# ALMOST AUTOMORPHIC SOLUTIONS OF VOLTERRA EQUATIONS ON TIME SCALES

CARLOS LIZAMA, JAQUELINE G. MESQUITA, RODRIGO PONCE, AND EDUARD TOON

Abstract. The existence and uniqueness of almost automorphic solutions for linear and semilinear nonconvolution Volterra equations on time scales is studied. The existence of asymptotically almost automorphic solutions is proved. Examples that illustrate our results are given.

## 1. Introduction

Several important and interesting models in diverse applied areas are described using differential and integral equations, or even difference and summation equations. However, in the last years, there are some recent studies which have been showed that these equations are not the best choices to describe most of the existent models. It happens, because most of phenomena in the environment do not involve only continuous aspects or only discrete aspects, but they feature elements of both the continuous and the discrete. These phenomena are called *hybrid processes* and there are several examples of them. See, for instance, [4, 5, 7, 13, 18, 28].

By these reasons, Stefan Hilger and Bernd Aulbach in 1988 introduced a theory to study in a unified way large classes of time scales. This theory encompasses the study of the differential equations, integral equations, difference equations, summation equations, among others. Therefore, using this theory, it is possible to describe in a more precise way the real-world problems, obtaining a more detailed analysis and description of specifics problems. For a detailed account, see [1, 2, 4, 5, 6, 7, 13, 15, 18, 19, 20, 23, 28] and references therein.

On the other hand, it is well known that Volterra integral equations play an important role in applications since they can describe several interesting phenomena. Some typical examples are provided by viscoelastic fluids and heat flow in materials of fading memory type. Due to this fact, these equations have been attracted the attention of several researchers from long ago. Some references are [3, 8, 17, 24, 27].

However, when we introduce into the study of Volterra equations with nonconvolution kernels, the study of both existence of solutions as well as qualitative behavior,

2010 *Mathematics Subject Classification.* 34N05; 45D05; 43A60.

*Key words and phrases.* Times scales; Volterra equation; almost automorphic; nonlinear; asymptotic.

The first author is partially supported by FONDECYT grant 1140258 and CONICYT under grant PBCT ACT 1416.

The second author is partially supported by FEMAT-Fundação de Estudos em Ciências Matemáticas Proc. 036/2016.

The third author is partially supported by FONDECYT grant 11130619.

is much more complicated than for convolution equations. Only in the scalar case and for the time scale $\mathbb{R}$, it has been possible to obtain interesting results on the asymptotic behavior of solutions. These results are based on monotonicity or sign properties of the kernel, see [16, Chapter 9] and in the combination of Schauder's fixed point theorem and the contraction mapping principle [16, Chapter 12].

The setting of qualitative properties of Volterra delta integral equations on time scales is not well-developed yet. There are few references concerning this topic and, to the best of our knowledge, there are no results concerning asymptotically almost automorphic solutions of nonconvolution Volterra delta integral equations on general time scales.

Motivated by this fact, our goal in this paper is to search for the existence and uniqueness of asymptotically almost automorphic solutions for the class of nonconvolution Volterra integral equation on time scales given by:

$$u(t) = \int_{t_0}^{t} a(t,\sigma(s))[u(s) + f(s,u(s))]\Delta s, \tag{1.1}$$

where $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ is almost automorphic in both variables and $f : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ is almost automorphic with respect to the first variable and satisfies a Lipschitz condition in the second variable.

It is worthwhile to observe that the concept of almost automorphic functions on time scales was introduced in the literature recently by Lizama and Mesquita [23] and since then, interesting new applications have appeared [21, 25, 29]. In contrast, this topic has been extensively studied mainly on the setting of functions defined only on $\mathbb{R}$. See, for instance, [9, 10, 11, 12, 14, 26].

To achieve our results, one important difficulty that arises is how to handle the concept of almost automorphy for the nonconvolution kernel $a(t,\sigma(s))$ in (1.1). Our approach in this paper is to adopt the idea of almost automorphy with respect to both variables. See Definition 2.25 below. This new concept shows to be efficient to undertake our investigation.

In particular, this definition provides new insights on the behavior of the exponential function on time scales. See Theorem 3.3 and its Corollary. We also find a practical condition that should be verified for the kernel in equation (1.1), namely

**(H1)** There exist positive constants $K, \gamma \in \mathbb{R}$ such that
$$\int_{-\infty}^{u} \|a(t,\sigma(s))\|\Delta s \leq \int_{-\infty}^{u} K e_{\ominus\gamma}(t,\sigma(s))\Delta s,$$
for every $t, u \in \mathbb{T}$.

Under this condition, we prove that if $\mathbb{T}$ is an invariant under translations time scale, then there exists a unique asymptotically almost automorphic solution of (1.1) either in case $t_0 \in \mathbb{T}_+$ or $t_0 = -\infty$, provided $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2(1+L)$, where $L$ is the Lipschitz constant of $f$. See Theorems 4.1 and 6.1. Moreover, if $f$ does not depend on the second variable, then $L$ can be chosen to be 0. See Theorems 3.6 and 5.1. Several examples along the text are also provided. Finally, it is worthwhile to remark that the results are essentially new when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.

## 2. Preliminaries

2.1. **Time scales.** In this section, we review some basic concepts and results concerning time scales which will be essential to prove our main results. For more details, the reader may consult [5, 6].

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T},\ s < t\}.$$

If $\sigma(t) > t$, we say that $t$ is *right-scattered*. Otherwise, $t$ is called *right-dense*. Analogously, if $\rho(t) < t$, then $t$ is called *left-scattered* whereas if $\rho(t) = t$, then $t$ is *left-dense*.

We also define the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ by

$$\mu(t) = \sigma(t) - t.$$

**Definition 2.1** ([5])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

**Definition 2.2** ([5])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$. If the function $f : \mathbb{T} \to \mathbb{R}$ is continuous at each right-dense point and each left-dense point, then the function $f$ is said to be *continuous* on $\mathbb{T}$. We denote the class of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.

Given a pair of numbers $a$, $b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T};\ a \leq t \leq b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R};\ a \leq t \leq b\}$.

We define the set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^\kappa = \mathbb{T} \backslash \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

**Definition 2.3** ([5])**.** For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* of $y$ to be the number (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|y(\sigma(t)) - y(s) - y^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

**Definition 2.4** ([5])**.** A *partition* of $[a, b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0, t_1, \dots, t_m\} \subset [a, b]_{\mathbb{T}}, \quad a = t_0 < t_1 < \dots < t_m = b.$$

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1, \dots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)$ for every $i \in \{1, \dots, m\}$. The set of all tagged partitions of $[a, b]_{\mathbb{T}}$ will be denoted by the symbol $D(a, b)$.

If $\delta > 0$, then $D_\delta(a, b)$ denotes the set of all tagged partitions of $[a, b]_{\mathbb{T}}$ such that for every $i \in \{1, \dots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. Note that in the last case, the only way to choose a tag in $[t_{i-1}, t_i)$ is to take $\xi_i = t_{i-1}$.

In the sequel, we present the definition of Riemann $\Delta$-integrals. See [5, 6], for instance.

**Definition 2.5.** We say that $f$ is *Riemann $\Delta$-integrable* on $[a,b]_{\mathbb{T}}$, if there exists a number $I$ with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_i f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon,$$

for every $P \in D_\delta(a,b)$ independently of $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ for $1 \leq i \leq m$. It is clear that such a number $I$ is unique and we call it the Riemann $\Delta$-integral of $f$ from $a$ to $b$.

In what follows, we present the concept of regressive functions, which will be important to define the generalized exponential function.

**Definition 2.6** ([5])**.** We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Suppose that $p, q \in \mathcal{R}$, then we define $p \oplus q$ and $\ominus p$ as follows:

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

and

$$(\ominus p)(t) := \frac{-p(t)}{1 + \mu(t)p(t)}, \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

By this definition, it is possible to prove that $(\mathcal{R}, \oplus)$ is an Abelian group. See [5]. In the sequel, we define the *generalized exponential function* $e_p(t,s)$.

**Definition 2.7** ([5])**.** If $p \in \mathcal{R}$, then we define the *generalized exponential function* by

$$e_p(t,s) = \exp\left( \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right) \quad \text{for} \quad s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is given by

$$\xi_h(z) = \frac{1}{h}\log(1 + zh),$$

where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

In the following, we present some important properties of the generalized exponential function.

**Theorem 2.8.** [5, Theorem 2.39] *If $p \in \mathcal{R}$ and $a,\ b,\ c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

**Theorem 2.9.** [23, Theorem 2.14] *If $\alpha > 0$, then $e_{\ominus\alpha}(t,s) \leq 1$ for $t,\ s \in \mathbb{T}$ such that $t > s$.*

**Theorem 2.10.** [20, Lemma 5.1] *Let* $\alpha > 0$*, then for any fixed* $s \in \mathbb{T}$ *and* $s = -\infty$*, one has the following:*

$$e_{\ominus\alpha}(t, s) \to 0, \quad t \to +\infty.$$

In what follows, we recall some definitions about matrix-valued functions on time scales.

**Definition 2.11** ([5])**.** Let $A$ be an $m \times n$ matrix-valued function on $\mathbb{T}$. We say that $A$ is *rd-continuous* on $\mathbb{T}$ if each entry of $A$ is rd-continuous on $\mathbb{T}$. We denote the class of all rd-continuous $m \times n$ matrix-valued function on $\mathbb{T}$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{m\times n})$.

We say that $A$ is *delta-differentiable* at $\mathbb{T}$ if each entry of $A$ is delta-differentiable on $\mathbb{T}$. And in this case, we have

$$A^\sigma(t) = A(t) + \mu(t)A^\Delta(t).$$

**Definition 2.12** ([5])**.** An $m \times n$ matrix-valued function $A$ on a time scale $\mathbb{T}$ is called *regressive* (with respect to $\mathbb{T}$) provided

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^\kappa,$$

and the class of all such regressive rd-continuous is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R}^{m\times n})$.

Assume $A$ and $B$ are regressive $n \times n$ matrix-valued functions on $\mathbb{T}$. Then, we define $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t), \qquad \forall t \in \mathbb{T}^\kappa,$$

and we define $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t), \qquad \forall t \in \mathbb{T}^\kappa.$$

It is clear that $(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}), \oplus)$ is a group (see [5]). We proceed by giving the definition of the matrix exponential function found in [5].

**Definition 2.13.** (Matrix Exponential Function) Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$ matrix valued function. The unique matrix-valued solution of the IVP

$$Y^\Delta(t) = A(t)Y(t), \qquad Y(t_0) = I,$$

where $I$ denotes, as usual, the $n \times n$-identity matrix, is called the *matrix exponential function* at $t_0$ and it is denoted by $e_A(\cdot, t_0)$.

In the sequel, we enunciate a result which describes the properties of the matrix exponential function. It can be found in [5, Theorem 5.21].

**Theorem 2.14.** *If* $A, B \in \mathcal{R}$ *are matrix-valued functions on* $\mathbb{T}$*, then*

(i) $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$*;*
(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$*;*
(iii) $e_A{}^{-1}(t, s) = e^*_{\ominus A^*}(t, s)$*;*
(iv) $e_A(t, s) = e_A{}^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$*;*
(v) $e_A(t, s)e_A(s, r) = e_A(t, r)$*;*
(vi) $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$ *if* $e_A(t, s)$ *and* $B(t)$ *commute.*

Using these notions, one can obtain the following result which resembles the variation of constants formula. This result can be found in [5, Theorem 5.24].

**Theorem 2.15.** *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} y^{\Delta}(t) = A(t)y(t) + f(t), \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

*has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by*

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

2.2. **Asymptotically almost automorphic and almost automorphic functions.** In this section, we remember several properties concerning asymptotically almost automorphic functions and almost automorphic functions defined on $\mathbb{T}$. For more details, the reader may consult [22, 23].

We start by presenting the definition of invariant under translations time scales.

**Definition 2.16** ([22])**.** A time scale $\mathbb{T}$ is called *invariant under translations* if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T},\ \forall t \in \mathbb{T}\} \neq \{0\}. \tag{2.2}$$

By this definition, it is easy to check that $\mathbb{R}$, $h\mathbb{Z}$, $\frac{1}{n}\mathbb{Z}$, $\mathbb{P}_{a,b}$, among others, are examples of invariant under translations time scales. Moreover, notice that by this definition, it follows that $\sup \mathbb{T} = +\infty$ and $\inf \mathbb{T} = -\infty$. From this definition, we obtain several immediate consequences, which will be essential to prove our main results. See below.

**Theorem 2.17.** [22, Lemma 3.3] *If $\mathbb{T}$ is invariant under translations and $\alpha \in \Pi$, then $k\alpha \in \Pi$ for every $k \in \mathbb{Z}$.*

**Theorem 2.18.** [22, Lemma 3.6] *If $\mathbb{T}$ is an invariant under translation time scale and $h \in \Pi$, then*

$$\sigma(t) + h = \sigma(t + h) \quad \text{and} \quad \sigma(t) - h = \sigma(t - h),$$

*for every $t \in \mathbb{T}$.*

**Theorem 2.19.** [22, Corollary 3.12] *If $\mathbb{T}$ is an invariant under translations time scale and $h \in \Pi$, then*

$$\mu(t + h) = \mu(t) = \mu(t - h),$$

*for every $t \in \mathbb{T}$.*

In the sequel, we recall the definition of an almost automorphic function $f : \mathbb{T} \to X$.

**Definition 2.20** ([23])**.** Let $X$ be (real or complex) Banach space and $\mathbb{T}$ be an invariant under translation time scale. Then, an rd-continuous function $f : \mathbb{T} \to X$

is called *almost automorphic* on $\mathbb{T}$ if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t)$$

exists and is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t),$$

exists and is well defined for every $t \in \mathbb{T}$.

We denote the space of all almost automorphic function on time scales $f : \mathbb{T} \to X$ by $AA_{\mathbb{T}}(X)$. We note that from [23, Theorem 3.18] is already known that $AA_{\mathbb{T}}(X)$ is a Banach space.

In the sequel, we present a result which describes several properties of almost automorphic functions defined on $\mathbb{T}$. For a proof of this result, see [23].

**Theorem 2.21.** *Let $\mathbb{T}$ be an invariant under translations time scale and suppose the rd-continuous functions $f, g : \mathbb{T} \to X$ are almost automorphic on time scales, then the following assertions hold.*

(i) *$f + g$ is almost automorphic function on time scales;*
(ii) *$cf$ is almost automorphic function on time scales for every scalar $c$;*
(iii) *For each $l \in \Pi$, the function $f_l : \mathbb{T} \to X$ defined by $f_l(t) := f(l+t)$ is almost automorphic on time scales.*
(iv) $\sup_{t\in\mathbb{T}} \|f(t)\| < \infty$*, that is, $f$ is a bounded function;*
(v) $\sup_{t\in\mathbb{T}} \|\bar{f}(t)\| \leq \sup_{t\in\mathbb{T}} \|f(t)\|$*, where*

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t) \quad \text{and} \quad \lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t).$$

**Theorem 2.22** ([23])**.** *Let $\mathbb{T}$ be invariant under translations time scale and the functions $f, u : \mathbb{T} \to X$ be almost automorphic on time scales, then the function $uf : \mathbb{T} \to X$ defined by $(uf)(t) = u(t)f(t)$ is almost automorphic on time scales.*

In what follows, we present a result which brings a property concerning composition of an almost automorphic function on time scales and a continuous function.

**Theorem 2.23** ([23])**.** *Let $\mathbb{T}$ be an invariant under translations time scale and let $X, Y$ be Banach spaces. Suppose $f : \mathbb{T} \to X$ is an almost automorphic function on time scales and $\phi : X \to Y$ is a continuous function, then the composite function $\phi \circ f : \mathbb{T} \to Y$ is an almost automorphic function on time scales.*

Now, we present the definition of a function $f : \mathbb{T} \times X \to X$ which is almost automorphic with respect to the first variable.

**Definition 2.24** ([23])**.** Let $X$ be a (real or complex) Banach space and $\mathbb{T}$ be an invariant under translations time scale. Then, an rd-continuous function $f : \mathbb{T} \times X \to X$ is called *almost automorphic at $t \in \mathbb{T}$ for each $x \in X$*, if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t+\alpha_n, x) = \bar{f}(t, x) \tag{2.3}$$

exists and is well defined for each $t \in \mathbb{T}$, $x \in X$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n, x) = f(t,x) \tag{2.4}$$

exists and is well-defined for every $t \in \mathbb{T}$ and $x \in X$.

In the sequel, we present the definition of a function $f : \mathbb{T} \times \mathbb{T} \to X$ which is almost automorphic with respect to both variables.

**Definition 2.25.** Let $X$ be a (real or complex) Banach space and $\mathbb{T}$ be an invariant under translations time scale. Then, an rd-continuous function $f : \mathbb{T} \times \mathbb{T} \to X$ is called *almost automorphic with respect to both variables* if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t+\alpha_n, s+\alpha_n) = \bar{f}(t,s) \tag{2.5}$$

exists and is well defined for each $t,\ s \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n, s-\alpha_n) = f(t,s) \tag{2.6}$$

exists and is well-defined for every $t,\ s \in \mathbb{T}$.

Now, we recall a result concerning the properties of almost automorphic functions on time scales with respect to the first variable. This result can be found in [23].

**Theorem 2.26.** *Let $\mathbb{T}$ be invariant under translations and $f, g : \mathbb{T} \times X \to X$ be almost automorphic functions on time scales in $t$ for each $x$ in $X$, then the following assertions hold.*

(i) *$f + g$ is almost automorphic function on time scales in $t$ for each $x$ in $X$.*
(ii) *$cf$ is almost automorphic function on time scales in $t$ for each $x$ in $X$, where $c$ is an arbitrary scalar.*
(iii) *$\sup_{t\in\mathbb{T}} \|f(t,x)\| = M_x < \infty$, for each $x$ in $X$.*
(iv) *$\sup_{t\in\mathbb{T}} \|\bar{f}(t,x)\| = N_x < \infty$, for each $x$ in $X$, where $\bar{f}$ is the function in the Definition 2.24.*

The following result will be essential to our purposes. A detailed proof can be found in [23].

**Theorem 2.27.** *Let $\mathbb{T}$ be invariant under translations and $f : \mathbb{T}\times X \to X$ be almost automorphic function on time scales for each $x \in X$ and if $f$ is Lipschitzian in $x$ uniformly in $t$, then $\bar{f}$ given by* (2.3) *and* (2.4) *satisfies the same Lipschitz condition in $x$ uniformly in $t$.*

**Theorem 2.28.** [23, Theorem 3.23] *Let $\mathbb{T}$ be an invariant under translations time scale and $f : \mathbb{T} \times X \to X$ be an almost automorphic function on time scales in $t$ for each $x \in X$ and satisfies Lipschitz condition in $x$ uniformly in $t$, that is,*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|,$$

*for all $x,\ y \in X$. Suppose $\phi : \mathbb{T} \to X$ is almost automorphic function on time scales, then the function $U : \mathbb{T} \to X$ defined by $U(t) = f(t, \phi(t))$ is almost automorphic on time scales.*

In the sequel, we remember the concept of asymptotically almost automorphic functions on time scales. See, for instance, [22].

**Definition 2.29.** Let $X$ be a (real or complex) Banach space, $\mathbb{T}$ be an invariant under translations time scale and $f : \mathbb{T} \to X$. We say that $f$ is an *asymptotically almost automorphic function on time scales* if there is an almost automorphic function $f_1 : \mathbb{T} \to X$ and an rd-continuous function $f_2 : \mathbb{T}_+ \to X$ such that $\lim_{t\to+\infty} \|f_2(t)\| = 0$ such that

$$f(t) = f_1(t) + f_2(t),$$

for every $t \in \mathbb{T}_+$. We say that $f_1$ and $f_2$ are called, respectively, the *principal* and *corrective* terms of the function $f$. We denote the set of all asymptotically almost automorphic functions $f : \mathbb{T}_+ \to X$ by $AAA_{\mathbb{T}}(X)$.

By this definition, it is clear that every almost automorphic function on time scale restricted to $\mathbb{T}_+$ is asymptotically almost automorphic. See [22], for instance.

The next result is essential to our purposes. The reader may find a proof of it in [22, Theorem 3.19].

**Theorem 2.30.** *Let $\mathbb{T}$ be an invariant under translations time scale. The decomposition of an asymptotically almost automorphic function $f : \mathbb{T}_+ \to X$ is unique.*

## 3. Almost automorphic solutions of linear Volterra Integral Equations on Time scales

In this section, we investigate the existence and uniqueness of almost automorphic solutions for class of linear Volterra integral equations on time scales given by:

$$u(t) = \int_{-\infty}^{t} a(t, \sigma(s))[u(s) + g(s)]\Delta s, \tag{3.1}$$

where $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n\times n}$ is an almost automorphic function with respect to both variables and $g : \mathbb{T} \to \mathbb{R}^n$ is an almost automorphic function. From now on, we consider the following hypothesis on the function $a$:

**(H1)** There exist positive constants $K, \gamma \in \mathbb{R}^+$ such that

$$\int_{-\infty}^{u} \|a(t, \sigma(s))\|\Delta s \leq \int_{-\infty}^{u} K e_{\ominus\gamma}(t, \sigma(s))\Delta s,$$

for every $t,\ u \in \mathbb{T}$.

**Remark 3.1.** The hypothesis (H1) remembers the property of exponential dichotomy on $\mathbb{T}$. But clearly (H1) is weaker than the exponential dichotomy condition on the time scale $\mathbb{R}$.

**Remark 3.2.** We point out that for the specific case $\mathbb{T} = \mathbb{R}$, the hypothesis (H1) can be rewritten as follows:

**($H1_{\mathbb{R}}$)** There exist positive constants $K, \gamma \in \mathbb{R}^+$ such that

$$\int_{-\infty}^{u} \|a(t, s)\|ds \leq \frac{K}{\gamma} e^{-\gamma(t-u)},$$

for every $t,\ u \in \mathbb{R}$.

On the other hand, for the specific case $\mathbb{T} = h\mathbb{Z}$, $h \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=-\infty}^{u/h-1} \|a(t,k+h)\|h &\leq K \sum_{k=-\infty}^{u/h-1} (1+h\ominus\gamma)^{(t/h-k)-1} h \\ &= K \sum_{k=-\infty}^{u/h-1} \left(1+h\frac{-\gamma}{1+h\gamma}\right)^{(t/h-k)-1} h \\ &= K \sum_{k=-\infty}^{u/h-1} h\left(\frac{1}{1+h\gamma}\right)^{(t/h-k)-1} \\ &= K \sum_{k=-\infty}^{u/h-1} h\,(1+h\gamma)^{k+1-t/h}\end{aligned}$$

for every $t,\ u \in h\mathbb{Z}$. Consequently, the hypothesis $(H1)$ reads as follows in this case
$(H1_{h\mathbb{Z}})$ There exist positive constants $K, \gamma \in \mathbb{R}^+$ such that

$$\sum_{k=-\infty}^{m-1} \|a(n,k+h)\| \leq K \sum_{k=-\infty}^{m-1} (1+h\gamma)^{k-n/h+1}$$

for every $m, n \in \mathbb{Z}$.

Before to proceed with our main result concerning the existence and uniqueness of almost automorphic solutions, we will need to prove some auxiliaries results. Our first result provide new insights on the exponential function on time scales.

**Theorem 3.3.** *Let $\mathbb{T}$ be an invariant under translations time scale and $a : \mathbb{T} \to \mathbb{R}$ be an almost automorphic and regressive function, then $e_{\ominus a}(t, \sigma(s))$ is also an almost automorphic function with respect to both variables.*

*Proof.* Since $a : \mathbb{T} \to \mathbb{R}$ is an almost automorphic function, then for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} a(t+\alpha_n) = \bar{a}(t)$$

exists and is well-defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{a}(t-\alpha_n) = a(t)$$

exists and is well-defined for each $t \in \mathbb{T}$.

Then, we have

$$\begin{aligned}e_{\ominus a}(t+\alpha_n, \sigma(s+\alpha_n)) &= e_{\ominus a}(t+\alpha_n, \sigma(s)+\alpha_n) \\ &= \exp\left(\int_{\sigma(s)+\alpha_n}^{t+\alpha_n} \frac{1}{\mu(\tau)} \log(1+\ominus a(\tau)\mu(\tau))\Delta\tau\right) \\ &= \exp\left(\int_{\sigma(s)+\alpha_n}^{t+\alpha_n} \frac{1}{\mu(\tau)} \log\left(1-\frac{a(\tau)}{1+\mu(\tau)a(\tau)}\mu(\tau)\right)\Delta\tau\right)\end{aligned}$$

$$= \exp\left(\int_{\sigma(s)}^{t} \frac{1}{\mu(\tau+\alpha_n)} \log\left(1 - \frac{a(\tau+\alpha_n)\mu(\tau+\alpha_n)}{1+\mu(\tau+\alpha_n)a(\tau+\alpha_n)}\right)\Delta\tau\right)$$
$$= \exp\left(\int_{\sigma(s)}^{t} \frac{1}{\mu(\tau)} \log\left(1 - \frac{a(\tau+\alpha_n)}{1+\mu(\tau)a(\tau+\alpha_n)}\mu(\tau)\right)\Delta\tau\right).$$

Applying the limit as $n \to +\infty$, we have:

$$\begin{aligned}
\lim_{n\to+\infty} e_{\ominus a}(t+\alpha_n, \sigma(s)+\alpha_n) &= \lim_{n\to+\infty} \exp\left(\int_{\sigma(s)}^{t} \frac{1}{\mu(\tau)} \log\left(1 - \frac{a(\tau+\alpha_n)\mu(\tau)}{1+\mu(\tau)a(\tau+\alpha_n)}\right)\Delta\tau\right) \\
&= \exp\left(\int_{\sigma(s)}^{t} \frac{1}{\mu(\tau)} \log\left(1 - \frac{\bar{a}(\tau)}{1+\mu(\tau)\bar{a}(\tau)}\mu(\tau)\right)\Delta\tau\right) \\
&= \exp\left(\int_{\sigma(s)}^{t} \frac{1}{\mu(\tau)} \log\left(1 + \ominus\bar{a}(\tau)\mu(\tau)\right)\Delta\tau\right) \\
&= \exp\left(\int_{\sigma(s)}^{t} \xi_{\mu(\tau)}(\ominus\bar{a})\Delta\tau\right) \\
&= e_{\ominus\bar{a}}(t, \sigma(s)),
\end{aligned}$$

for each $t,\ s \in \mathbb{T}$.

Also, it is not difficult to prove that

$$\lim_{n\to+\infty} e_{\ominus\bar{a}}(t-\alpha_n, \sigma(s)-\alpha_n) = e_{\ominus a}(t, \sigma(s))$$

exists and is well-defined for every $t,\ s \in \mathbb{T}$. And the result follows as well. □

As an immediate consequence, we have the following result.

**Corollary 3.4.** *Let $\mathbb{T}$ be an invariant under translations time scale, $\alpha \in \mathbb{R}^+$, then $e_{\ominus\alpha}(t, \sigma(s))$ is almost automorphic with respect to both variables.*

Finally, we present an important auxiliary result, which brings a very important property concerning the limit function of the almost automorphic function $a$.

**Lemma 3.5.** *Let $\mathbb{T}$ be an invariant under translations time scale and $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ be almost automorphic on time scales with respect to both variables. More precisely, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that $\lim_{n\to\infty} a(t+\alpha_n, \sigma(s)+\alpha_n) = \bar{a}(t, \sigma(s))$ exists and is well-defined for each $t,\ s \in \mathbb{T}$ and $\lim_{n\to\infty} \bar{a}(t-\alpha_n, \sigma(s)-\alpha_n) = a(t, \sigma(s))$ exists and is well-defined for each $t,\ s \in \mathbb{T}$. If the function $a$ satisfies the condition (H1), then $\bar{a} : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ also satisfies the condition (H1) for the same constants $K,\ \gamma > 0$.*

*Proof.* Let $t,\ s \in \mathbb{T}$ and $\varepsilon > 0$ be given. Then, by the almost automorphicity of the function $a$ and by the definition of $\bar{a}$, we have that for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} a(t+\alpha_n, \sigma(s)+\alpha_n) = \bar{a}(t, \sigma(s))$$

exists and is well-defined for each $t,\ \sigma(s) \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{a}(t-\alpha_n, \sigma(s)-\alpha_n) = a(t, \sigma(s))$$

exists and is well-defined for each $t,\ \sigma(s) \in \mathbb{T}$.

Therefore, we have

$$\int_{-\infty}^{t+\alpha_n} \|a(t+\alpha_n, \sigma(s))\|\Delta s \leq \int_{-\infty}^{t+\alpha_n} K e_{\ominus\gamma}(t+\alpha_n, \sigma(s))\Delta s.$$

Hence,

$$\int_{-\infty}^{t} \|a(t+\alpha_n, \sigma(s+\alpha_n))\|\Delta s \leq \int_{-\infty}^{t} K e_{\ominus\gamma}(t+\alpha_n, \sigma(s+\alpha_n))\Delta s.$$

Therefore,

$$\int_{-\infty}^{t} \|a(t+\alpha_n, \sigma(s)+\alpha_n)\|\Delta s \leq \int_{-\infty}^{t} K e_{\ominus\gamma}(t+\alpha_n, \sigma(s)+\alpha_n)\Delta s.$$

Applying the limit as $n \to \infty$ and using Theorem 3.3, we have

$$\int_{-\infty}^{t} \|\bar{a}(t, \sigma(s))\|\Delta s \leq \int_{-\infty}^{t} K e_{\ominus\gamma}(t, \sigma(s))\Delta s$$

and we have the desired result. □

Now, we are able to prove our main result in this section concerning existence and uniqueness of almost automorphic solution of the linear Volterra equation (3.1).

**Theorem 3.6.** *Consider the linear Volterra integral equation* (3.1). *Also, consider that the time scale* $\mathbb{T}$ *is invariant under translations, the function* $a : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n\times n}$ *satisfies hypothesis (H1) with the positive constants* $\gamma$ *and* $K$ *being such that* $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2$, *where* $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$ *and that the function* $g : \mathbb{T} \to \mathbb{R}^n$ *is almost automorphic. Then, the Volterra equation* (3.1) *has a unique almost automorphic solution.*

*Proof.* At first, we notice that by hypothesis (H1), we have:

$$\begin{aligned} \int_{-\infty}^{t} \|a(t, \sigma(s))\|\Delta s &\leq \int_{-\infty}^{t} K e_{\ominus\gamma}(t, \sigma(s))\Delta s \\ &= \frac{K}{|\ominus\gamma|}[e_{\ominus\gamma}(t, -\infty) - e_{\ominus\gamma}(t,t)] \\ &\leq \frac{2K}{|\ominus\gamma|} = \frac{2K(1+\mu(t)\gamma)}{\gamma} \leq \frac{2K(1+\tilde{\mu}\gamma)}{\gamma} \end{aligned} \tag{3.2}$$

where the first equality follows from Theorem 2.8 and the second inequality follows from Theorem 2.9.

Define an operator $T : AA_{\mathbb{T}}(\mathbb{R}^n) \to AA_{\mathbb{T}}(\mathbb{R}^n)$ as follows:

$$(Tu)(t) = \int_{-\infty}^{t} a(t, \sigma(s))[u(s) + g(s)]\Delta s,$$

for all $u \in AA_{\mathbb{T}}(\mathbb{R}^n)$.

Now, we have to show that $T$ is well defined. In fact, by the almost automorphicity of the functions $g(t)$, $a(t,\sigma(s))$ and $u(t)$, it follows that for every sequence $(\alpha_n')\in\Pi$, there exists a subsequence $(\alpha_n)\subset(\alpha_n')$ such that

$$\lim_{n\to\infty} g(t+\alpha_n)=\overline{g}(t),$$
$$\lim_{n\to\infty} a(t+\alpha_n,\sigma(s)+\alpha_n)=\overline{a}(t,\sigma(s)) \quad \text{and}$$
$$\lim_{n\to\infty} u(t+\alpha_n)=\overline{u}(t)$$

exists and is well-defined for every $t,\sigma(s)\in\mathbb{T}$ and

$$\lim_{n\to\infty} \overline{g}(t-\alpha_n)=g(t),$$
$$\lim_{n\to\infty} \overline{a}(t-\alpha_n,\sigma(s)-\alpha_n)=a(t,\sigma(s)) \quad \text{and}$$
$$\lim_{n\to\infty} \overline{u}(t-\alpha_n)=u(t)$$

exists and is well-defined for every $t,\sigma(s)\in\mathbb{T}$.

Define a function $h:\mathbb{T}\to\mathbb{R}^n$ as follows:

$$h(t):=\int_{-\infty}^{t}\overline{a}(t,\sigma(s))[\overline{u}(s)+\overline{g}(s)]\Delta s.$$

Then, we have

$$\begin{aligned}
\|(Tu)(t+\alpha_n)-h(t)\| &= \Big\|\int_{-\infty}^{t+\alpha_n} a(t+\alpha_n,\sigma(s))[u(s)+g(s)]\Delta s\\
&- \int_{-\infty}^{t}\overline{a}(t,\sigma(s))[\overline{u}(s)+\overline{g}(s)]\Delta s\Big\|\\
&= \Big\|\int_{-\infty}^{t} a(t+\alpha_n,\sigma(s+\alpha_n))[u(s+\alpha_n)+g(s+\alpha_n)]\Delta s\\
&- \int_{-\infty}^{t}\overline{a}(t,\sigma(s))[\overline{u}(s)+\overline{g}(s)]\Delta s\Big\|\\
&= \Big\|\int_{-\infty}^{t} a(t+\alpha_n,\sigma(s)+\alpha_n)[u(s+\alpha_n)+g(s+\alpha_n)]\Delta s\\
&- \int_{-\infty}^{t}\overline{a}(t,\sigma(s))[\overline{u}(s)+\overline{g}(s)]\Delta s\Big\|\\
&= \Big\|\int_{-\infty}^{t}\overline{a}(t,\sigma(s))[u(s+\alpha_n)-\overline{u}(s)]\Delta s\\
&+ \int_{-\infty}^{t}[a(t+\alpha_n,\sigma(s)+\alpha_n)-\overline{a}(t,\sigma(s))]u(s+\alpha_n)\Delta s\\
&+ \int_{-\infty}^{t}\overline{a}(t,\sigma(s))[g(s+\alpha_n)-\overline{g}(s)]\Delta s\\
&+ \int_{-\infty}^{t}[a(t+\alpha_n,\sigma(s)+\alpha_n)-\overline{a}(t,\sigma(s))]g(s+\alpha_n)\Delta s\Big\|.
\end{aligned}$$

Applying the limit as $n \to \infty$ and using the boundedness of the functions $g$ and $u$ (by Theorem 2.21 (iv)), the fact that $\bar{a}$ satisfies the condition (H1) (by Lemma 3.5) and the almost automorphicity of the functions $a$, $g$ and $u$, we obtain

$$\lim_{n\to\infty} (Tu)(t+\alpha_n) = h(t),$$

for every $t \in \mathbb{T}$. Similarly, one can prove that

$$\lim_{n\to\infty} h(t-\alpha_n) = (Tu)(t),$$

for every $t \in \mathbb{T}$. From these facts, we conclude that $Tu$ is an almost automorphic function. Thus, $Tu$ is well-defined.

Now, let us prove that $T$ is a contraction.

$$\begin{aligned} \|Tu - Tv\| &= \left\| \int_{-\infty}^{t} a(t,\sigma(s))[u(s)+g(s)]\Delta s - \int_{-\infty}^{t} a(t,\sigma(s))[v(s)+g(s)]\Delta s \right\| \\ &= \left\| \int_{-\infty}^{t} a(t,\sigma(s))[u(s)-v(s)]\Delta s \right\| \\ &\leq \int_{-\infty}^{t} \|a(t,\sigma(s))\|\Delta s \|u-v\|_\infty \\ &\leq \frac{2K(1+\tilde{\mu}\gamma)}{\gamma}\|u-v\|_\infty \end{aligned}$$

where the last inequality follows by (3.2).Therefore, $T$ is a contraction, then by Banach fixed-point Theorem, $T$ has a unique fixed point. By the definition of $T$, we obtain that the integral equation (3.1) has a unique solution which is almost automorphic. □

In what follows, we present some examples to illustrate our result.

**Example 3.7.** Consider the following integral equation:

$$x(t) = \int_{-\infty}^{t} Ke_{\ominus b}(t,\sigma(s))[x(s)+f(s)]\Delta s.$$

Let us suppose that both $b$ and $f$ are almost automorphic functions on $\mathbb{T}$. Also, suppose that the function $b$ is lower bounded, that is, there exists a constant $\gamma > 0$ such that $\gamma < |b(t)|,\ t \in \mathbb{T}$.

By Theorem 3.3, the function $a(t,\sigma(s)) = e_{\ominus b}(t,\sigma(s))$ is almost automorphic with respect to both variables.

Also, it is easy to see that the hypothesis (H1) is satisfied with

$$\int_{-\infty}^{t} e_{\ominus b}(t,\sigma(s))\Delta s \leq \int_{-\infty}^{t} e_{\ominus\gamma}(t,\sigma(s))\Delta s.$$

Also, suppose that $\dfrac{K\gamma}{1+\tilde{\mu}\gamma} > 2$, where $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$. Therefore, all the hypotheses of Theorem 3.6 are satisfied and we conclude that the delta integral equation

$$x(t) = \int_{-\infty}^{t} e_{\ominus b}(t,\sigma(s))[x(s)+f(s)]\Delta s$$

has an almost automorphic solution.

**Example 3.8.** Consider the following integral equation

$$x(t) = \int_{-\infty}^{t} a(t,\sigma(s))(x(s)+g(s))\Delta s, \tag{3.3}$$

where $g : \mathbb{T} \to \mathbb{R}^n$ is an almost automorphic function and

$$a(t,\sigma(s)) = \begin{pmatrix} \dfrac{e_{\ominus 2}(t,\sigma(s))}{4} & 0 \\ 0 & \dfrac{e_{\ominus 2}(t,\sigma(s))}{4} \end{pmatrix}.$$

By Corollary 3.4, it follows that $a$ is almost automorphic with respect to both variables. Moreover,

$$\begin{aligned} \int_{-\infty}^{t} \|a(t,\sigma(s))\|\Delta s &= \int_{-\infty}^{t} \left\| \begin{pmatrix} \frac{e_{\ominus 2}(t,\sigma(s))}{4} & 0 \\ 0 & \frac{e_{\ominus 2}(t,\sigma(s))}{4} \end{pmatrix} \right\| \Delta s \\ &\leq \int_{-\infty}^{t} \frac{\sqrt{2}}{4} e_{\ominus 2}(t,\sigma(s))\Delta s, \end{aligned}$$

which shows that $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ satisfies the hypothesis **(H1)**. Also, suppose that $\dfrac{8}{\sqrt{2}(1+\tilde{\mu}2)} > 2$, where $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$. Therefore, all the hypotheses of Theorem 3.6 are satisfied, and hence the integral equation (3.3) has a unique almost automorphic solution.

## 4. Almost automorphic solutions of semilinear Volterra Integral Equations on Time scales

In this section, our goal is to investigate the existence and uniqueness of almost automorphic solutions of the semilinear Volterra integral equation on time scales given by:

$$u(t) = \int_{-\infty}^{t} a(t,\sigma(s))[u(s)+f(s,u(s))]\Delta s, \tag{4.1}$$

where $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ is almost automorphic with respect to both variables and $f : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ is almost automorphic with respect to the first variable and satisfies a Lipschitz condition in the second variable. As in the previous section, we consider that the hypothesis **(H1)** is satisfied.

**Theorem 4.1.** *Suppose that the time scale $\mathbb{T}$ is invariant under translations, the function $f : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ is almost automorphic on time scales in $t$ for each $x \in \mathbb{R}^n$ and satisfies a Lipschitz condition in $x$ uniformly in $t$, that is,*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|,$$

*for all $x,\ y \in \mathbb{R}^n$, the function $a : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^{n\times n}$ is almost automorphic with respect to both variables and satisfies hypothesis (H1) with the positive constants $\gamma$ and $K$*

*being such that* $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2(1+L)$, *where* $L$ *is the Lipschitz constant. Then, the equation* (4.1) *possesses a unique almost automorphic solution.*

*Proof.* We start by defining an operator $T : AA_{\mathbb{T}}(\mathbb{R}^n) \to AA_{\mathbb{T}}(\mathbb{R}^n)$ as follows:

$$(Tu)(t) = \int_{-\infty}^{t} a(t,\sigma(s))[u(s) + f(s,u(s))]\Delta s,$$

for all $u \in AA_{\mathbb{T}}(\mathbb{R}^n)$.

At first, we will show that $T$ is well defined. In order to do this, notice that by the almost automorphicity of $f$ and $u$ and by the Lipschitz condition, it follows that $f(\cdot,u(\cdot)) \in AA_{\mathbb{T}}(\mathbb{R}^n)$, by Theorem 2.28.

By the almost automorphicity of functions $a(t,\sigma(s))$, $u(t)$ and $f(\cdot,u(\cdot))$, we have that for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} a(t+\alpha_n, \sigma(s)+\alpha_n) = \overline{a}(t,\sigma(s)), \quad \lim_{n\to\infty} u(t+\alpha_n) = \overline{u}(t)$$
$$\text{and} \quad \lim_{n\to\infty} f(t+\alpha_n, u(t+\alpha_n)) = \overline{f}(t,\overline{u}(t))$$

exists and is well-defined for every $t, \sigma(s) \in \mathbb{T}$ and

$$\lim_{n\to\infty} \overline{a}(t-\alpha_n, \sigma(s)-\alpha_n) = a(t,\sigma(s)), \quad \lim_{n\to\infty} \overline{u}(t-\alpha_n) = u(t)$$
$$\text{and} \quad \lim_{n\to\infty} \overline{f}(t-\alpha_n, \overline{u}(t-\alpha_n)) = f(t,u(t))$$

exists and is well-defined for every $t, \sigma(s) \in \mathbb{T}$.

Define the function $M : \mathbb{T} \to \mathbb{R}^n$ as follows:

$$M(t) = \int_{-\infty}^{t} \overline{a}(t,\sigma(s))[\overline{u}(s) + \overline{f}(s,\overline{u}(s))]\Delta s.$$

Then, we have

$$\begin{aligned}
\|(Tu)(t+\alpha_n) - M(t)\| &= \bigg\| \int_{-\infty}^{t+\alpha_n} a(t+\alpha_n,\sigma(s))[u(s) + f(s,u(s))]\Delta s \\
&- \int_{-\infty}^{t} \overline{a}(t,\sigma(s))[\overline{u}(s) + \overline{f}(s,\overline{u}(s))]\Delta s \bigg\| \\
&= \bigg\| \int_{-\infty}^{t} a(t+\alpha_n,\sigma(s+\alpha_n))[u(s+\alpha_n) + f(s+\alpha_n,u(s+\alpha_n))]\Delta s \\
&- \int_{-\infty}^{t} \overline{a}(t,\sigma(s))[\overline{u}(s) + \overline{f}(s,\overline{u}(s))]\Delta s \bigg\| \\
&= \bigg\| \int_{-\infty}^{t} a(t+\alpha_n,\sigma(s)+\alpha_n)[u(s+\alpha_n) + f(s+\alpha_n,u(s+\alpha_n))]\Delta s \\
&- \int_{-\infty}^{t} \overline{a}(t,\sigma(s))[\overline{u}(s) + \overline{f}(s,\overline{u}(s))]\Delta s \bigg\| \\
&= \bigg\| \int_{-\infty}^{t} \overline{a}(t,\sigma(s))[u(s+\alpha_n) - \overline{u}(s)]\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{-\infty}^{t} [a(t+\alpha_n, \sigma(s)+\alpha_n) - \overline{a}(t, \sigma(s))] u(s+\alpha_n)\Delta s \\
&+ \int_{-\infty}^{t} \overline{a}(t, \sigma(s))[f(s+\alpha_n, u(s+\alpha_n)) - \overline{f}(s, \overline{u}(s))]\Delta s \\
&+ \int_{-\infty}^{t} [a(t+\alpha_n, \sigma(s)+\alpha_n) - \overline{a}(t, \sigma(s))] f(s+\alpha_n, u(s+\alpha_n))\Delta s \Bigg\|.
\end{aligned}
$$

Applying the limit as $n \to \infty$ and using the boundedness of the functions $f$ and $u$ (by Theorem 2.21 (iv)), the fact that $\bar{a}$ satisfies condition (H1) (by Lemma 3.5) and the almost automorphicity of the functions $a$, $f$ and $u$, we obtain that

$$\lim_{n\to\infty} (Tu)(t+\alpha_n) = M(t)$$

exists and is well-defined for every $t \in \mathbb{T}$. Similarly, one can prove that

$$\lim_{n\to\infty} M(t-\alpha_n) = (Tu)(t)$$

exists and is well-defined for every $t \in \mathbb{T}$. Therefore, we conclude that $Tu$ is an almost automorphic function and thus, $Tu$ is well-defined.

Now, let us prove that $T$ is a contraction.

$$
\begin{aligned}
\|Tu - Tv\| &= \left\| \int_{-\infty}^{t} a(t,\sigma(s))[u(s) + f(s,u(s))]\Delta s - \int_{-\infty}^{t} a(t,\sigma(s))[v(s) + f(s,v(s))]\Delta s \right\| \\
&= \left\| \int_{-\infty}^{t} a(t,\sigma(s))[u(s) - v(s) + f(s,u(s)) - f(s,v(s))]\Delta s \right\| \\
&\le \int_{-\infty}^{t} \|a(t,\sigma(s))\|\Delta s(\|u-v\|_\infty + L\|u-v\|_\infty) \\
&\le \frac{2K(1+\tilde{\mu}\gamma)}{\gamma}(L+1)\|u-v\|_\infty,
\end{aligned}
$$

where the last inequality follows from hypothesis (H1), (3.2) and the hypothesis that $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2(1+L)$. It follows that $T$ is a contraction, then by Banach fixed-point Theorem, $T$ has a unique fixed point. By the definition of $T$, we obtain that the delta integral equation (4.1) has a unique solution which is almost automorphic. □

In the sequel, let us present an example to illustrate our result.

**Example 4.2.** Consider the following integral equation on time scale

$$x(t) = \int_{-\infty}^{t} Ke_{\ominus\gamma}(t,\sigma(s))(x(s) + \eta(s)e^{-\alpha x(s)})\Delta s, \tag{4.2}$$

where $K$, $\gamma$ and $\alpha$ are real and positive constants.

Also, suppose that $\eta$ is an almost automorphic function. Therefore, it follows that $\eta$ is a bounded function, so denote $\tilde{\eta} = \sup_{t\in\mathbb{T}} |\eta(t)|$. Then, we have

$$
\begin{aligned}
|g(t,x(t)) - g(t,y(t))| &:= |\eta(t)e^{-\alpha x(t)} - \eta(t)e^{-\alpha y(t)}| \\
&\le \tilde{\eta}\alpha|x(t) - y(t)|.
\end{aligned}
$$

Therefore, $g$ satisfies the Lipschitz condition. Moreover, let us consider $2(1+\tilde{\eta}\alpha) < \dfrac{\gamma}{K(1+\tilde{\mu}\gamma)}$, where $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$.

Finally, it follows from Corollary 3.4 that $Ke_{\ominus\gamma}(t,\sigma(s))$ is an almost automorphic function with respect to both variables. Therefore, all the hypotheses of Theorem 4.1 are satisfied, then it follows that the delta integral equation (4.2) has a unique almost automorphic solution.

## 5. Asymptotically almost automorphic solutions of linear equations

In this section, we present a necessary condition for the existence of asymptotically almost automorphic solutions of the linear Volterra delta integral equation:

$$u(t) = \int_{t_0}^{t} a(t,\sigma(s))[u(s)+g(s)]\Delta s, \tag{5.1}$$

with $t_0 \in \mathbb{T}_+$, $t > t_0$, where $a : \mathbb{T}_+ \times \mathbb{T}_+ \to \mathbb{R}^{n\times n}$ is almost automorphic with respect to both variables and $g : \mathbb{T}_+ \to \mathbb{R}^n$ is asymptotically almost automorphic.

The definition of asymptotically almost automorphic functions on time scales makes clear why we are considering the Volterra integral equations on the form of the equation (5.1) instead of considering equations on the form of the equation (3.1). Also, let us recall the following hypothesis under the function $a : \mathbb{T}_+ \times \mathbb{T}_+ \to \mathbb{R}^{n\times n}$:

**(H1)** There exist real and positive constants $K$ and $\gamma$ such that

$$\int_{t_0}^{u} \|a(t,\sigma(s))\|\Delta s \leq \int_{t_0}^{u} Ke_{\ominus\gamma}(t,\sigma(s))\Delta s, \quad \text{for every } u,\ t,\ t_0 \in \mathbb{T}_+.$$

The following theorem ensures the existence of an asymptotically almost automorphic solution of (5.1). The proof is inspired in the proof of [22, Theorem 4.2] and the proof of [14, Lemma 2.6].

**Theorem 5.1.** *Suppose that the time scale $\mathbb{T}$ is invariant under translations, $a : \mathbb{T}_+ \times \mathbb{T}_+ \to \mathbb{R}^{n\times n}$ satisfies the hypothesis (H1) with $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2$, where $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$ and the function $g : \mathbb{T}_+ \to \mathbb{R}^n$ is asymptotically almost automorphic. Then, the integral equation* (5.1) *possesses an asymptotically almost automorphic solution.*

*Proof.* Since $g : \mathbb{T} \to \mathbb{R}^n$ is an asymptotically almost automorphic function, then we can write the function $g$ as follows:

$$g(t) = g_1(t) + g_2(t),$$

where $g_1$ and $g_2$ are, respectively, the principal and corrective terms of the function $g$. Thus, equation (5.1) can be written as:

$$u(t) = \int_{t_0}^{t} a(t,\sigma(s))[u(s)+g_1(s)+g_2(s)]\Delta s,$$

for every $t \in \mathbb{T}_+$, with $t > t_0$. Hence, by Theorem 3.6, we have

$$h_1(t) := \int_{t_0}^{t} a(t, \sigma(s))[u(s) + g_1(s)]\Delta s \in AA_{\mathbb{T}}(\mathbb{R}^n).$$

Therefore, it remains to prove that the function $h_2 : \mathbb{T} \to \mathbb{R}^n$ defined by

$$h_2(t) := \int_{t_0}^{t} a(t, \sigma(s))g_2(s)\Delta s,$$

for every $t \in \mathbb{T}_+$, with $t > t_0$, is an rd-continuous function and satisfies

$$\lim_{t\to+\infty} \|h_2(t)\| = 0.$$

Notice that from these facts, we obtain that the solution of (5.1) is asymptotically almost automorphic. The rd-continuity of the function $h_2$ follows immediately from the definition. Let us show that $\lim_{t\to+\infty} \|h_2(t)\| = 0$. In fact,

$$\begin{aligned} \|h_2(t)\| &= \left\| \int_{t_0}^{t} a(t, \sigma(s))g_2(s)\Delta s \right\| \\ &\leq \int_{t_0}^{t} \|a(t, \sigma(s))\| \|g_2(s)\|\Delta s \leq \int_{t_0}^{t} K e_{\ominus\gamma}(t, \sigma(s))\|g_2(s)\|\Delta s \end{aligned}$$

Since $\lim_{t\to+\infty} \|g_2(t)\| = 0$, then for every $\varepsilon > 0$, there exists a $T > 0$ sufficiently large such that for every $t > T$, we have

$$\|g_2(t)\| < \varepsilon.$$

Hence, we obtain

$$\begin{aligned} \|h_2(t)\| &\leq \int_{t_0}^{T} K e_{\ominus\gamma}(t, \sigma(s))\|g_2(s)\|\Delta s + \int_{T}^{t} K e_{\ominus\gamma}(t, \sigma(s))\|g_2(s)\|\Delta s \\ &\leq \tilde{g}K \frac{[e_{\ominus\gamma}(t, T) - e_{\ominus\gamma}(t, t_0)]}{|\ominus \gamma|} + \varepsilon K \frac{[e_{\ominus\gamma}(t, t) - e_{\ominus\gamma}(t, T)]}{|\ominus \gamma|}, \end{aligned}$$

where $\tilde{g} = \sup_{t\in[t_0,T]_{\mathbb{T}}} \|g(t)\|$. Applying the limit as $t \to +\infty$ and by Theorem 2.10, we have

$$\lim_{t\to+\infty} \|h_2(t)\| = \frac{\varepsilon K}{|\ominus \gamma|}.$$

Since $\varepsilon$ is arbitrary, the result follows as well. □

The following result gives us a characterization of the asymptotically almost automorphic solution of (5.1) whenever it exists. It is an immediate consequence of the previous theorem and Theorem 2.30.

**Corollary 5.2.** *If $u : \mathbb{T}_+ \to \mathbb{R}^n$ is an asymptotically almost automorphic solution of* (5.1) *and $g$ is an asymptotically almost automorphic function, then the principal and corrective terms of $u$ are given, respectively, by*

$$\int_{t_0}^{t} a(t, \sigma(s))[u(s) + g_1(s)]\Delta s \quad \textit{and} \quad \int_{t_0}^{t} a(t, \sigma(s))g_2(s)\Delta s,$$

*where $g_1$ and $g_2$ are, respectively, the principal and the corrective terms of $g$.*

Notice that from the previous corollary, the existence of an asymptotically almost automorphic solution of (5.1) implies the uniqueness of this solution by Theorem 2.30.

We finish this section by presenting an example to illustrate our result.

**Example 5.3.** Let us consider the following integral equation

$$x(t) = \int_{-\infty}^{t} Ke_{\ominus\alpha}(t, \sigma(s)) \left( x(s) + 2 + \frac{1}{\sigma(s)+1} \right) \Delta s, \tag{5.2}$$

where $\alpha,\ K > 0$ are real constants and $t_0 \in \mathbb{T}_+$. It is not difficult to see that

$$a(t, \sigma(s)) := Ke_{\ominus\alpha}(t, \sigma(s))$$

is almost automorphic with respect to both variables (see Corollary 3.4). Also, it is clear that the function satisfies the hypothesis **(H1)**.

Moreover, the function $g : \mathbb{T}_+ \to \mathbb{R}^n$ given by

$$g(s) = 2 + \frac{1}{\sigma(s)+1}$$

is clearly asymptotically almost automorphic, where the principal term is

$$g_1(s) = 2$$

and the corrective term is

$$g_2(s) = \frac{1}{\sigma(s)+1}.$$

Indeed, $g_2$ is rd-continuous (see [5, Theorem 1.6]) and $\lim_{s\to+\infty} \|g_2(s)\| = 0$, since $\sigma(s)+1$ is an increasing function. Assume that $\frac{\alpha}{K(1+\tilde{\mu}\alpha)} > 2$, where $\tilde{\mu} = \sup_{t\in\mathbb{T}} |\mu(t)|$. Therefore, all the hypotheses of Theorem 5.1 are satisfied, then the integral equation (5.2) has an asymptotically almost automorphic solution $u : \mathbb{T}_+ \to \mathbb{R}^n$ and by Corollary 5.2, the solution $u : \mathbb{T}_+ \to \mathbb{R}^n$ has the following principal and corrective terms

$$u_1(t) = \int_{t_0}^{t} Ke_{\ominus\alpha}(t, \sigma(s))(u(s) + 2)\Delta s$$

$$u_2(t) = \int_{t_0}^{t} Ke_{\ominus\alpha}(t, \sigma(s)) \left( \frac{1}{\sigma(s)+1} \right) \Delta s,$$

respectively.

## 6. Asymptotically almost automorphic solutions of semilinear equations

In this section, we present a sufficient condition for the existence of asymptotically almost automorphic solutions of the following semilinear Volterra delta integral equation:

$$u(t) = \int_{t_0}^{t} a(t, \sigma(s))[u(s) + f(s, u(s))]\Delta s, \tag{6.1}$$

with $t_0 \in \mathbb{T}_+$, $t > t_0$, where $a : \mathbb{T}_+ \times \mathbb{T}_+ \to \mathbb{R}^{n\times n}$ is almost automorphic with respect to both variables and $f : \mathbb{T}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is an almost automorphic function with respect to the first variable and satisfying the Lipschitz condition in the second variable.

To study the existence and uniquenes of asymptotically almost automorphic to the semilinear problem, we assume that the function $f$ satisfies a Lipschitz condition with respect to the second argument.

**Theorem 6.1.** *Consider the semilinear Volterra integral equation* (6.1) *and let* $\mathbb{T}$ *be an invariant under translations time scale. Assume that the function* $f : \mathbb{T}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is almost automorphic on time scales in* $t$ *for each* $x \in \mathbb{R}^n$ *and satisfies Lipschitz condition in* $x$ *uniformly in* $t$*, that is,*

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|, \tag{6.2}$$

*for all* $x,\ y \in \mathbb{R}^n$. *Moreover, suppose that the function* $a : \mathbb{T}_+ \times \mathbb{T}_+ \to \mathbb{R}^{n\times n}$ *is almost automorphic with respect to both variables and satisfies the hypothesis (H1) with the positive constants* $\gamma$ *and* $K$ *being such that* $\dfrac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2(1+L)$, *where* $L$ *is the Lipschitz constant of* $f$. *Then, there exists a unique asymptotically almost automorphic solution of* (6.1).

*Proof.* We start by defining an operator $T : AAA_{\mathbb{T}}(\mathbb{R}^n) \to AAA_{\mathbb{T}}(\mathbb{R}^n)$ as follows:

$$(Tu)(t) = \int_{t_0}^{t} a(t,\sigma(s))(u(s) + f(s,u(s))\Delta s,$$

for all $u \in AAA_{\mathbb{T}}(\mathbb{R}^n)$.

Let us show that $T$ is well-defined. In fact, since $u$ is an asymptotically almost automorphic function, we can write

$$u(t) = u_1(t) + u_2(t)$$

where $u_1$ and $u_2$ are, respectively, the principal and corrective terms of $u$. Then, we have

$$\begin{aligned}(Tu)(t) &= \int_{t_0}^{t} a(t,\sigma(s))[u(s) + f(s,u(s))]\Delta s \\ &= \int_{t_0}^{t} a(t,\sigma(s))[u_1(s) + u_2(s) + f(s,u(s))]\Delta s \\ &= \int_{t_0}^{t} a(t,\sigma(s))[u_1(s) + f(s,u_1(s))]\Delta s \\ &+ \int_{t_0}^{t} a(t,\sigma(s))[u_2(s) + f(s,u(s)) - f(s,u_1(s))]\Delta s.\end{aligned}$$

By Theorem 4.1, we have

$$\int_{t_0}^{t} a(t,\sigma(s))[u_1(s) + f(s,u_1(s))]\Delta s \in AA_{\mathbb{T}}(\mathbb{R}^n).$$

Hence, it remains to prove that

$$h(t) := \int_{t_0}^{t} a(t,\sigma(s))[u_2(s) + f(s,u(s)) - f(s,u_1(s))]\Delta s,$$

for every $t \in \mathbb{T}_+$, with $t > t_0$ is an rd-continuous function and satisfies

$$\lim_{t\to+\infty} \|h(t)\| = 0.$$

The rd-continuity of the function $h$ follows directly by the definition. Let us show that $\lim_{t\to+\infty} \|h(t)\| = 0$:

$$\begin{aligned}
\|h(t)\| &= \left\| \int_{t_0}^{t} a(t,\sigma(s))[u_2(s) + f(s,u(s)) - f(s,u_1(s))]\Delta s \right\| \\
&\leq \int_{t_0}^{t} \|a(t,\sigma(s))\| \|u_2(s) + f(s,u(s)) - f(s,u_1(s))\| \Delta s \\
&\leq \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s))[\|u_2(s)\| + \|f(s,u(s)) - f(s,u_1(s))\|]\Delta s \\
&\leq \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s))[\|u_2(s)\| + L\|u(s) - u_1(s)\|]\Delta s \\
&= \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s))[\|u_2(s)\| + L\|u_2(s)\|]\Delta s \\
&= (1+L) \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s)) \|u_2(s)\| \Delta s.
\end{aligned}$$

Using a similar argument as in the proof of Theorem 5.1, one can conclude that $\lim_{t\to+\infty} \|h(t)\| = 0$. Therefore, the operator $T$ is well-defined. Now, let us show that $T$ is a contraction.

$$\begin{aligned}
\|Tu - Tv\| &= \left\| \int_{t_0}^{t} a(t,\sigma(s))[u(s) - v(s) + f(s,u(s)) - f(s,v(s))]\Delta s \right\| \\
&\leq \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s)) \left[\|u(s) - v(s)\| + L\|u(s) - v(s)\|\right] \Delta s \\
&= \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s)) \left[(1+L)\|u(s) - v(s)\|\right] \Delta s \\
&\leq \|u - v\|_\infty \int_{t_0}^{t} K e_{\ominus\gamma}(t,\sigma(s))(1+L)\Delta s \\
&= \|u - v\|_\infty (1+L) \left[ \frac{K}{|\ominus\gamma|} \|e_{\ominus\gamma}(t,t_0) - e_{\ominus\gamma}(t,t) \right] \\
&\leq \|u - v\|_\infty (1+L) \left[ \frac{K}{|\ominus\gamma|} 2 \right] \\
&= \|u - v\|_\infty (1+L) \frac{2K(1+\mu(t)\gamma)}{\gamma}
\end{aligned}$$

$$\leq \quad \|u-v\|_\infty \frac{2K(1+\tilde{\mu}\gamma)}{\gamma}(1+L).$$

Therefore, $T$ is a contraction, then by Banach fixed point Theorem, $T$ has a unique fixed point. By the definition of $T$, we obtain that the delta integral equation (6.1) has a unique solution which is asymptotically almost automorphic. □

We notice that the condition $\frac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2$ in Theorem 5.1 allows to prove that the solution to equation (5.1) is asymptotically almost automorphic, and the assumption $\frac{\gamma}{K(1+\tilde{\mu}\gamma)} > 2(1+L)$ in Theorem 6.1 gives a sufficient condition to apply the Banach fixed point theorem to ensure the existence and uniqueness of asymptotically almost automorphic solutions to Eq. (6.1).

## References


[1] M. Adıvar and Y. N. Raffoul. Existence results for periodic solutions of integro-dynamic equations on time scales. *Ann. Mat. Pura Appl. (4)*, 188(4):543–559, 2009.

[2] M. Adivar and Y. N. Raffoul. Existence of resolvent for Volterra integral equations on time scales. *Bull. Aust. Math. Soc.*, 82(1):139–155, 2010.

[3] J. A. D. Appleby, I. Győri, and D. W. Reynolds. On exact rates of decay of solutions of linear systems of Volterra equations with delay. *J. Math. Anal. Appl.*, 320(1):56–77, 2006.

[4] F. M. Atici, D. C. Biles, and A. Lebedinsky. An application of time scales to economics. *Math. Comput. Modelling*, 43(7-8):718–726, 2006.

[5] M. Bohner and A. Peterson. *Dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[6] M. Bohner and A. Peterson, editors. *Advances in dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[7] D. Brigo and F. Mercurio. Discrete time vs continuous time stock-price dynamics and implications for option pricing. *Finance and Stochastics*, 4:147–159, 2000.

[8] J. Caballero, B. López, and K. Sadarangani. On monotonic solutions of an integral equation of Volterra type with supremum. *J. Math. Anal. Appl.*, 305(1):304–315, 2005.

[9] J. Campos and M. Tarallo. Almost automorphic linear dynamics by Favard theory. *J. Differential Equations*, 256(4):1350–1367, 2014.

[10] T. Caraballo and D. Cheban. Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I. *J. Differential Equations*, 246(1):108–128, 2009.

[11] T. Caraballo and D. Cheban. Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. II. *J. Differential Equations*, 246(3):1164–1186, 2009.

[12] T. Caraballo and D. Cheban. Levitan/Bohr almost periodic and almost automorphic solutions of second order monotone differential equations. *J. Differential Equations*, 251(3):708–727, 2011.

[13] F. B. Christiansen and T. M. Fenchel. *Theories of populations in biological communities*. Springer-Verlag, Berlin, 1977.

[14] H.-S. Ding, T.-J. Xiao, and J. Liang. Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions. *J. Math. Anal. Appl.*, 338(1):141–151, 2008.

[15] S. R. Grace, J. R. Graef, S. Panigrahi, and E. Tunc. On the oscillatory behavior of Volterra integral equations on time-scales. *Panamer. Math. J.*, 23(2):35–41, 2013.

[16] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[17] M. L. Heard. On asymptotic behavior and periodic solutions of a certain Volterra integral equation. *J. Differential Equations*, 6:172–186, 1969.
[18] I. Klapper and H. Qian. Remarks on discrete and continuous large-scale models of dna dynamics. *Biophysical Journal*, 74:2504–2514, 1998.
[19] T. Kulik and C. C. Tisdell. Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains. *Int. J. Difference Equ.*, 3(1):103–133, 2008.
[20] Y. Li and C. Wang. Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales. *Adv. Difference Equ.*, pages 2012:77, 24, 2012.
[21] Y. Li and L. Yang. Almost automorphic solution for neutral type high-order hopfield neural networks with delays in leakage terms on time scales. *Applied Mathematics and Computation*, 242:679–693, 2014.
[22] C. Lizama and J. G. Mesquita. Asymptotically almost automorphic solutions of dynamic equations on time scales. *submitted*.
[23] C. Lizama and J. G. Mesquita. Almost automorphic solutions of dynamic equations on time scales. *J. Funct. Anal.*, 265(10):2267–2311, 2013.
[24] S.-O. Londen. On a nonlinear Volterra integral equation. *J. Differential Equations*, 14:106–120, 1973.
[25] G. Mophou, G. N'Guérékata, and A. Milce. Almost automorphic functions of order n and applications to dynamic equations on time scales. *Discrete Dynamics in Nature and Society*, 2014(410210), 2014.
[26] G. M. N'Guérékata. *Topics in almost automorphy*. Springer-Verlag, New York, 2005.
[27] J. Prüss. *Evolutionary integral equations and applications*, volume 87 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1993.
[28] C. C. Tisdell and A. Zaidi. Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling. *Nonlinear Anal.*, 68(11):3504–3524, 2008.
[29] C. Wang. Almost periodic solutions of impulsive bam neural networks with variable delays on time scales. *Communications in Nonlinear Science and Numerical Simulation*, 19(8):2828–2842, 2014.

Universidad de Santiago de Chile, Facultad de Ciencia, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, CHILE
*E-mail address*: carlos.lizama@usach.cl

Universidade de Brasilia, Departamento de Matemática, Campus Universitario Darcy Ribeiro, Asa Norte 70910-900, Brasilia-DF, BRAZIL
*E-mail address*: jgmesquita@unb.br

Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca, CHILE
*E-mail address*: rponce@inst-mat.utalca.cl

Universidade Federal de Juiz de Fora, Instituto de Ciências Exatas, Departamento de Matemática, 36036-900, Juiz de Fora MG, BRAZIL
*E-mail address*: eduard.toon@ufjf.edu.br