ON THE COMPACTNESS OF FRACTIONAL RESOLVENT OPERATOR FUNCTIONS.

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Abstract. We study and characterize the compactness of resolvent families of operators associated to fractional differential equations. We show an application in the study of existence of mild solutions for a class of semilinear fractional differential equations with non-local initial conditions.

1. Introduction

Of concern in this paper is a remarkable class of families of bounded and linear operators which have proved to be useful in the study of abstract models for Partial Differential Equations describing anomalous diffusion. We mean here compact fractional resolvent operator functions. A fractional resolvent operator function endows the solution operator, defined by the nonhomogeneous equation

$$D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad 0 < \alpha \le 2, \tag{1.1}$$

by means of the variation of constants formula, with the compactness property, comparable with the finite-dimensional counterpart.

For $\alpha = 1$, the well known criteria for compactness of C_0-semigroups (see e.g. [17, Theorem 3.3, Chapter 2]), assert that a C_0-semigroup $\{T(t)\}_{t\ge 0}$ generated by A is compact (for $t > 0$) if and only if $T(t)$ is continuous in the uniform operator topology for $t > 0$ and the resolvent operator $(\lambda - A)^{-1}$ is compact for all $\lambda \in \rho(A)$, the resolvent set of A. This criteria has great importance in the study of existence of mild solutions for (1.1), because arguments to solve (1.1) using fixed points Theorems of Schauder's type can be applied.

In case $\alpha = 2$ we find a similar situation assuming that A is the generator of a strongly continuous sine family $\{S(t)\}_{t\ge 0}$. In this case, the compactness criteria of sine family (see [22]), asserts that a sine family $S(t)$ is compact for all $t > 0$, if and only if the resolvent operator $(\lambda^2 - A)^{-1}$ is compact for every $\lambda \in \rho(A)$. Observe that, in infinite dimensional Banach spaces, a cosine family $\{C(t)\}_{t\ge 0}$ cannot be compact.

In the last decade, the fractional differential equation (1.1) where the fractional derivative is understood in the Caputo sense, has been extensively studied. Equations with memory, of type (1.1) are of interest in connection with several applications in Physics and Viscoelasticity theory (see [18, 21] and references therein). The solution to equation (1.1) in case $0 < \alpha < 1$ is

2000 *Mathematics Subject Classification.* 47D06; 47D09; 47B07; 35R11.

Key words and phrases. Compactness; C_0-Semigroup; Cosine family; Fractional resolvent families.

C. Lizama is partially supported by FONDECYT Grant #1140258.

R. Ponce is partially supported by FONDECYT Grant #11130619.

essentially given by

$$u(t) = S_\alpha(t)u(0) + \int_0^t S_\alpha(t-s)f(s,u(s))ds, \tag{1.2}$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the $(\alpha,1)$-resolvent family generated by A. Several properties of $\{S_\alpha(t)\}_{t\geq 0}$ have been studied in [4, 11, 12] among others. The compactness of $\{S_\alpha(t)\}_{t>0}$ was first studied by subordination methods, i.e. A is supposed to be a generator of a compact semigroup, and then compactness of the family $\{S_\alpha(t)\}_{t>0}$ is obtained, see Prüss [20, Corollary 2]. After that, Wang, Chen and Xiao [23], assuming that A is an almost sectorial operator and $(\lambda^\alpha - A)^{-1}$ is compact, proved that the family $\{S_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology for $t > 0$ [23, Theorem 3.2] and compact [23, Theorem 3.5]. The method relies in the use of functional calculus. Very recently, and under the hypothesis continuity in the uniform operator topology for $t > 0$, Fan [7] found that the compactness of the resolvent operator $(\lambda^\alpha - A)^{-1}$ is necessary and sufficient for compactness of $\{S_\alpha(t)\}_{t>0}$. The proof follows a direct method having in mind the case $\alpha = 1$. However, the necessary condition has a mistake in their proof (see Remark 3.6 below), and therefore the problem of characterization of compactness remains open. The objective of this paper is to provide a completely new approach to Fan's result, and to provide a complete characterization in the complementary case $1 < \alpha \leq 2$ for the associated family $R_\alpha(t) = (g_{\alpha-1} * S_\alpha)(t)$ that corresponds to the fractional counterpart of the sine functions for $\alpha = 2$ and that has not been studied previously in the literature. We finish this paper with a new application to semilinear fractional abstract equations with nonlocal initial conditions.

2. Preliminaries

Let X be a complex Banach space and denote by $L^1_{\text{loc}}(\mathbb{R}_+, X)$ the Banach space of all locally (Bochner) integrable vector-valued functions. The Laplace transform of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+, X)$ is defined by

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad \text{Re}\lambda > \omega,$$

whenever the integral is absolutely convergent for $\text{Re}\lambda > \omega$. We denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X.

The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D_t^\alpha f(t) := (g_{m-\alpha} * f^{(m)})(t) := \int_0^t g_{m-\alpha}(t-s)f^{(m)}(s)ds,$$

where m is the smallest integer greater that or equal to α, and for $\beta > 0$

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Applying the properties of the Laplace transform, an easy computation shows that for $0 < \alpha < 1$,

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0).$$

For details in fractional calculus, we refer the reader to [9, 10, 16]. The following definition was first introduced in [3] although implicitly in [21] and [13].

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an $(\alpha, 1)-$ resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subseteq \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \mathrm{Re}\lambda > \omega, x \in X.$$

In this case the family $\{S_\alpha(t)\}_{t\geq 0}$ is called an $(\alpha, 1)$-*resolvent family generated by* A.

The next definition was introduced in [1] after previous work in [3].

Definition 2.2. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $1 \leq \alpha \leq 2$. We call A the generator of an (α, α)-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $R_\alpha : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subseteq \rho(A)$ and*

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} R_\alpha(t) x dt, \quad \mathrm{Re}\lambda > \omega, x \in X.$$

In this case the family $\{R_\alpha(t)\}_{t\geq 0}$ is called an (α, α)-resolvent family generated by A.

Because of the uniqueness of the Laplace transform, a $(1, 0)$-times resolvent family is the same as a C_0-semigroup, a $(2, 1)$-resolvent family corresponds to the concept of sine family and a $(2, 0)$-resolvent family is a cosine family, see [2].

Remark 2.3. *If A is the generator of an $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ then by* [13, Proposition 3.1 and Lemma 2.2] *we have that the family $\{S_\alpha(t)\}_{t\geq 0}$ verifies the following properties:*

$S1$) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;

$S2$) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \geq 0$;

$S3$) for $x \in D(A)$, the resolvent equation

$$S_\alpha(t)x = x + \int_0^t g_\alpha(t-s) S_\alpha(s) A x ds$$

holds for all $t \geq 0$.

Similarly, an (α, α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ verifies:

$R1$) $R_\alpha(t)$ is strongly continuous for $t \geq 0$ and $R_\alpha(0) = g_\alpha(0)$;

$R2$) $R_\alpha(t)A \subset AR_\alpha(t)$ for $t \geq 0$;

$R3$) for $x \in D(A)$, the resolvent equation

$$R_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s) R_\alpha(s) A x ds$$

holds for all $t \geq 0$.

Finally, we recall that a strongly continuous family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is said to be of type (M, ω) or *exponentially bounded* if there exist constants $M > 0$ and $\omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

3. A characterization

The next Theorem, was proved recently in [7, Theorem 3.6]. Unfortunately the proof in [7] does not allow to obtain that the compactness of the resolvent $(\lambda^\alpha - A)^{-1}$ implies the compactness of the $(\alpha, 1)$-resolvent family $\{S_\alpha(t)\}_{t>0}$, because there is a logical mistake in the proof (see Remark 3.6 below). Here we prove, by a completely different method, the desired characterization.

Our method of proof relies in two main ingredients. The first of them is a Theorem due to Weis [24] that asserts - roughly speaking - that the integral of a family of compact operators is a compact operator. The second ingredient is a Theorem due to Hasse [8] that gives direct inversion of the Laplace transform for one-parameter families of operators, when the family is regularized by finite convolution with a locally integrable kernel.

Theorem 3.4. *Let $0 < \alpha \le 1$ and $\{S_\alpha(t)\}_{t\ge 0}$ be an $(\alpha,1)$-resolvent family of type (M,ω) generated by A. Suppose that $S_\alpha(t)$ is continuous in the uniform operator topology for all $t>0$. Then the following assertions are equivalent*

i) *$S_\alpha(t)$ is a compact operator for all $t>0$.*
ii) *$(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $\{S_\alpha(t)\}_{t>0}$ is compact and let $\lambda > \omega$ be fixed. Then we have

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t} S_\alpha(t)dt,$$

where the integral in the right-hand side exists in the Bochner sense, because $\{S_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology, by hypothesis. Then, by [24, Corollary 2.3] we conclude that $(\lambda^\alpha - A)^{-1}$ is a compact operator.

$(ii) \Rightarrow (i)$ The case $\alpha = 1$ follows from [17]. Let $t>0$ be fixed. Since $\alpha<1$, it follows that $g_{1-\alpha} \in L^1_{\text{loc}}[0,\infty)$ and therefore, by [8, Proposition 2.1] we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\omega - iN}^{\omega+iN} e^{\lambda t}(\widehat{g_{1-\alpha} * R_\alpha})(\lambda) d\lambda = (g_{1-\alpha} * R_\alpha)(t) = S_\alpha(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda = S_\alpha(t),$$

where Γ is the path consisting of the vertical line $\{\omega + it : t \in \mathbb{R}\}$. By hypothesis and [24, Corollary 2.3], we conclude that $S_\alpha(t)$ is compact. □

Remark 3.5. *Theorem 3.4 extends the compactness criteria for semigroup operator functions, see e.g.* [17], [6, Chapter II,Theorem 4.29] *and* [5].

Remark 3.6. *The proof of* [7, Theorem 3.6] *in $(ii) \implies (i)$ uses* [7, Lemma 3.4]. *However, one of the hypothesis of such Lemma is precisely (i).*

Remark 3.7. *Useful criteria for continuity of $S_\alpha(t)$ in the uniform operator topology can be found in the work of Fan* [7]. *For example, this property is true for the class of analytic resolvents, see* [7, Lemma 3.8].

Our second main result completely characterizes the compactness of (α,α)-resolvent families in the range $1<\alpha\le 2$. In contrast with the case $0<\alpha\le 1$, it is remarkable that we obtain here a characterization solely in terms of properties of their generator A.

Theorem 3.8. *Let $1<\alpha\le 2$ and A be the generator of an $(\alpha,1)$-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ of type (M,ω). Then A generates an (α,α)-resolvent family $\{R_\alpha(t)\}_{t\ge 0}$ of type $(\frac{M}{\alpha-1},\omega)$ and the following assertions are equivalent*

i) $R_\alpha(t)$ *is a compact operator for all* $t > 0$.
ii) $(\mu - A)^{-1}$ *is a compact operator for all* $\mu > \omega^{1/\alpha}$.

Proof. We first prove that A generates an (α, α)-resolvent family $\{R_\alpha(t)\}_{t\geq 0}$ of type $(\frac{M}{\alpha-1}, \omega)$. Indeed, let $\alpha > 1$ be given. By hypothesis we have $\|S_\alpha(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Define

$$R_\alpha(t) := (g_{\alpha-1} * S_\alpha)(t),$$

for all $t \geq 0$. We obtain

$$\begin{aligned}\|R_\alpha(t)\| &\leq M\int_0^t g_{\alpha-1}(t-s)e^{\omega s}ds \leq M\int_0^t g_{\alpha-1}(s)e^{\omega(t-s)}ds \\ &\leq Me^{\omega t}\int_0^t \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}e^{-\omega s}ds \leq \frac{Me^{\omega t}}{\Gamma(\alpha-1)}\int_0^\infty s^{\alpha-2}e^{-\omega s}ds \leq \frac{Me^{\omega t}}{\omega^{\alpha-1}}.\end{aligned}$$

In particular, we conclude that $R_\alpha(t)$ is Laplace transformable and, for $\lambda > \omega$, we have

$$\hat{R}_\alpha(\lambda) = \frac{1}{\lambda^{\alpha-1}}\hat{S}_\alpha(\lambda) = (\lambda^\alpha - A)^{-1},$$

and hence, by definition, A is generator of $R_\alpha(t)$ and it is an (α, α)-resolvent family. This proves the claim.

$(i) \Rightarrow (ii)$ Suppose that $\{R_\alpha(t)\}_{t>0}$ is compact. We prove that $R_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$. In fact, we can assume that $\omega > 0$. First, observe that for $t > s$, we have

$$R_\alpha(t) - R_\alpha(s) = \int_s^t g_{\alpha-1}(t-r)S_\alpha(r)dr + \int_0^s [g_{\alpha-1}(t-r) - g_{\alpha-1}(s-r)]S_\alpha(r)dr =: I_1 + I_2,$$

where

$$\|I_1\| \leq \int_s^t g_{\alpha-1}(t-r)\|S_\alpha(r)\|dr \leq Me^{\omega t}\int_s^t g_{\alpha-1}(t-r)dr.$$

Because $\alpha > 1$, we have $g_\alpha(0) = 0$ and we obtain

$$\|I_1\| \leq Me^{\omega t}g_\alpha(t-s). \tag{3.3}$$

On the other hand,

$$\begin{aligned}\|I_2\| &\leq \int_0^s |g_{\alpha-1}(t-r) - g_{\alpha-1}(s-r)|\|S_\alpha(r)\|dr \\ &\leq Me^{\omega s}\int_0^s |g_{\alpha-1}(t-r) - g_{\alpha-1}(s-r)|dr \\ &= Me^{\omega s}\int_0^s |g_{\alpha-1}(t-s+r) - g_{\alpha-1}(r)|dr.\end{aligned}$$

Note that $g_{\alpha-1}$ is decreasing for $\alpha < 2$, therefore $g_{\alpha-1}(r) - g_{\alpha-1}(t-s+r) > 0$, obtaining

$$\|I_2\| \leq Me^{\omega s}\int_0^s [g_{\alpha-1}(r) - g_{\alpha-1}(t-s+r)]dr = Me^{\omega s}[g_\alpha(s) - g_\alpha(t) + g_\alpha(t-s)]. \tag{3.4}$$

Observe that in the border case $\alpha = 2$ we have $I_2 \equiv 0$ because $g_1(t) \equiv 1$. Combining (3.3) and (3.4), we obtain the assertion. Then for $\lambda > \omega$ fixed we have

$$(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t}R_\alpha(t)dt,$$

where the integral in the right-hand side exists in the Bochner sense, because $\{R_\alpha(t)\}_{t>0}$ is continuous in the uniform operator topology. Then, by [24, Corollary 2.3] we conclude that $(\lambda^\alpha - A)^{-1}$ is a compact operator.

$(ii) \Rightarrow (i)$ Let $t > 0$ be fixed. Since $\alpha > 1$, it follows that $g_{\alpha-1} \in L^1_{\text{loc}}[0,\infty)$ and therefore, by [8, Proposition 2.1] we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}\widehat{(g_{\alpha-1} * S_\alpha)}(\lambda)d\lambda = (g_{\alpha-1} * S_\alpha)(t) = R_\alpha(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha - A)^{-1}d\lambda = R_\alpha(t),$$

where Γ is the path consisting of the vertical line $\{\omega + it : t \in \mathbb{R}\}$. By hypothesis and [24, Corollary 2.3], we conclude that $R_\alpha(t)$ is a compact operator. □

Remark 3.9. *In case* $\alpha = 2$ *the preceding Theorem extends the compactness criteria for sine operator functions in* [22]*. See also* [19, Theorem 10.1.1].

4. An application to a semilinear problem with non-local initial condition

In this section, we present one example which do not aim at generality but indicate how our theorems can be applied to more concrete problems. For other examples, see Fan [7, Theorem 4.1] and Wang-Chen-Xiao [23, Theorem 5.3].

Recall that the Riemann-Liouville fractional integral of order $0 < \beta < 1$ is defined as follows

$$J_t^\beta u(t) = \int_0^t g_\beta(t-s)u(s)ds, \quad u \in L^1(\mathbb{R}_+), \quad t > 0.$$

Let $T > 0$ be given. We study the semilinear problem

$$D_t^\alpha u(t) = Au(t) + J_t^{1-\alpha} f(t,u(t)), \quad 0 < \alpha < 1, \quad 0 \le t \le T, \tag{4.5}$$

with nonlocal initial condition $u(0) + g(u) = u_0$ where $f : [0,T] \times X \to X$ and $g : C(I,X) \to C(I,X)$ are continuous. Here D^α denotes Caputo fractional derivative. The concept of nonlocal initial condition has been introduced to extend the study of classical initial value problems. This notion is more precise for describing nature phenomena than the classical notion because additional information is taken into account. For the importance of nonlocal conditions in different fields, the reader is referred to [15] and the references cited therein.

Let A be the generator of an $(\alpha,1)$ - resolvent family $S_\alpha(t)$. Then it is well known that the mild solution of (4.5) is defined by means of the variation-of-constant formula.

$$u(t) = S_\alpha(t)[u_0 - g(u)] + \int_0^t S_\alpha(t-s)f(s,u(s))ds, \quad t \in I := [0,T].$$

See e.g. [7, Section 4]. We will make the following assumptions

- **H1.** f satisfies the Carathéodory condition, that is $f(\cdot,u)$ is strongly measurable for each $u \in X$ and $f(t,\cdot)$ is continuous for each $t \in I$.
- **H2.** There exists a continuous function $\mu : I \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le \mu(t)\|u\|, \quad \forall\, t \in I,\ u \in C(I,X).$$

- **H3.** $g : C(I,X) \to C(I,X)$ is continuous and there exists $L_g > 0$ such that

$$\|g(u) - g(v)\| < L_g \|u - v\|, \ \forall u, v \in C(I,X).$$

We prove the following existence theorem. The method of proof combines ideas from [7] and [14].

Theorem 4.10. *Let A be the generator of an $(\alpha,1)$-resolvent family $S_\alpha(t)$ of type (M,ω). If $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega$ and $S_\alpha(t)$ is continuous in the uniform operator topology for all $t > 0$, then, under assumptions H1-H3, Equation (4.5) has at least one mild solution.*

Proof. Define the operator $\Gamma : C(I,X) \to C(I,X)$ by

$$(\Gamma u)(t) := S_\alpha(t)[u_0 - g(u)] + \int_0^t S_\alpha(t-s) f(s,u(s)) ds$$

Let $B_r := \{u \in C(I,X) : \|u\| \leq r\}$. The proof will be conducted into several steps

Step 1.

We first show that Γ sends bounded sets of $C(I,X)$ into bounded sets of $C(I,X)$; in other words for any given $r > 0$ there exists $\xi > 0$ such that $\Gamma B_r \subset B_\xi$. So let $u \in B_r$ and $G := \sup_{u \in B_r} \|g(u)\|$. Then

$$\begin{aligned}
\|\Gamma u(t)\| &\leq M\|S_\alpha(t)\|(\|u_0\| + \|g(u)\|) + M\int_0^t \|S_\alpha(t-s)\|\|f(s,u(s))\| ds \\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M\int_0^t \|S_\alpha(t-s)\|\|f(s,u(s))\| ds \\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M\int_0^t e^{\omega(t-s)}\|f(s,u(s))\| ds \\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M\int_0^t e^{\omega(t-s)}\mu(s)\|u(s)\| ds \\
&\leq Me^{\omega T}(\|u_0\| + G) + Mr\|\mu\|\tfrac{e^{\omega T}}{\omega} = \xi.
\end{aligned} \tag{4.6}$$

Thus $\Gamma B_r \subset B_\xi$.

Step 2. Let's show that Γ is a continuous operator.

Let $u_n, u \in B_r$ such that $u_n \to u$ in $C(I,X)$. Then we have

(4.7)
$$\begin{aligned} \|\Gamma u_n(t) - \Gamma u(t)\| &\leq \|S_\alpha(t)\|(\|g(u_n) - g(u)\|) + \int_0^t \|S_\alpha(t-s)\|\|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\leq Me^{\omega t} L_g \|u_n - u\| + M\int_0^t e^{\omega(t-s)}\|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\leq Me^{\omega T} L_g \|u_n - u\| + M\int_0^t e^{\omega(t-s)}\|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\leq Me^{\omega t} L_g \|u_n - v\| + M\int_0^t e^{\omega(t-s)}\mu(s)(\|u_n(s)\| + \|u(s)\|)ds \\ &\leq Me^{\omega t} L_g \|u_n - u\| + 2rM\int_0^t e^{\omega(t-s)}\mu(s)ds. \end{aligned}$$

Choose n large enough such that $\|u_n - u\| < \epsilon$. Also note that $e^{\omega(t-s)}\mu(s)$ is integrable on I. So by the Lebesgue's Dominated Convergence Theorem, $\int_0^t e^{\omega(t-s)}\|f(s,u_n(s)) - f(s,u(s))\|ds \to 0$ as $n \to \infty$; which shows that Γ is continuous.

Step 3 Γ sends bounded sets of $C(I,X)$ into equicontinuous sets of $C(I,X)$.

Let $u \in B_r$, with $r > 0$ and take $t_1, t_2 \in I$ with $t_2 < t_1$. Then we have

$$\begin{aligned} \|\Gamma u(t_1) - \Gamma u(t_2)\| &\leq \|(S_\alpha(t_1) - S_\alpha(t_2))(u_0 - g(u))\| + \int_{t_2}^{t_1} \|S_\alpha(t_1 - s)f(s,u(s))\|ds \\ &+ \int_0^{t_2} \|(S_\alpha(t_1 - s) - S_\alpha(t_2 - s))f(s,u(s))\|ds \qquad (4.8) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We have
$$I_1 \leq \|(S_\alpha(t_1) - S_\alpha(t_2))\|\|(u_0 - g(u))\|.$$
Using the uniform continuity of $S_\alpha(t)$ for $t > 0$, we obtain that $\lim_{t_1 \to t_2} I_1 = 0$.

Next we have
$$I_2 \leq \int_{t_2}^{t_1} e^{\omega(t_1 - s)}\mu(s)\|u(s)\|ds \leq r\|\mu\|e^{\omega T}(t_1 - t_2).$$
Thus $\lim_{t_1 \to t_2} I_2 = 0$. Finally we have

$$\begin{aligned} I_3 &\leq \int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\|\|f(s,u(s))\|ds \\ &\leq \int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\|\mu(s)\|u(s)\|ds \qquad (4.9) \\ &\leq r\int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\|\mu(s)ds. \end{aligned}$$

Now observe that

$$\|S_\alpha(t_1-\cdot)-S_\alpha(t_2-\cdot)\|\mu(s)\leq 2Me^{\omega T}\mu(\cdot)\in L^1(I,\mathbb{R}),$$

and $S_\alpha(t_1-s)-S_\alpha(t_2-s)\to 0$ in $\mathcal{B}(X)$, as $t_1\to t_2$. Thus $\lim_{t_1\to t_2} I_3=0$ by the Lebesgue's dominated convergence theorem.

Step 4. Γ maps B_r into relatively compact sets in X.

Indeed in view of the hypothesis and Theorem 3.4 we have that $S_\alpha(t)$ is compact for all $s>0$, and hence we deduce that the set $\mathcal{K}=\{S_\alpha(t-s)f(s,u(s)):u\in C(I,X),0\leq s\leq t\}$ is relatively compact for each $t\in I$ (see the proof of [7, Theorem 4.1] for details). Then the set $\overline{conv}\mathcal{K}$ is compact. Moreover, for $u\in B_r$, using the Mean-Value Theorem for the Bochner integral, we obtain

$$\Gamma(u(t))\in t\,\overline{conv}\mathcal{K},\ \forall t\in[0,T].$$

Therefore the set $\{\Gamma u(t);u\in B_r\}$ is relatively compact in X for every $t\in[0,T]$. From Steps 1-4, we deduce that Γ is continuous and compact by the Arzela-Ascoli's theorem.

Step 5. Consider the set

$$\Omega:=\{u\in B_r:u=\lambda\Gamma u,\ 0<\lambda<1\}.$$

Clearly $\Omega\neq\emptyset$ since $0\in\Omega$. So let $u\in\Omega$. Then we have

$$\begin{aligned}\|u(t)\|&\leq\lambda[Me^{\alpha t}(\|u_0\|+\|g(u)\|)+M\int_0^t e^{\omega(t-s)}\|f(s,u(s)\|ds]\\&\leq\lambda[Me^{\alpha t}(\|u_0\|+G)+Mr\int_0^t e^{\omega(t-s)}\mu(s)ds]\\&\leq[Me^{\alpha t}(\|u_0\|+G)+Mr\|\mu\|\frac{e^{\omega T}}{\omega}]\end{aligned}$$

Thus Ω is bounded. So by the Leray-Schauder theorem Γ has a fixed point. The proof is complete.

□

References

[1] Araya, D., Lizama, C.: Almost automorphic mild solutions to fractional differential equations, Nonlinear Anal. **69** (2008), 3692-3705.
[2] Arendt, W., Batty, C., Hieber, M., Neubrander, F.: Vector-Valued Laplace transforms and Cauchy problems, Monogr. Math., vol. 96, Birkhäuser, Basel, 2001.
[3] E. Bazhlekova: Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, 2001.
[4] Chen, C., Li, M.: On fractional resolvent operator functions, Semigroup Forum **80** (2010), 121-142.
[5] Goldstein, J. A. Semigroups of linear operators and applications. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985.
[6] Engel, K.J., Nagel, R.: One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.
[7] Fan, Z.: Characterization of compactness for resolvents and its applications. Appl. Math. Comput. 232 (2014), 60-67.
[8] Haase, M.: The complex inversion formula revisited, J. Aust. Math. Soc. **84** (2008), no. 1, 73-83.
[9] Hilfer, R.: Applications of fractional calculus in physics, Edited by R. Hilfer, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
[10] Kilbas, A., Srivastava, H., Trujillo, J.: Theory and applications of fractional differential equations, North-Holland Mathematics studies 204, Elsevier Science B.V., Amsterdam, 2006.

[11] Li, M., Chen, C., Li, F.: On fractional powers of generators of fractional resolvent families, J. Funct. Anal. 259 (2010), no. **10**, 2702-2726.
[12] Li, K., Peng, J., Jia, J.: Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. J. Funct. Anal. **263** (2012), no. 2, 476-510.
[13] Lizama, C.: Regularized solutions for abstract Volterra equations, J. Math. Anal. Appl. **243** (2000), 278-292.
[14] Lizama, C., N'Guérékata, G. M.: Mild solutions for abstract fractional differential equations. Applicable Analysis, 92 (8) (2013), 1731-1754.
[15] Lizama, C., Pozo, J.C., Existence of mild solutions for semilinear integrodifferential equations with nonlocal conditions. Abstract and Applied Analysis, Volume 2012 (2012), Article ID 647103, 15 pages doi:10.1155/2012/647103.
[16] Miller, K., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York 1993.
[17] Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.
[18] Podlubny, I., Fractional Differential Equations. Academic Press, San Diego, 1999.
[19] Vasilév, V., Piskarev, S. Differential equations in Banach spaces. II. Theory of cosine operator functions. Functional analysis. J. Math. Sci. (N. Y.) 122 (2004), no. 2, 3055-3174.
[20] Prüss, J.: Positivity and regularity of hyperbolic Volterra equations in Banach spaces. Math. Ann., 279 (1987), 317-344.
[21] Prüss, J.: Evolutionary Integral Equations and Applications. Monographs Math., **87**, Birkhäuser Verlag, 1993.
[22] Travis, C. C., Webb, G. F. Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. Houston J. Math. 3 (1977), no. 4, 555-567.
[23] Wang, R.N., Chen D.H., Xiao, T.J.: Abstract fractional Cauchy problems with almost sectorial operators. J. Diff. Equations 252 (2012), 202-235.
[24] Weis, L. W.: A generalization of the Vidav-Jörgens perturbation theorem for semigroups and its application to transport theory, J. Math. Anal. Appl. **129** (1988), no. 1, 6-23.

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