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HÖLDER–LEBESGUE REGULARITY AND ALMOST PERIODICITY FOR SEMIDISCRETE EQUATIONS WITH A FRACTIONAL LAPLACIAN

Carlos Lizama*

Departamento de Matemática y Ciencia de la Computación, Facultad de Ciencias
Universidad de Santiago de Chile
Casilla 307, Correo 2, Santiago, Chile

Luz Roncal

BCAM - Basque Center for Applied Mathematics
Alameda de Mazarredo 14, 48009 Bilbao, Spain

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Abstract. We study the equations

$$\partial_t u(t,n) = Lu(t,n) + f(u(t,n),n); \partial_t u(t,n) = iLu(t,n) + f(u(t,n),n)$$

and

$$\partial_{tt} u(t,n) = Lu(t,n) + f(u(t,n),n),$$

where $n \in \mathbb{Z}$, $t \in (0,\infty)$, and L is taken to be either the discrete Laplacian operator $\Delta_{\mathrm{d}} f(n) = f(n+1)-2f(n)+f(n-1)$, or its fractional powers $-(-\Delta_{\mathrm{d}})^{\sigma}$, $0<\sigma<1$. We combine operator theory techniques with the properties of the Bessel functions to develop a theory of analytic semigroups and cosine operators generated by Δ_{d} and $-(-\Delta_{\mathrm{d}})^{\sigma}$. Such theory is then applied to prove existence and uniqueness of almost periodic solutions to the above-mentioned equations. Moreover, we show a new characterization of well-posedness on periodic Hölder spaces for linear heat equations involving discrete and fractional discrete Laplacians. The results obtained are applied to Nagumo and Fisher–KPP models with a discrete Laplacian. Further extensions to the multidimensional setting $\mathbb{Z}^N$ are also accomplished.

1. Introduction: background and main results.

The study of lattice differential equations has experienced a rapid increase in the last few years. The literature in the subject is very vast, and we will try to give just a glimpse. For instance, it is remarkable the thorough investigation of topics related to traveling waves in spatially discrete dynamical systems, such as the study of their existence, and monotonicity and stability properties, see [10, 28, 32, 33, 47] and references

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* Corresponding author: Carlos Lizama.

therein. Studies on qualitative properties, like the relationship between exponential dichotomies of nonautonomous difference equations and admissibility for classes of weighted bounded solutions have been developed in [52]. Spatial dynamics of lattice differential equations with delay are investigated in [29]. Also, the existence of wave train solutions for the semidiscrete wave equation with a discrete Laplacian in $\mathbb{Z}^2$ was considered in [51]. Quite recently, a study related to the semidiscrete transport equation has been carried out in [2]. Finally, harmonic analysis associated to a discrete Laplacian and in the context of ultraspherical expansions has been also developed in [11] and [7], respectively.

On the other hand, lattice differential equations with a fractional discrete Laplacian arise to understand the behavior of processes related to small objects where the continuum limit cannot describe events on length scales comparable to nanometers [42]–[46]. The lattice approach gives a possible microstructural basis for anomalous diffusion in media that are characterized by the non-locality of power law type. Examples of such behavior can be found on the deformations and diffusion processes in solid objects like nanocrystalline and ultrafine grain polycrystals, both with and without external forces [3], nanomechanics [44] and N-dimensional physical lattices with long-range particle interactions [45]. For instance, a 3D lattice Fokker-Plank equation to describe space-fractional diffusion processes was proposed in [43]. In all of these cases, the main advantage of the semidiscrete analysis over the continuum is the better accuracy in the description of the phenomena [9]. However, much work remains to be done, especially concerning properties of the solutions either in the linear or nonlinear cases and qualitative and regularity properties of such solutions. It is worth to mentioning the analysis of nonlocal discrete equations driven by fractional powers of the discrete Laplacian on a mesh performed in [13, 14], where some properties concerning this operator are studied, as well as error estimates for the approximation problem and regularity estimates in Hölder spaces. On the other hand, a relation between fractional powers of the discrete Laplacian with a fractional derivative in the sense of Liouville was investigated in [12].

1.1. **Background.** To begin with, the typical equation describing semidiscrete diffusion models is the semidiscrete heat equation

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = \Delta_{\mathrm{d}} u(t,n) + g(t,n), \quad t>0,\ n\in\mathbb{Z}, \\ u(0,n) = \varphi(n), \quad n\in\mathbb{Z}, \end{cases} \tag{1}$$

where u is the unknown function, the sequence $\varphi = \{\varphi(n)\}_{n\in\mathbb{Z}}$ is the initial datum, g is a function representing a forcing term, and Δ_{d} is the second order central difference operator

$$\Delta_{\mathrm{d}} f(n) = f(n-1) - 2f(n) + f(n+1),$$

which we will call the discrete Laplacian. Analogously, we could formulate the semidiscrete Schrödinger equation and the wave equation. In many cases, one has to study nonlinear models associated to the above-mentioned processes.

The solution to the homogeneous version of the equation (1) (i.e., $g\equiv 0$) is known to be *the semidiscrete heat semigroup*, which is represented by the series

$$\mathcal{T}_t\varphi(n) = \sum_{m\in\mathbb{Z}} T_t(n-m)\varphi(m), \quad t\geq 0, \quad n\in\mathbb{Z},$$

where the kernel is given by

$$T_t(n) = e^{-2t} I_n(2t), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{2}$$

and I_n denotes the modified Bessel function. This result is known within a formulation of probability theory in the framework of birth-and-death processes and random walks, cf. [22, Ch. XVII.5] and [23, Ch. II.7 and Ch. XIV.6]. It has been also found by means of statistical methods [48] or by using Fourier series techniques [19]. Very recently, it has been used in different contexts. For instance, in the study of physical models [26, 27], mutation models in population biology [9, Section 4] or for the development of harmonic analysis associated to Δ_{d}, see [11]. Furthermore, one more way to find the kernel (2) as the fundamental solution of the semidiscrete heat equation has been rediscovered recently [29], where a lattice differential equation model describing growth and spread of species in a shifting habitat is studied.

Let $0 < \sigma < 1$. The fractional powers of the discrete Laplacian can be defined as

$$(-\Delta_{\mathrm{d}})^{\sigma} f(n) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} \big(\mathcal{T}_t f(n) - f(n)\big) \frac{dt}{t^{1+\sigma}}, \quad n \in \mathbb{Z},$$

see [11, Section 3], [12, Theorem 2], [13], and [41]. Let us consider equation (1) with the fractional powers of the discrete Laplacian $-(-\Delta_{\mathrm{d}})^{\sigma}$, instead of Δ_{d}, namely

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = -(-\Delta_{\mathrm{d}})^{\sigma} u(t,n) + g(t,n), \quad t > 0,\ n \in \mathbb{Z}, \\ u(0,n) = \varphi(n), \quad n \in \mathbb{Z}. \end{cases} \tag{3}$$

It seems that basic properties for the solutions of the non-homogeneous equation (3) involving the fractional discrete Laplacian either in the linear or in the nonlinear cases, and qualitative and regularity properties of such solutions are not known. The motivation for such study relies on recent formulations of fractional calculus for N-dimensional lattices that showed to be helpful in applications where long-range physical particle interactions appear, as well as in the field of nanomechanics. The references [44, 45, 46] cover several of these phenomena and demonstrate the importance of fractional discrete equations.

On the basis of analytic semigroups and cosine operators theory, we analyze the heat, Schrödinger, and wave models, both in the case of integer and in the case of fractional spatial discretization. We will show that this approach facilitates the analysis of qualitative properties of the model, providing directly their main general properties.

More precisely, the equations under investigation in this paper are

$$\frac{\partial u(t,n)}{\partial t} = Lu(t,n) + G(t,n), \quad u(0,n) = \varphi(n), \tag{4}$$

$$\frac{\partial w(t,n)}{\partial t} = iLw(t,n) + G(t,n), \quad w(0,n) = \psi(n), \tag{5}$$

and

$$\frac{\partial^2 v(t,n)}{\partial t^2} = Lv(t,n) + G(t,n), \quad v(0,n) = \xi(n),\ v_t(0,n) = \eta(n), \tag{6}$$

where $n \in \mathbb{Z}$. The notation L means either the discrete Laplacian Δ_{d} or the fractional powers $-(-\Delta_{\mathrm{d}})^{\sigma}$, and the sequences φ, ψ, ξ and η are the initial data at $t = 0$. The function $G(t,n)$ is the forcing term. We will consider both linear and nonlinear type equations, namely, we will study the equations above in the case in which either $G(t,n)$ is a function of the form $g(t,n)$, $t \geq 0$, or of the form $g(u(t,n),n)$, $g(w(t,n),n)$, or $g(v(t,n),n)$, with $t \in J := [0,T]$.

Our first aim is, therefore, to develop a detailed theory of analytic semigroups and cosine operators generated by the operators Δ_{d} and $-(-\Delta_{\mathrm{d}})^{\sigma}$. We will merge some techniques from operator theory and a careful use of properties of Bessel functions. Actually these results, contained in Theorems 1.1, 1.2, and 1.3, are the core of the paper.

Our second goal will be to consider the existence of almost periodic solutions to the equations (4), (5), and (6). The existence of solutions to the linear versions of (4), (5) and (6) will follow from the general theory of semigroups and cosine operators. We will prove, under appropriate conditions on the forcing term, an almost periodic behavior of these solutions which is not present in the continuous case, see Theorem 1.5. Concerning nonlinear models, we will show the existence and uniqueness of solutions to the equations (4), (5) and (6) when $G(t,n)$ is taken to be either $g(u(t,n),n)$, $g(w(t,n),n)$, or $g(v(t,n),n)$, see Theorem 1.6. As in the linear models, under suitable conditions on the forcing terms, the solutions present an almost periodic behavior. The proofs will be based on the previous results for the solutions to the linear models, together with an application of the Banach fixed point theorem.

As a third aim, we will show a completely new characterization of well-posedness on periodic Hölder spaces for the linear model

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = \Delta_{\mathrm{d}} u(t,n) + r u(t,n) + f(t,n), \quad t \in [0,2\pi], \ n \in \mathbb{Z}, \\ u(0,n) = u(2\pi,n), \ n \in \mathbb{Z}, \end{cases} \tag{7}$$

on periodic Hölder spaces, where $r > 0$. Namely, we will prove that the problem (7) is well-posed in $C^{\alpha}_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ if and only if $r > 4$, see Section 5 for the definitions and the proof of the results. This characterization is particularly important for the analysis of general nonlinear models by means of fixed point arguments using the implicit function theorem. In the case in which the discrete Laplacian is replaced by the fractional discrete Laplacian, we will also study the well-posedness of the corresponding problem. Maximal regularity estimates will be deduced as corollaries.

The theory of analytic semigroups and cosine operators generated by Δ_{d} and $-(-\Delta_{\mathrm{d}})^{\sigma}$ can be also described in the multidimensional setting $\mathbb{Z}^N$. This will be done in Section 6. The results for the one-dimensional case and the use of the multidimensional discrete Fourier transform will be the ingredients to develop the corresponding multidimensional theory of analytic semigroups and cosine functions. With this extension, most of the results stated in Theorems 1.5 and 1.6 for the solutions of the equations in the one-dimensional case will remain valid in the multidimensional case. We notice that the multidimensional discrete heat semigroup was implicitly used in the study of asymptotic estimates concerning the fractional discrete Laplacian (see [13, Section 5]).

Finally, we will call the attention on some nonhomogeneous and fractional versions of reaction-diffusion equations that are of particular interest, for instance in Zinner's model for the well known Fisher–KPP equation,

$$\frac{\partial u(t,n)}{\partial t} = \Delta_{\mathrm{d}} u(t,n) + r u(t,n) - r u^2(t,n) + g(t,n), \quad t > 0, \ n \in \mathbb{Z},$$

where $r > 0$ and g denotes an external forcing term. Another example is provided by the semidiscrete Nagumo equation

$$\frac{\partial u(t,n)}{\partial t} = \Delta_{\mathrm{d}} u(t,n) - au(t,n) + (1+a)u^2(t,n) - u^3(t,n) + f(t,n), \quad t > 0,\ n \in \mathbb{Z},$$

where $0 < a < 1$ and once again f is a forcing term. As applications of the results in Sections 4 and 5, we will provide results on the existence and uniqueness of almost discrete solutions to the equations above, and information on the $C^\alpha - \ell^p$ regularity, see Section 7.

The structure of the paper is the following. We present the statements of the main results in Subsection 1.2. The results on the theory of analytic semigroups and cosine operators generated by Δ_{d} and $-(-\Delta_{\mathrm{d}})^\sigma$ are stated in Theorems 1.1, 1.2, 1.3, and 1.4, and their proofs will be developed, respectively, in Sections 2 and 3. Such theories will be applied to prove the existence and uniqueness of solutions to the nonhomogeneous semidiscrete heat, Schrödinger and wave equations. These results are stated in Theorems 1.5 and 1.6, and the proofs are contained in Section 4. The results of well-posedness and maximal regularity for the linear semidiscrete heat equations are stated in Theorems 1.7 and 1.8, and Section 5 will be devoted to their proofs. In Section 6 we will show the extension of the theory of semigroups and cosine operators to the multidimensional setting. In Section 7 we will present the applications of our results to the semidiscrete Nagumo and Fisher–KPP equations. Finally, we collect facts and results concerning analytic semigroups and cosine operators, Bessel functions, almost periodic sequences and discrete almost periodic functions, and the discrete Fourier transform in Appendices A, B, C, and D. These are the general tools needed to develop the rest of the results, and we consider convenient to move them to the end of the paper, for easy reading.

1.2. **Main results.** We collect in this subsection the main results of the paper.

For every $1 \le p \le \infty$, we will denote by $\ell^p(\mathbb{Z})$, or just ℓ^p, the space of complex sequences $\{a(n)\}_{n\in\mathbb{Z}}$ such that

$$\|a\|_{\ell^p} := \Big(\sum_{n\in\mathbb{Z}} |a(n)|^p\Big)^{1/p} < \infty,\ \ 1 \le p < \infty, \quad \|a\|_{\ell^\infty} := \sup_{n\in\mathbb{Z}} |a(n)| < \infty, \quad p = \infty.$$

If X is a Banach space, $\mathcal{B}(X)$ will denote the collection of all bounded linear operators from X into itself. With an abuse of notation, we will often just write a or $a(n)$ instead of $\{a(n)\}_{n\in\mathbb{Z}}$ when referring to sequences.

Let us define $\Sigma_\pi := \{z \in \mathbb{C} : |\arg z| < \pi\}$. We write

$$T_z(n) := e^{-2z} I_n(2z), \quad z \in \mathbb{C}, \tag{8}$$

where $I_k(z)$ is the Bessel function of imaginary argument (see Appendix B for the definition and properties). We define the operator

$$\mathcal{T}_z\varphi(n) := (T_z * \varphi)(n) = \sum_{m\in\mathbb{Z}} T_z(n-m)\varphi(m), \quad z \in \mathbb{C}, \tag{9}$$

where the convolution is taken on $\mathbb{Z}$. In Theorem 1.1 we will show that the familiy of operators $\{\mathcal{T}_z\}_{z\in\Sigma_\pi}$ form an *analytic semigroup* in ℓ^p, see Definition A.1. Recall also that the definitions and facts concerning almost periodic functions are collected in Appendix C.

Theorem 1.1. *(Semidiscrete heat analytic semigroup) Given $z \in \Sigma_\pi$, let $\mathcal{T}_z$ be the operators defined in (9).*

(i) *We have* $\{\mathcal{T}_z\}_{z\in\Sigma_\pi} \subset \mathcal{B}(\ell^p)$ *and it is an analytic semigroup in* Σ_π*, with bounded generator* $\Delta_{\rm d}$ *on* ℓ^p, $1 \le p \le \infty$*. Moreover, for all* $\varphi \in \ell^p$*:*
  1. $\|(-\Delta_{\rm d})\varphi\|_{\ell^p} \le 4\|\varphi\|_{\ell^p}$.
  2. $\|\mathcal{T}_z\varphi\|_{\ell^p} \le e^{4|z|}\|\varphi\|_{\ell^p}$ *for all* $z \in \Sigma_\pi$.

(ii) *Concerning the sequence* $T_z(n)$*, we have:*
  1. $\|T_z\|_{\ell^p} \le e^{4|z|}$*, for* $1 \le p \le \infty$.
  2. $T_z(n)$ *is almost periodic in the variable* n*, for each* $z \in \Sigma_\pi$.

(iii) *The spectrum of the operators* $\Delta_{\rm d}$ *and* $\mathcal{T}_z$ *is, respectively,*

$$\sigma(\Delta_{\rm d}) = \{-4\sin^2\theta/2\}_{\theta\in(-\pi,\pi]} = [-4,0], \qquad \textit{and} \qquad \sigma(\mathcal{T}_z) = \{e^{-4z\sin^2\theta/2}\}_{\theta\in(-\pi,\pi]}.$$

From Theorem 1.1, by the general semigroup theory, we have that the solution to the homogeneous version of (5) with $L = \Delta_{\rm d}$ is given by

$$\mathcal{U}_t\psi(n) = \sum_{m\in\mathbb{Z}} e^{-2it} I_{n-m}(2it)\psi(m).$$

Let us call

$$C_t(n) := J_{2n}(2t), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}, \tag{10}$$

where J_k is the Bessel function of first kind (see Appendix B for the definition and properties) and consider the operator

$$\mathcal{C}_t\xi(n) := (C_t * \xi)(n) = \sum_{m\in\mathbb{Z}} C_t(n-m)\xi(m), \quad t \in \mathbb{R}. \tag{11}$$

We shall prove in the following result that $\{\mathcal{C}_t\}_{t\in\mathbb{R}}$ is a *cosine operator family* (see Definition A.2) which is also bounded on ℓ^p and we provide several properties for the sequence $C_t(n)$.

Theorem 1.2. *(Semidiscrete wave cosine family) Given* $t \in \mathbb{R}$*, let* $\mathcal{C}_t$ *be the operators defined in* (11).

(i) *We have* $\{\mathcal{C}_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ *and it is an uniformly continuous cosine family, with bounded generator* $\Delta_{\rm d}$ *on* ℓ^p, $1 \le p \le \infty$.

(ii) *Concerning the sequence* $C_t(n)$*, we have*
  1. $\|C_t\|_{\ell^p} \le (2\cosh|t|^p - 1)^{1/p}$*, for* $1 \le p < \infty$.
  2. $C_t(n)$ *is almost periodic in the variable* n*, for each* $t \in \mathbb{R}$*, and also almost periodic in the variable* t*, for each* $n \in \mathbb{Z}$.
  3. *The spectrum of* $\mathcal{C}_t$ *is* $\sigma(\mathcal{C}_t) = \{\cos(2t\sin\theta/2)\}_{\theta\in(-\pi,\pi]}$.

It follows from Theorem 1.2 and the general theory of cosine operators ([5, 20]) that $\mathcal{C}_t\xi$ is the solution to the homogeneous semidiscrete wave equation (6) with $\eta = 0$. We observe that this fact is already known from the work by H. Bateman [6, Formula (4.2) p. 502] (see also C. E. Pearson [37]). Here we just put that result in a framework of operator theory.

Moreover, the relation

$$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-s^2/4t} J_{2n}(2s)ds = e^{-2t} I_n(2t)$$

holds, see Remark 3. This is understood as a discrete version of the abstract Weierstrass formula, which is well-known from the theories of cosine and semigroups of operators.

Given $z \in \Sigma_\pi$ let us define

$$\mathcal{L}_z^\sigma\varphi := e^{-z(-\Delta_{\rm d})^\sigma}\varphi, \qquad \varphi \in \ell^p. \tag{12}$$

We will show that the family of operators $\mathcal{L}_z^\sigma$ is an uniformly continuous and analytic semigroup of angle π, as well as other qualitative properties of such operators, with the associated kernel

$$L_t^\sigma(n) = \int_0^\infty e^{-2\lambda} I_n(2\lambda) f_{t,\sigma}(\lambda) d\lambda, \quad t \geq 0, \quad n \in \mathbb{Z}, \quad 0 < \sigma < 1,$$

where $f_{t,\sigma}(\lambda)$ is the stable Lévy distribution [50]. Moreover, the latter expression is positive due to the positivity of (2). An analytical expression is not available in general, but in the case $\sigma = 1/2$ we have the precise formula

$$f_{t,1/2}(\lambda) = \frac{t}{\sqrt{4\pi\lambda^3}} e^{-t^2/4\lambda},$$

corresponding to the Van der Waals profile (or the probability density function of the Lévy distribution over the domain $\lambda \geq 0$ with scale parameter $t^2/2$) used commonly in spectroscopy. Analogously to the case of the discrete Laplacian $\Delta_{\rm d}$, we will also study the solutions to the fractional Schrödinger and wave semidiscrete equations, that we will denote, respectively, by $\mathcal{U}_t^\sigma$ and $\mathcal{C}_t^\sigma$.

Theorem 1.3. *(Fractional discrete semigroup) Let $0 < \sigma < 1$. Given $z \in \Sigma_\pi$, let $\mathcal{L}_z^\sigma$ be the operators defined in (12).*

(i) We have $\{\mathcal{L}_z^\sigma\}_{z \in \Sigma_\pi} \subset \mathcal{B}(\ell^p)$ and it is an analytic semigroup, with bounded generator $-(-\Delta_{\rm d})^\sigma$ on ℓ^p, $1 \leq p \leq \infty$. Moreover, for all $\varphi \in \ell^p$:
   1. *$\|(-\Delta_{\rm d})^\sigma \varphi\|_{\ell^p} \leq \frac{2\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2} \|\varphi\|_{\ell^p}$.*
   2. *$\|\mathcal{L}_z^\sigma \varphi\|_{\ell^p} \leq e^{\frac{2\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}|z|} \|\varphi\|_{\ell^p}$, for all $z \in \Sigma_\pi$.*

(ii) We have the following representation for $\mathcal{L}_z^\sigma$ as a convolution operator with a kernel. For each $z \in \Sigma_\pi$ and $n \in \mathbb{Z}$:

$$\mathcal{L}_z^\sigma \varphi(n) = (L_z^\sigma * \varphi)(n) = \sum_{k \in \mathbb{Z}} L_z^\sigma(n-k)\varphi(k),$$

where

$$\begin{aligned} L_z^\sigma(n) &:= (-1)^n \sum_{k=1}^\infty (-1)^k \frac{z^k}{k!} \frac{\Gamma(2k\sigma + 1)}{\Gamma(1 + k\sigma + n)\Gamma(1 + k\sigma - n)} + \delta_{0,n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-z\left(4\sin^2\frac{\theta}{2}\right)^\sigma} e^{-in\theta}\, d\theta, \end{aligned}$$

and $\delta_{0,n}$ is the Kronecker delta function.

(iii) The spectrum of the operators $-(-\Delta_{\rm d})^\sigma$ and $\mathcal{L}_z^\sigma$ is, respectively,

$$\sigma\big(-(-\Delta_{\rm d})^\sigma\big) = \{-\big(4\sin^2\theta/2\big)^\sigma\}_{\theta \in (-\pi,\pi]} = [-4^\sigma, 0],$$

and

$$\sigma(\mathcal{L}_z^\sigma) = \{e^{-z(4\sin^2\theta/2)^\sigma}\}_{\theta \in (-\pi,\pi]}.$$

(iv) Concerning the sequence L_z^σ, we have:
   1. *$\|L_z^\sigma\|_{\ell^p} \leq e^{2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}|z|}$, for all $z \in \Sigma_\pi$, $1 \leq p \leq \infty$.*
   2. *$L_z^\sigma(n)$ is discrete almost periodic in the variable n, for each $z \in \Sigma_\pi$.*

(v) In the case $z = t \geq 0$, we have
   1. *$\mathcal{L}_t^\sigma \varphi(n) \geq 0$ if $\varphi \geq 0$ and $L_t^\sigma(n) \geq 0$, for each $n \in \mathbb{Z}$.*
   2. *$\sum_{n \in \mathbb{Z}} L_t^\sigma(n) = 1$, and $\mathcal{L}_t^\sigma$ is a Markovian semigroup.*

Let $0<\sigma<1$ be given. For any $\xi \in \ell^p$, $1\le p\le\infty$, we define:

$$\mathcal{C}_t^\sigma \xi := \frac{\mathcal{U}_t^{\sigma/2}\xi + \mathcal{U}_{-t}^{\sigma/2}\xi}{2}, \quad t\in\mathbb{R}. \tag{13}$$

Concerning the fractional discrete cosine function generated by the fractional discrete Laplacian, we have the following.

Theorem 1.4. *(Fractional discrete cosine function) Let $0<\sigma<1$. Given $t\in\mathbb{R}$, let $\mathcal{C}_t^\sigma$ be the operators defined in (13).*

(i) We have $\{\mathcal{C}_t^\sigma\}_{t\in\mathbb{R}}\subset\mathcal{B}(\ell^p)$ and it is an uniformly continuous cosine family, with bounded generator $-(-\Delta_{\mathrm{d}})^\sigma$ on ℓ^p, $1\le p\le\infty$.

(ii) The following identity holds

$$\mathcal{C}_t^\sigma\xi(n) = (C_t^\sigma * \xi)(n) = \sum_{m\in\mathbb{Z}} C_t^\sigma(n-m)\xi(m), \quad t\in\mathbb{R}, \tag{14}$$

where

$$C_t^\sigma(n) := \frac{2}{\pi}\int_0^{\pi/2}\cos\left(t(4\sin^2\theta)^{\sigma/2}\right)\cos(2n\theta)d\theta. \tag{15}$$

(iii) The spectrum of $\mathcal{C}_t^\sigma$ is $\sigma(\mathcal{C}_t^\sigma)=\{\cos\left(t(4\sin^2\theta/2)^{\sigma/2}\right)\}_{\theta\in(-\pi,\pi]}$.

(iv) Concerning the sequence $C_t^\sigma(n)$, we have

1. $\|C_t^\sigma\|_{\ell^p}\le e^{2\frac{\Gamma(1+\sigma)}{\Gamma(1+\sigma/2)^2}|t|}$, $t\in\mathbb{R}$.

2. $C_t^\sigma(n)$ is almost periodic in the variable n, for each $t\in\mathbb{R}$, and also almost periodic in the variable t, for each $n\in\mathbb{Z}$.

The following result gives explicit solutions of the non homogeneous linear equations (4), (5) and (6), each one in both cases $L=\Delta_{\mathrm{d}}, -(-\Delta_{\mathrm{d}})^\sigma$ and establishes qualitative properties of such solutions. Let us define

$$\mathcal{S}_t\eta = \int_0^t \mathcal{C}_s\eta\, ds, \quad \eta\in\ell^p,\ t\ge 0,$$

and

$$\mathcal{S}_t^\sigma\eta = \int_0^t \mathcal{C}_s^\sigma\eta\, ds, \quad \eta\in\ell^p,\ t\ge 0.$$

Recall that the definitions and facts on discrete almost periodic functions are collected in the Subsection C in Appendix B.

Theorem 1.5. *Let $0<\sigma<1$. Let $\varphi,\psi,\xi,\eta\in\ell^\infty$ and $g:(0,\infty)\times\mathbb{Z}\to\mathbb{C}$ be continuous in the first variable and such that $g(t,\cdot)\in\ell^\infty$ for each $t\in[0,\infty)$, and $g(\cdot,n)\in L^\infty((0,\infty))$, for each $n\in\mathbb{Z}$. Then, for $t\ge0$ and $n\in\mathbb{Z}$, the functions*

$$u(t,n)=\mathcal{T}_t\varphi(n)+\int_0^t\mathcal{T}_{t-s}g(s,n)\,ds, \quad w(t,n)=\mathcal{U}_t\psi(n)+\int_0^t\mathcal{U}_{t-s}g(s,n)\,ds,$$

and

$$v(t,n)=\mathcal{C}_t\xi(n)+\mathcal{S}_t\eta(n)+\int_0^t\mathcal{S}_{t-s}g(s,n)\,ds;$$

and

$$u_\sigma(t,n)=\mathcal{L}_t^\sigma\varphi(n)+\int_0^t\mathcal{L}_{t-s}^\sigma g(s,n)\,ds, \quad w_\sigma(t,n)=\mathcal{U}_t^\sigma\psi(n)+\int_0^t\mathcal{U}_{t-s}^\sigma g(s,n)\,ds,$$

and

$$v_\sigma(t,n)=\mathcal{C}_t^\sigma\xi(n)+\mathcal{S}_t^\sigma\eta(n)+\int_0^t\mathcal{S}_{t-s}^\sigma g(s,n)\,ds,$$

are well-defined for every $(t,n) \in (0,\infty)\times\mathbb{Z}$ *and they solve the problems* (4), (5), *and* (6), *with* $L = \Delta_{\rm d}$ *or* $-(-\Delta_{\rm d})^{\sigma}$, *respectively, when* $G(t,n) = g(t,n)$.

Moreover, if $\varphi, \psi, \xi, \eta \in \ell^1$, *then the following statements also hold:*

1. *If* $g(t,\cdot)$ *is discrete almost periodic in the second variable, for each* $t \geq 0$, *then* $u(t,\cdot)$, $w(t,\cdot)$, $v(t,\cdot)$, $u_\sigma(t,\cdot)$, $w_\sigma(t,\cdot)$, *and* $v_\sigma(t,\cdot)$ *are also discrete almost periodic in the second variable.*
2. *If* $g(\cdot,n)$ *is almost periodic in the first variable, for each* $n \in \mathbb{Z}$, *then* $w(\cdot,n)$, $v(\cdot,n)$, $w_\sigma(\cdot,n)$, *and* $v_\sigma(\cdot,n)$ *are also almost periodic in the first variable.*
3. *Let* $c > 4$. *If* $g(t,\cdot) \in \ell^1$ *is such that* $\sum_{n\in\mathbb{Z}} |g(t,n)| \leq Me^{-ct}$, *for some* $M > 0$, *then both* $u(t,n) \to 0$ *and* $u_\sigma(t,n) \to 0$ *as* $t \to \infty$, *for each fixed* $n \in \mathbb{N}$.

We now treat nonlinear versions of the semidiscrete equations presented in Subsection 1.1. Recall that, given $0 < T < \infty$, we denote $J := [0,T]$.

Let us introduce the following definitions. Let $\varphi, \psi, \xi, \eta \in \ell^\infty$ and $g : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ be continuous in the second variable and such that $g(x,\cdot) \in \ell^\infty$, $\forall x \in \mathbb{R}$. We say that $u(t,n)$, $w(t,n)$, $v(t,n)$, $u_\sigma(t,n)$, $w_\sigma(t,n)$, and $v_\sigma(t,n)$ are a solution of (4), (5), and (6), with $L = \Delta_{\rm d}$ or $-(-\Delta_{\rm d})^{\sigma}$, in the case in which $G(t,n)$ is either $g(u(t,n),n)$, $g(w(t,n),n)$, $g(v(t,n),n)$, $g(u_\sigma(t,n),n)$, $g(w_\sigma(t,n),n)$, or $g(v_\sigma(t,n),n)$, respectively, if they solve

$$u(t,n) = \mathcal{T}_t\varphi(n) + \int_0^t \mathcal{T}_{t-s} g(u(s,n),n)\,ds,$$
$$w(t,n) = \mathcal{U}_t\psi(n) + \int_0^t \mathcal{U}_{t-s} g(w(s,n),n)\,ds,$$

and

$$v(n,t) = \mathcal{C}_t\xi(n) + \mathcal{S}_t\eta(n) + \int_0^t \mathcal{S}_{t-s} g(v(s,n),n)\,ds;$$

and

$$u_\sigma(t,n) = \mathcal{L}_t^\sigma\varphi(n) + \int_0^t \mathcal{L}_{t-s}^\sigma g(u_\sigma(s,n),n)\,ds,$$
$$w_\sigma(t,n) = \mathcal{U}_t^\sigma\psi(n) + \int_0^t \mathcal{U}_{t-s}^\sigma g(w_\sigma(s,n),n)\,ds,$$

and

$$v_\sigma(t,n) = \mathcal{C}_t^\sigma\xi(n) + \mathcal{S}_t^\sigma\eta(n) + \int_0^t \mathcal{S}_{t-s} g(v_\sigma(s,n),n)\,ds;$$

for $t \in J$, $n \in \mathbb{Z}$. The main result concerning existence and uniqueness of solutions to the nonlinear equations is the following.

Theorem 1.6. *Let* $0 < \sigma < 1$. *Let* $\varphi, \psi, \xi, \eta \in \ell^\infty$, $T > 0$ *and* $g : [0,T]\times\mathbb{Z} \to \mathbb{C}$ *be such that* $g(t,\cdot) \in \ell^\infty$ *for each* $t \in [0,T]$. *Let us assume that there exists* $0 < C_L < \infty$ *such that*

$$|g(s,n) - g(u,n)| \leq C_L|s-u|, \quad s,u \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{16}$$

If $g(t,\cdot)$ *is discrete almost periodic in the second variable, for each* $t \in \mathbb{R}$, *then there exists a unique solution* u, w, v, u_σ, w_σ, *and* v_σ *on* $[0,T]\times\mathbb{Z}$ *to the problems* (4), (5), *and* (6), *with* $L = \Delta_{\rm d}$ *or* $-(-\Delta_{\rm d})^{\sigma}$, *respectively, when* $G(t,n)$ *is one of the above, that is also discrete almost periodic in the second variable.*

The proofs of Theorems 1.5 and 1.6 are contained in Section 4.

Let $r \in \mathbb{R}_+$. We now focus on the problem

$$\big(P_{\mathrm{per}}(f)\big) \quad \begin{cases} \dfrac{\partial u(t,n)}{\partial t} = (\Delta_{\mathrm{d}} + r)u(t,n) + f(t,n), & \text{in } [0,2\pi]\times\mathbb{Z}, \\ u(0,n) = u(2\pi,n), & \text{on } \mathbb{Z}. \end{cases} \tag{17}$$

The aim is to characterize well-posedness (or maximal regularity) of the solution of (17) in the sense of Hölder continuity on periodic times, and ℓ^p summability in a lattice space. Observe that such characterizations have been studied earlier by others authors with different methods [21, 30].

Let us define $C_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ the space of continuous functions $F:[0,2\pi]\to\ell^p(\mathbb{Z})$ with $F(0)=F(2\pi)$. Let $0<\alpha<1$. We denote by $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ the space of α-Hölder continuous functions F taking values on $\ell^p(\mathbb{Z})$, such that $F(0)=F(2\pi)$. The seminorm on $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ is given by

$$\|F\|_\alpha := \sup_{\substack{t\neq s\\ t,s\in(0,2\pi]}} \frac{\|F(t)-F(s)\|_{\ell^p}}{|t-s|^\alpha} = \sup_{\substack{t\neq s\\ t,s\in[0,2\pi]}} \frac{\big(\sum_{n\in\mathbb{Z}}|f(t,n)-f(s,n)|^p\big)^{1/p}}{|t-s|^\alpha}.$$

Moreover, $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ is a Banach space endowed with the norm

$$\|F\|_{C^\alpha} := \|F\|_\alpha + \|F(0)\|_{\ell^p}.$$

We will also define the space $C^{1,\alpha}_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ as the space of functions $F\in C^1\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ such that $F(0)=F(2\pi)$, and $F'\in C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ (actually we could define, more generally, the space of functions $C^{k,\alpha}_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$, for each $k\in\mathbb{N}$, but this is enough for our purposes).

We say that the problem (P_{per}) in (17) is *well-posed* in $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ if, for each $f\in C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$, there is a unique classical solution $u\in C^{1,\alpha}_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ of $(P_{\mathrm{per}}(f))$.

We will characterize the $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$-well-posedness of the problem (17) only in terms of the parameter r. We remark that this is a notable qualitative property that is certainly not present in the continuous setting. The key of the proof is the fact that the spectrum of the operator $A=\Delta_{\mathrm{d}}+rI$ is bounded and therefore the resolvent set of A can be defined on the left side of the complex plane. This is a particular property of the discrete case; in the situation of the continuous Laplacian this is not true because the Laplacian is an unbounded operator and its spectrum is located in the entire negative real semiaxis.

Theorem 1.7. *Let $0<\alpha<1$, $1\le p\le\infty$, and $r\in\mathbb{R}_+$. The problem (P_{per}) in (17) is well-posed in $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ if and only if $r>4$.*

Given $0<\sigma<1$, we also consider the problem

$$\big(P^\sigma_{\mathrm{per}}(f)\big) \quad \begin{cases} \dfrac{\partial u(t,n)}{\partial t} = (-(-\Delta_{\mathrm{d}})^\sigma + r)u(t,n) + f(t,n), & \text{in } [0,2\pi]\times\mathbb{Z}, \\ u(0,n) = u(2\pi,n), & \text{on } \mathbb{Z}. \end{cases} \tag{18}$$

For this problem, we will state a result of well-posedness as well.

Theorem 1.8. *Let $0<\alpha<1$, $1\le p\le\infty$, and $0<\sigma<1$. Let $r>2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$. Then the problem $(P^\sigma_{\mathrm{per}})$ in (18) is well-posed in $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$.*

Theorems 1.7 and 1.8 will be proved in Section 5 and $C^\alpha-\ell^p$ maximal regularity results will be obtained as corollaries, see also Section 5.

2. **Analytic semigroups and cosine operators generated by Δ_{d}.** In this section we will prove the main results on the theory of analytic semigroups and cosine operators generated by the discrete Laplacian Δ_{d}. We emphasize that this operator is trivially bounded on ℓ^p, $1 \le p \le \infty$, unlike the continuous Laplacian, and this fact allows to employ known tools from functional calculus together with the general theory of semigroups and cosine operators to obtain several particular properties in an efficient way.

2.1. **The semidiscrete heat analytic semigroup.** Observe that, from the formula [38, p. 456, 2.5.40 (3)]

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos n\theta\, d\theta, \quad |\arg z| < \pi,\ n \in \mathbb{Z},$$

we can obtain

$$T_z(n) = e^{-2z} I_n(2z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4z\sin^2\theta/2} e^{-in\theta}\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} e^{-4z\sin^2\theta}\cos(2n\theta)\, d\theta. \tag{19}$$

Proof of Theorem 1.1. It is clear that $\varphi \mapsto \mathcal{T}_z\varphi$ is a linear map. Let us see the boundedness on ℓ^p. Because $I_n(z) = I_{-n}(z)$, $I_n(z) = i^{-n}J_n(iz)$ and the upper bound (44), we deduce

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |e^{-2z}I_n(2z)| &\le 2\sum_{n=1}^{\infty} |e^{-2z}I_n(2z)| + |e^{-2z}I_0(2z)| \\
&= 2\sum_{n=1}^{\infty} |e^{-2z}i^{-n}J_n(2iz)| + |e^{-2z}J_0(2iz)| \\
&\le 2e^{-2\operatorname{Re} z}\sum_{n=1}^{\infty} \frac{|iz|^n e^{|\operatorname{Im} 2iz|}}{\Gamma(n+1)} + e^{-2\operatorname{Re} z}e^{|\operatorname{Im} 2iz|} \\
&= 2e^{-2\operatorname{Re} z}\sum_{n=0}^{\infty} \frac{|iz|^n e^{|\operatorname{Im} 2iz|}}{\Gamma(n+1)} - 2e^{-2\operatorname{Re} z}e^{|\operatorname{Im} 2iz|} + e^{-2\operatorname{Re} z}e^{|\operatorname{Im} 2iz|} \\
&\le 2e^{-2\operatorname{Re} z}e^{|z|}e^{|\operatorname{Im} 2iz|} - e^{-2\operatorname{Re} z}e^{|\operatorname{Im} 2iz|} \\
&\le e^{-2\operatorname{Re} z}e^{|\operatorname{Im} 2iz|}(2e^{|z|}-1) =: C_z.
\end{aligned}$$

Hence by Minkowski's integral inequality we obtain

$$\|T_z * \varphi\|_{\ell^p} \le \|T_z\|_{\ell^1}\|\varphi\|_{\ell^p} \le C_z\|\varphi\|_{\ell^p}.$$

The function $z \mapsto \mathcal{T}_z$ is analytic in Σ_π because T_z is analytic in Σ_π. In order to check that the semigroup properties hold, it is enough to observe that $\mathcal{T}_0\varphi = \varphi$ and that $\mathcal{T}_{z_1}\mathcal{T}_{z_2}\varphi = \mathcal{T}_{z_1+z_2}\varphi$. But the first one was just proved in [11, Proposition 1] and the second one can be shown analogously as it was proved in [11, Proposition 1], just noticing that the Neumann's identity (50) is valid for $z_1, z_2 \in \Sigma_\pi$.

Now we shall prove that the family has the bounded generator Δ_{d}. We first observe that Minkowski's inequality shows that $\|\Delta_{\mathrm{d}}\|_{\mathcal{B}(\ell^p)} \le 4$, proving (i)-(1). It suffices to check that $\Delta_{\mathrm{d}}\varphi = \frac{\partial}{\partial z}\mathcal{T}_z\varphi|_{z=0}$, for an arbitrary sequence $\varphi \in \ell^p$. Indeed, the modified Bessel function $I_k(z)$ satisfies $I_k'(z) = \frac{1}{2}(I_{k+1}(z) + I_{k-1}(z))$. From here it follows

$$\frac{\partial}{\partial z}\big(e^{-2z}I_k(2z)\big) = e^{-2z}\big(I_{k+1}(2z) - 2I_k(2z) + I_{k-1}(2z)\big).$$

Then, we have

$$\mathcal{T}_z'\varphi(n) = e^{-2z}\sum_{m\in\mathbb{Z}}\big(I_{n-m+1}(2z) - 2I_{n-m}(2z) + I_{n-m-1}(2z)\big)\varphi(m).$$

Therefore, taking into account (49), we conclude the desired statement. By uniqueness of the generator we conclude that $T_z = e^{z\Delta_{\mathrm{d}}}$ for all $z \in \Sigma_\pi$. This last identity together with (i)-(1) shows the inequality (i)-(2).

Concerning (ii)-(1), we take $u(n) = \delta_{0,n}$ in (i)-(2) obtaining in this way $\|T_z\|_{\ell^p} \le e^{4|z|}$, for $1 < p \le \infty$.

Now we prove (ii)-(2). Let us assume that $\operatorname{Re} z \ge 0$. Otherwise, the argument below is valid after slight modifications. The sequence $\{f_\theta(n)\}_{n\in\mathbb{Z}}$, where $f_\theta(n) := \cos(2\pi n\theta)$ is almost periodic in n for any real valued θ, see [8]. Then, given $\varepsilon > 0$, there exists a positive integer $M(\varepsilon)$ such that any set consisting of $M(\varepsilon)$ consecutive integers contains at least one $p \in \mathbb{Z}$ such that

$$\sup_{n\in\mathbb{Z}} |f_\theta(n+p) - f_\theta(n)| < \varepsilon.$$

It follows from here and (19) that

$$\begin{aligned}|T_z(n+p) - T_z(n)| &\le \frac{2}{\pi}\int_0^{\pi/2} |e^{-4z\sin^2\theta}||\cos(2(n+p)\theta) - \cos 2n\theta|\, d\theta \\ &= 2\int_0^{1/2} e^{-4\operatorname{Re} z\sin^2(\pi\theta)}|f_\theta(n+p) - f_\theta(n)|\, d\theta.\end{aligned}$$

By Dominated Convergence Theorem, we thus have

$$\sup_{n\in\mathbb{Z}}|T_z(n+p) - T_z(n)| < 2\varepsilon\int_0^{1/2} e^{-4\operatorname{Re} z\sin^2(\pi\theta)}\, d\theta \le \varepsilon,$$

and the proof is complete.

Finally, part (iii) follows from the definition of the corresponding operators with the discrete Fourier transform and the spectral mapping theorem for the case of $\mathcal{T}_z$ (see e.g. [18, Lemma 3.13, p.19]). □

Remark 1. *By taking $z = t \in (0,\infty)$ in (9) we are led to the semidiscrete heat semigroup, which is given by the series*

$$\mathcal{T}_t\varphi(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t)\varphi(m), \quad \varphi \in \ell^p.$$

It was proved (see [11, Proposition 1]*) that $\{\mathcal{T}_t\}_{t\ge 0}$ is a positive Markovian diffusion semigroup, and is bounded on ℓ^p for $1 \le p \le \infty$. In particular, $\|T_t\|_{\ell^1} = 1$, due to (51). Observe that since $I_n(z)$ is an entire function on z, it follows that the semigroup $\mathcal{T}_t$ generated by Δ_{d} is in fact a group. In other words, it can be defined for all $t \in \mathbb{R}$. In particular, the identity $e^{-t\Delta_{\mathrm{d}}}\varphi = T_{-t} * \varphi$ holds for $t \ge 0$ and for each $\varphi \in \ell^p$, $1 \le p \le \infty$, and we also have*

$$\|\mathcal{T}_{-t}\varphi\|_{\ell^p} \le e^{4t}\|\varphi\|_{\ell^p}.$$

Remark 2. *As a by-product, from the proof of Theorem 1.1 we also obtain that*

$$\|T_z\|_{\ell^1} \le e^{-2\operatorname{Re} z} e^{|\operatorname{Im} 2iz|}(2e^{|z|} - 1),$$

which is valid for $z \in \mathbb{C}$.

It turns out that when considering $z = it$, $t \in \mathbb{R}$, the operators

$$\mathcal{U}_t := \mathcal{T}_{it},$$

form also a *group*. This is contained in the following theorem.

Theorem 2.1. *Let $t \in \mathbb{R}$.*

(i) The family $\{\mathcal{U}_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ and it forms a group with bounded generator $i\Delta_{\mathrm{d}}$, $1 \le p \le \infty$. Moreover, $\|\mathcal{U}_t\| \le e^{4t}$.

(ii) $T_{it}(n)$ is almost periodic in the variable t, for each $n \in \mathbb{Z}$.

Proof. Concerning part (i), by taking into account (8), from the very definition of the Bessel function $I_k(z)$ in (47) we have that $S_t(n) := T_{-t}(n) = e^{2t} I_n(-2t)$. Observe that the identity $2I_n'(z) = I_{n-1}(z) + I_{n+1}(z)$ imply

$$S_t'(n) = -e^{2t}\big(I_{n+1}(2t) - 2I_n(2t) + I_{n-1}(2t)\big),$$

so that $S_t'(0) = -\Delta_{\mathrm{d}}\delta_{0,n}$. By [36, Th. 6.3, p. 23], we get the conclusion. The second part follows from Theorem 1.1, part (i)-(2).

On the other hand, it can be proved that the function $T_{it}(n)$ in (19) (with the choice $z = it$ there) is almost periodic in t by following an argument parallel to the one developed in Theorem 1.1, part (ii)-(2). This time we have to take into account that $e^{-4it\sin^2\theta}$ is almost periodic in t, for any real valued θ. $\square$

2.2. **The semidiscrete wave cosine family.** Let $C_t(n)$ be defined in (10). From ([25, p. 420, Formula 3.7.15 (9)]) we have

$$C_t(n) = \frac{2}{\pi}\int_0^{\pi/2} \cos(2t\sin x)\cos(2nx)dx. \tag{20}$$

Proof of Theorem 1.2. We begin with part (i). By (42) and the upper bound (44) we have, for each t fixed,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |J_{2m}(2t)\varphi(n-m)|^p &\le \|\varphi\|_{\ell^p} \sum_{m\in\mathbb{Z}} |J_{2m}(2t)| \\ &\le \|\varphi\|_{\ell^p}\Big(2\sum_{m=1}^{\infty}\frac{|t|^{2m}}{\Gamma(2m+1)} + |J_0(2t)|\Big) \\ &\le \|\varphi\|_{\ell^p}\big(2(\cosh|t|-1) + |J_0(2t)|\big) \le (2\cosh|t|-1)\|\varphi\|_{\ell^p},\end{aligned}$$

so $\mathcal{C}_t$ is a linear and bounded operator on ℓ^p. Note that, by (43), we obtain

$$\mathcal{C}_0\varphi(n) = \sum_{m\in\mathbb{Z}} C_0(n-m)\varphi(m) = \sum_{m\in\mathbb{Z}} J_{2n-2m}(0)\varphi(m) = \varphi(n).$$

On the other hand, by using (46) and (42) it is easily verified that

$$\begin{aligned}J_{2n}(2t-2s) &= \sum_{m\in\mathbb{Z}} J_{2n+2m}(2t)J_{2m}(2s) = \sum_{m\in\mathbb{Z}} J_{2n-2m}(2t)J_{-2m}(2s) \\ &= \sum_{m\in\mathbb{Z}} J_{2n-2m}(2t)J_{2m}(2s)(-1)^{2m} = \sum_{m\in\mathbb{Z}} J_{2n-2m}(2t)J_{2m}(2s).\end{aligned}$$

Moreover, (45) shows that

$$J_{2n}(2t+2s) = \sum_{m\in\mathbb{Z}} J_{2n-2m}(2t)J_{2m}(2s).$$

Therefore it follows that

$$J_{2n}(2t+2s)+J_{2n}(2t-2s)=2\sum_{m\in\mathbb{Z}}J_{2n-2m}(2t)J_{2m}(2s).$$

Since $C_t(n)=J_{2n}(2t)$ we arrive at

$$\begin{aligned}\mathcal{C}_t\mathcal{C}_s\varphi(n)&=\mathcal{C}_t\Big(\sum_{m\in\mathbb{Z}}C_s(\cdot-m)\varphi(m)\Big)(n)=\sum_{k\in\mathbb{Z}}C_t(n-k)\Big(\sum_{m\in\mathbb{Z}}C_s(k-m)\varphi(m)\Big)\\&=\sum_{m\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}C_t(n-k)C_s(k-m)\Big)\varphi(m)\\&=\frac{1}{2}\sum_{m\in\mathbb{Z}}\big(\mathcal{C}_{t+s}(n-m)+\mathcal{C}_{t-s}(n-m)\big)\varphi(m)=\frac{1}{2}(\mathcal{C}_{t+s}+\mathcal{C}_{t-s})\varphi(n).\end{aligned}$$

Now we will check that the family of cosine operators $\{\mathcal{C}_t\}_{t\in\mathbb{R}}$ has the bounded generator Δ_{d} and, in particular, is uniformly continuous. To that end, we will see that $\Delta_{\mathrm{d}}\varphi=\frac{\partial^2}{\partial t^2}\mathcal{C}_t|_{t=0}\varphi$, for an arbitrary sequence $\varphi\in\ell^p$. Indeed, the Bessel function $J_k(z)$ satisfies $J_k'(z)=-\frac{1}{2}(J_{k+1}(z)-J_{k-1}(z))$. With this, we readily obtain

$$\frac{\partial^2}{\partial z^2}J_k(2z)=J_{k+2}(2z)-2J_k(2z)+J_{k-2}(2z).$$

Then, we have

$$\frac{\partial^2}{\partial t^2}\mathcal{C}_t\varphi(n)=\sum_{m\in\mathbb{Z}}\big(J_{2(n-m+1)}(2t)-2J_{2(n-m)}(2t)+J_{2(n-m-1)}(2t)\big)\varphi(m).$$

Therefore, taking into account (43), we conclude the desired statement.

Now we pass to part (ii). To prove (ii)-(1) we follow an analogous reasoning to the one used in part (i) that leads to

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|J_{2n}(2t)|^p&\le 2\sum_{n=1}^{\infty}\Big(\frac{|t|^{2n}}{\Gamma(2n+1)}\Big)^p+|J_0(2t)|^p\\&\le 2\sum_{n=1}^{\infty}\frac{|t|^{2np}}{\Gamma(2n+1)}+|J_0(2t)|^p\le(2\cosh|t|^p-1).\end{aligned}$$

Part (2) follows immediately from the integral representation (20) for C_t.

Finally, part (3) follows from Theorem 1.1 (iii) and the spectral mapping theorem for cosine operator functions, i.e., the identity

$$\sigma(\mathcal{C}_t)=\cosh(t\sqrt{\sigma(\Delta_{\mathrm{d}})}).$$

Observe that we take into account that $\cosh(iz)=\cos(z)$. The theorem is proved. $\square$

Remark 3 (Weierstrass formula)**.** *The following identity can be found in the literature: for* $\operatorname{Re}p>0$, $\operatorname{Re}\nu>-1$, $|\arg c|<\pi$, *one has (see* [39, p. 186, 2.12.9. 1.]*)*

$$\int_0^{\infty}e^{-ps^2}J_\nu(cs)\,ds=\frac{1}{2}\sqrt{\frac{\pi}{p}}\exp\Big(-\frac{c^2}{8p}\Big)I_{\nu/2}\Big(\frac{c^2}{8p}\Big).$$

Then, if we take above $p=\frac{1}{4t}$, $\nu=2n$ *and* $c=2$, *it holds*

$$\frac{1}{\sqrt{\pi t}}\int_0^{\infty}e^{-s^2/4t}C_s(n)\,ds=T_t(n),\quad t>0,\quad n\in\mathbb{Z},$$

where C_s and T_t are the ones in (10) and (8), which yields

$$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-s^2/4t}\mathcal{C}_s\varphi(n)\,ds=\mathcal{T}_t\varphi(n),\quad t>0,\quad n\in\mathbb{Z}.$$

So the identity can be reinterpreted as the Weierstrass formula in the discrete setting. Observe that Weierstrass formula is valid in general in the framework of cosine and semigroup theory.

3. **Analytic semigroups and cosine operators generated by $-(-\Delta_{\mathrm{d}})^\sigma$.** We proceed in this section to construct the theory of analytic semigroups and cosine operators generated by the fractional discrete Laplacian $-(-\Delta_{\mathrm{d}})^\sigma$.

3.1. **The fractional discrete Laplacian.** We begin this subsection by defining the fractional powers of the discrete Laplacian. The first approach will be by means of the discrete Fourier transform, see Appendix D. We define the following function of order $\sigma>0$:

$$K^\sigma(n):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(4\sin^2(\theta/2)\right)^\sigma e^{-in\theta}\,d\theta,\quad n\in\mathbb{Z}.\tag{21}$$

A computation using [38, 2.5.12 (22), p. 402] shows that the function $K^\sigma(n)$ can be explicitly written as (see [12, Remark 1])

$$K^\sigma(n)=\frac{(-1)^n\Gamma(2\sigma+1)}{\Gamma(1+\sigma+n)\Gamma(1+\sigma-n)},\quad n\in\mathbb{Z},\tag{22}$$

for each $\sigma\in(0,\infty)\backslash\mathbb{N}$. This kernel has been also considered earlier by other authors. See e.g. [44, formula (26)] and references therein. It can be checked, by Stirling's approximation, that

$$|K^\sigma(n)|\sim\frac{\Gamma(2\alpha+1)}{\pi}|n|^{-2\alpha-1},\quad n\to\pm\infty,$$

so that the following expression can be well defined for $f\in\ell^\infty$ (see [12, Section 2]),

$$(-\Delta_{\mathrm{d}})^\sigma f(n):=\sum_{k\in\mathbb{Z}}K^\sigma(n-k)f(k),\quad n\in\mathbb{Z},\quad \sigma\in(0,\infty)\setminus\mathbb{N}\tag{23}$$

and is called the fractional discrete Laplacian. It is clear from (21) that

$$\mathcal{F}_{\mathbb{Z}}\big((-\Delta_{\mathrm{d}})^\sigma f\big)(\theta)=(4\sin^2(\theta/2))^\sigma\mathcal{F}_{\mathbb{Z}}(f)(\theta),$$

where $\mathcal{F}_{\mathbb{Z}}$ denotes the discrete Fourier transform, which is defined in Appendix D.

On the other hand, we can also define, for good enough functions, the fractional discrete Laplacian with the approach of the semidiscrete heat semigroup (9), as follows:

$$(-\Delta_{\mathrm{d}})^\sigma f(n)=\frac{1}{\Gamma(-\sigma)}\int_0^\infty(\mathcal{T}_tf(n)-f(n))\frac{dt}{t^{1+\sigma}},\quad n\in\mathbb{Z},\tag{24}$$

(see [11, 13, 41]). With this semigroup approach, we can derive another pointwise explicit formula, see [13, Theorem 1.2]:

$$(-\Delta_{\mathrm{d}})^\sigma f(n)=\sum_{k\neq n}\big(f(n)-f(k)\big)R^\sigma(k-n),\tag{25}$$

where

$$R^\sigma(n)=\frac{4^\sigma\Gamma(1/2+\sigma)\Gamma(|n|-\sigma)}{\sqrt{\pi}|\Gamma(-\sigma)|\Gamma(|n|+1+\sigma)},\quad n\in\mathbb{Z}\setminus\{0\},\quad R^\sigma(0)=0.\tag{26}$$

Remark 4. *It is proved in* [13] *that, if f is bounded then* $\lim_{\sigma\to 1^-}(-\Delta_{\rm d})^\sigma f(n) = -\Delta_{\rm d} f(n)$*, for each* $n\in\mathbb{Z}$.

The pointwise formulae (23) and (25) are equivalent, and the kernels (22) and (26) are related. In order to prove this equivalence, we present a previous technical lemma.

Lemma 3.1. *Let* $a\in(0,\infty)\setminus\mathbb{Z}$*. Then we have the identity*

$$\sum_{n=1}^{\infty}\frac{\Gamma(n-a)}{\Gamma(n+a+1)}=\frac{\Gamma(1-a)}{\Gamma(a+1)2a}.$$

Proof. Let us write the sum above as

$$\sum_{n=1}^{\infty}\frac{\Gamma(n-a)}{\Gamma(n+a+1)}=\sum_{n=0}^{\infty}\frac{\Gamma(n-a)\Gamma(n+1)}{\Gamma(n+a+1)n!}-\frac{\Gamma(-a)}{\Gamma(a+1)},$$

and the sum $\sum_{n=0}^\infty\frac{\Gamma(n-a)\Gamma(n+1)}{\Gamma(n+a+1)n!}$ in the right hand side is the absolute convergent Gauss hypergeometric series ${}_2F_1(-a,1;a+1;1)$ times $\frac{\Gamma(-a)}{\Gamma(a+1)}$, see [1, 15.1.1]. This particular series can be explicitly computed, see [1, 15.1.20], so that

$${}_2F_1(-a,1;a+1;1)=\frac{\Gamma(1+a)\Gamma(2a)}{\Gamma(1+2a)\Gamma(a)}=\frac{1}{2}.$$

Therefore we get

$$\sum_{n=1}^{\infty}\frac{\Gamma(n-a)}{\Gamma(n+a+1)}=-\frac{1}{2}\frac{\Gamma(-a)}{\Gamma(a+1)}=\frac{\Gamma(1-a)}{\Gamma(a+1)2a},$$

as desired. □

The following proposition shows that the kernel $K^\sigma(n)$ is mass preserving [16, Section 3] and coincides essentially with the kernel $R^\sigma(n)$ except for a change of sign.

Proposition 1. *Let* $0<\sigma<1$*. Then* $K^\sigma(n)=-R^\sigma(n)$*, for all* $n\in\mathbb{Z}\setminus\{0\}$*. Moreover,* $R^\sigma(0)=\sum_{n\in\mathbb{Z}}K^\sigma(n)=0$.

Proof. The pointwise formula (23) can be rewritten as

$$\begin{aligned}(-\Delta_{\rm d})^\sigma f(n)&=\sum_{k\neq n}K^\sigma(n-k)f(k)+K^\sigma(0)f(n)\\&=-\sum_{k\neq n}K^\sigma(n-k)\big(f(n)-f(k)\big)+\sum_{k\neq n}K^\sigma(n-k)f(n)+K^\sigma(0)f(n)\\&=-\sum_{k\neq 0}K^\sigma(k)\big(f(n)-f(n-k)\big)+f(n)\sum_{k\in\mathbb{Z}}K^\sigma(k).\end{aligned}$$

Moreover,

$$\sum_{k\in\mathbb{Z}}K^\sigma(k)=0. \tag{27}$$

Indeed, we split the sum into

$$\sum_{k\in\mathbb{Z}}K^\sigma(k)=2\sum_{k=1}^{\infty}K^\sigma(k)+K^\sigma(0).$$

On one hand, by Lemma 3.1,

$$\begin{aligned}
2\sum_{k=1}^{\infty}\frac{(-1)^k}{\Gamma(1+\sigma+k)\Gamma(1+\sigma-k)} &= 2\sum_{k=1}^{\infty}\frac{(-1)^k}{\Gamma(1+\sigma+k)\Gamma(1-(k-\sigma))} \\
&= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\Gamma(k-\sigma)\sin\big(\pi(k-\sigma)\big)}{\Gamma(1+\sigma+k)} \\
&= -\frac{2\sin(\pi\sigma)}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k-\sigma)}{\Gamma(1+\sigma+k)} \\
&= -\frac{\sin(\pi\sigma)}{\pi}\frac{\Gamma(1-\sigma)}{\Gamma(1+\sigma)\sigma}.
\end{aligned}$$

On the other hand,

$$K^\sigma(0) = \frac{\Gamma(2\sigma+1)}{\Gamma(1+\sigma)\Gamma(1+\sigma)} = \frac{\sin(\pi\sigma)}{\pi}\frac{\Gamma(1-\sigma)}{\Gamma(1+\sigma)\sigma},$$

so we conclude (27).

Now, we will see that $K^\sigma(k) = -R^\sigma(k)$ for all $k \in \mathbb{Z}\setminus\{0\}$. Since both kernels are even, it suffices to consider $k > 0$. Indeed, by applying in (22) the duplication formula $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\Gamma(2z)\sqrt{\pi}$ with $z = \frac{1+2\sigma}{2}$ and then Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ with $z = 1+\sigma-k$, we get

$$\begin{aligned}
-K^\sigma(k) &= \frac{(-1)^{k+1}\Gamma(2\sigma+1)}{\Gamma(1+\sigma+k)\Gamma(1+\sigma-k)} = \frac{(-1)^{k+1}\Gamma(\sigma+1/2)\Gamma(\sigma+1)4^\sigma}{\Gamma(1+\sigma+k)\Gamma(1+\sigma-k)\sqrt{\pi}} \\
&= \frac{(-1)^{k+1}\Gamma(\sigma+1/2)\Gamma(\sigma+1)4^\sigma\Gamma(k-\sigma)\sin[\pi(1+\sigma-k)]}{\Gamma(1+\sigma+k)\sqrt{\pi}\pi} \\
&= \frac{(-1)\Gamma(\sigma+1/2)\Gamma(\sigma+1)4^\sigma\Gamma(k-\sigma)\sin[\pi(1+\sigma)]}{\Gamma(1+\sigma+k)\sqrt{\pi}\pi} \\
&= \frac{(-1)\Gamma(\sigma+1/2)\Gamma(\sigma+1)4^\sigma\Gamma(k-\sigma)\pi}{\Gamma(1+\sigma+k)\sqrt{\pi}\pi\Gamma(-\sigma)\Gamma(1+\sigma)} \\
&= \frac{\Gamma(\sigma+1/2)4^\sigma\Gamma(k-\sigma)}{\Gamma(1+\sigma+k)\sqrt{\pi}|\Gamma(-\sigma)|} = R^\sigma(k),
\end{aligned}$$

and the proof is complete. □

We finish this subsection with the explicit computation of $\|K^\sigma\|_{\ell^1}$, which will be useful later on.

Lemma 3.2. *Let $0 < \sigma < 1$. Then*

$$\|K^\sigma\|_{\ell^1} = 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}.$$

Proof. Observe that, by (22),

$$\|K^\sigma\|_{\ell^1} = \Gamma(1+2\sigma)\sum_{n\in\mathbb{Z}\setminus\{0\}}\left|\frac{(-1)^n}{\Gamma(1+\sigma+n)\Gamma(1+\sigma-n)}\right| + \frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)\Gamma(1+\sigma)}.$$

Let us analyze the sum on the right hand side above. By Lemma 3.1 we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}}\Big|\frac{(-1)^n}{\Gamma(1+\sigma+n)\Gamma(1+\sigma-n)}\Big| &= 2\sum_{n=1}^{\infty}\frac{1}{\Gamma(1+\sigma+n)|\Gamma(1+\sigma-n)|}\\ &= 2\sum_{n=1}^{\infty}\frac{1}{\Gamma(1+\sigma+n)|\Gamma(1-(n-\sigma))|}\\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(n-\sigma)|\sin\big(\pi(n-\sigma)\big)|}{\Gamma(1+\sigma+n)}\\ &= \frac{2\sin(\pi\sigma)}{\pi}\sum_{n=1}^{\infty}\frac{\Gamma(n-\sigma)}{\Gamma(1+\sigma+n)}\\ &= \frac{\sin(\pi\sigma)}{\pi}\frac{\Gamma(1-\sigma)}{\Gamma(1+\sigma)\sigma}\end{aligned}$$

On the other hand, by the reflection formula for the Gamma function, we get $\frac{\sin(\pi\sigma)\Gamma(1-\sigma)}{\sigma\pi} = \frac{1}{\Gamma(1+\sigma)}$. Putting everything together, we conclude that

$$\|K^\sigma\|_{\ell^1} = 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2},$$

and the proof is finished.

□

3.2. **The fractional discrete semigroup.**

Proof of Theorem 1.3. Given a sectorial operator A, the spectral mapping theorem for fractional powers states that (see for instance [34, Ch. 5.3])

$$\sigma(A^\sigma) = \{z^\sigma : z\in\sigma(A)\},\quad \sigma>0.$$

If, in addition, the operator A is bounded, then for any $\sigma>0$ the operator A^σ is bounded as well. Thus, $(-\Delta_{\mathrm{d}})^\sigma$ is bounded on ℓ^p, for $1\le p\le\infty$. On the other hand, the spectrum is $\sigma\big((-\Delta_{\mathrm{d}})^\sigma\big) = \sigma\big((-\Delta_{\mathrm{d}})\big)^\sigma$, obtaining in this way part (iii), just by using Theorem 1.1, part (iii). We can say even more. According to (23) the fractional discrete Laplacian is defined as a convolution operator on $\mathbb{Z}$ as $(-\Delta_{\mathrm{d}})^\sigma f = K^\sigma * f$. Then by Minkowski's integral inequality we have that $\|(-\Delta_{\mathrm{d}})^\sigma f\|_{\ell^p}\le\|K^\sigma\|_{\ell^1}\|f\|_{\ell^p}$, for $1\le p\le\infty$. By Lemma 3.2, we conclude that the estimate (i)-(1) holds. Therefore, the family of operators $\mathcal{L}_z^\sigma\varphi$ defined in (12) is contained in $\mathcal{B}(\ell^p)$, it is an analytic semigroup and the estimate (i)-(2) holds.

Concerning the proof of (ii) we obtain, by using (23),

$$
\begin{aligned}
e^{-z(-\Delta_{\rm d})^\sigma}\varphi(n) &= \sum_{k=0}^{\infty}(-1)^k\frac{z^k}{k!}(-\Delta_{\rm d})^{k\sigma}\varphi(n)\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{z^k}{k!}\sum_{m\in\mathbb{Z}}(-1)^m\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+m)\Gamma(1+k\sigma-m)}\varphi(n-m)\\
&= \sum_{m\in\mathbb{Z}}\varphi(n-m)(-1)^m\sum_{k=0}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+m)\Gamma(1+k\sigma-m)}\\
&= \sum_{m\in\mathbb{Z}}\varphi(n-m)(-1)^m\sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+m)\Gamma(1+k\sigma-m)}\\
&\qquad+\sum_{m\in\mathbb{Z}}\varphi(n-m)(-1)^m\frac{1}{\Gamma(1+m)\Gamma(1-m)}\\
&= \sum_{m\in\mathbb{Z}}\varphi(n-m)(-1)^m\sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+m)\Gamma(1+k\sigma-m)}+\varphi(n).
\end{aligned}
$$

Let us prove the second equality in part (ii), and we start with the integral expression. With the change of variable $\theta/2=\pi/2-u$, it follows

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-z\left(4\sin^2\frac{\theta}{2}\right)^\sigma}e^{-in\theta}\,d\theta &= \frac{1}{\pi}\int_0^{\pi}e^{-z\left(4\sin^2\frac{\theta}{2}\right)^\sigma}\cos(n\theta)\,d\theta\\
&= \frac{2}{\pi}\int_0^{\pi/2}e^{-z\left(4\sin^2(\frac{\pi}{2}-u)\right)^\sigma}\cos\big(2n(\pi/2-u)\big)\,du\\
&= (-1)^n\frac{2}{\pi}\int_0^{\pi/2}e^{-z4^\sigma(\cos u)^{2\sigma}}\cos(2nu)\,du.
\end{aligned}
$$

Expand the exponential function and use the formula in [38, p. 402, 2.5.12 (22)]

$$
\int_0^{\pi/2}(\cos x)^{\nu-1}\cos(bx)\,dx = 2^{-\nu}\pi\frac{\Gamma(\nu)}{\Gamma\big(\frac{1+\nu+b}{2}\big)\Gamma\big(\frac{1+\nu-b}{2}\big)},\quad \operatorname{Re}\nu>0,\quad |\arg b|<\pi,
$$

with $\nu=2k\sigma+1$, $b=2n$, to get the desired result. Indeed,

$$
\begin{aligned}
\int_0^{\pi/2}e^{-z4^\sigma(\cos u)^{2\sigma}}\cos(2nu)\,du &= \int_0^{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(z4^\sigma(\cos u)^{2\sigma}\right)^k}{k!}\cos(2nu)\,du\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{(z4^\sigma)^k}{k!}\int_0^{\pi/2}(\cos u)^{2\sigma k}\cos(2nu)\,du\\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+n)\Gamma(1+k\sigma-n)}\\
&\qquad+\int_0^{\pi/2}\cos(2nu)\,du\\
&= \frac{\pi}{2}\sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\sigma+1)}{\Gamma(1+k\sigma+n)\Gamma(1+k\sigma-n)}+\delta_{0,n}.
\end{aligned}
$$

Let us prove now part (iv)-(1). Since the operator $(-\Delta_{\rm d})^\sigma$ is bounded on ℓ^p, a computation using the representation in series of $e^{-z(-\Delta_{\rm d})^\sigma}$ yields

$$\|e^{-z(-\Delta_{\rm d})^\sigma}\varphi\|_{\ell^p}\le e^{\|K^\sigma\|_{\ell^1} Re z}\|\varphi\|_{\ell^p},\qquad \text{for all } \varphi\in\ell^p.$$

We choose $\{\varphi(n)\}_{n\in\mathbb{Z}}=\{\delta_{0,n}\}_{n\in\mathbb{Z}}\in\ell^p$. Then, by part (ii) we obtain $e^{-z(-\Delta_{\rm d})^\sigma}\delta_{0,n}=L_z^\sigma(n)$ for each n and consequently by Lemma 3.2 we obtain the desired estimate.

In order to prove part (iv)-(2) we observe that, for any $\theta\in\mathbb{R}$, the sequence $\{\cos 2\pi n\theta\}_{n\in\mathbb{Z}}$ is almost periodic. Then, in view of the integral expression for L_z^σ in part (ii), by the definition of discrete almost periodic function, and Dominated Convergence Theorem, we have that L_z^σ is discrete almost periodic in $n\in\mathbb{Z}$ for each $z\in\Sigma_\pi$.

Proof of part (v)-(1). We recall the following representation describing a stable Levy process, see [50, Ch. IX.11, Proposition1],

$$e^{-ta^\sigma}=\int_0^\infty e^{-\lambda a}f_{t,\sigma}(\lambda)\,d\lambda,\qquad t>0,\quad a>0 \tag{28}$$

where $f_{t,\sigma}(\lambda)$ is a transition probability density that satisfies, see [50, Ch. IX.11, Propositions 2 and 3],

$$f_{t,\sigma}(\lambda)\ge 0\quad \text{for all } \lambda>0,\qquad \int_0^\infty f_{t,\sigma}(\lambda)\,d\lambda=1.$$

Take any $\varphi\ge 0$. Then by the representation (28) we have that

$$e^{-t(-\Delta_{\rm d})^\sigma}\varphi(n)=\int_0^\infty e^{\lambda\Delta_{\rm d}}\varphi(n)f_{t,\sigma}(\lambda)\,d\lambda\ge 0.$$

Let us choose $\varphi(n)=\delta_{0,n}$. It follows immediately that

$$0\le\int_0^\infty e^{\lambda\Delta_{\rm d}}\delta_{0,n}f_{t,\sigma}(\lambda)\,d\lambda=e^{-t(-\Delta_{\rm d})^\sigma}\delta_{0,n}=L_t^\sigma(n),$$

where the second equality is implied by part (ii).

Finally, for the proof of part (v)-(2), we use the representation (28) applied to the sequence constantly equal to 1 (which we denote $\mathbf{1}(n)$). Therefore we have

$$\mathcal{L}_t^\sigma\mathbf{1}(n)=\int_0^\infty e^{\lambda\Delta_{\rm d}}\mathbf{1}(n)f_{t,\sigma}(\lambda)\,d\lambda=\mathbf{1}(n)\int_0^\infty f_{t,\sigma}(\lambda)\,d\lambda=\mathbf{1}(n),$$

because $e^{\lambda\Delta_{\rm d}}=\mathcal{T}_\lambda$ is Markovian. Moreover, the representation of $\mathcal{L}_z^\sigma$ as a convolution in part (ii) shows that

$$\mathcal{L}_t^\sigma\mathbf{1}(n)=\sum_{k\in\mathbb{Z}}L_t^\sigma(k)\mathbf{1}(n-k)=\sum_{k\in\mathbb{Z}}L_t^\sigma(k).$$

This yields $\sum_{k\in\mathbb{Z}}L_t^\sigma(k)=1$.

The proof of the theorem is complete. □

Remark 5. *From the fact that* $e^{-z(-\Delta_{\rm d})^\sigma}\delta_{0n}=L_z^\sigma(n)$ *we deduce, for* $z,w\in\Sigma_\pi$ *and each* $n\in\mathbb{Z}$, *that*

$$(L_z^\sigma * L_w^\sigma)(n)=L_{z+w}^\sigma(n). \tag{29}$$

Remark 6. *Note that, for* $\sigma=1$, *we recover the modified Bessel function, i.e.,* $L_z^1(n)=e^{-2z}I_n(2z)$. *We could say that* L_z^σ *is some sort of generalized modified Bessel function.*

For $z = it, t \in \mathbb{R}$, let us define $\mathcal{U}_t^\sigma$ as

$$\mathcal{U}_t^\sigma \psi(n) = \sum_{m\in\mathbb{Z}} L_{it}^\sigma(n-m)\psi(m), \tag{30}$$

where $\psi \in \ell^p$.

Theorem 3.3. *The family $\{\mathcal{U}_t^\sigma\}_{t\in\mathbb{R}}$ forms a group of bounded and linear operators on ℓ^p with bounded generator $-i(-\Delta_{\rm d})^\sigma$. Moreover, the spectrum is $\sigma(\mathcal{U}_t^\sigma) = \{e^{-it(4\sin^2\theta/2)^\sigma}\}_{\theta\in(-\pi,\pi]}$.*

Proof. The group property follows from identity (29) applied to $z = it$, $w = is$, with $t, s \in \mathbb{R}$. Since $L_{it}^\sigma \in \ell^1$, then $\mathcal{U}_t^\sigma$ is a well-defined operator from ℓ^p into itself. In addition, $\|\mathcal{U}_t^\sigma\|_{\mathcal{B}(\ell^p)} \leq \|L_{it}^\sigma\|_{\ell^1}$. The spectrum is a particular case of Theorem 1.3. □

3.3. The fractional discrete cosine function.

Proof of Theorem 1.4. Let us begin with part (i). Theorem 1.3, part (i), shows that $\{\mathcal{C}_t^\sigma\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ and it is an uniformly continuous family of operators. The D'Alembert functional equation follows from the fact that $\mathcal{U}_t^{\sigma/2}$ is a group. From (21) and discrete Fourier transform, it follows that $K^{\sigma/2} * K^{\sigma/2} = K^\sigma$. Therefore, formula (23) gives

$$(-\Delta_{\rm d})^{\sigma/2}(-\Delta_{\rm d})^{\sigma/2} = (-\Delta_{\rm d})^\sigma.$$

The latter identity shows that $\mathcal{C}_t^\sigma$ is an uniformly continuous operator cosine function with generator $-(-\Delta_{\rm d})^\sigma$. Indeed, we have

$$2[\mathcal{C}_t^\sigma]' = -i(-\Delta_{\rm d})^{\sigma/2}\mathcal{U}_t^{\sigma/2} + i(-\Delta_{\rm d})^{\sigma/2}\mathcal{U}_{-t}^{\sigma/2}$$

and hence

$$\begin{aligned} 2[\mathcal{C}_t^\sigma]'' &= -(-\Delta_{\rm d})^{\sigma/2}(-\Delta_{\rm d})^{\sigma/2}\mathcal{U}_t^{\sigma/2} - (-\Delta_{\rm d})^{\sigma/2}(-\Delta_{\rm d})^{\sigma/2}\mathcal{U}_{-t}^{\sigma/2} \\ &= -(-\Delta_{\rm d})^\sigma\mathcal{U}_t^{\sigma/2} - (-\Delta_{\rm d})^\sigma\mathcal{U}_{-t}^{\sigma/2} = -(-\Delta_{\rm d})^\sigma[\mathcal{U}_t^{\sigma/2} + \mathcal{U}_{-t}^{\sigma/2}]. \end{aligned}$$

This implies

$$[\mathcal{C}_t^\sigma]'' = -(-\Delta_{\rm d})^\sigma \mathcal{C}_t^\sigma,$$

and, after evaluating in $t = 0$, the conclusion follows.

For the proof of (ii) it is enough to note, from Theorem 1.3 part (ii), that we have the identity

$$\frac{L_{it}^{\sigma/2}(n) + L_{-it}^{\sigma/2}(n)}{2} = C_t^\sigma(n). \tag{31}$$

Part (iii) follows from (15). From identity (31) above one can conclude part (iv)-(1), by taking into account Theorem 1.3 part (iv)-(1). Finally, part (iv)-(2) is true by (15). □

Remark 7. *Observe that in the limit case $\sigma = 1$ we obtain*

$$\mathcal{C}_t^1\varphi = \frac{\mathcal{L}_{it}^{1/2}\varphi + \mathcal{L}_{-it}^{1/2}\varphi}{2}, \quad t \in \mathbb{R}.$$

with generator $-(-\Delta_{\rm d})^1 = \Delta_{\rm d}$. We will check that

$$\mathcal{C}_t^1 = \mathcal{C}_t$$

where $\mathcal{C}_t$ was defined in (11). Indeed, it is enough to prove that

$$C_t(n) = \frac{L_{it}^{1/2}(n) + L_{-it}^{1/2}(n)}{2},$$

but this follows from formula (20) and the integral expression for $L_z^\sigma(n)$ in Theorem 1.3, part (ii). This fact also justifies the definition of discrete cosine function given in Subsection 2.2.

Remark 8. *From Theorem 1.3 in case $\sigma = 1/2$ we obtain $\|(-\Delta_{\mathrm{d}})^{1/2}\| \le \frac{4}{\sqrt{\pi}}$. Since $\cosh[(-\Delta_{\mathrm{d}})^{1/2}t]$ is a cosine operator function with generator Δ_{d} it follows from uniqueness that $\mathcal{C}_t = \cosh[(-\Delta_{\mathrm{d}})^{1/2}t]$ for all $t \in \mathbb{R}$, where $\mathcal{C}_t$ is the semidiscrete wave cosine family defined in the previous section. In particular, we obtain*

$$\|\mathcal{C}_t\| \le e^{\frac{4}{\sqrt{\pi}}t},$$

which complements Theorem 1.2 and provides the estimate

$$\|C_t\|_{\ell^p} \le e^{\frac{4}{\sqrt{\pi}}t},$$

for the corresponding kernel. Compare this estimate with the one in Theorem 1.2, part (ii)-(1).

4. Existence and uniqueness of almost periodic solutions: proofs of Theorems 1.5 and 1.6.

We devote this section to the proofs of Theorems 1.5 and 1.6 concerning the existence and uniqueness of almost periodic solutions to nonhomogeneous heat, Schrödinger, and wave equations involving a discrete Laplacian, and some of their properties. The research carried out in the previous Theorems 1.1, 1.2, 1.3 and 1.4 and in the corresponding proofs in Sections 2 and 3 will be the key to prove the results concerning these solutions. In order to unify the presentation, we will show the problems in a general form. Therefore, recall from Subsection 1.1 that we will study the following equations:

$$\frac{\partial u(t,n)}{\partial t} = Lu(t,n) + G(t,n), \quad u(0,n) = \varphi(n), \tag{32}$$

$$\frac{\partial w(t,n)}{\partial t} = iLw(t,n) + G(t,n), \quad w(0,n) = \psi(n), \tag{33}$$

and

$$\frac{\partial^2 v(t,n)}{\partial t^2} = Lv(t,n) + G(t,n), \quad v(0,n) = \xi(n),\ v_t(0,n) = \eta(n), \tag{34}$$

where $n \in \mathbb{Z}$, L means either the discrete Laplacian Δ_{d} or the fractional powers $-(-\Delta_{\mathrm{d}})^\sigma$, the sequences φ, ψ, ξ and η are the initial data at $t = 0$ and G is a function either of the form $g(t,n)$, $t \ge 0$, or of the form $g(u(t,n),n)$, $g(w(t,n),n)$, or $g(v(t,n),n)$, with $t \in J := [0,T]$ and $0 < T < \infty$ is given.

For easy reading, we will also collect some ingredients that have been already presented in the previous sections. When $L = \Delta_{\mathrm{d}}$, the solution to the homogeneous version of the equation (32) is represented by the series

$$\mathcal{T}_t\varphi(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t)\varphi(m), \quad t \ge 0, \quad n \in \mathbb{Z}.$$

Then, from Theorem 2.1 and semigroup theory we have that the solution to the homogeneous version of (33) is given by

$$w(t,n) = \mathcal{U}_t\psi(n) = \sum_{m\in\mathbb{Z}} e^{-2it} I_{n-m}(2it)\psi(m).$$

Recall that $U_t := T_{it}$, where T_z is the one in (8). It is clear that

$$w(t,n) = (U_t * \psi)(n).$$

In its turn, let us define $\mathcal{S}_t$, the one parameter family of operators in ℓ^p by

$$\mathcal{S}_t f = \int_0^t \mathcal{C}_s f\,ds, \quad f\in\ell^p,\ t\geq 0,$$

where $\mathcal{C}_s$ is the given in (11). From Theorem 1.2 and cosine operator theory, we have that

$$v(t,n) = \mathcal{C}_t\xi(n) + \mathcal{S}_t\eta(n)$$

is the solution to the homogenous version of (34) and it is explicitly given by

$$v(t,n) = \sum_{m\in\mathbb{Z}} J_{2(n-m)}(2t)\xi(m) + \sum_{m\in\mathbb{Z}}\Big(\int_0^t J_{2(n-m)}(2s)\,ds\Big)\eta(m).$$

Let us define $\mathcal{S}_t^\sigma$ by

$$\mathcal{S}_t^\sigma f = \int_0^t \mathcal{C}_s^\sigma f\,ds, \quad f\in\ell^p,\ t\geq 0.$$

It happens that, from semigroup and cosine operator theory, the functions $\mathcal{L}_t^\sigma\varphi(n)$, $\mathcal{U}_t^\sigma\psi(n)$, and $\mathcal{C}_t^\sigma\xi(n)+\mathcal{S}_t^\sigma\eta(n)$, where $\mathcal{L}_t^\sigma$, $\mathcal{U}_t^\sigma$, and $\mathcal{C}_t^\sigma$ are defined, respectively, in Theorem 1.3, (30), and (14), are the solutions to the homogenous versions of (32), (33) and (34) when $L = -(-\Delta_{\mathrm{d}})^\sigma$. See e.g. [5, Proposition 3.1.16 and Corollary 3.14.8].

4.1. **The linear cases: proof of Theorem 1.5.** It follows directly from semigroup theory (see e.g. [36]) that $u(t,n)$ is indeed the solution to (32). The proof for the solution $w(t,n)$ follows from Theorem 1.1, and concerning the function $v(t,n)$ the statement follows immediately from cosine operator theory shown in Theorem 1.2. The proof for $u_\sigma(t,n)$, $w_\sigma(t,n)$ and $v_\sigma(t,n)$ is immediate from the results contained in Theorem 1.3, Theorem 3.3, and Theorem 1.4.

Now let us prove (1). By Theorem 1.1, we know that $T_t(n)$ is a discrete almost periodic function in the variable n and $T_t\in\ell^1$, for each $t\in(0,\infty)$. Then, under the hypothesis that $\varphi\in\ell^1$, by Theorem C.6 we have that $\sum_{k\in\mathbb{Z}} T_t(n-k)\varphi(k)$ is also discrete almost periodic. On the other hand, again by Theorem C.6, the function $F(t,s,n)$ defined by

$$F(t,s,n) := \sum_{k\in\mathbb{Z}} T_{t-s}(n-k)g(s,k)$$

is discrete almost periodic. Let

$$\Psi(t,n) := \int_0^t F(t,s,n)\,ds.$$

Given $\varepsilon>0$, there exists a positive integer $M(\varepsilon)$ such that any set consisting of $M(\varepsilon)$ consecutive integers contains at least an integer p for which $|F(t,s,n+p)-F(t,s,n)|<\varepsilon$, for each $n\in\mathbb{Z}$ and each $t,s\in(0,\infty)$. Then we obtain immediately $|\Psi(t,n+p)-\Psi(t,n)|<\varepsilon$. This completes the proof of (1) for $u(t,\cdot)$. The proof for $w(t,\cdot)$, $v(t,\cdot)$, $u_\sigma(t,\cdot)$, $w_\sigma(t,\cdot)$, and $v_\sigma(t,\cdot)$ is exactly the same.

Following analogous arguments as the ones in part (1) we can conclude that $w(\cdot,n)$, $v(\cdot,n)$, $w_\sigma(\cdot,n)$, and $v_\sigma(\cdot,n)$ are also almost periodic in the first variable, for each $n\in\mathbb{Z}$. One just has to consider, respectively, the expression in (19) for $T_{it}(n)$, the expression (20) for $C_t(n)$, Theorem 1.3 part (ii) for L^σ_{it}, and Theorem 1.4, (15), for C^σ_t. Then the result in (2) follows immediately from Dominated Convergence Theorem.

Finally, let us prove part (3). On one hand, it follows that

$$\sum_{m\in\mathbb{Z}}|T_t(n-m)\varphi(m)|\le \sup_{n\in\mathbb{Z}}|T_t(n)|\sum_{m\in\mathbb{Z}}|\varphi(m)|.$$

Then, in virtue of the representation (19) and since $\varphi\in\ell^1$, we easily deduce that the quantity above vanishes as $t\to\infty$. On the other hand we have, again from (19), that

$$\begin{aligned}\int_0^t \Big|\sum_{m\in\mathbb{Z}}T_{t-s}(n-m)g(s,m)\Big|\,ds &\le \int_0^t \sup_{n\in\mathbb{Z}}|T_{t-s}(n)|\sum_{m\in\mathbb{Z}}\big|g(s,m)\big|\,ds\\ &\le \frac{2}{\pi}\int_0^t\Big(\int_0^{\pi/2}e^{-4(t-s)\sin^2\theta}\,d\theta\Big)\sum_{m\in\mathbb{Z}}\big|g(s,m)\big|\,ds\\ &= \frac{2}{\pi}\int_0^{\pi/2}e^{-4t\sin^2\theta}\Big(\int_0^t e^{4s\sin^2\theta}\sum_{m\in\mathbb{Z}}\big|g(s,m)\big|\,ds\Big)\,d\theta.\end{aligned}$$

Then, under the assumption on g, we readily conclude the proof of part (3) for $u(t,n)$. The analogous statement for $u_\sigma(t,n)$ also follows, taking into account the integral representation in Theorem 1.3 part (ii). Theorem 1.5 is proved.

Remark 9. *Observe that, under the assumption that $\varphi,\psi,\xi,\eta\in \mathrm{AP}_{\mathrm{d}}(\mathbb{Z})$, we can also prove that $u(t,\cdot)$, $w(t,\cdot)$, $v(t,\cdot)$, $u_\sigma(t,\cdot)$, $w_\sigma(t,\cdot)$, and $v_\sigma(t,\cdot)$ are discrete almost periodic in the first variable, for $t\ge 0$. The reasoning is the same as the one in Theorem 1.5 part (3), taking into account that $T_z, C_t, L^\sigma_z, C^\sigma_t\in\ell^1$.*

Remark 10. *In the particular case $g\equiv 0$, if we assume that $k\mapsto T_t(n-k)\varphi(k)$ is summable for each $n\in\mathbb{Z}$, the solutions u,w,v,u_σ in Theorem 1.5 are automatically almost periodic independently of the regularity of the initial conditions φ,ψ,ξ,η.*

4.2. **The nonlinear cases: proof of Theorem 1.6.** We will show in detail the proof concerning the semidiscrete nonlinear heat equation. For the rest of the cases we will give the appropriate hints to complete the proofs.

We define a map $F:\mathrm{AP}_{\mathrm{d}}(J\times\mathbb{Z})\to \mathrm{AP}_{\mathrm{d}}(J\times\mathbb{Z})$ by

$$F(u)(t,n):=\mathcal{T}_t\varphi(n)+\int_0^t \mathcal{T}_{t-s}g(u(s,n),n)\,ds,\quad t\in J. \tag{35}$$

Note that, by Theorem 1.5, F is well defined under the hypotheses for φ and g.

In order to prove the existence and uniqueness, we will apply a well known extension of the contraction principle. Recall that

$$\|F(u)-F(v)\|_{\mathrm{AP}_{\mathrm{d}}(J\times\mathbb{Z})}=\sup_{(n,t)\in J\times\mathbb{Z}}|F(u)(t,n)-F(v)(t,n)|.$$

Let us analyze $|F(u)(t,n)-F(v)(t,n)|$. In view of (16), we have

$$
\begin{aligned}
|F(u)(t,n)-F(v)(t,n)| &\le \sum_{k\in\mathbb{Z}}\int_0^t T_{t-s}(n-k)|g(u(s,k),k)-g(v(s,k),k)|\,ds\\
&\le C_L\sum_{k\in\mathbb{Z}}\int_0^t T_{t-s}(n-k)|u(s,k)-v(s,k)|\,ds\\
&\le C_L\sum_{k\in\mathbb{Z}}\int_0^t T_{t-s}(n-k)\sup_{(s,k)\in J\times\mathbb{Z}}|u(s,k)-v(s,k)|\,ds\\
&= C_L\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})}\int_0^t\sum_{k\in\mathbb{Z}}T_s(k)\,ds\\
&\le tC_L\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})}.
\end{aligned}
$$

Using the last inequality, (35) and induction on m it follows easily that

$$
|F^m(u)(t,n)-F^m(v)(t,n)|\le \frac{t^m {C_L}^m}{m!}\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})},
$$

whence

$$
\|F^m(u)-F^m(v)\|_{\mathrm{AP_d}(J\times\mathbb{Z})}\le \frac{(TC_L)^m}{m!}\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})}.
$$

For m large enough we have that $\frac{(TC_L)^m}{m!}<1$, and we can conclude that F has a unique fixed point in $\mathrm{AP_d}(J\times\mathbb{Z})$.

In order to prove the theorem for the case of the Schrödinger equation, we proceed exactly as in the proof in the latter case. Observe that, by Remark 2, we have that

$$
\int_0^t\sum_{k\in\mathbb{Z}}|T_{is}(k)|\,ds\le\int_0^t\big(2e^s-1\big)\,ds\le(2e^T-1)t,\quad t\in J.
$$

This yields

$$
|F(u)(t,n)-F(v)(t,n)|\le t(2e^T-1)C_L\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})},\quad t\in J,
$$

and we can conclude analogously as in the case of the semidiscrete heat equation.

Finally, for the case of the wave equation, proceeding as in the previous cases, by Theorem 1.2, we arrive at

$$
\begin{aligned}
\int_0^t\sum_{k\in\mathbb{Z}}\Big|\int_0^s C_u(k)\,du\Big|\,ds &\le \int_0^t\int_0^s(2\cosh u-1)\,du\,ds\\
&=\int_0^t(2\sinh s-s)\,ds\le(2\sinh T-T)t,\quad t\in J,
\end{aligned}
$$

since $(2\sinh s-s)$ is an increasing function of s. This yields

$$
|F(u)(t,n)-F(v)(t,n)|\le t(2\sinh T-T)C_L\|u-v\|_{\mathrm{AP_d}(J\times\mathbb{Z})},\quad t\in J,
$$

and we proceed as in the previous cases.

In the case of $u_\sigma(t,n)$, we obtain an analogous result to the one for $u(t,n)$, due to Theorem 1.3 part (v)-(2). Concerning $w_\sigma(t,n)$, recall that $\|L_z^\sigma\|_{\ell^p}\le e^{2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}|z|}$, again by Theorem 1.3. Then we conclude the result with the constant

$$
\frac{\Gamma(1+\sigma)^2}{2\Gamma(1+2\sigma)}\big(e^{2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}|T|}-1\big)C_L.
$$

Analogously, for the solution $v_\sigma(t,n)$, we have $\|C_t^\sigma\|_{\ell^p} \leq e^{2\frac{\Gamma(1+\sigma)}{\Gamma(1+\sigma/2)^2}|t|}$, by Theorem 1.4. Therefore, the result for $v_\sigma(t,n)$ holds with the constant

$$\frac{\Gamma(1+\sigma/2)^2}{2\Gamma(1+\sigma)}\Big(\frac{\Gamma(1+\sigma/2)^2}{2\Gamma(1+\sigma)}\big(e^{2\frac{\Gamma(1+\sigma)}{\Gamma(1+\sigma/2)^2}|T|}-1\big)-T\Big)C_L.$$

Indeed, we just have to observe that $\frac{e^{At}-1-At}{A}$ is an increasing function of t, for $A>0$. Theorem 1.6 is proved.

5. $C^\alpha - \ell^p$ **maximal regularity: proof of Theorems 1.7 and 1.8.** In order to prove Theorem 1.7, we will use a result by W. Arendt, C. Batty, and S. Bu [4], established in a more general context. Let X be a Banach space. We denote by $C_{\rm per}([0,2\pi];X)$ the space of all continuous functions $F:[0,2\pi]\to X$ such that $F(0)=F(2\pi)$. Let $0<\alpha<1$, and

$$\|F\|_\alpha := \sup_{\substack{t\neq s\\ t,s\in(0,2\pi]}} \frac{\|F(t)-F(s)\|}{|t-s|^\alpha}.$$

Define the space of α-Hölder continuous periodic functions taking values on X as

$$C^\alpha_{\rm per}([0,2\pi];X) := \{F\in C_{\rm per}([0,2\pi];X) : \|F\|_\alpha<\infty\},$$

and

$$C^{1,\alpha}_{\rm per}([0,2\pi];X) := \{F\in C^1([0,2\pi];X) : F(0)=F(2\pi), F'\in C^\alpha_{\rm per}([0,2\pi];X)\}.$$

Let A be a closed operator on X. For $f\in C([0,2\pi];X)$ we consider the problem

$$\big(P_{A,\rm per}(f)\big)\begin{cases} u'(t)=Au(t)+f(t), & t\in[0,2\pi],\\ u(0)=u(2\pi).\end{cases}$$

The problem $(P_{A,\rm per})$ is said to be well-posed in $C^\alpha_{\rm per}([0,2\pi];X)$ if for each $f\in C^\alpha_{\rm per}([0,2\pi];X)$ there exists a unique classical solution $u\in C^{1,\alpha}_{\rm per}([0,2\pi];X)$ of $(P_{A,\rm per}(f))$. In the sequel, $\rho(A)$ denotes the resolvent set of an operator A.

Theorem 5.1 (See [4, Theorem 4.2])**.** *Let $0<\alpha<1$. The following assertions are equivalent:*

(i) the problem $(P_{A,\rm per})$ is well-posed in $C^\alpha_{\rm per}([0,2\pi];X)$;
(ii) $\{ik\}_{k\in\mathbb{Z}}\subseteq\rho(A)$ and $\sup_{k\in\mathbb{Z}}\|k(ik-A)^{-1}\|<\infty$.

We take $X=\ell^p$, $1\leq p\leq\infty$, and $A:=\Delta_{\rm d}+rI$, where I is the identity operator. The corresponding definitions were already introduced in Subsection 1.2.

Proof of Theorem 1.7. Let us first assume that the problem (17) is well-posed in the Hölder space $C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$. By Theorem 5.1 we have that $\{ik\}_{k\in\mathbb{Z}}\in\rho(\Delta_{\rm d}+rI)$. This implies that $-r\in\rho(\Delta_{\rm d})$. By Theorem 1.1,

$$\sigma(\Delta_{\rm d})=\{-4\sin^2\theta/2\}_{\theta\in(-\pi,\pi]}=[-4,0].$$

From here, we easily deduce that $-r<-4$, proving one implication.

For the converse, we again observe that $\sigma(\Delta_{\rm d})=[-4,0]$. From $r>4$ we necessarily deduce $\{-r+ik\}_{k\in\mathbb{Z}}\subseteq\rho(\Delta_{\rm d})$ or, equivalently, $\{ik\}_{k\in\mathbb{Z}}\subseteq\rho(\Delta_{\rm d}+rI)$, so the first condition in part (ii) of Theorem 5.1 is satisfied. Now, observe that the identity $ik(ik-A)^{-1}=I+A(ik-A)^{-1}$ implies

$$\|k(ik-A)^{-1}\|\leq 1+\|A\|\,\|(ik-A)^{-1}\|,\quad k\in\mathbb{Z}.$$

Since A is bounded, then both A and $-A$ generate semigroups or, equivalently, A generates a group. Therefore, it is enough to show that

$$\sup_{k\in\mathbb{Z}} \|(ik+A)^{-1}\| < \infty.$$

Indeed, we have

$$(ik+A)^{-1} = \big((ik+r)+\Delta_{\mathrm{d}}\big)^{-1} = \int_0^\infty e^{-(ik+r)s}e^{-s\Delta_{\mathrm{d}}}ds$$

so that, by Theorem 1.1, part (i)-(2),

$$\Big\|\int_0^\infty e^{-(ik+r)s}e^{-s\Delta_{\mathrm{d}}}ds\Big\| \le \int_0^\infty e^{-rs}e^{s\|\Delta_{\mathrm{d}}\|}ds \le \int_0^\infty e^{-rs}e^{4s}ds = \frac{1}{r-4},$$

valid for all $r > 4$. This implies the claim and finishes the proof of the theorem. □

As a corollary of Theorem 1.7, we obtain the following result on estimates of maximal regularity in the Hölder space $C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$.

Corollary 1. *Let $0 < \alpha < 1$ and $A = \Delta_{\mathrm{d}} + rI$. Let $1 \le p \le \infty$. If $r > 4$ then $u', Au \in C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ and there exists a constant $C > 0$ independent of $f \in C^\alpha_{\mathrm{per}}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ such that the following estimate holds:*

$$\|u'\|_{C^\alpha} + \|Au\|_{C^\alpha} \le C\|f\|_{C^\alpha}.$$

Proof of Theorem 1.8. By Theorem 1.3, part (iii), we have that $\sigma(-(-\Delta_{\mathrm{d}})^\sigma) = [-4^\sigma, 0]$. Under the assumption, $r > 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2} > 4^\sigma$, so we obtain $\{ik\}_{k\in\mathbb{Z}} \subseteq \rho(-(-\Delta_{\mathrm{d}})^\sigma + rI)$, which is the first condition in part (ii) of Theorem 5.1. On the other hand, following an analogous reasoning as in Theorem 1.7, and by Theorem 1.3, part (i)-(2), we end up with

$$\begin{aligned}\Big\|\int_0^\infty e^{-(ik+r)s}e^{s(-\Delta_{\mathrm{d}})^\sigma}ds\Big\| &\le \int_0^\infty e^{-rs}e^{s\|(-\Delta_{\mathrm{d}})^\sigma\|}ds\\ &\le \int_0^\infty e^{-rs}e^{s2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}}ds = \frac{1}{r-2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}},\end{aligned}$$

which is valid under the restriction $r > 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$. □

In view of Theorem 1.8, we also obtain a result of maximal regularity involving the operator $-(-\Delta_{\mathrm{d}})^\sigma + rI$.

Corollary 2. *Let $0 < \sigma < 1$ and $A = -(-\Delta_{\mathrm{d}})^\sigma + rI$. If $r > 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$ then the same conclusion of Corollary 1 remains true.*

Remark 11. *For the case of $p = 2$ the condition $r > 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$ can be replaced by $r > 4^\sigma$. This is a consequence of the fact that in such case the spectral radius of the fractional discrete Laplacian $(-\Delta_{\mathrm{d}})^\sigma$, which is clearly a bounded and normal operator on the Hilbert space ℓ^2, coincides with its norm. Moreover, this last condition on r turns out to be a characterization in such case. The proof of this assertion is similar to that of Theorem 1.7. The problem of the extension of this characterization to the general case of ℓ^p spaces seems to be a difficult task that relies on sharp estimates of the fractional discrete Laplacian, and therefore is left open.*

6. **Extension to higher dimensions.** Our aim in this section is to extend to higher dimensions the results contained in Theorems 1.1, 1.2, 1.3 and 1.4.

Let $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N$. We are going to consider the *multidimensional* discrete Laplacian $\Delta_{\mathrm{d},N}$, defined as

$$\Delta_{\mathrm{d},N}\varphi(\mathbf{n}) = \sum_{j=1}^{N} \big(\varphi(\mathbf{n}+e_j) - 2\varphi(\mathbf{n}) + \varphi(\mathbf{n}-e_j)\big), \tag{36}$$

where e_j denotes the unit vector in the positive direction of the j-th coordinate. It is easy to see that the operator $\Delta_{\mathrm{d},N}$ maps $\ell^p(\mathbb{Z}^N)$ into itself boundedly for all $1 \le p \le \infty$.

The discrete and inverse discrete Fourier transforms in the multidimensional setting are, respectively,

$$\mathcal{F}_{\mathbb{Z}^N}(f)(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\mathbb{Z}^{\mathbb{N}}} f(\mathbf{n}) e^{i\mathbf{n}\cdot\boldsymbol{\theta}}, \quad \boldsymbol{\theta} \in \mathbb{T}^N, \tag{37}$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$, and

$$\mathcal{F}^{-1}_{\mathbb{Z}^N}(\varphi)(\mathbf{n}) = \frac{1}{(2\pi)^N} \int_{[-\pi,\pi]^N} \varphi(\boldsymbol{\theta}) e^{-i\mathbf{n}\cdot\boldsymbol{\theta}}\, d\boldsymbol{\theta}, \quad \mathbf{n} \in \mathbb{Z}^N.$$

Lemma 6.1. *Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$. We have*

1. $\mathcal{F}_{\mathbb{Z}^N}\big((-\Delta_{\mathrm{d},N})\varphi\big)(\boldsymbol{\theta}) = \sum_{j=1}^N 4\sin^2(\theta_j/2)\mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta})$.
2. $\mathcal{F}_{\mathbb{Z}^N}\big(\mathbb{T}_z\varphi\big)(\boldsymbol{\theta}) = e^{-z\sum_{j=1}^N(4\sin^2(\theta_j/2))}\mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta})$, *for each $z \in \Sigma_\pi$.*

Proof. The proof is an exercise. For the first one, apply the discrete Fourier transform (37) to $\Delta_{\mathrm{d},N}\varphi(\mathbf{n})$ in (36). A direct computation yields the result. With a similar computation, by taking into account (38) and (19), we get the second one. □

Let us define, for each $z \in \Sigma_\pi$,

$$\mathbb{T}_z\varphi(\mathrm{n}) := \sum_{\mathbf{k}\in\mathbb{Z}^N} \mathbf{T}_z(\mathbf{n}-\mathbf{k})\varphi(\mathbf{k}) = (\mathbf{T}_z * \varphi)(\mathbf{n}) \quad \varphi \in \ell^p(\mathbb{Z}^N),$$

where

$$\mathbf{T}_z(\mathbf{n}) := \prod_{j=1}^N T_z(n_j) = \prod_{j=1}^N e^{-2z} I_{n_j}(2z). \tag{38}$$

We denote the multidimensional Kronecker delta by $\delta_{\mathbf{0n}} = \delta_{0n_1} \cdot \ldots \cdot \delta_{0n_N}$.

Theorem 6.2. *Let $z \in \Sigma_\pi$. Then, $\{\mathbb{T}_z\}_{z\in\Sigma\pi} \subset \mathcal{B}(\ell^p(\mathbb{Z}^N))$ and it is an uniformly continuous analytic semigroup in Σ_π with bounded generator $\Delta_{\mathrm{d},N}$ on $\ell^p(\mathbb{Z}^N)$, $1 \le p \le \infty$. Moreover, the spectrum of the operators $\Delta_{\mathrm{d},N}$ and $\mathbb{T}_z$ is, respectively,*

$$\sigma(\Delta_{\mathrm{d},N}) = \Big\{ -\sum_{j=1}^N 4\sin^2(\theta_j/2)\Big\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N} = [-4N, 0],$$
$$\sigma(\mathbb{T}_z) = \{e^{-z\sum_{j=1}^N 4\sin^2(\theta_j/2)}\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}.$$

Proof. First, it is clear that the kernel $\mathbf{T}_z$ in (38) is in $\ell^p(\mathbb{Z}^N)$, by Theorem 1.1. Moreover, given $z, w \in \Sigma_\pi$ and $\mathbf{n} \in \mathbb{Z}^N$, it holds

$$(\mathbf{T}_z * \mathbf{T}_w)(\mathbf{n}) = \mathbf{T}_{z+w}(\mathbf{n}).$$

Indeed, from (29),

$$\begin{aligned}(\mathbf{T}_z * \mathbf{T}_w)(\mathbf{n}) &= \sum_{\mathbf{k}\in\mathbb{Z}^N} \mathbf{T}_z(\mathbf{n}-\mathbf{k})\mathbf{T}_w(\mathbf{n}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \prod_{j=1}^N T_z(n_j-k_j)\prod_{i=1}^N T_w(k_i)\\ &= \sum_{\mathbf{k}\in\mathbb{Z}^N} \prod_{j=1}^N T_z(n_j-k_j)T_w(k_j)\\ &= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\cdots\sum_{k_N\in\mathbb{Z}} \prod_{j=1}^N T_z(n_j-k_j)T_w(k_j)\\ &= \prod_{j=1}^N \sum_{k_j\in\mathbb{Z}} T_z(n_j-k_j)T_w(k_j) = \prod_{j=1}^N (T_z * T_w)(n_j)\\ &= \prod_{j=1}^N T_{z+w}(n_j) = \mathbf{T}_{z+w}(\mathbf{n}).\end{aligned}$$

Now we compute the generator. We derive $\mathbb{T}_z$ with respect to z and evaluate in $z=0$, so that

$$\mathbb{T}'_z\varphi(\mathbf{n})\big|_{z=0} = (\mathbf{T}'_z * \varphi)(\mathbf{n})\big|_{z=0} = \Delta_{\mathrm{d},N}\delta_{\mathbf{0}\mathbf{n}},$$

where the last equality follows from the fact that $T_0(n_i)=\delta_{0n_i}$, for each $n_i\in\mathbb{Z}$.

The statement concerning the spectra of the operators follows immediately from Lemma 6.1. $\square$

Remark 12. *If we take $z=t\in(0,\infty)$ in (38), we have that*

$$\begin{aligned}\|\mathbf{T}_t\|_{\ell^1(\mathbb{Z}^N)} &= \sum_{\mathbf{n}\in\mathbb{Z}^N}\prod_{j=1}^N T_t(n_j) = \sum_{\mathbf{n}\in\mathbb{Z}^N}\prod_{j=1}^N e^{-2t}I_{n_j}(2t)\\ &= \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\cdots\sum_{n_N\in\mathbb{Z}}\prod_{j=1}^N e^{-2t}I_{n_j}(2t)\\ &= \prod_{j=1}^N\sum_{n_j\in\mathbb{Z}} e^{-2t}I_{n_j}(2t) = 1.\end{aligned}$$

For $z\in\mathbb{C}$, with a similar computation as in the proof of Theorem 1.1 (see also Remark 2) adapted to the multidimensional setting, we get

$$\|\mathbf{T}_z\|_{\ell^1(\mathbb{Z}^N)} \le e^{-2N\operatorname{Re} z}e^{N|\operatorname{Im} 2iz|}(2e^{|z|}-1)^N.$$

Remark 13. *Observe that, since $\Delta_{\mathrm{d},N}$ is bounded on $\ell^p(\mathbb{Z}^N)$, the expression $e^{z\Delta_{\mathrm{d},N}}$ has sense, and by uniqueness we conclude that*

$$e^{z\Delta_{\mathrm{d},N}} = \mathbb{T}_z. \tag{39}$$

A multidimensional result analogous to Theorem 2.1 follows immediately. Let $\mathbb{U}_{it}\varphi := e^{-it(-\Delta_{\mathrm{d},N})}\varphi$.

Theorem 6.3. *The family $\{\mathbb{U}_{it}\}_{t\in\mathbb{R}}\subset\mathcal{B}(\ell^p(\mathbb{Z}^N))$ and it forms an uniformly continuous group with bounded generator $i\Delta_{\mathrm{d},N}$, $1\le p\le\infty$.*

On the other hand, we define

$$\mathbb{C}_t := \frac{\mathbb{U}_{it}^{1/2} + \mathbb{U}_{-it}^{1/2}}{2} = \cos\big(t(-\Delta_{\mathrm{d},N})^{1/2}\big),$$

where the second identity follows from (39), and we have the multidimensional result analogous to Theorem 1.2.

Theorem 6.4. *Let $0 < \sigma < 1$. We have $\{\mathbb{C}_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ and it is an uniformly continuous cosine family, with bounded generator $\Delta_{\mathrm{d},N}$ on $\ell^p(\mathbb{Z}^N)$, $1 \le p \le \infty$. Moreover, the spectrum of $\mathbb{C}_t$ is $\sigma(\mathbb{C}_t) = \Big\{ \cos\big[2t\big(\sum_{j=1}^N \sin^2(\theta_j/2)\big)^{1/2}\big]\Big\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}$.*

Let us define

$$(-\Delta_{\mathrm{d},N})^\sigma\varphi(\mathrm{n}) = \frac{1}{\Gamma(-\sigma)}\int_0^\infty \big(\mathbb{T}_t\varphi(\mathrm{n}) - \varphi(\mathrm{n})\big)\frac{dt}{t^{1+\sigma}}.$$

We can prove the following.

Lemma 6.5. *Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$ and $0 < \sigma < 1$. Then*

$$\mathcal{F}_{\mathbb{Z}^N}\big((-\Delta_{\mathrm{d},N})^\sigma\varphi\big)(\boldsymbol{\theta}) = \Big(\sum_{j=1}^N 4\sin^2(\theta_j/2)\Big)^\sigma \mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta}).$$

Proof. We use the representation (24) and Lemma 6.1, part (2) with $z = t \ge 0$, then

$$\begin{aligned}\mathcal{F}_{\mathbb{Z}^N}\big((-\Delta_{\mathrm{d},N})^\sigma\varphi\big)(\boldsymbol{\theta}) &= \frac{1}{\Gamma(-\sigma)}\int_0^\infty (\mathcal{F}_{\mathbb{Z}^N}\big(\mathbb{T}_t\varphi\big)(\boldsymbol{\theta}) - \mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta})\frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(-\sigma)}\int_0^\infty (e^{-t\sum_{j=1}^N(4\sin^2(\theta_j/2))} - 1)\frac{dt}{t^{1+\sigma}}\mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta}) \\ &= \Big(\sum_{j=1}^N 4\sin^2(\theta_j/2)\Big)^\sigma \mathcal{F}_{\mathbb{Z}^N}(\varphi)(\boldsymbol{\theta}),\end{aligned}$$

so we get the conclusion. ☐

Observe that, by the spectral mapping theorem, $(-\Delta_{\mathrm{d},N})^\sigma$ is bounded on $\ell^p(\mathbb{Z}^N)$ and the spectrum is $\sigma\big((-\Delta_{\mathrm{d},N})^\sigma\big) = \big(\sigma(-\Delta_{\mathrm{d},N})\big)^\sigma$, being the latter bounded. Then, the expression

$$\mathbb{L}_z^\sigma\varphi := e^{-z(-\Delta_{\mathrm{d},N})^\sigma}\varphi, \quad z \in \Sigma_\pi, \tag{40}$$

has sense.

From the considerations above, we conclude the following.

Theorem 6.6. *Let $0 < \sigma < 1$. Given $z \in \Sigma_\pi$, let $\mathbb{L}_z^\sigma$ be the operators defined in (40). We have $\{\mathbb{L}_z^\sigma\}_{z\in\Sigma_\pi} \subset \mathcal{B}\big(\ell^p(\mathbb{Z}^N)\big)$ and it is an uniformly continuous and analytic semigroup in Σ_π, with bounded generator $(-\Delta_{\mathrm{d},N})^\sigma$ on $\ell^p(\mathbb{Z}^N)$, $1 \le p \le \infty$. Moreover, the spectrum of the operators $(-\Delta_{\mathrm{d},N})^\sigma$ and $\mathbb{L}_z^\sigma$ is, respectively,*

$$\sigma((-\Delta_{\mathrm{d},N})^\sigma) = \Big\{ -\Big(\sum_{j=1}^N 4\sin^2(\theta_j/2)\Big)^\sigma\Big\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}$$

$$\sigma(\mathbb{L}_z^\sigma) = \{e^{-z(\sum_{j=1}^N 4\sin^2(\theta_j/2))^\sigma}\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}.$$

Taking into account Theorem 6.6, we can write an integral representation for the semigroup $e^{-t(-\Delta_{\mathrm{d},N})^\sigma}$, by using the identity in (28).

Theorem 6.7. *Let $0<\sigma<1$. We have, for $\varphi \in \ell^p(\mathbb{Z}^N)$ and $\mathbf{n} \in \mathbb{Z}^N$,*

$$e^{-t(-\Delta_{\mathrm{d},N})^\sigma}\varphi(\mathbf{n}) = \int_0^\infty \sum_{\mathbf{k}\in\mathbb{Z}^N} \prod_{j=1}^N T_\lambda(n_j-k_j)\varphi(k_j) f_{t,\sigma}(\lambda)\, d\lambda, \quad t \geq 0.$$

From the latter result, we obtain the following corollary.

Corollary 3. *Let $0<\sigma<1$ and $t \geq 0$. Then $\mathbb{L}_t^\sigma\varphi := e^{-t(-\Delta_{\mathrm{d},N})^\sigma}$ is an uniformly continuous and Markovian semigroup.*

We observe easily that $\mathbb{U}_{it}^\sigma = e^{-it(-\Delta_{\mathrm{d},N})^\sigma}$ is a group with generator $-i(-\Delta_{\mathrm{d},N})^\sigma$.

Theorem 6.8. *Let $0<\sigma<1$. The family $\{\mathbb{U}_{it}^\sigma\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p(\mathbb{Z}^N))$, and it forms a uniformly continuous group with bounded generator $-i(-\Delta_{\mathrm{d},N})^\sigma$, $1 \leq p \leq \infty$.*

Finally, we define

$$\mathbb{C}_t^\sigma := \frac{\mathbb{U}_{it}^{\sigma/2} + \mathbb{U}_{-it}^{\sigma/2}}{2} = \cos\left(t(-\Delta_{\mathrm{d},N})^{\sigma/2}\right),$$

where the second identity follows from (40). We have the corresponding multidimensional result.

Theorem 6.9. *Let $0<\sigma<1$. We have $\{\mathbb{C}_t^\sigma\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ and it is an uniformly continuous cosine family, with bounded generator $(-\Delta_{\mathrm{d},N})^\sigma$ on $\ell^p(\mathbb{Z}^N)$, $1 \leq p \leq \infty$. Moreover $\sigma(\mathbb{C}_t^\sigma) = \left\{\cos\left[t\left(4\sum_{j=1}^N \sin^2(\theta_j/2)\right)^{\sigma/2}\right]\right\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}$.*

Proof. The first assertion is clear. For the identity concerning the spectrum is enough to observe that the uniform continuity of $\mathbb{C}_t^\sigma$ implies, by Theorem 6.6 and the spectral mapping theorem, that

$$\sigma(\mathbb{C}_t^\sigma) = \cosh\left(t\sqrt{\sigma\left((-\Delta_{\mathrm{d},N})^\sigma\right)}\right) = \left\{\cos\left[t\left(4\sum_{j=1}^N \sin^2(\theta_j/2)\right)^{\sigma/2}\right]\right\}_{\boldsymbol{\theta}\in(-\pi,\pi]^N}.$$

□

Remark 14. *From the theory developed in the present section, some of the results concerning existence and uniqueness of solutions to linear and nonlinear semidiscrete equations can be also stated on $\mathbb{Z}^N$. Nevertheless, since we do not have information on the norm of the multidimensional cosine operator or of the multidimensional fractional operators (in particular we cannot give quantitative information about their $\ell^1(\mathbb{Z}^N)$ norm), some of the statements are left open. Namely, Theorem 1.5 remains completely valid except for part (3), which can be proved, with the tools in the present section, for the case of semidiscrete heat and Schrödinger equations when $L = \Delta_{\mathrm{d},N}$. Also, with the multidimensional theory in this section, one can prove Theorem 1.6 on $\mathbb{Z}^N$ for the semidiscrete heat and Schrödinger equations when $L = \Delta_{\mathrm{d},N}$.*

Remark 15. *In the special case of $\ell^2(\mathbb{Z}^N)$, in view of the given characterizations of the spectrum, we obtain the following identities*

1. $\|\Delta_{\mathrm{d},N}\| = 4N$
2. $\|\mathbb{T}_z\| = e^{4N Re(z)}$
3. $\|\mathbb{C}_t\| = 1$
4. $\|(-\Delta_{\mathrm{d},N})^\sigma\| = (4N)^\sigma$
5. $\|\mathbb{T}_z^\sigma\| = e^{(4N)^\sigma Re(z)}$

6. $\|\mathbb{C}_t^\sigma\| = 1$.

Assertions (1), (2), (4) and (5) follow from Theorem 6.6 and the fact that on a Hilbert space, the spectral radius of the fractional discrete Laplacian coincides with its norm (because this is a bounded and normal operator) and the spectral mapping theorem holds. Assertions (3) and (6) follow from Theorem 6.9 by the same reasoning.

7. **Applications to Nagumo and Fisher–KPP equations.** In this section we will apply the results obtained in the previous sections to Nagumo and Fisher–KPP models involving a discrete Laplacian, or the fractional powers of a discrete Laplacian. More precisely, we consider the following problems

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = Lu(t,n) + u(t,n)(u(t,n)-a)(1-u(t,n)), & \text{in } (0,\infty)\times\mathbb{Z}, \\ u(0,n) = f(n), \quad \text{on } \mathbb{Z}, \end{cases}$$

where $0 < a < 1$, and

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = Lu(t,n) + ru(t,n)(1-u(t,n)), & \text{in } (0,\infty)\times\mathbb{Z}, \\ u(0,n) = f(n), \quad \text{on } \mathbb{Z}, \end{cases}$$

where $r > 0$ and the operators L above are any of the following: Δ_{d}, $i\Delta_{\mathrm{d}}$, $-(-\Delta_{\mathrm{d}})^\sigma$, or $-i(-\Delta_{\mathrm{d}})^\sigma$.

Our main results in this section are theorems on local existence of solutions for the discrete Nagumo and Fisher equations. We highlight again that the qualitative behavior concerning almost periodicity in the discrete variable is a remarkable property that is not present in the continuous setting. It is also interesting to observe that the same conclusion is valid also for the fractional case, i.e. when the discrete Laplacian is replaced by the fractional discrete Laplacian.

Theorem 7.1. *Let $0 < \sigma \le 1$ and $f \in \ell^\infty$ be given. For each $T > 0$ there exists a unique solution $u : [0,T]\times\mathbb{Z} \to \mathbb{C}$ to the fractional discrete Fisher equation*

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = -(-\Delta_{\mathrm{d}})^\sigma u(t,n) + ru(t,n)(1-u(t,n)), & \text{in } (0,T]\times\mathbb{Z}, \\ u(0,n) = f(n), \quad \text{on } \mathbb{Z}. \end{cases}$$

Moreover, $u(t,n)$ is discrete almost periodic in the second variable.

Proof. Define $g(s,n) = rs(1-s)$ and observe that $g(s,\cdot) \in \ell^\infty$ and that it is obviously almost periodic in the discrete variable. It is easy to check that g satisfy condition (16) with Lipchitz constant $C_L = r(1+2T)$. Then, according to Theorem 1.6 we can conclude that there exists a unique solution which is discrete almost periodic in the second variable. □

With a similar proof, we can prove the following result.

Theorem 7.2. *Let $0 < \sigma \le 1$ and $f \in \ell^\infty$ be given. For each $T > 0$ there exists a unique solution $u : [0,T]\times\mathbb{Z} \to \mathbb{C}$ to the fractional discrete Nagumo equation*

$$\begin{cases} \dfrac{\partial u(t,n)}{\partial t} = -(-\Delta_{\mathrm{d}})^\sigma u(t,n) + u(t,n)(u(t,n)-a)(1-u(t,n)), & \text{in } (0,T]\times\mathbb{Z}, \\ u(0,n) = f(n), \quad \text{on } \mathbb{Z}. \end{cases}$$

Moreover, $u(t,n)$ is discrete almost periodic in the second variable.

It is worthwhile to notice that similar results holds in the case that $(-\Delta_{\rm d})^\sigma$ is replaced by $i(-\Delta_{\rm d})^\sigma,\ 0<\sigma\le 1$.

We finish this paper with a general result on non linear equations, which includes as a particular case a non homogeneous version of the fractional discrete Fisher equation. It corresponds to an application of Theorem 1.8 and gives information about the $C^\alpha-\ell^q$ regularity of solutions.

Theorem 7.3. *Let $0<\alpha<1$, $1\le p\le\infty$ and $r>2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$ for $0<\sigma<1$ (respectively $r>4$ for $\sigma=1$). Suppose that $G: C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)\to C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ satisfies $G(0)=0$ is continuously Fréchet differentiable at $u=0$ and $G'(0)=0$. Then there exists $\rho^*>0$ such that the equation*

$$\frac{\partial u(t,n)}{\partial t}=(-(-\Delta_{\rm d})^\sigma+r)u(t,n)+G(u)(t,n)+\rho f(t,n),\quad \text{in } [0,2\pi]\times\mathbb{Z}, \tag{41}$$

is solvable for each $\rho\in[0,\rho^*)$, with solution $u=u_\rho\in C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$.

Proof. We first prove the result for $0<\sigma<1$. Define

$$\mathcal{Z}:=C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)\cap C^{1,\alpha}_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big).$$

Observe that the space $\mathcal{Z}$ becomes a Banach space under the norm $|||w|||:=\|(-\Delta)^\sigma w+rw\|+\|w'\|$. Define the operator $\mathcal{A}u(t,n)=\partial_t u(t,n)+(-\Delta_{\rm d})^\sigma u(t,n)-ru(t,n)$ with $D(\mathcal{A})=C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$. Then $\mathcal{A}$ is an isomorphism onto. Indeed, by Corollary 2 we have $|||u|||\le C||\mathcal{A}u||$. On the other hand, by definition of the operator $\mathcal{A}$ we have $\|\mathcal{A}u\|\le|||u|||$. Therefore $\mathcal{A}$ is an isomorphism. By Theorem 1.8 we obtain that $\mathcal{A}$ is onto, proving the claim.

Let $\rho\in(0,1)$ and let us define the one parameter family of problems:

$$H[u,\rho]=-\mathcal{A}u+G(u)+\rho f.$$

By hypothesis it is clear that $H[0,0]=0$, H is continuously differentiable at $(0,0)$ and the partial Fréchet derivative is $H^1_{(0,0)}=-\mathcal{A}$, which is invertible by the preceding argument. We now apply the implicit function theorem (see e.g. [24, Theorem 17.6]) and then, we can find ρ^* such that for all $\rho\in[0,\rho^*)$ there exists $u=u_\rho\in C^\alpha_{\rm per}\big([0,2\pi];\ell^p(\mathbb{Z})\big)$ which satisfies (41). The proof of the case $\sigma=1$ is analogous with the obvious modifications. □

Remark 16. *From the theory developed in Section 6, and with obvious modifications, the results provided in this section can be extended to the multidimensional case.*

Appendix A. Analytic semigroups and cosine operators. Let us define the sector

$$\Sigma_\pi:=\{z\in\mathbb{C}:|\arg z|<\pi\}.$$

Definition A.1. *Let X be a Banach space. A family $\{\mathcal{W}_z\}_{z\in\Sigma_\pi}\subseteq\mathcal{B}(X)$ is said to be an analytic semigroup in Σ_π (see [36, Ch. 2, 2.5]) if*

(i) $z\mapsto\mathcal{W}_z x$ is analytic in Σ_π for each $x\in X$.
(ii) $\mathcal{W}_0=I$ and $\lim_{\substack{z\to 0\\ z\in\Sigma_\pi}}\mathcal{W}_z x=x$ for every $x\in X$.
(iii) $\mathcal{W}_{z_1+z_2}x=\mathcal{W}_{z_1}\mathcal{W}_{z_2}x$, for $z_1,z_2\in\Sigma_\pi$ and $x\in X$.

The semigroup $\mathcal{W}_z$ so defined will be just called analytic. The restriction of an analytic semigroup to the real axis is a C_0-semigroup.

More information about the well established theory of semigroups of operators can be found in the extensively bibliography on this topic. We refer the reader, for instance, to the monograph [5, Section 3].

Definition A.2. *A family of operators $\{\mathcal{C}_t\}_{t\in\mathbb{R}} \subseteq \mathcal{B}(X)$ is called a cosine operator function (see* [20, Ch. II.3]*) if*

(i) $\mathcal{C}_0 = I$.
(ii) $\mathcal{C}_{s+t}x + \mathcal{C}_{s-t}x = 2\mathcal{C}_s x \mathcal{C}_t x$, *for all* $s, t \in \mathbb{R}$ *and* $x \in X$.
(iii) $t \mapsto \mathcal{C}_t x$ *is continuous in* $\mathbb{R}$ *for each* $x \in X$.

About the theory of cosine operator functions the reader can see the classical monograph [20], or the recent reference [5, Sections 3.14 and 3.15].

Appendix B. **Properties of the Bessel functions J_k and I_k.** In this subsection we list some definitions and properties related to Bessel functions. We provide references of many of them and refer the reader to [31, Chapter 5], [35] and [49] for further issues concerning these functions.

The Bessel function of the first kind and order $k \in \mathbb{Z}$, $J_k(z)$, can be defined by

$$J_k(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+k+1)} \left(\frac{z}{2}\right)^{2m+k}, \quad |z| < \infty.$$

Since k is an integer (and $1/\Gamma(n)$ is taken to be equal zero if $n = 0, -1, -2, \ldots$), the function J_k is defined in the whole real line, and even in the whole complex plane, where J_k is an entire function. Therefore, if not otherwise indicated, from now on we will consider $z \in \mathbb{C}$. The Bessel function $J_k(z)$ satisfies (see [49, Ch. II, 2.1, p. 15])

$$J_{-k}(z) = (-1)^k J_k(z) \tag{42}$$

for each $k \in \mathbb{Z}$. It is also clear that

$$J_0(0) = 1 \quad \text{and} \quad J_k(0) = 0 \quad \text{for} \quad k \neq 0. \tag{43}$$

The following upper bound, for ν real and greater than $-1/2$, will be useful, see [49, Ch. III, 3.31, p. 49] or [35, Formula 10.14.4]

$$|J_\nu(z)| \leq \frac{|\frac{1}{2}z|^\nu e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)}. \tag{44}$$

Observe that (44) remains valid for all $n \in \mathbb{Z}$, due to (42).

In [49, Ch. II. 2.4, p. 30] we find the following addition formula for Bessel functions, also known as *Neumann's identity*

$$J_k(z_1 + z_2) = \sum_{m\in\mathbb{Z}} J_m(z_1) J_{k-m}(z_2), \quad z_1, z_2 \in \mathbb{C}, \tag{45}$$

where $k \in \mathbb{Z}$ but this formula is still valid for k unrestricted (see [49, Ch. V. 5.3, p. 143]). Analogously, see [49, Ch. V, 5.3, p. 145] it can be proved that

$$J_k(z_1 - z_2) = \sum_{m\in\mathbb{Z}} J_m(z_1) J_{k+m}(z_2), \quad z_1, z_2 \in \mathbb{C}. \tag{46}$$

The addition formulae above are consequences of the generating function (see [49, Ch. II, 2.1, p. 14] and also [35, formula 10.12.1])

$$e^{\frac{1}{2}z(u-u^{-1})} = \sum_{k\in\mathbb{Z}} u^k J_k(z), \quad z \in \mathbb{C}, \ u \in \mathbb{C} \setminus \{0\},$$

that serves also as a definition for J_k.

Now we pass to several properties and identities concerning the Bessel functions $I_k(z)$. The modified Bessel function of the first kind and order $k \in \mathbb{Z}$, or Bessel function of imaginary argument, $I_k(z)$, is defined by

$$I_k(z) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+k+1)} \left(\frac{z}{2}\right)^{2m+k}. \tag{47}$$

From the very definition of I_k we deduce that

$$I_k(t) \geq 0,$$

for every $k \in \mathbb{Z}$ and $t \geq 0$. Analogous observations as for the Bessel function $J_k(z)$ can be done. From now on we will also consider $z \in \mathbb{C}$ unless otherwise stated. The Bessel functions $I_k(z)$ and $J_k(z)$ are related by (see [35, 10.27.6])

$$I_k(2z) = e^{-ik\pi/2} J_k(2iz). \tag{48}$$

It is verified that $I_{-k}(z) = I_k(z)$ for each $k \in \mathbb{Z}$. From (47) it is clear that

$$I_0(0) = 1 \quad \text{and} \quad I_k(0) = 0 \quad \text{for} \quad k \neq 0. \tag{49}$$

The following identity, which is the addition formula analogous to the one for J_k, follows from Neumann's identity (45) and the relationship between the Bessel functions I_k and J_k in (48)

$$I_k(z_1 + z_2) = \sum_{m\in\mathbb{Z}} I_m(z_1) I_{k-m}(z_2) \quad \text{for} \quad k \in \mathbb{Z},\ z_1, z_2 \in \mathbb{C}; \tag{50}$$

this formula is also an easy consequence of the generating function, which is valid for $z \in \mathbb{C}$ and $u \in \mathbb{C} \setminus \{0\}$,

$$e^{\frac{1}{2}z(u+u^{-1})} = \sum_{k\in\mathbb{Z}} u^k I_k(z)$$

see, for instance, [35, formula 10.35.1]. By taking $u = 1$ in the generating function and changing z into $2z$, we obtain

$$\sum_{k\in\mathbb{Z}} e^{-2z} I_k(2z) = 1. \tag{51}$$

Appendix C. **Discrete almost periodic functions.** Let X be a (real or complex) Banach space equipped with a norm $\|\cdot\|$. First we recall the definition of almost periodic sequences (see e.g. [15, Ch. I. 6.]).

Definition C.1. *A sequence $F : \mathbb{Z} \to X$ is called almost periodic if for every $\varepsilon > 0$, there exists a positive integer $M(\varepsilon)$ such that any set consisting of $M(\varepsilon)$ consecutive integers contains at least one integer p for which*

$$\|F(n+p) - F(n)\| < \varepsilon, \quad n \in \mathbb{Z}.$$

We will denote by $\mathrm{AP_d}(X)$ the set of almost periodic sequences. Observe that each almost periodic sequence is bounded, i.e., $\sup_{n\in\mathbb{Z}} |F(n)| = C < \infty$.

Remark 17. *Actually, the definition of almost periodic sequence above is the analogous to the definition of almost periodic functions given by H. Bohr, see* [15, Ch. VI, 1]. *Throughout the paper we also consider almost periodic functions without further comment on their definition.*

We will deal with a more general definition of almost periodic sequences. Previously, we need the following.

Definition C.2. *A function $F : X \times \mathbb{Z} \to \mathbb{C}$ is said to be uniformly continuous on each bounded subset of X uniformly in $n \in \mathbb{Z}$ if for every $\varepsilon > 0$ and every bounded subset $K \subset X$, there exists $\delta(\varepsilon, K) > 0$ such that*

$$\|F(x,n) - F(y,n)\| < \varepsilon, \quad \text{for all } n \in \mathbb{Z},$$

and all $x, y \in X$ with $\|x - y\| \leq \delta(\varepsilon, K)$.

Definition C.3. *A function $F : X \times \mathbb{Z} \to \mathbb{C}$ is said to be discrete almost periodic in $n \in \mathbb{Z}$ for each $x \in X$ if for every $\varepsilon > 0$, there exists a positive integer $M(\varepsilon)$ such that any set consisting of $M(\varepsilon)$ consecutive integers contains at least one integer p for which*

$$\|F(x, n+p) - F(x,n)\| < \varepsilon, \quad n \in \mathbb{Z},$$

for each $x \in X$. If $K \subset X$ is a compact set, we will denote by $\mathrm{AP_d}(K \times \mathbb{Z})$ the set of all discrete almost periodic functions in $n \in \mathbb{Z}$ for each $x \in K$. The space $\mathrm{AP_d}(K \times \mathbb{Z})$ is provided with the norm

$$\|F\|_{\mathrm{AP_d}(K\times\mathbb{Z})} := \sup_{(n,x)\in K\times\mathbb{Z}} \|F(n,x)\|.$$

Remark 18. *The definitions above can be stated analogously for almost periodic functions $F : X \times \mathbb{R} \to \mathbb{C}$.*

Theorem C.4 below follows analogously to the classical result for almost periodic functions depending on parameters contained in [15, Ch. II, Th. 2.8].

Theorem C.4. *Let $u : \mathbb{Z} \to X$ be a discrete almost periodic sequence. Let $F : X \times \mathbb{Z} \to X$ be a discrete almost periodic function in $n \in \mathbb{Z}$ for each $x \in X$, such that $F(x,n)$ is uniformly continuous in each bounded subset of X uniformly in $k \in \mathbb{Z}$. Then, the Nemytskii operator $U : \mathbb{Z} \to X$ defined by $U(n) = F(u(n), n)$ is discrete almost periodic.*

As a consequence of Theorem C.4, we obtain the following composition theorem for discrete almost periodic functions satisfying a global Lipschitz condition.

Theorem C.5. *Let $u : \mathbb{Z} \to X$ be a discrete almost periodic sequence. Let $F : X \times \mathbb{Z} \to X$ be a discrete almost periodic function in $n \in \mathbb{Z}$ for each $x \in X$, such that satisfies a global Lipschitz condition in $x \in X$ uniformly in $n \in \mathbb{Z}$; that is, there is a constant $L > 0$ such that*

$$\|F(x,n) - F(y,n)\| \leq L\|x - y\|, \quad \text{for all } x, y \in X, \ n \in \mathbb{Z}.$$

Then, the Nemytskii operator $U : \mathbb{Z} \to X$ defined by $U(n) = F(u(n), n)$ is discrete almost periodic.

Finally, we have the following result on convolution of discrete almost periodic functions with functions $v : X \times \mathbb{Z} \to \mathbb{C}$ that are summable in the first variable.

Theorem C.6. *Let $v : X \times \mathbb{Z} \to \mathbb{C}$ be summable in the first variable, that is, for any $x \in X$*

$$\sum_{n\in\mathbb{Z}} |v(x,n)| < \infty.$$

Then for any discrete almost periodic function $u : X \times \mathbb{Z} \to \mathbb{C}$ the function $w(x,n)$ defined for each $x \in X$ by

$$w(x,n) = \sum_{k\in\mathbb{Z}} v(x,k)u(x,n-k), \quad n \in \mathbb{Z},$$

is also discrete almost periodic.

Proof. Let us assume w.l.o.g. that $\|v(x,\cdot)\|_{\ell^1} = 1$ for each $x \in X$. Given $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that any set consisting of $M(\varepsilon)$ consecutive integers contains at least one $p \in \mathbb{Z}$ with $\|u(x,n+p) - u(x,n)\| < \varepsilon$, for each $x \in X$ and for all $n \in \mathbb{Z}$. Now, for all $n \in \mathbb{Z}$, we have that

$$\|w(x,n+p) - w(x,n)\| \le \sum_{k\in\mathbb{Z}} |v(x,k)|\|u(x,n+p) - u(x,n)\| < \varepsilon,$$

so w is discrete almost periodic.

□

Remark 19. *Observe that, although the definitions and results in this subsection are stated for $\mathbb{Z}$, they can be analogously stated for $\mathbb{Z}^N$ without critical modifications.*

Appendix D. **The discrete Fourier transform.** For a given sequence $f \in \ell^1$, we define the discrete Fourier transform

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T},$$

where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ is the unidimensional torus, that we identify with the interval $(-\pi,\pi]$. We describe the integration over $\mathbb{T}$ by means of Lebesgue integration over $(-\pi,\pi]$. The operator $f \mapsto \mathcal{F}_{\mathbb{Z}}(f)$ can be extended as an isometry from ℓ^2 into $L^2(\mathbb{T})$, where the inverse discrete Fourier transform is obtained by the formula

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)e^{-in\theta}\,d\theta, \quad n \in \mathbb{Z}.$$

Therefore

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\mathbb{Z}}(f)(\theta)e^{-in\theta}\,d\theta, \quad n \in \mathbb{Z},$$

and it is easily verified that

$$\mathcal{F}_{\mathbb{Z}}(f * g)(\theta) = \mathcal{F}_{\mathbb{Z}}(f)(\theta)\mathcal{F}_{\mathbb{Z}}(g)(\theta).$$

Although the definitions in this subsection are enough for our purposes, we refer the reader to [17, 40] for details on discrete Fourier transform and an abstract theory of discrete distributions.

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E-mail address: carlos.lizama@usach.cl
E-mail address: lroncal@bcamath.org