A CONNECTION BETWEEN ALMOST PERIODIC FUNCTIONS DEFINED ON TIME SCALES AND $\mathbb{R}$

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Abstract. In this paper, we prove a strong connection between almost periodic functions on time scales and almost periodic functions on $\mathbb{R}$. An application to difference equations on $\mathbb{T} = h\mathbb{Z}$ is given.

1. Introduction

In the last few decades, the theory of discrete processes has been attracting the attention of several researchers. Basic problems such as stability and other asymptotic behavior, the presence of oscillations in such systems, and many other features related to the solutions of difference equations have been investigated successfully by many scientists. The applications of such topics to various fields have been a concern of researchers, and a rich body of literature has appeared relatively recently. See, for instance [1, 4, 5, 8, 9, 10, 11, 12, 13, 20, 24, 25].

It is interesting to point out the fact that a certain parallelism takes place between the theory of almost periodic functions in the case of continuous time and that in the case of discrete time. In [6, Theorem 1.27] (see also [7, Proposition 3.35]), C. Corduneanu investigated a surprising and strong connection between almost periodic functions defined on $\mathbb{Z}$ and almost periodic functions defined on $\mathbb{R}$, namely: A necessary and sufficient condition for $f \in AP(\mathbb{Z})$ is the existence of a function $\varphi \in AP(\mathbb{R})$ such that $f(n) = \varphi(n), n \in \mathbb{Z}$. From this fact, a natural and more general open question which appears is the following: Let $\mathbb{T}$ be an arbitrary time scale, i.e. a closed and nonempty subset of $\mathbb{R}$. For a given almost periodic function $g : \mathbb{T} \to \mathbb{R}$, is there an almost periodic function $F_g : \mathbb{R} \to \mathbb{R}$ such that the restriction of F_g to $\mathbb{T}$ coincides with g? In this paper, we will answer this question affirmatively. In addition, we prove that this fact turns out to be a very useful tool to provide criteria for the almost periodicity of solutions to diamond-α dynamic equations on time scales. Indeed, we will prove the following (Theorem 4.1): Let $g \in AP(h\mathbb{Z})$ be given and suppose that $x(t), t \in \mathbb{R}$ is a solution of the problem $x'(t) = Ax(t) + F_g(t), t \in \mathbb{R}$. Then the restriction of $x(t)$ to the set $h\mathbb{Z}$ is the solution of the equation:

$$x^{\diamondsuit_\alpha}(t) = Ax(t) + g(t), \quad t \in h\mathbb{Z}$$

in case $\alpha = 1/2$.

We observe that the concept of almost periodic functions on time scales is very recent in the literature and it was introduced in [14]. After that, several papers were written about

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this topic, since it has be shown a powerful tool for applications. We can cite, for instance, the applications of this theory in the study of Hopfield neural networks with delays on time scales. See, for instance, [14, 15, 16, 17].

The outline of this paper is as follows: In the second section, we present some basic concepts concerning the theory of time scales. In the third section, we prove some auxiliaries lemmas and our main result. Finally, we present an application to difference equations on $\mathbb{T} = h\mathbb{Z}$ in the last section of this paper.

2. Preliminaries

In this section, we present some basic concepts and results about the theory of time scales. They will be essential to prove our main result. For details, see [2], [3], [21] and [22].

Let $\mathbb{T}$ be a time scale, that is, closed and nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, we define the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T},\ s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is *right-scattered*. Otherwise, t is called *right-dense*. Analogously, if $\rho(t) < t$, then t is called *left-scattered* whereas if $\rho(t) = t$, then t is *left-dense*.

We also define the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$ and the *backward graininess function* $\nu : \mathbb{T} \to \mathbb{R}^+$, respectively, by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t).$$

Definition 2.1. [2] A function $f : \mathbb{T} \to \mathbb{R}$ continuous at each right-dense point and at each left-dense point is said to be *continuous* on $\mathbb{T}$. We denote by $C(\mathbb{T}, \mathbb{R})$ the set of all continuous functions $f : \mathbb{T} \to \mathbb{R}$.

Given a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T};\ a \le t \le b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R};\ a \le t \le b\}$.

We define the set $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$ which are derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum n, then $\mathbb{T}_\kappa = \mathbb{T} - \{n\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, we define $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. See [2] and [21] for more information related with the above sets.

Definition 2.2. [2] For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* of f at t to be the number $f^\Delta(t)$ (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(t) - f^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

In the sequel, we present the definition of nabla derivatives similarly as before.

Definition 2.3. [2] For $t \in \mathbb{T}_\kappa$, we define the *nabla-derivative* of f at t to be the number $f^\nabla(t)$, if one exists, such that for all $\varepsilon > 0$, there exists a neighborhood V of t such that for all $s \in V$,

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \varepsilon|\rho(t) - s|.$$

We say that f is *nabla differentiable* on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

For example, note that in the case $\mathbb{T} = \mathbb{R}$, then $f^\nabla(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\nabla(t) = f(t) - f(t-1)$. Now, we introduce the diamond-α derivative. For details, see [21] and [22] .

Definition 2.4. Let $\mu_{ts} = \sigma(t) - s$ and $\eta_{ts} = \rho(t) - s$, then we define $f^{\diamond_\alpha}(t)$ to be the value, if one exists, such that for all $\varepsilon > 0$, there exists a neighborhood U of t such that for all $s \in U$, we have

$$|\alpha[f^\sigma(t) - f(s)]\eta_{ts} + (1-\alpha)[f^\rho(t) - f(s)]\mu_{ts} - f^{\diamond_\alpha}(t)\mu_{ts}\eta_{ts}| < \varepsilon|\mu_{ts}\eta_{ts}|.$$

A function f is said *diamond-α differentiable* on $\mathbb{T}_\kappa^\kappa$ provided $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}_\kappa^\kappa$. Let $0 \le \alpha \le 1$. If f is differentiable on $t \in \mathbb{T}_\kappa^\kappa$ both in the delta and nabla senses, then f is diamond-α differentiable at t and the dynamic derivative $f^{\diamond_\alpha}(t)$ is given by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1-\alpha)f^\nabla(t).$$

In what follows, we present a definition of time scale which are invariant under translations. See [18].

Definition 2.5. A time scale $\mathbb{T}$ is called *invariant under translations* if

$$\Pi := \{\tau \in \mathbb{R} : \quad t \pm \tau \in \mathbb{T}, \quad \forall t \in \mathbb{T}\} \neq \{0\}.$$

The following three results describe important properties of the points in $\mathbb{T}$, forward jump operator and of the graininess function, respectively, whenever the time scale is invariant under translations. They can be found in [19]. We repeat the proofs here for the reader's convenience.

Lemma 2.6. *Let $\mathbb{T}$ be an invariant under translations time scales. If t is right-dense, then for every $h \in \Pi$, $t + h$ is right-dense. Similarly, if t is right-scattered, then for every $h \in \Pi$, $t + h$ is right-scattered.*

Proof. In fact, if t is right-dense, then there exists $t_n \in \mathbb{T}$ such that

$$\lim_{n\to\infty} t_n = t \quad \text{and} \quad t_n > t.$$

Since $\mathbb{T}$ is invariant under translations, we obtain

$$t_n + h \in \mathbb{T} \quad \text{and} \quad t_n + h > t + h,$$

for every $h \in \Pi$. Then, we have

$$\lim_{n\to\infty} t_n + h = t + h$$

and, thus, it follows that $\sigma(t+h) = t + h$, which implies that $t + h$ is right-dense.

Now, suppose that t is right-scattered and $t + h$ is right-dense for some $h \in \Pi$. Then, there exists $t_n + h \in \mathbb{T}$ such that

$$\lim_{n\to\infty} t_n + h = t + h \quad \text{and} \quad t_n + h > t + h.$$

By the invariance of the time scale, it follows that $t_n \in \mathbb{T}$. And by the definition of t_n, we have

$$t_n > t \quad \text{and} \quad \lim_{n\to\infty} t_n = t.$$

It implies that $\sigma(t) = t$, contradicting the fact that t is right-scattered. Therefore, $t + h$ is right-scattered for every $h \in \Pi$. $\square$

Lemma 2.7. *If $\mathbb{T}$ is an invariant under translation time scale and $h \in \Pi$, then*

$$\sigma(t) + h = \sigma(t + h) \quad \textit{and} \quad \sigma(t) - h = \sigma(t - h),$$

for every $t \in \mathbb{T}$.

Proof. If t is right-dense, it follows immediately, by Lemma 2.6, the equalities

$$\sigma(t) + h = \sigma(t + h) \quad \text{and} \quad \sigma(t) - h = \sigma(t - h),$$

for every $t \in \mathbb{T}$.

On the other hand, if t is right-scattered, then $t + h$, for $h \in \Pi$, is also right-scattered, by Lemma 2.6, and $\sigma(t + h) > t + h$, by the definition. Then,

$$\sigma(t + h) - h > t, \quad \text{for every} \quad t \in \mathbb{T}. \tag{2.1}$$

Since $\mathbb{T}$ is invariant under translations and $h \in \Pi$, we have that $\sigma(t+h)-h \in \mathbb{T}$. Thus, by equation (2.1) and the definition of forward jump operator, it follows that $\sigma(t) \leq \sigma(t+h)-h$, which implies $\sigma(t + h) \geq \sigma(t) + h$.

Reciprocally, since t is right-scattered, we have that $\sigma(t) > t$. Then,

$$\sigma(t) + h > t + h. \tag{2.2}$$

Again, it is clear that $\sigma(t) + h \in \mathbb{T}$, then by (2.2) and from the definition of forward jump operator, it follows that $\sigma(t) + h \geq \sigma(t + h)$, for every $t \in \mathbb{T}$.

Combining these two inequalities, we obtain the desired result. Similarly, one can prove the other equality $\sigma(t) - h = \sigma(t - h)$. $\square$

Corollary 2.8. *If $\mathbb{T}$ is an invariant under translations time scale and $h \in \Pi$, then*

$$\mu(t + h) = \mu(t) = \mu(t - h),$$

for every $t \in \mathbb{T}$.

Proof. In fact, by the definition of the graininess function and by Lemma 2.7, we have

$$\mu(t + h) = \sigma(t + h) - (t + h) = \sigma(t + h) - t - h = \sigma(t) + h - t - h = \mu(t),$$

for every $h \in \Pi$ and $t \in \mathbb{T}$. Similarly, we have

$$\mu(t - h) = \sigma(t - h) - (t - h) = \sigma(t) - h - t + h = \mu(t)$$

for every $t \in \mathbb{T}$ and $h \in \Pi$. $\square$

Throughout this paper, $\mathbb{E}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $\|\cdot\|$ be the norm in $\mathbb{E}^n$.

In what follows, we present a definition of almost periodic function on time scales, which can be found in [14].

Definition 2.9. Let $\mathbb{T}$ be an invariant under translation time scale. A function $f \in C(\mathbb{T}, \mathbb{E}^n)$ is called an *almost periodic function on time scale* if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\tau \in \Pi; \|f(t + \tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{T}\}$$

is relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$. In other words, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau = \tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$\|f(t+\tau) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{T},$$

where τ is called ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

Also, we have the following definition of a normal function on time scales, which was introduced in [14].

Definition 2.10. Let $\mathbb{T}$ be invariant under translations. Then, a continuous function f is called *normal* if, for any sequence $(\alpha_n') \subset \Pi$ there exists a subsequence $(\alpha_n) \subset (\alpha_n')$ such that $\lim_{n\to\infty} f(t+\alpha_n)$ exists uniformly on $\mathbb{T}$.

In [14], the authors proved that the Definitions 2.9 and 2.10 are equivalent. The result which describes this equivalence follows for the case of a function $f : \mathbb{T} \to \mathbb{E}^n$. For details, see [14], Theorems 3.13 and 3.14.

Theorem 2.11. *A continuous function $f : \mathbb{T} \to \mathbb{E}^n$ is normal, if and only if, the function f is almost periodic in $t \in \mathbb{T}$.*

Remark 2.12. We point out that the equivalence described in Theorem 2.11 is well-known for the case $\mathbb{T} = \mathbb{R}$ and it was proved previously. See [6], Theorems 1.9, 1.10 and 1.11, for instance.

3. Main result

In this section, our goal is to prove a result which describes an interesting connection between almost periodic functions on $\mathbb{T}$ and almost periodic functions on $\mathbb{R}$.

In order to present our result, we need to introduce the following notation. Similarly as in [23], for a given $t \in \mathbb{T}$ we define t_* by

$$t_* = \sup\{s \in \mathbb{T}; s \le t\}.$$

In the sequel, we present some auxiliaries results which will be fundamental to prove our main theorem and that can be of independent interest.

Lemma 3.1. *If $\mathbb{T}$ is an invariant under translations time scale such that $\mathbb{T} \neq \mathbb{R}$ and if $t \in \mathbb{R} \setminus \mathbb{T}$, then $t + p \in \mathbb{R} \setminus \mathbb{T}$, for every $p \in \Pi$.*

Proof. Suppose by contradiction that $t + p \in \mathbb{T}$. Then, by invariance of $\mathbb{T}$, we obtain

$$(t+p) - p \in \mathbb{T},$$

which implies that $t \in \mathbb{T}$, contradicting the hypothesis. □

Lemma 3.2. *If $\mathbb{T}$ is an invariant under translations time scale, then $(t+p)_* = t_* + p$, for every $p \in \Pi$, for every $t \in \mathbb{R}$.*

Proof. If $t \in \mathbb{T}$, it follows immediately by invariance of $\mathbb{T}$ that $t + p \in \mathbb{T}$. Also, we get

$$(t+p)_* = t + p = t_* + p.$$

Now, suppose that $t \in \mathbb{R} \setminus \mathbb{T}$, then by Lemma 3.1, $t + p \in \mathbb{R} \setminus \mathbb{T}$. Thus, we get

$$t_* + p \le \sup\{s \in \mathbb{T}; s \le t + p\} = (t+p)_*, \tag{3.1}$$

since $t_* + p < t + p$, by the definition of t_* and by using the fact that $t_* + p \in \mathbb{T}$, by the invariance of $\mathbb{T}$.

Suppose the contrary, i.e., $(t+p)_* \neq t_* + p$. By equation (3.1), it means to suppose that $(t+p)_* > t_* + p$. Also, denote by

$$\gamma := (t+p)_* = \sup\{s \in \mathbb{T}; s \leq t + p\}.$$

Therefore, by the definition of γ and using the fact that $t + p \in \mathbb{R} \setminus \mathbb{T}$, we get

$$t_* + p < \gamma < t + p$$

which implies that

$$t_* < \gamma - p < t.$$

It is a contradiction by the definition of t_*, because $\gamma - p \in \mathbb{T}$, by the invariance of the time scale. Thus, $t_* + p = \gamma$. $\square$

Lemma 3.3. *If $\mathbb{T}$ is invariant under translations and $t \in \mathbb{R} \setminus \mathbb{T}$, then t_* is right-scattered.*

Proof. Suppose that t_* is right-dense, then there exists

$$t_n = t_* + \frac{1}{n} \in \mathbb{T}$$

such that

$$t_n > t_* \quad \text{and} \quad \lim_{n\to\infty} t_n = t_*.$$

Since $t \in \mathbb{R} \setminus \mathbb{T}$, then $t > t_* + \frac{1}{n} = t_n$, but it contradicts the definition of t_* and we have the desired result. $\square$

Finally, we present our main result.

Theorem 3.4. *If $\mathbb{T}$ is invariant under translations, a necessary and sufficient condition for a continuous function $g : \mathbb{T} \to \mathbb{E}^n$ to be almost periodic on $\mathbb{T}$ is the existence of an almost periodic function $f : \mathbb{R} \to \mathbb{E}^n$ such that $f(t) = g(t)$ for every $t \in \mathbb{T}$.*

Proof. The condition is sufficient, because if such function exists, then by Theorem 2.11 for the case $\mathbb{T} = \mathbb{R}$, we obtain that for any sequence $\{f(t + \alpha_n)\}$, $\alpha_n \in \Pi$, one can extract a subsequence $\{f(t + \alpha_{n_k})\}$ converging uniformly for $t \in \mathbb{R}$. Consequently, taking $t \in \mathbb{T}$, the sequence $\{f(t + \alpha_{n_k})\}$ also converges uniformly with respect to t as $k \to \infty$. Then, $f : \mathbb{T} \to \mathbb{E}^n$ is a normal function, which implies, by Theorem 2.11, that $f : \mathbb{T} \to \mathbb{E}^n$ is almost periodic on $\mathbb{T}$.

Reciprocally, suppose that a continuous function $g : \mathbb{T} \to \mathbb{E}^n$ is almost periodic on $\mathbb{T}$. Then, define a function $f : \mathbb{R} \to \mathbb{E}^n$ as follows:

$$f(t) = \begin{cases} \left(1 - \dfrac{(t - t_*)}{\mu(t_*)}\right) g(t_*) + \dfrac{t - t_*}{\mu(t_*)} g(\sigma(t_*)), & t \in \mathbb{R} \setminus \mathbb{T}, \\ g(t), \quad t \in \mathbb{T}. \end{cases}$$

By Lemma 3.3, it follows that the function f is well-defined.

Let us prove that the function f is continuous on $\mathbb{R}$. In order to do this, consider two cases:

(i) $t \in \mathbb{R} \setminus \mathbb{T}$;
(ii) $t \in \mathbb{T}$;

If $t \in \mathbb{R}\setminus\mathbb{T}$, then we can construct a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t$ uniformly. Notice that there exists $n_0 \in \mathbb{N}$ sufficiently large such that for $n > n_0$, we obtain that $t_n \in \mathbb{R}\setminus\mathbb{T}$.

Indeed, notice that any sequence which converges to t does not belong to $\mathbb{T}$, since $\mathbb{T}$ is a closed subset of $\mathbb{R}$ and $t \in \mathbb{R}\setminus\mathbb{T}$. Moreover, it is not difficult to see that there exists $n_0 \in \mathbb{N}$ sufficiently large such that for $n > n_0$, we have $(t_n)_* = t_*$.

Therefore, we obtain

$$\begin{aligned}
\lim_{n\to\infty} f(t_n) &= \lim_{n\to\infty}\left(1-\frac{(t_n-(t_n)_*)}{\mu((t_n)_*)}\right)g((t_n)_*)+\frac{t_n-(t_n)_*}{\mu((t_n)_*)}g(\sigma((t_n)_*))\\
&= \lim_{n\to\infty}\left(1-\frac{(t_n-t_*)}{\mu(t_*)}\right)g(t_*)+\frac{t_n-t_*}{\mu(t_*)}g(\sigma(t_*))\\
&= \left(1-\frac{(t-t_*)}{\mu(t_*)}\right)g(t_*)+\frac{t-t_*}{\mu(t_*)}g(\sigma(t_*))\\
&= f(t),
\end{aligned}$$

which implies that f is continuous on t.

Now, let us consider that $t \in \mathbb{T}$. We can consider two cases: t is right-scattered or t is right-dense.

At first, consider t is right-scattered. Notice that $t \in \mathbb{T}$, which implies that $t = t_*$. By the fact that t is right-scattered, we obtain that there exists a sequence $(t_n) \in \mathbb{R}\setminus\mathbb{T}$ such that $t_n \to t$ uniformly, satisfying $t_n > t$ and $(t_n)_* = t_* = t$. Thus, we get

$$\begin{aligned}
f(t_n) &= \left(1-\frac{(t_n-(t_n)_*)}{\mu((t_n)_*)}\right)g((t_n)_*)+\frac{t_n-(t_n)_*}{\mu((t_n)_*)}g(\sigma((t_n)_*))\\
&= \left(1-\frac{(t_n-t_*)}{\mu(t_*)}\right)g(t_*)+\frac{t_n-(t_*)}{\mu((t_*)}g(\sigma(t_*))\\
&= \left(1-\frac{(t_n-t)}{\mu(t)}\right)g(t)+\frac{t_n-t}{\mu((t)}g(\sigma(t)).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\lim_{n\to\infty} f(t_n) &= \lim_{n\to\infty}\left(1-\frac{(t_n-t)}{\mu(t)}\right)g(t)+\frac{t_n-t}{\mu((t)}g(\sigma(t))\\
&= g(t)=f(t).
\end{aligned}$$

Thus, f is continuous on t. Now, consider that t is right-dense, then the continuity of the function f follows immediately from the continuity of the function g, using the definition of f for this specific case.

Now, let $p \in \Pi$. Notice that, by the Definition 2.9, we have that $p \in E\{\varepsilon, g\}$, since g is an almost periodic function. Then, by Lemma 3.1, we have to consider two cases:

Case 1: $t, t+p \in \mathbb{R}\setminus\mathbb{T}$;
Case 2: $t, t+p \in \mathbb{T}$.

We start by considering the Case 1, that is, $t, t+p \in \mathbb{R} \setminus \mathbb{T}$. Then, we get

$$\begin{aligned} f(t+p) - f(t) &= g((t+p)_*) - \frac{(t+p-(t+p)_*)}{\mu((t+p)_*)} g((t+p)_*) \\ &+ \frac{(t+p-(t+p)_*)}{\mu((t+p)_*)} g(\sigma((t+p)_*)) - g(t_*) \\ &+ \frac{(t-t_*)}{\mu(t_*)} g(t_*) - \frac{(t-t_*)}{\mu(t_*)} g(\sigma(t_*)). \end{aligned}$$

By Lemma 3.2, we obtain

$$\begin{aligned} f(t+p) - f(t) &= g(t_*+p) - \frac{(t-t_*)}{\mu(t_*+p)} g(t_*+p) + \frac{(t-t_*)}{\mu(t_*+p)} g(\sigma(t_*+p)) + \\ &- g(t_*) + \frac{(t-t_*)}{\mu(t_*)} g(t_*) - \frac{(t-t_*)}{\mu(t_*)} g(\sigma(t_*)). \end{aligned}$$

Then, we have

$$\begin{aligned} f(t+p) - f(t) &= [g(t_*+p) - g(t_*)] + \frac{(t-t_*)}{\mu(t_*+p)} [g(\sigma(t_*+p)) - g(t_*+p)] + \\ &- \frac{(t-t_*)}{\mu(t_*)} [g(\sigma(t_*)) - g(t_*)] \\ &= [g(t_*+p) - g(t_*)] + \frac{(t-t_*)}{\mu(t_*+p)} [g(\sigma(t_*+p)) - g(\sigma(t_*))] + \\ &- \frac{(t-t_*)}{\mu(t_*)} [g(t_*+p) - g(t_*)] \\ &+ (t-t_*) \left[\frac{1}{\mu(t_*+p)} - \frac{1}{\mu(t_*)} \right] (g(\sigma(t_*)) - g(t_*+p)) \end{aligned}$$

Since $t \in \mathbb{R} \setminus \mathbb{T}$ and by Lemma 3.3, we get

$$t_* < t < \sigma(t_*),$$

which implies

$$0 < t - t_* < \sigma(t_*) - t_* = \mu(t_*) \quad \Rightarrow \quad 0 < \frac{t-t_*}{\mu(t_*)} < 1.$$

Similarly, by Lemmas 2.6 and 3.3, we have

$$t_* + p < t + p < \sigma(t_*+p) \quad \Rightarrow \quad 0 < t - t_* < \sigma(t_*+p) - (t_*+p) = \mu(t_*+p)$$

and thus,

$$0 < \frac{t-t_*}{\mu(t_*+p)} < 1.$$

Combining these facts and the equality above, we obtain

$$\|f(t+p) - f(t)\| \leq \|g(t_*+p) - g(t_*)\| + \left| \frac{t-t_*}{\mu(t_*+p)} \right| \|g(\sigma(t_*+p)) - g(\sigma(t_*))\| +$$

$$+ \left\| \frac{(t-t_*)}{\mu(t_*)} (g(t_*+p) - g(t_*)) \right\| + \left\| (t-t_*) \left[\frac{1}{\mu(t_*+p)} - \frac{1}{\mu(t_*)} \right] (g(\sigma(t_*)) - g(t_*+p)) \right\|.$$

Therefore, using Lemma 2.7, Corollary 2.8 and by the almost periodicity of the function g, we get

$$\begin{aligned} \|f(t+p)-f(t)\| &< \varepsilon+\varepsilon+\varepsilon+\left\|(t-t_*)\frac{(\mu(t_*)-\mu(t_*+p))}{\mu(t_*+p)\mu(t_*)}(g(\sigma(t_*))-g(t_*+p))\right\| \\ &< \varepsilon+\varepsilon+\varepsilon=3\varepsilon, \end{aligned}$$

since $\mu(t_*+p)=\mu(t_*)$ by Corollary 2.8.

Now, consider the second case. If $t, t+p\in\mathbb{T}$, then by the definition of function f, we get

$$|f(t+p)-f(t)|\leq|g(t+p)-g(t)|<\varepsilon,$$

using the almost periodicity of the function g.

Combining these two cases, we obtain that f is almost periodic on $\mathbb{R}$, by definition and the desired result follows as well. $\square$

4. Applications

One of the most common sources for discrete processes is the theory of difference equations. In this section, our goal is to present an application of our main result presented in the previous Section to a difference equation defined on the set $h\mathbb{Z}$.

Theorem 4.1. *Let A be a nonsingular square matrix of order n. Let $g: h\mathbb{Z}\to\mathbb{E}^n$ be an almost periodic function, then if $x:\mathbb{R}\to\mathbb{E}^n$ is the almost periodic solution of*

$$x'(t)=-Ax(t)+f(t),\quad t\in\mathbb{R}, \tag{4.1}$$

where $f:\mathbb{R}\to\mathbb{E}^n$ is defined by

$$f(t)=\begin{cases}\left(1-\dfrac{(t-t_*)}{\mu(t_*)}\right)g(t_*)+\dfrac{t-t_*}{\mu(t_*)}g(\sigma(t_*)), & t\in\mathbb{R}\setminus h\mathbb{Z},\\ g(t), \quad t\in h\mathbb{Z}, & \end{cases}$$

then the restriction of x to the time scale $\mathbb{T}=h\mathbb{Z}$, i.e. $x: h\mathbb{Z}\to\mathbb{E}^n$, is the almost periodic solution of

$$x^{\diamond_\alpha}(t)=-Ax(t)+g(t),\quad t\in h\mathbb{Z}, \tag{4.2}$$

for $\alpha=\frac{1}{2}$.

Proof. It is well known and easy to prove, that

$$x(t)=\int_{-\infty}^{t}e^{-A(t-s)}f(s)ds \tag{4.3}$$

defines the unique solution of (4.1) for t on $\mathbb{R}$.

Since $h\mathbb{Z}$ is an enumerable set, we have, by integral property,

$$\int_{(-\infty,t]}e^{-A(t-s)}f(s)ds=\int_{(-\infty,t]\setminus h\mathbb{Z}}e^{-A(t-s)}f(s)ds.$$

Using this fact, without loss of generality, we can calculate $x(t)$ considering only the values of the function $f(t)$ for $t \in \mathbb{R} \setminus h\mathbb{Z}$ in the formula (4.3). Thus, we obtain:

$$\begin{aligned}
x(t) &= \int_{-\infty}^{t} e^{-A(t-s)} f(s)ds \\
&= \int_{-\infty}^{t} e^{-A(t-s)} \left[\left(1 - \frac{s-n}{h}\right) g(n) + \frac{s-n}{h} g(n+h) \right] ds \\
&= g(n) \int_{-\infty}^{t} e^{-A(t-s)} ds + \int_{-\infty}^{t} e^{-A(t-s)} \frac{s-n}{h} [g(n+h) - g(n)] ds \\
&= A^{-1} g(n) + \frac{(g(n+h) - g(n))}{h} \int_{-\infty}^{t} e^{-A(t-s)} s ds - \frac{n(g(n+h) - g(n))}{h} \int_{-\infty}^{t} e^{-A(t-s)} ds \\
&= A^{-1} g(n) + \frac{(g(n+h) - g(n))}{h} \int_{0}^{\infty} (t-r) e^{-Ar} dr - \frac{(g(n+h) - g(n))}{h} n A^{-1} \\
&= g(n) A^{-1} + \frac{(g(n+h) - g(n))}{h} \left(tA^{-1} - \int_{0}^{\infty} r e^{-Ar} \right) - \frac{(g(n+h) - g(n))}{h} n A^{-1} \\
&= g(n) A^{-1} + \frac{(g(n+h) - g(n))}{h} \left(tA^{-1} - A^{-2} \right) - \frac{(g(n+h) - g(n))}{h} n A^{-1}
\end{aligned}$$

Therefore, we obtain

$$x(t) = g(n)A^{-1} + \frac{(g(n+h) - g(n))}{h} \left(tA^{-1} - A^{-2} - nA^{-1} \right).$$

Now, consider that

$$x(n+h) - x(n-h) = \frac{(g(n+h) - g(n))}{h} \left(2hA^{-1} \right) = 2A^{-1}(g(n+h) - g(n))$$

and also

$$x(n) = g(n)A^{-1} + \frac{(g(n+h) - g(n))}{h} \left(-A^{-2} \right).$$

Then, notice that

$$x^{\diamond_\alpha}(n) = \frac{x(n+h) - x(n-h)}{2h}$$

and

$$\begin{aligned}
-Ax(n) + g(n) &= g(n) + A^{-1} \frac{(g(n+h) - g(n))}{h} + g(n) \\
&= A^{-1} \frac{(g(n+h) - g(n))}{h} = \frac{x(n+h) - x(n)}{2h} = x^{\diamond_\alpha}(n),
\end{aligned}$$

for every $n \in h\mathbb{Z}$ and we obtain that x is a solution of (4.2). Since $x : \mathbb{R} \to \mathbb{E}^n$ is almost periodic function, it follows by Theorem 3.4 that $x : h\mathbb{Z} \to \mathbb{E}^n$ is almost periodic function and the desired result follows as well. $\square$

Remark 4.2. Notice that Theorem 4.1 is not valid if we replace the equation (4.2) by the one of following equations:

$$x^{\Delta}(t) = -Ax(t) + g(t), \quad t \in h\mathbb{Z} \tag{4.4}$$

or

$$x^{\nabla}(t) = -Ax(t) + g(t), \quad t \in h\mathbb{Z}. \tag{4.5}$$

One can easily check this fact by following the same steps of the proof of Theorem 4.1.

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