$$S_\alpha(t) = \int_0^\infty \Phi_{\alpha/2}(s)S_2(st^{\alpha/2})ds,$$

is the α-resolvent family generated by A.

To complete the picture, we observe that for the Riemann-Liouville fractional derivative we have the following representation of the solutions:

$$u(t) = L_{\alpha,\alpha-1}(t)u_0 + \int_0^t L_{\alpha,\alpha-1}(s)u_1 ds + \int_0^t R_\alpha(t-s)f(s)ds \tag{12}$$

where $\{L_{\alpha,\alpha-1}(t)\}_{t>0}$ is an $(\alpha,\alpha-1)$-resolvent family generated by A and $R_\alpha(t) := (g_1 * L_{\alpha,\alpha-1})(t)$ is an (α,α)-resolvent family. Note that if A is a bounded operator, then we have

$$L_{\alpha,\alpha-1}(t) = t^{\alpha-2}E_{\alpha,\alpha-1}(At^\alpha), \quad t > 0, \quad 1 < \alpha \leq 2.$$

The article [32] investigates the solvability of the fractional order inhomogeneous Cauchy problem (6) using integrated α-resolvent families for $0 < \alpha \leq 1$. When A is the generator of a β-times integrated semigroup on a Banach space X, with $\beta \geq 0$, explicit representations of mild and classical solutions of the above problem in terms of the integrated semigroup are derived. Then, the existence, uniqueness and regularity of the solutions of problem (6) is studied. The results are applied to the fractional diffusion equation with non-homogeneous, Dirichlet, Neumann and Robin boundary conditions and to the time fractional order Schrödinger equation $\mathbb{D}_t^\alpha u(t,x) = e^{i\theta}\Delta_p u(t,x) + f(t,x), t > 0,\ x \in \mathbb{R}^N$ where $\pi/2 \leq \theta < (1-\alpha/2)\pi$ and Δ_p is a realization of the Laplace operator on $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. In case of $1 < \alpha \leq 2$ analogous results are proved in the paper [31]. The solvability of the fractional order Cauchy problem (6) with Riemann-Liouville fractional derivative has been studied in the paper [65] by Mophou and N'Guérékata.

3.4 Comments and Open Problems

A theory of resolvent families in order to consider fractional abstract equations in the form

$$\begin{cases} {}^C D^{\alpha} u(t) + c_1 \ {}^C D^{\beta_1} u(t) + ... + c_d \ {}^C D^{\beta_d} u(t) = Au(t) & t > 0 \\ u^{(k)}(0) = x_j, & k = 0, 1, ..., m-1 \end{cases} \tag{13}$$

where $\alpha > \beta_1 ... > \beta_d > 0, c_j$ are constants and m is the smallest integer greater that or equal to α, has been introduced by G.G. Li, Kostic, M. Li and Piskarev [43]. They derive generation theorems, algebraic equations and approximation theorems for such resolvent families.

Since the domain of the generator of (α, α)-resolvent families is dense in the case $0 < \alpha = \beta < 1$ it would be interesting to know if the same property remains true in case $0 < \alpha < 1$ and $0 < \beta < 1$.

It should be noted that the theory of (a, k)-regularized families introduced in [50] covers most of the results about (α, β)-resolvents. A more general notion of (local) (a, k)-regularized C-resolvent family was introduced by Kostić in [36]. If $a(t) = g_{\alpha}(t)$ and $k(t) = 1$ we obtain the class of α-times C-regularized resolvent families introduced in 2010 by Chen and Li [14]. A connection of this general theory with abstract time-fractional equations appears in [37].

It is well known that both strongly continuous semigroups and strongly continuous cosine functions are necessarily exponentially bounded. However, whether an (α, β)-resolvent family is exponentially bounded is unknown in general.

When $1 < \alpha \leq 2$, a careful study of $(\alpha, \alpha - 1)$-resolvent families seems to be missing in the literature, except for the general properties stated in this Chapter. They are necessary in order to study the existence, uniqueness and qualitative behavior of solutions for the fractional abstract Cauchy problem with Riemann-Lioville fractional order derivative for both, the linear and nonlinear cases. For instance, an explicit representation in terms of an strongly continuous cosine family generated by A and some special density function should be very useful.

Recall that a family of bounded and linear operators $\{S(t)\}_{t \geq 0}$ is called exponentially stable if there are constants $M > 0$ and $\omega > 0$ such that

$$\|S(t)\| \leq M e^{-\omega t}, \quad t \geq 0. \tag{14}$$

The next result shows that when $0 < \alpha < 1$, an α-resolvent family is never exponentially stable.

Theorem 3.6. *[42, Proposition 2.7] Suppose that* $\alpha \in (0, 1)$ *and* $\{S_\alpha(t)\}_{t\geq 0}$ *is a strongly continuous* α*-resolvent family generated by* A*, then* $\{S_\alpha(t)\}_{t\geq 0}$ *does not decay exponentially.*

The same happens in case of $\alpha = 2$ because of D'Alembert functional equation [75].

A criteria for stability of α-resolvent families is unknown, except in the finite dimensional case, where the most well known criteria is Matignon's stability theorem [62]. An extension of Matignon's result to the case $1 < \alpha < 2$ is given in [73]. The general abstract case, when A is generator of an α-resolvent family, remains widely open and seems to be a very difficult task. The study of spectral mapping theorems for (α, β)-resolvent families, which are also missing in the literature, could be a remarkable advance to such objective.

There is a large amount of literature dealing with the subject of this chapter, see for instance [33, 34, 35] and references therein. However, the list of references do not escape the usual rule of being incomplete. In general, I have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and I apologize for omissions.

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