NONLOCAL INTEGRATED SOLUTIONS FOR A CLASS OF ABSTRACT EVOLUTION EQUATIONS

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Abstract. In this article we show the existence of at least one integrated solution of a semilinear second order differential equation with an extra convolution term and nonlocal initial conditions. As main tool we use properties of the Hausdorff measure of noncompactness and fixed point theorems.

1. Introduction

In this paper we study the following equation

$$\begin{aligned} u''(t)+Au(t)-(k*Au)(t) =& f(t,u(t)), \quad t\in[0,b], \\ u(0)=g(u), \quad u'(0) =& h(u), \end{aligned} \tag{1}$$

where X is a Banach space, $A : D(A) \subseteq X \to X$ the generator of a resolvent family $S(t)$ with integrated kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, where $k \in L^1(\mathbb{R}_+)$. Here, $g, h : \mathcal{C}([0,b];X) \to X$ are continuous maps, and $f : [0,b]\times X \to X$ satisfies Carathéodory type conditions, which will be described in the Section 4. The objective in this work is to establish a result concerning the existence of integrated solutions for the above problem, using the theory of measure of noncompactness and fixed point theorems.

The problem of the existence of solutions to second order problems has been the subject of many studies in the last years, since they have a great importance in concrete models from mathematical physics, viscoelasticity theory, mechanics, among others. For more details, the reader can see [3, 22, 32, 29]. This study has been done using different tools. For example, Arendt and Batty in [4] studied existence of a distinguished class of solutions based on the factorisation method. Bureau in [13] and Banas and Chlebowicz in [7] used the method of ascent to study integrable solutions. Henríquez, Poblete and Pozo [23] studied a non-autonomous abstract differential equation of second order obtaining results based on the properties of evolution operators and measure of noncompactness. In Hilbert spaces, the problem (1) with local conditions has been studied by Prüss in [32] and Chang in [16]. Prüss obtain energy estimates and the optimal decay rate for the solutions of the linear problem, through of frequency domain methods, and Chang studied the existence of S-asymptotically ω-periodic solutions to the semilinear equation. The asymptotic behavior of the solutions of the following

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second order evolution equation in Hilbert spaces is studied in [15]

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)ds = \nabla F(u(t)) + f(t), \quad t \geq 0,$$

where A is a positive operator on a Hilbert space X, with dense domain $D(A)$, and ∇F denotes the gradient of a Gâteaux differentiable functional $F : D(A^{1/2}) \to \mathbb{R}$.
In [24] a more general equation than (1) with local condition was studied. In that work, by means of the theorem on approximation of resolvents, the authors gave sufficient conditions under which the mild solution becomes the strong one. Recently, in [28] Luong found decay mild solutions for a class of second order evolution equations with memory and nonlocal conditions using measure of noncompactness on the space of solutions, and proved the existence of a compact set containing decay mild solutions to the problem in Hilbert spaces.

Therefore, although there exists a wide literature about to the second order abstract Cauchy problem, the existence of solutions with nonlocal initial conditions for the equation (1) with damping by a convolution term, in Banach spaces, has not been studied in the literature.

We consider nonlocal initial conditions motivated by the observation that this type of conditions is more practical than classical conditions when treating physical problems. For instance, the sum

$$u(x,0) + \sum_{k=1}^{n} \beta_k(x) u(x, T_k) \tag{2}$$

is more accurate to measurement of a state than $u(x,0)$ alone. This approach was used by Deng in [17] to describe the diffusion phenomenon of a small amount of gas in a tube. If there is too little gas at the initial time, the measurement (2) of the sum of the amounts of the gas is more reliable than the measurement $u(x,0)$ of the amount of the gas at the instant $t = 0$. For more information we refer the reader to the articles [1, 14, 38, 37] and references therein.

In this paper, we prove the existence of integrated solutions for the local and nonlocal initial value problem. In the local case, we use methods described in [20] to show the existence of integrated solutions under conditions of compactness of the resolvent $S(t)$ generated by A. Concerning the nonlocal case, we follow ideas of Lizama and Pozo [27], using properties of the measure of noncompactness as the main tool. This concept has been studied widely in [6, 7, 8, 9, 10, 11].

We observe that the approach based on the use of measure of noncompactness for abstract Cauchy problems allows us to remove stronger assumptions, like Lipschitz type conditions on the external forcing term f in (1) employed in the paper [32, Section 6] and so obtain more general results in comparison with other methods. Although this method has been employed in the last years by several authors for the study of existence of solutions to ordinary differential equations, their application to abstract evolution equations remain underdeveloped. Following ideas of [27] (see also the references therein), we will assume that the term f satisfies the following set of conditions

(i) There exists a function $m \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq m(t)\phi(\|x\|+1),$$

for all $x \in X$ and almost all $t \in [0,b]$.

(ii) There exists a function $H \in L^1([0,b];\mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t,S)) \leq H(t)\xi(S), \tag{3}$$

for almost all $t \in [0,b]$, where ξ denote the Hausdorff measure of noncompactness defined in X.

Under the above assumptions, we prove our main result on the existence of integrated solutions to equation (1). See Theorem 4.2 below. Finally, in Section 5, we include constructive examples to illustrate the feasibility of the given hypothesis. In the second example, we included a function f which satisfies the condition (3), but it is not Lipschitzian.

We remark that our results can be applied to models for viscoelastic materials, so our work is a interesting way to research existence of solutions to second order problems considering additional mechanical properties. A future research is study results that can provide sufficient conditions to ensure regularity and polynomial decay of the solutions, following ideas in [16] and [28], for instance.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. We denote the space of all bounded linear operators from X into X by $\mathcal{B}(X)$. If A is a closed linear operator on X, we denote by $D(A)$ the domain of A equipped with the graph norm $|\cdot|_A$ of A, i.e. $|x|_A = |x| + |Ax|$, $\rho(A)$ denote the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$.

Throughout this paper, $\mathcal{C}([0,b];X)$ and $L^p(\mathbb{R}_+, X)$ for $1 \leq p < \infty$, denote the vector space of all continuous functions $f : [0,b] \to X$ and the vector- valued space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ such that $\|f\|_p := \left(\int_0^\infty \|f(t)\|^p dt\right)^{1/p} < \infty$, respectively.

Let Y be a Banach space. We recall that a map $G : X \to Y$ is said to be compact if the image under G of any bounded subset of X is a relatively compact subset in Y.

Theorem 2.1 (Schauder fixed point theorem)**.** *Let C be a nonempty, closed, bounded, and convex subset of a Banach space X. Suppose that $T : C \to C$ is a compact operator. Then T has at least a fixed point in C.*

Theorem 2.2. [18, Theorem 6.4] *Let (Z,d) be a metric space and Y an arbitrary space. Assume $\mathcal{F} \subset \mathcal{C}(Y;Z)$. satisfies:*

(i) *$\mathcal{F}$ is equicontinuous on Y.*

(ii) *for each $y \in Y$, $\{f(y), f \in \mathcal{F}\}$ is relatively compact in Y.*

Then $\overline{\mathcal{F}}$ is compact and equicontinuous on Y.

In what follows we consider the integral equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in [0,b], \tag{4}$$

where $f \in \mathcal{C}([0,b];X)$. We recall the following definition.

Definition 2.3. [34, Definition 1.1] Let A be a closed linear operator with domain $D(A)$ defined in a Banach space X, $a \in L^1_{loc}(\mathbb{R}_+)$ be a scalar kernel and $f \in \mathcal{C}([0,b];X)$. A function $u \in \mathcal{C}([0,b],X)$ is called

(i) strong solution of (4) on $[0,b]$ if $u \in \mathcal{C}([0,b],D(A))$ and (4) holds on $[0,b]$,

(ii) mild solution of (4) on $[0,b]$ if $a*u \in \mathcal{C}([0,b],D(A))$ and $u(t) = f(t)+A(a*u)(t)$ on $[0,b]$,

where the star indicates the finite convolution, i.e.

$$(k*u)(t) = \int_0^t k(t-s)u(s)ds, \quad t \geq 0.$$

The following concept of resolvent family will be fundamental in our considerations.

Definition 2.4. [34, Definition 1.3] Let A be a closed linear operator with domain $D(A)$ defined in a Banach space X, and $a \in L^1_{loc}(\mathbb{R}_+)$. A family $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ of bounded linear operators in X is called a resolvent family generated by A (or of (4)) if the following conditions are satisfied.

(i) $S(t)$ is strongly continuous on $\mathbb{R}_+$, (i.e. $S(\cdot)x$ are continuous from $\mathbb{R}_+$ into X for every $x \in X$) and $S(0) = I$;

(ii) $S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(iii) the resolvent equation holds

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds \quad \text{for all} \quad x \in D(A), t \geq 0.$$

Definition 2.5. A resolvent family $\{S(t)\}_{t\geq 0}$ is called exponentially bounded if there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$. The pair (M,ω) is called a type of $S(t)$.

Theorem 2.6. [34, Proposition 1.2] *Let A be a closed linear operator in a Banach space X with domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$. If $\{S(t)\}_{t\geq 0}$ is a resolvent family generated by A, and $f \in W^{1,1}([0,b];D(A))$, then*

$$u(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds, \quad t \in [0,b],$$

is a strong solution of (4).

In the following definitions $\hat{a}$ denote the Laplace transform of a.

Definition 2.7. [34, p. 90] A infinitely differentiable function $a : (0,\infty) \to \mathbb{R}$ is called completely monotonic if $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0, n \in \mathbb{N}_0$.

Definition 2.8. [33, p. 326] Let $a \in L^1_{loc}(\mathbb{R}_+)$ be such that a is Laplace transformable, a is called completely positive if and only if $\frac{1}{\lambda\hat{a}(\lambda)}$ and $\frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2}$, with $\lambda > 0$ are completely monotone functions.

The concept of measure of noncompactness will be used for one of our results, consequently we recall the following definition. For more details, we refer the reader to [8].

Definition 2.9. Let B be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness of B is defined by

$$\chi_H(B) = \inf\{\epsilon > 0 : B \text{ has a finite cover by balls of radius } \epsilon\}.$$

The Hausdorff measure of noncompactness has some useful properties, now we list some of them that we will require in this paper. See [8, 2, 9] for more details. Let B_1, B_2 be bounded subsets of a normed space Y. Then

(i) $\chi_H(B_1) \leq \chi_H(B_2)$ if $B_1 \subseteq B_2$,
(ii) $\chi_H(B_1) = \chi_H(\overline{B_1})$, where $\overline{B_1}$ denotes the closure of B_1,
(iii) $\chi_H(B_1) = 0$ if and only if B_1 is totally bounded,
(iv) $\chi_H(\lambda B_1) = |\lambda|\chi_H(B_1)$ with $\lambda \in \mathbb{R}$,
(v) $\chi_H(B_1 \cup B_2) = \max\{\chi_H(B_1), \chi_H(B_2)\}$,
(vi) $\chi_H(B_1 + B_2) \leq \chi_H(B_1) + \chi_H(B_2)$, where $B_1 + B_2 = \{b_1 + b_2 : b_1 \in B_1, b_2 \in B_2\}$,
(vii) $\chi_H(B_1) = \chi_H(\overline{co}(B_1))$, where $(\overline{co}(B_1))$ is the closed convex hull of B_1.

In what follows, we denote by ξ the Hausdorff measure of noncompactness defined in X, by γ the Hausdorff measure of noncompactness on $\mathcal{C}([0, b]; X)$. The following lemma on measure of noncompactness will allow us to prove our main findings.

Lemma 2.10. [9, Lemma 5.1] *Let $G : X \to X$ be a Lipschitz continuous map with constant k. Then $\xi(G(B)) \leq k\xi(B)$ for any bounded subset B of X.*

Lemma 2.11. [39, Property 1.1] *Let $W \subseteq \mathcal{C}([0, b]; X)$ be a subset of continuous functions. If W is bounded and equicontinuous on $[0, b]$, then the set $\overline{co}(W)$ is also bounded and equicontinuous on $[0, b]$.*

Next, we set forth some lemmas that will play an important part in the proof of our main result in Section 4.

Lemma 2.12. [9, Lemma 5.3] *Let $W \subseteq \mathcal{C}([0, b]; X)$ be a bounded set. Then $\xi(W(t)) \leq \gamma(W)$ for all $t \in [0, b]$. If W is equicontinuous on $[0, b]$, then $\xi(W(t))$ is continuous on $[0, b]$, and*

$$\gamma(W) = sup\{\xi(W(t)) : t \in [0, b]\},$$

where $W(t) = \{w(t) : w \in W\}$.

Lemma 2.13. [9, Lemma 5.4] *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1([0, b], X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \to \xi\big(\{u_n(t)\}_{n\in\mathbb{N}}\big)$ is measurable and*

$$\xi\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \leq 2\int_0^t \xi(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.14. [12, Theorem 2] *Let Y be a Banach space. If $W \subseteq Y$ is a bounded set, then for each $\epsilon > 0$, there exist a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that*

$$\xi(W) \leq 2\xi(\{u_n\}_{n=1}^{\infty}) + \epsilon. \tag{5}$$

Lemma 2.15. [21, Theorem 3.1] *For all* $0 \le m \le n$, *denote by* $C_m^n = \frac{n!}{m!(n-m)!}$. *If* $0 < \epsilon < 1, h > 0$ *and let*

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \dots + \frac{h^n}{n!}, \quad n \in \mathbb{N}, \tag{6}$$

then $\lim_{n\to\infty} S_n = 0$.

Next, we recall an important fixed-point theorem obtained in [21].

Lemma 2.16. [21, Lemma 2.4] *Let* S *be a closed and convex subset of a complex Banach space* Y, $F : S \to S$ *be a continuous operator such that* $F(S)$ *is a bounded set. Define*

$$F^1(S) = F(S)$$

and

$$F^n(S) = F(\overline{co}(F^{n-1}(S))), \quad n = 2, 3, \dots$$

If there exist a constant $0 \le r < 1$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\xi(F^{n_0}(S)) \le r\xi(S),$$

then F *has a fixed point in the set* S.

3. A criteria for existence of integrated solutions.

Let X be a Banach space, $A : D(A) \subseteq X \to X$ be a closed linear operator that generates a resolvent $\{S(t)\}_{t\ge 0}$ with kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, where $k \in L^1(\mathbb{R}_+)$. In this section, we want to study the existence of solutions for the following semilinear problem.

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t, u(t)), \quad t \in [0, b], \\ u(0) = u_0, u'(0) =& u_1, \quad u_0, u_1 \in X. \end{aligned} \tag{7}$$

Here $f : \mathbb{R}_+ \times X \to X$ is locally integrable. Let us consider the associated linear problem

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t), \quad t \in [0, b], \\ u(0) = u_0, u'(0) =& u_1, \quad u_0, u_1 \in X. \end{aligned} \tag{8}$$

Observe that integrating (8) twice, we obtain the following equivalent representation,

$$u(t) - A(a * u)(t) = (g_2 * f)(t) + tu_1 + u_0, \tag{9}$$

where $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $\alpha > 0$ and $a(t) = (g_2 * k - g_2)(t)$. Here $\Gamma(\alpha)$ denotes the Gamma function. Then, by Theorem 2.6, we have that

$$u(t) = S(t)h(0) + \int_0^t S(t-s)h'(s)ds,$$

solves (9) with $h(t) := (g_2 * f)(t) + tu_1 + u_0$, thus

$$u(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s)ds, \tag{10}$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$ solves (8) whenever $S(t)$ and the initial data are regular enough.

Motivated by this observation the following definition is meaningful.

Definition 3.1. Suppose that A is the generator of a resolvent family $\{S(t)\}_{t\geq 0}$ with kernel $a(t) = \int_0^t (t-s)k(s)ds - t$. Let $u_0, u_1 \in X$ be given. We say that $u \in \mathcal{C}([0,b];X)$ is an integrated solution of (7) if u satisfy the integral equation

$$u(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in [0,b],$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$.

Borrowing ideas from [20, Lemma 3.4 and Lemma 3.5] we obtain the following result.

Lemma 3.2. *Let $\{S(t)\}_{t\geq 0}$ be a resolvent family with generator A. Suppose that*

(i) $\{S(t)\}_{t\geq 0}$ is continuous in the uniform operator topology for all $t > 0$.

(ii) $S(t)$ is compact for each $t > 0$.

Then

(a) $\lim_{h\to 0} \|S(t+h) - S(h)S(t)\| = 0$ for all $t > 0$;

(b) $\lim_{h\to 0} \|S(t) - S(h)S(t-h)\| = 0$ for all $t > 0$.

Proof. We first prove (a). Let $x \in X$ with $\|x\| \leq 1$, $t > 0$ and $\epsilon > 0$ be given. From (ii) we deduce that the set $W_t := \{S(t)x : \|x\| \leq 1\}$ is also compact. Thus, there exists a finite family $\{S(t)x_1, S(t)x_2, ..., S(t)x_m\} \subset W_t$ such that for any x with $\|x\| \leq 1$, there exists $x_i (1 \leq i \leq m)$ such that

$$\|S(t)x - S(t)x_i\| \leq \frac{\epsilon}{3(M+1)}, \tag{11}$$

where $M = \sup\limits_{t\in[0,b]} \|S(t)\| < \infty$. From the strong continuity of $S(t)$, there exists $0 < h_i < \min\{t, b\}$ such that

$$\|S(t)x_i - S(h)S(t)x_i\| \leq \frac{\epsilon}{3}, \tag{12}$$

for all $0 \leq h \leq h_i$ and $1 \leq i \leq m$. On the other hand, from (i), there exists $0 < h_2 < \min\{t, b\}$ such that

$$\|S(t+h)x - S(t)x\| \leq \frac{\epsilon}{3}, \tag{13}$$

for all $0 \leq h \leq h_2$ and $\|x\| \leq 1$. Thus, for $0 \leq h \leq \min\{h_1, h_2\}$ and $\|x\| \leq 1$, it follows from (11)–(13) that

$$\begin{aligned}
&\|S(t+h)x - S(h)S(t)x\| \\
&\qquad \leq \|S(t+h)x - S(t)x\| + \|S(t)x - S(t)x_i\| + \|S(t)x_i - S(h)S(t)x_i\| \\
&\qquad + \|S(h)S(t)x_i - S(h)S(t)x\| \\
&\qquad \leq \|S(t+h)x - S(t)x\| + (M+1)\|S(t)x - S(t)x_i\| + \|S(t)x_i - S(h)S(t)x_i\| \leq \epsilon,
\end{aligned}$$

which implies (a).

We prove (b). Let $t > 0$ and $0 < h < \min\{t, b\}$. Then, there exist $M > 0$ such that

$$\begin{aligned} & \|S(t) - S(h)S(t-h)\| \\ & \leq \|S(t) - S(t+h)\| + \|S(t+h) - S(h)S(t)\| + \|S(h)S(t) - S(h)S(t-h)\| \\ & \leq \|S(t) - S(t+h)\| + \|S(t+h) - S(h)S(t)\| + M\|S(t) - S(t-h)\| \end{aligned} \tag{14}$$

which implies the desired result by (a) and (i). □

Remark 3.3. In contrast with the theory of C_0-semigroups, where compactness of the semigroup implies their continuity in the uniform operator topology for $t > 0$ [31, Theorem 3.2], the compactness of a resolvent family alone is not enough to guarantee their continuity in $\mathcal{B}(X)$, except in particular cases of the kernel $a(t)$. See [33, Corollary 2 and Theorem 7].

The following is the main result of this section.

Theorem 3.4. *Suppose that A is the generator of a resolvent $\{S(t)\}_{t\geq 0}$ exponentially bounded of type (M, ω) and kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, that is in addition compact and continuous in the uniform operator topology for all $t > 0$. Let $f : [0, b] \times X \to X$ be continuous with respect to the second variable and assume that the function $f(\cdot, x)$ is measurable for all $x \in X$, and $\|f(t, x)\| \leq \alpha(t)(\|x\| + 1)$ for a.e. $t \in [0, b]$ and $x \in X$, with $\alpha \in L^1([0, b]; X)$. Then there is at least one integrated solution of (7), provided that $Mbe^{\omega b}\|\alpha\|_{L^1} < 1$.*

Proof. Fix $u_0, u_1 \in X$ and define the operator $G : \mathcal{C}([0, b], X) \to \mathcal{C}([0, b], X)$ by

$$Gu(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s, u(s))ds, \qquad t \in [0, b],$$

where $R(t)x := \int_0^t S(\tau)xd\tau$, $x \in X$, and $\|S(t)\| \leq Me^{\omega b}$ for all $t \in [0, b]$, where $M > 0$ and w.l.o.g. $\omega > 0$. Then G is clearly well defined, and u is an integrated solution of (7) if and only if it is a fixed point of operator G. Now we will show that the mapping G is continuous on $\mathcal{C}([0, b], X)$. Let $\{u_n\}_{n\geq 1}$ be a sequence in $\mathcal{C}([0, b], X)$ with $\lim_{n\to\infty} u_n = u$ in $\mathcal{C}([0, b], X)$. Then

$$\|(Gu_n)(t) - (Gu)(t)\| \leq Mbe^{\omega b} \int_0^t \|f(s, u_n(s)) - f(s, u(s))\|ds, \quad t \in [0, b].$$

Since $f(s, u_n(s))$ converges to $f(s, u(s))$ in X for $s \in [0, b]$, and

$$\|f(s, u_n(s)\| \leq \alpha(s)(\|u_n(s)\| + 1),$$

where $\alpha \in L^1([0, b], \mathbb{R})$ then, by the Lebesgue dominated convergence theorem, we obtain that $G(u_n) \to G(u)$, as $n \to \infty$. This proves the claim.

We denote

$$W_R = \{u \in \mathcal{C}([0, b], X) : \|u(t)\| \leq R, \quad \text{for all} \quad t \in [0, b]\},$$

where $R > 0$ is given. We claim that there exists $r > 0$ such that G maps W_r into itself. We choose $r > 0$ such that

$$(Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1})(1 - Mbe^{\omega b}\|\alpha\|_{L^1})^{-1} < r.$$

Note that the last inequality implies

$$Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1}(r+1) < r.$$

Then by definition of G

$$\|G(u)\| \leq Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1}(r+1) < r.$$

This proves the claim.

Now, we will prove that $G : W_r \to W_r$ is a compact operator. Indeed, by the Arzela- Ascoli theorem, we have to show that the set $GW_r := \{Gu : u \in W_r\}$ is equicontinuous, and the set $\{Gu(t) : u \in W_r\}$ is relatively compact in X for each $t \in [0, b]$. Let $0 \leq t_1 \leq t_2 \leq b$ and $u \in W_r$, we have

$$\begin{aligned}
\|(Gu)(t_2) - (Gu)(t_1)\| &\leq \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&\quad + \|\int_0^{t_2} R(t_2 - s)f(s, u(s))ds - \int_0^{t_1} R(t_1 - s)f(s, u(s))ds\| \\
&\leq \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&\quad + \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\|f(s, u(s))ds\|ds \\
&\quad + \int_{t_1}^{t_2} \|R(t_2 - s)f(s, u(s))\|ds \\
&\leq \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&\quad + \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\alpha(s)rds + Mbe^{\omega b}(r+1)\int_{t_1}^{t_2} \alpha(s)ds. \qquad (15)
\end{aligned}$$

If $t_1 = 0$, then

$$\lim_{t_2 \to 0} \|(Gu)(t_2) - (Gu)(t_1)\| = 0, \quad \text{uniformly for} \quad u \in W_r.$$

If $0 < t_1 < b$. Note that $R(t)x = \int_0^t S(\tau)xd\tau$ and hence $R(t)$ is a norm continuous operator. Then, from (15) we obtain

$$\lim_{|t_1 - t_2| \to 0} \|(Gu)(t_2) - (Gu)(t_1)\| = 0, \quad \text{uniformly for } u \in W_r.$$

Then, the set GW_r is equicontinuous on $\mathcal{C}([0, b], X)$, proving the first part of the claim. Now, we will show that the set $M(t) := \{(Gu)(t) : u \in W_r\}$ is relatively compact in X for every $t \in [0, b]$. If $t = 0$ then the set $M(0)$ is clearly relatively compact in X. We denote $g(s, u) := \displaystyle\int_0^s f(t, u(t))dt$, $0 \leq s \leq b$, $u \in \mathcal{C}([0, b], X)$, and note that the hypothesis implies

$$\|g(s, u)\| \leq \|\alpha\|_{L^1}(r+1) \quad \text{for all } 0 \leq s \leq b, \quad u \in W_r.$$

Moreover, integration by parts shows the identity

$$\int_0^t R(t-s)f(s,u(s))ds = \int_0^t S(t-s)\Big[\int_0^s f(\tau,u(\tau))d\tau\Big]ds = \int_0^t S(t-s)g(s,u)ds.$$

Let $0 < t \le b$ be given and $0 < \epsilon < t$. We first observe that the set $\{\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_{r}\}$ is bounded. Indeed, for all $u \in W_r$

$$\| \int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds\| \le Mbe^{\omega b}\|\alpha\|_{L^1}(r+1).$$

Thus $\{S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$ is relatively compact, since $S(\epsilon)$ is compact, and the set $\{\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$ is bounded.
Moreover, for all $u \in W_r$

$$\begin{aligned}&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\|\\ \le &\int_0^{t-\epsilon} \|S(\epsilon)S(t-s-\epsilon) - S(t-s)\|\|\alpha\|_{L^1}(r+1)ds\end{aligned}$$

Moreover, since $S(t)$ is compact and continuous in the uniform operator topology for all $t > 0$. Then by part (b) of Lemma 3.2, we have that

$$S(\epsilon)S(t-s-\epsilon) - S(t-s) \to 0, \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad s \in [0,t-\epsilon]$$

and

$$\int_0^{t-\epsilon} \|S(\epsilon)S(t-s-\epsilon) - S(t-s)\|ds \le ((Me^{\omega b})^2 + Me^{\omega b})b.$$

From the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\epsilon\to 0} \|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| = 0.$$

Moreover,

$$\begin{aligned}&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\|\\ \le&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\|\\ &+\|\int_0^{t-\epsilon} S(t-s)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\|\\ \le&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| + \int_{t-\epsilon}^t Me^{\omega b}\|\alpha\|_{L^1} r ds.\end{aligned}$$

Thus,

$$\lim_{\epsilon\to 0} \|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\| = 0.$$

Then, $\{\int_0^t R(t-s)f(s,u(s))ds : u \in W_r\} = \{\int_0^t S(t-s)g(s,u)ds : u \in W_r\}$ is relatively compact in X by using the relative compactness of the set $\{S(\epsilon)\int_0^{t-\epsilon} S(t-s-$

$\epsilon)g(s,u)ds : u \in W_r\}$. Hence, the set $M(t) = \{(Gu)(t) : u \in W_r\}$ is relatively compact in X for each $t \in [0,b]$, proving the claim.

Hence, by Schauder fixed point theorem, we conclude that (7) has an integrated solution. □

Remark 3.5. The hypothesis $\|f(s,x)\| \leq \alpha(s)(\|x\|+1)$, $\alpha \in L^1([0,b],\mathbb{R})$, also has been considered previously in [2, Theorem 5.2.2] in order to prove the existence of mild solutions for a class of semilinear differential inclusions in a Banach space X.

4. Nonlocal initial conditions

Let X be a Banach space, $A : D(A) \subseteq X \to X$ be closed and lineal operator and $k \in L^1(\mathbb{R}_+)$ be a scalar memory kernel. We consider the problem

$$
\begin{aligned}
u''(t) + Au(t) - (k * Au)(t) &= f(t,u(t)), \quad t \in [0,b], \\
u(0) = g(u), u'(0) &= h(u),
\end{aligned}
\tag{16}
$$

where $g, h : \mathcal{C}([0,b];X) \to X$ are continuous maps and $f : [0,b] \times X \to X$. We set the following conditions.

- (H1) A generates a exponentially bounded resolvent $\{S(t)\}_{t\geq 0}$ of type (M,ω) and kernel $a(t) = \int_0^t (t-s)k(s)ds - t$.
- (H2) g, h are compact maps.
- (H3) The function $f(\cdot,x)$ is measurable for all $x \in X$ and $f(t,\cdot)$ is continuous for almost all $t \in [0,b]$.
- (H4) There exists a function $m \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$\|f(t,x)\| \leq m(t)\phi(\|x\|),$$
for all $x \in X$ and almost all $t \in [0,b]$.
- (H5) There exists a function $H \in L^1([0,b];\mathbb{R}^+)$ such that for any bounded $S \subseteq X$
$$\xi(f(t,S)) \leq H(t)\xi(S),$$
for almost all $t \in [0,b]$.

We introduce the following definition.

Definition 4.1. A function $u \in \mathcal{C}([0,b],X)$ is called a nonlocal integrated solution of the equation (16) if u satisfies

$$u(t) = R(t)h(u) + S(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in [0,b], \tag{17}$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$.

The following is the main result of this section.

Theorem 4.2. *Suppose that A satisfies* ***(H1)****, $g, h : \mathcal{C}([0,b];X) \to X$ satisfies* ***(H2)****, $f : [0,b] \times X \to X$ satisfies* ***(H3)-(H5)****, and there exists a constant $R > 0$ such that*

$$Me^{\omega b}(h_R + bg_R + b\phi(R)\int_0^b m(s)ds) \leq R, \tag{18}$$

where $g_R := \sup\{\|g(u)\| : \|u\|_\infty \leq R\} < \infty$, *and* $h_R := \sup\{\|h(u)\| : \|u\|_\infty \leq R\} < \infty$. *Then the problem* (16) *has at least one nonlocal integrated solution.*

Proof. Define the operator $F : \mathcal{C}([0,b],X) \to \mathcal{C}([0,b],X)$ by

$$Fu(t) = S(t)h(u) + R(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in [0,b],$$

where $R(t)x := \displaystyle\int_0^t S(\tau)xd\tau$, $x \in X$. We will show that the mapping F is continuous on $\mathcal{C}([0,b],X)$. Indeed, let $\{u_n\}_{n\geq 1}$ be a sequence in $\mathcal{C}([0,b],X)$ with $\lim_{n\to\infty} u_n = u$, for the norm of uniform convergence. Then

$$\begin{aligned}\|(Fu_n)(t) - (Fu)(t)\| \leq & Me^{\omega b}\|h(u_n) - h(u)\| + Mbe^{\omega b}\|g(u_n) - g(u)\| \\ & + Mbe^{\omega b}\int_0^t \|f(s,u_n(s)) - f(s,u(s))\|ds, \quad t \in [0,b],\end{aligned}$$

where w.l.o.g. $\omega > 0$. Since g, h are continuous maps, we obtain $g(u_n) \to g(u)$, and $h(u_n) \to h(u)$ as $n \to \infty$. Moreover, since f satisfies hypotheses **(H3)**, by the Lebesgue dominated convergence theorem, we obtain that $F(u_n) \to F(u)$, as $n \to \infty$. This proves the claim.

Define the set

$$B_R = \{u \in \mathcal{C}([0,b],X) : \|u(t)\| \leq R \quad \text{for all} \quad t \in [0,b]\}.$$

Then

$$\begin{aligned}\|Fu(t)\| \leq & \|S(t)h(u)\| + \|R(t)g(u)\| + \int_0^t \|R(t-s)f(s,u(s))\|ds \\ \leq & Me^{\omega b}(h_R + bg_R + b\phi(R)\int_0^b m(s)ds) \leq R.\end{aligned}$$

Then, F maps B_R into itself, and $F(B_R)$ is a bounded set. On the other hand, as in the proof of Theorem 3.4, we get that the set $F(B_R)$ is an equicontinuous set of functions.

Now, we define the set $\mathfrak{A} := \overline{co}(F(B_R))$. By Lemma 2.11 such set $\mathfrak{A}$ is equicontinuous. Since $\mathfrak{A} \subseteq B_R$, we conclude that the map $F : \mathfrak{A} \to \mathfrak{A}$ is continuous and $F(\mathfrak{A})$ is a bounded set of functions.

Let $\epsilon > 0$ be fixed. By Lemma 2.14 there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq F(\mathfrak{A})$ such that $\xi(F(\mathfrak{A})) \leq 2\xi(\{v_n(t)\}_{n=1}^\infty) + \epsilon \leq 2\xi\big(\int_0^t \{R(t-s)f(s,u_n(s))\}_{n=1}^\infty ds\big) + \epsilon$, where in the second inequality we have used the compactness of h and g. By hypotheses **(H4)** and **(H5)**, we have that

$$\begin{aligned}\xi(F(\mathfrak{A})(t)) \leq & 4Mbe^{\omega b}\int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^\infty)ds + \epsilon \\ & \leq 4Mbe^{\omega b}\int_0^t H(s)\xi(\{u_n(s)\}_{n=1}^\infty)ds + \epsilon \\ & \leq 4Mbe^{\omega b}\gamma(\mathfrak{A})\int_0^t H(s)ds + \epsilon.\end{aligned}$$

By the hypotheses **(H5)** we have $H \in L^1([0,b];\mathbb{R}^+)$. Then for $\alpha < \frac{1}{4Mbe^{\omega b}}$ there exist $\varphi \in \mathcal{C}([0,b],\mathbb{R}^+)$ such that $\int_0^b |H(s)-\varphi(s)|ds < \alpha$. Hence

$$\begin{aligned}\xi(F(\mathfrak{A})(t)) \leq& 4Mbe^{\omega b}\gamma(\mathfrak{A})\big[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\big] + \epsilon \\ \leq& 4Mbe^{\omega b}\gamma(\mathfrak{A})[\alpha + Nt] + \epsilon,\end{aligned}$$

where $N = \|\varphi\|_\infty$. Then, we have

$$\xi(F(\mathfrak{A})(t)) \leq (a+ct)\gamma(\mathfrak{A}), \quad \text{where} \quad a = 4\alpha Mbe^{\omega b} \quad \text{and} \quad c = 4MNbe^{\omega b}. \tag{19}$$

Let $\epsilon > 0$ be given, then by Lemma 2.14 there exist a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(\mathfrak{A}))$ such that

$$\begin{aligned}\xi(F^2(\mathfrak{A})(t)) \leq& 2\xi\big(\int_0^t \{R(t-s)f(s,w_n(s))\}_{n=1}^\infty ds\big) + \epsilon \\ \leq& 4Mbe^{\omega b}\int_0^t \xi\{f(s,w_n(s))\}_{n=1}^\infty ds + \epsilon \\ \leq& 4Mbe^{\omega b}\int_0^t H(s)\xi(\overline{co}(F^1(\mathfrak{A})(s))) + \epsilon \\ =& 4Mbe^{\omega b}\int_0^t H(s)\xi(F^1(\mathfrak{A})(s)) + \epsilon.\end{aligned}$$

By (19) we have that

$$\begin{aligned}\xi(F^2(\mathfrak{A})(t)) \leq& 4Mbe^{\omega b}\int_0^t [|H(s)-\varphi(s)| + |\varphi(s)|](a+cs)\gamma(\mathfrak{A})ds + \epsilon \\ &\leq 4Mbe^{\omega b}(a+ct)\gamma(\mathfrak{A})\int_0^t |H(s)-\varphi(s)|ds + 4MNbe^{\omega b}\gamma(\mathfrak{A})\big(at + \frac{ct^2}{2}\big) + \epsilon \\ &\leq (a(a+ct) + c(at + \frac{ct^2}{2}))\gamma(\mathfrak{A}) + \epsilon \\ &\leq (a^2 + 2act + \frac{(ct)^2}{2})\gamma(\mathfrak{A}).\end{aligned}$$

By induction, for all $n \in \mathbb{N}$,

$$\xi(F^n(\mathfrak{A})(t)) \leq \big(a^n + C_1^n a^{n-1}ct + C_2^n a^{n-2}\frac{(ct)^2}{2!} + ... + \frac{(ct)^n}{n!}\big)\gamma(\mathfrak{A}),$$

where C_m^n denotes the binomial coefficient $\binom{m}{n}$, for $0 \leq m \leq n$. Moreover, $F^n(\mathfrak{A})$ is an equicontinuous set of functions for all $n \in \mathbb{N}$. Thus, by Lemma 2.12

$$\gamma(F^n(\mathfrak{A})) \leq \big(a^n + C_1^n a^{n-1}c + C_2^n a^{n-2}\frac{c^2}{2!} + ... + \frac{c^n}{n!}\big)\gamma(\mathfrak{A}).$$

Since $0 \leq a < 1$ and $c > 0$, by Lemma 2.13 there exist $n_0 \in \mathbb{N}$ such that

$$(a^{n_0} + C_1^{n_0}a^{n_0-1}ct + C_2^{n_0}a^{n_0-2}\frac{(ct)^2}{2!} + ... + \frac{(ct)^{n_0}}{n_0!}) = r < 1.$$

Therefore, $\gamma(F^{n_0}(\mathfrak{A})) \leq r\gamma(\mathfrak{A})$. Then F has a fixed point in $\mathfrak{A}$ by Lemma 2.16.

□

5. Examples

Example 5.1.

We set $X = L^2([0,1])$, and we consider $Az(\xi) = \frac{d^2 z(\xi)}{d\xi^2}$ with domain $D(A) = \{z \in H^2[0,1] : z(0) = z(1) = 0\}$. It is well known that A generates an bounded analytic semigroup on $L^2[0,1]$, see [19, Example 4.8]. Let $a(t) = 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}e^{-\alpha t}$ where $-\alpha \leq \beta \leq 0 < \alpha$. We will prove that $a(t)$ is completely positive. See Definition 2.8.
Indeed, we have

$$\hat{a}(\lambda) = \frac{\lambda + \alpha + \beta}{\lambda(\lambda + \alpha)},$$

then

$$\frac{1}{\lambda \hat{a}(\lambda)} = \frac{\lambda + \alpha}{\lambda + \alpha + \beta} \quad \text{and} \quad \frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2} = \frac{\lambda^2 + 2(\alpha+\beta)\lambda + \alpha\beta + \alpha^2}{(\lambda + \alpha + \beta)^2}.$$

If we denote $c_1(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)}$ and $c_2(\lambda) = \frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2}$, then we obtain that

$$c_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta n!}{(\lambda+\alpha+\beta)^{n+1}} \quad \text{and} \quad c_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha+\beta)(n+1)!}{(\lambda+\alpha+\beta)^{n+2}} \quad \text{for} \quad n \in \mathbb{N}.$$

Since $-\alpha \leq \beta \leq 0 < \alpha$, c_1 and c_2 are completely monotone. We conclude that a is completely positive.

Let $p : [0,1] \times [0,1] \to \mathbb{R}$ be Hilbert Schmidt, i.e.

$$\int_0^1 \int_0^1 |p(x,y)|^2 dxdy = c < \infty. \tag{20}$$

Given $-\alpha \leq \beta \leq 0 < \alpha$, and $s > \frac{1}{2Mc}$, we consider the following problem
(21)

$$\begin{cases} \dfrac{\partial^2 u(t,\xi)}{\partial t^2} + \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \alpha\beta e^{-\alpha(t-s)} \dfrac{\partial^2 u(s,\xi)}{\partial \xi^2} ds = t \int_0^1 \sin(u(t,s))ds, \ t, \xi \in [0,1], \\ u(t,0) = u(t,1) = 0, \quad t \in [0,1], \\ u(0,\xi) = \displaystyle\int_0^1 sp(\xi,y)u(1/2,y)dy, \quad \xi \in [0,1], \\ \dfrac{\partial u}{\partial t}(0,\xi) = \displaystyle\int_0^1 sp(\xi,y)u(1/2,y)dy, \quad \xi \in [0,1]. \end{cases}$$

The problem (21) can be rewritten as

$$\begin{cases} u''(t) + Au(t) - (k * Au)(t) = f(t,u(t)), \quad t \in [0,1] \\ u(0) = g(u), \\ u'(0) = h(u), \end{cases} \tag{22}$$

with $k(t) = -\alpha\beta e^{-\alpha t} \in L^1(\mathbb{R}_+)$, and $g, h : \mathcal{C}([0,1];X) \to X$ are explicitly given by

$$g(u) = sk_g(u(1/2)), \quad h(u) = sk_h(u(1/2)), \tag{23}$$

with $(k_g v)(\xi) = (k_h v)(\xi) = \int_0^1 p(\xi,y)v(y)dy$, for $v \in L^2[0,1]$, $\xi \in [0,1]$ and $f(t,\phi) = t\int_0^1 \sin(\phi(s))ds$.

We will prove that the hypothesis **(H1) - (H5)** are satisfied.

(H1) From the above, a is completely positive. Thus, by [34, Theorem 4.2] the operator A generates a resolvent operator $S(t)$, which is exponentially bounded and of type (M,ω).

(H2) By [35, Theorem 8.83] g and h are compact maps.

(H3) Is clear.

(H4) Note that,

$$\|f(t,\phi)\| \leq \|t\int_0^1 \sin(\phi(s))ds\| \leq m(t)\phi(\|x\|),$$

where $m(t)=|t|$, and $\phi(z)\equiv 1$.

(H5) Since

$$\|f(t,\phi_1)-f(t,\phi_2)\| = \|t\int_0^1 (\sin(\phi_1(s))-\sin(\phi_2(s)))ds\| \leq |t|\|\phi_1-\phi_2\| \leq \|\phi_1-\phi_2\|.$$

Thus, by Lemma 2.10,

$$\xi(f(t,S)) \leq \xi(S) \leq H(t)\xi(S),$$

with $H(t)=1$, for all bounded $S\subset X, t\in[0,1]$.

Then, we have that conditions (H1)- (H5) are satisfied. Now, we will prove that the inequality (18) is satisfied. Since $m(t)=|t|$, and $\phi(z)\equiv 1$, then we have to find $R>0$ such that

$$M\Big(h_R+g_R+\phi(R)\int_0^1 m(s)ds\Big) = M(h_R+g_R+\frac{1}{2}) < R.$$

Indeed, by [35, Lemma 8.20], we have $\|k_g(v)\|\leq c\|v\|$, for $v\in X$, where c is given in (20). Then

$$\begin{aligned} g_R := \sup\{\|g(u)\| : \|u\|_\infty \leq R\} &= \sup\{\|sk_g(u(1/2))\| : \|u\|_\infty \leq R\} \\ &\leq \sup_{\|u\|\leq R} sc\|(u(1/2))\| \leq scR. \end{aligned}$$

Thus, $g_R+h_R\leq 2Rcs$, so $M(h_R+g_R+\frac{1}{2}) \leq M(2Rcs+\frac{1}{2})$. Since $s<\frac{1}{2Mc}$, we have that for all $R>\frac{M}{2(1-2Mcs)}$ the inequality $M(h_R+g_R+\frac{1}{2})<R$ is fulfilled. Then, all the hypothesis of Theorem 4.2 are satisfied and we conclude that the problem (21) has at least one nonlocal integrated solution.

Example 5.2.

We set $X=c_0(\mathbb{N})$, and we consider $Az=M_q z=q\cdot z$ where, $q:\mathbb{N}\to\mathbb{C}$ with real part bounded above, and domain $D(A)=D(M_q)=\{z\in X : qz\in X\}$. Then, A generates an strongly continuous semigroup S of type $(M;\omega)$ on X, see [19, Lemma, pag 65]. Let $a(t)=1+\frac{\beta}{\alpha}-\frac{\beta}{\alpha}e^{-\alpha t}$ where $-\alpha\leq\beta\leq 0<\alpha$. Then $a(t)$ is completely positive by Example 5.1. Given $-\alpha\leq\beta\leq 0<\alpha$, we consider the following problem

$$\begin{cases} u''(t)+Au(t)-(k*Au)(t)=f(t,u(t)), \quad t\in[0,1] \\ u(0)=g(u), \\ u'(0)=h(u), \end{cases} \tag{24}$$

where $f : [0,1] \times X \to X$ is given by

$$f(t,x) = m(t)\Big\{ln(|x_k|+1) + \frac{t}{k^2}\Big\}_{k=1}^{\infty}, \text{ for } t \in [0,1],\ x = \{x_k\}_k \in c_0, \tag{25}$$

$k(t) = -\alpha\beta e^{-\alpha t} \in L^1(\mathbb{R}_+)$, $m \in L^1([0,1];\mathbb{R}^+)$, such that $\int_0^1 m(s)ds \neq \frac{1}{2M}$, and c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm $\|x\|_\infty = \sup_k |x_k|$.
Let $x = (\frac{1}{4M}, 0, 0, 0...)$. Define $g, h : \mathcal{C}([0,1];X) \to X$ are explicitly given by

$$g(u)_k = u(1/2)_k x_k, \quad h(u)_k = u(1/3)_k x_k, \quad k \in \mathbb{N}. \tag{26}$$

We will prove that the hypothesis **(H1) - (H5)** are satisfied.

- (H1) For the above, a is completely positive. Thus, by [34, Theorem 4.2] the operator A generates a resolvent operator $S(t)$, which is exponentially bounded of type (M,ω).
- (H2) Since that g and h are bounded with finite rank, then g and h are compact maps.
- (H3) Is clear.
- (H4) Note that,

$$\begin{aligned}\|f(t,x)\|_\infty =& m(t)\|(\ln(|x_k|+1) + \frac{t}{k^2})_k\|_\infty \le m(t)(\sup_k |x_k| + t)\\ \le& m(t)(\|x\|_\infty + 1) := m(t)\phi(\|x\|_\infty),\end{aligned}$$

  where $\phi(z) = z + 1$. This shows that (H4) holds.
- (H5) The Hausdorff measure of noncompacness ξ in the space c_0 can be computed by means of the formula

$$\xi(B) = \lim_{n\to\infty} \sup_{x\in B} \|(I - P_n)x\|_\infty,$$

  where B is a bounded subset in c_0 and P_n is the projection onto the linear span of the first n vectors in the standard basis. The reader can see [2, 1.1.9, p. 5]. Analogously to [9, Example 5.1, p. 227] we obtain

$$\xi(f(t,B)) \le m(t)\xi(B). \tag{27}$$

Then, we have that conditions (H1)- (H5) are satisfied. Now, we will prove that the inequality (18) is satisfied. Since $\phi(z) = z + 1$, then we have to find $R > 0$ such that

$$M\Big(h_R + g_R + \phi(R)\int_0^1 m(s)ds\Big) = M\Big(h_R + g_R + (R+1)\int_0^1 m(s)ds\Big) < R. \tag{28}$$

Note that,

$$g_R := \sup\{\|g(u)\| : \|u\|_\infty \le R\} \le \frac{R}{4M},\ h_R := \sup\{\|h(u)\| : \|u\|_\infty \le R\} \le \frac{R}{4M}.$$

Therefore, since that $\int_0^1 m(s)ds \neq \frac{1}{2M}$, then for all $R > \frac{2M\int_0^1 m(s)ds}{1-2M\int_0^1 m(s)ds}$ the inequality (28) is fulfilled. Then, all the hypothesis of Theorem 4.2 are satisfied and we conclude that the problem (24) has at least one nonlocal integrated solution.

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