THE SEMIDISCRETE DAMPED WAVE EQUATION WITH A FRACTIONAL LAPLACIAN

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Abstract. In this paper we completely solve the open problem of finding the fundamental solution of the semidiscrete fractional-spatial damped wave equation. We combine operator theory and Laplace transform methods with properties of Bessel functions to show an explicit representation of the solution when initial conditions are given. Our findings extend known results from the literature and also provide new insights into the qualitative behavior of the solutions for the studied model. As an example, we show the existence of almost periodic solutions as well as their profile in the homogeneous case.

1. Introduction

In this article, we are concerned with the Cauchy problem for the semidiscrete strongly damped wave equation given by:

$$\frac{\partial^2 u}{dt^2}(n,t) = -a(-\Delta_d)^\alpha \frac{\partial u}{dt}(n,t) - c(-\Delta_d)^\alpha u(n,t) + f(n,t), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $(-\Delta_d)^\alpha$ denotes the unidimensional discrete fractional Laplacian defined by

$$(-\Delta_d)^\alpha f(n) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{t\Delta_d} f(n) - f(n)) \frac{dt}{t^{1+\alpha}}, \quad n \in \mathbb{Z}, \tag{2}$$

see [18] and references therein. We note that the unidimensional discrete fractional Laplacian has been widely examined in [4, 5, 18] and can be used, for example, to model the non-local motion of an electron in a one-dimensional chain with atoms [19]. The N-dimensional case has been treated in [28].

The study of semidiscrete equations is an old topic that can be already found in the works of H. Bateman [2]. In recent years, the investigation of this type of equations incorporating the discrete fractional Laplacian has experienced a growing interest because, on the one hand, they are capable of better describing the dynamics of physical and probabilistic processes behind them [11, 13, 24] and, on the other hand, because they give a better understanding

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of the dynamics beyond the continuous case [23]. For recent developments, we refer to [17, 14, 15] and references therein.

It is well-known that, in general, initial value problems for semidiscrete equations like (1) do not have unique solutions, see e.g. [25, 26] and its references. However, for well-behaved initial data, there exists a unique solution. The explicit representation of these solutions is also known in some cases. For example, for the semidiscrete diffusion equation

$$\frac{\partial u}{dt}(n,t) = a\Delta_d u(n,t), \quad t \geq 0, \quad n \in \mathbb{Z},$$

satisfying $u(n,0) = \varphi(n)$, where $\varphi(n)$ is a bounded sequence, it is well-known that the unique solution is given by the superposition formula

$$u(n,t) = e^{-2at} \sum_{k \in \mathbb{Z}} I_{n-k}(2at)\varphi(k), \quad t \geq 0, \quad n \in \mathbb{Z},$$

where I_α is the modified Bessel function of the first kind of order α. See e.g. [1] and references therein.

The fundamental solution for the equation (1) in case $a = 0$ and $\alpha = 1$, i.e. the semidiscrete wave equation, appears in the works by H. Bateman and C. E. Pearson [2, 20]. More recently, in the reference [18] combining operator theory techniques with the properties of the Bessel functions it is proven that the second order equation

$$\frac{\partial^2 u}{dt^2}(n,t) = \Delta_d u(n,t), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{3}$$

with initial conditions $u(n,0) = \varphi(n)$, $\frac{\partial u}{dt}(n,0) = \psi(n)$, has a unique solution given by [18, p.1386]

$$u(n,t) = \sum_{k \in \mathbb{Z}} J_{2(n-k)}(2t)\varphi(n) + \sum_{k \in \mathbb{Z}} \left(\int_0^t J_{2(n-k)}(2s)ds \right) \psi(n), \quad t \geq 0, \quad n \in \mathbb{Z},$$

where J_k is the Bessel function of the first kind.

When the discrete Laplacian in (3) is replaced by the discrete fractional Laplacian $-(-\Delta_d)^\alpha$, $0 < \alpha \leq 1$, it was found in [18, Theorem 1.4, Formulae (14) and (15)] that the Bessel function must be replaced by the fundamental solution

$$C_t^\alpha(n) := \frac{2}{\pi} \int_0^{\pi/2} \cos(t(4\sin^2\theta)^{\alpha/2}) \cos(2n\theta) d\theta, \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{4}$$

where it is worth mentioning that in the case $\alpha = 1$, by [16, Formula 8.411(2)], the above expression coincides with $J_{2k}(2t)$.

However, an explicit representation of the solution of the damped wave equation (1), even in case $\alpha = 1$, remains as an open problem.

In general, this is a difficult problem. Until now, there is no known representations even for the prototypical case $\alpha = 1$ that corresponds to the so called strongly damped wave equation without mass term [21], or $\alpha = 2$ that corresponds to the damped extensible beam equation [12] which is called the Kelvin-Voigt model [29]. We note that the importance of considering the fractional Laplacian in (1) for a continuous spatial domain has been recently pointed out in the references [7, 8].

In this article, we completely solve this problem. We find that whenever $4^{\alpha-1}a^2 \leq c$ the fundamental solution of (1) is given by

$$D_{2t}^{\alpha}(2k) := \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-aa_{2\theta}t}\left(\cos t\sqrt{a_{2\theta}(4c-a^2a_{2\theta})} - \frac{aa_{2\theta}\sin t\sqrt{a_{2\theta}(4c-a^2a_{2\theta})}}{\sqrt{a_{2\theta}(4c-a^2a_{2\theta})}}\right)\cos(2k\theta)d\theta,$$

where $a_\theta = (4\sin^2(\theta/2))^{\alpha}$. It is notable that this expression is comparable to recent results stated in [9, Theorem 1.1] for the continuous Laplacian, therefore providing new insights. We observe that $D_{2t}^{\alpha}(2k)$ reduces to (4) when $a = 0$ and $c = 1$. Then, we have succeeded in showing that the unique solution for (1), bounded on $\mathbb{Z} \times [0, T]$ for each $T > 0$, with initial conditions $u(n, 0) = \varphi(n)$, $\frac{\partial u}{dt}(n, 0) = \psi(n)$, can be expressed as follows:

$$\begin{aligned} u(n,t) = {} & \sum_{k\in\mathbb{Z}} D_t^{\alpha}(n-k)\varphi(n) + \sum_{k\in\mathbb{Z}} S_t^{\alpha}(n-k)[\psi(n) + a(-\Delta_d)^{\alpha}\varphi(n)] \\ & + \sum_{k\in\mathbb{Z}}\int_0^t S_{t-s}^{\alpha}(n-k)f(k,s)ds, \quad t \geq 0, \quad n \in \mathbb{Z}, \end{aligned}$$

where $S_t^{\alpha} := \int_0^t D_s^{\alpha} ds$. In particular, this result generalizes [18, Theorem 1.5].

Our main result could be used to provide new insights on the behavior and properties of the solutions, even in the multidimensional case, generalizing results of Slavik [26, 27], or to search for additional qualitative behavior, as done e.g., in the reference [18]. As an example, we show in Remark 3.5 the profile of the solutions of the homogeneous equation (1), and in Theorem 3.6 we prove the existence of almost periodic solutions to the equation (1) under appropriate conditions on the forcing term. We note that such qualitative behavior is not present in the continuous case.

2. Preliminaries

We recall the following definition.

Definition 2.1. *Let X be a complex Banach space. The Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ will be denoted by*

$$\hat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t)dt, \quad \Re(\lambda) > \omega,$$

whenever the integral is absolutely convergent for $\Re(\lambda) > \omega$.

An equivalent formulation to the definition given in (2), but valid for any $\alpha > 0$, is the following [18, Section 3.1].

Theorem 2.2. *Let $1 \leq p \leq \infty$. The discrete fractional Laplacian of order $\alpha > 0$ of a given sequence $f \in \ell_p(\mathbb{Z})$ can be written as*

$$(-\Delta_d)^{\alpha} f(n) = \sum_{j\in\mathbb{Z}} K^{\alpha}(n-j)f(j), \quad n \in \mathbb{Z} \tag{5}$$

where

$$K^\alpha(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(4\sin^2(\theta/2))^\alpha e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(4\sin^2(\theta/2))^\alpha \cos(n\theta)d\theta, \quad n\in\mathbb{Z}. \tag{6}$$

From [18, Theorem 1.3 (ii)] we know that

$$e^{-(-\Delta_d)^\alpha t}\varphi(n) = \sum_{k\in\mathbb{Z}} L_t^\alpha(n-k)\varphi(k), \quad t\geq 0, \quad n\in\mathbb{Z}, \quad \varphi\in\ell_p(\mathbb{Z}), \tag{7}$$

is an analytic semigroup, where

$$L_t^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-t(4\sin^2\frac{\theta}{2})^\alpha} e^{-in\theta}d\theta, \quad t\geq 0, \quad n\in\mathbb{Z}. \tag{8}$$

We will also need the following formula, which can be obtained from [10, Chapter IV, Section 4.1, Formula (36)] with $\nu = 0$.

Lemma 2.3. *Let $f\in L^1_{loc}(\mathbb{R}_+, X)$ and $C>0$ be given. The Laplace transform of*

$$F(t) = \int_0^t J_0[2\sqrt{C}\sqrt{(t-s)s}]f(s)ds, \quad t\geq 0,$$

is given by $\dfrac{1}{\lambda}\hat{f}\left(\lambda+\dfrac{C}{\lambda}\right)$ where $J_0(t)$ denotes the Bessel function.

We now recall the notion of discrete almost periodic functions. For more information about almost periodic functions see [6].

Definition 2.4. *Let $(X, ||\cdot||)$ be a Banach space. A function $f:\mathbb{Z}\times X\to\mathbb{C}$ is discrete almost periodic in $n\in\mathbb{Z}$ if for every $x\in X$ and every $\epsilon>0$, there exists a positive integer $N(\epsilon)$ such that any set consisting of $N(\epsilon)$ consecutive integers contains at least one integer p with the property*

$$|f(n+p,x)-f(n,x)|<\epsilon, \quad n\in\mathbb{Z}.$$

The following result stated in [18] shows discrete almost periodicity for the convolution of discrete almost periodic functions with functions $v: X\times\mathbb{Z}\to\mathbb{C}$ which are summable in the first variable.

Theorem 2.5. *Let $v:\mathbb{Z}\times X\to\mathbb{C}$ be a summable function in the first variable. If $u:\mathbb{Z}\times X\to\mathbb{C}$ is a discrete almost periodic function in $n\in\mathbb{Z}$, then for each $x\in X$ the function*

$$w(n,x) := \sum_{k\in\mathbb{Z}} v(k,x)u(n-k,x), \quad n\in\mathbb{Z}$$

is also discrete almost periodic in $n\in\mathbb{Z}$.

3. Main results

Let $c>0$ and $a\in\mathbb{R}$. We consider the second order damped wave equation given by:

$$(9)\qquad \begin{cases} \dfrac{\partial^2 u}{dt^2}(n,t) &= -a(-\Delta_d)^\alpha \dfrac{\partial u}{dt}(n,t) - c(-\Delta_d)^\alpha u(n,t) + f(n,t), \quad t\geq 0, \quad n\in\mathbb{Z}, \\ u(n,0) &= \varphi(n), \quad n\in\mathbb{Z}, \\ u'(n,0) &= \psi(n), \quad n\in\mathbb{Z}, \end{cases}$$

where $\alpha>0$. Suppose that $a\neq 0$ and we define the operator

$$(10)\qquad \mathcal{S}_t^\alpha\varphi(n) := e^{-(c/a)t}\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{2(c/a)s}e^{-as(-\Delta_d)^\alpha}\varphi(n)ds, \quad \varphi\in\ell_p(\mathbb{Z}).$$

We will show that the above family of operators have several qualitative properties, together with the associated kernel

$$(11)\qquad S_t^\alpha(n) := e^{-(c/a)t}\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{2(c/a)s}L_{as}^\alpha(n)ds, \quad t\geq 0, \quad n\in\mathbb{Z},$$

where L_{as}^α is defined in (8).

In what follows, we denote $a_\theta := (4\sin^2(\theta/2))^\alpha$.

Theorem 3.1. *Let $\{\mathcal{S}_t^\alpha\}_{t\geq 0}$ be the operators defined in* (10).

(i) We have $\{\mathcal{S}_t^\alpha\}_{t\geq 0}\subset\mathcal{B}(\ell_p(\mathbb{Z}))$ for each $1\leq p\leq\infty$;

(ii) $\|\mathcal{S}_t^\alpha\varphi\|_{\ell_p}\leq e^{(c/a+aa_\alpha)t}\|\varphi\|_{\ell_p}$ for all $\varphi\in\ell_p(\mathbb{Z})$, where $a_\alpha := \frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}$;

(iii) If $4^{\alpha-1}a^2\leq c$ then we have the following representation for $\mathcal{S}_t^\alpha$ as a convolution operator with a kernel:

$$(12)\qquad \mathcal{S}_t^\alpha\varphi(n) := (S_t^\alpha * \varphi)(n) = \sum_{m\in\mathbb{Z}} S_t^\alpha(n-m)\varphi(m), \quad t\geq 0, \quad n\in\mathbb{Z},$$

where

$$(13)\qquad S_t^\alpha(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{\frac{-aa_\theta t}{2}}\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}\cos(n\theta)d\theta, \quad t\geq 0, \quad n\in\mathbb{Z}.$$

In particular, $S_0^\alpha(n)=0$, $n\in\mathbb{Z}$.

(iv) $\|S_t^\alpha\|_{\ell_p}\leq e^{(c/a+aa_\alpha)t}$, for $1\leq p\leq\infty$.

Proof. Since by [18, Theorem 1.3 (i)] we have $\mathcal{L}_t^\alpha\varphi := e^{-t(-\Delta_d)^\alpha}\varphi\in\ell_p(\mathbb{Z})$ for all $\varphi\in\ell_p(\mathbb{Z})$ we deduce (i). Moreover, the following inequality [18, Theorem 1.3 (i)-1]

$$\|\mathcal{L}_t^\alpha\varphi\|_{\ell_p}\leq e^{\frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}t}\|\varphi\|_{\ell_p}, \quad t\geq 0,$$

and the property $|J_0(x)|\leq 1$ for all $x\geq 0$ shows (ii).

Next, suppose that $4^{\alpha-1}a^2 \le c$. Inserting (7) in (10), we obtain

$$\begin{aligned}\mathcal{S}_t^\alpha \varphi(n) &= e^{-(c/a)t}\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{2(c/a)s}\sum_{k\in\mathbb{Z}} L_{as}^\alpha(n-k)\varphi(k)ds\\ &= \sum_{k\in\mathbb{Z}} S_t^\alpha(n-k)\varphi(k), \quad t\ge 0, \quad n\in\mathbb{Z},\end{aligned}$$

where

$$\begin{aligned}S_t^\alpha(n) &= e^{-(c/a)t}\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{2(c/a)s}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-as(4\sin^2(\theta/2))^\alpha}e^{-in\theta}d\theta\right)ds\\ &= e^{-(c/a)t}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{(2(c/a)-a(4\sin^2(\theta/2))^\alpha)s}ds\right)e^{-in\theta}d\theta\right]\\ &= e^{-(c/a)t}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]g(s)ds\right)e^{-in\theta}d\theta\right]\end{aligned} \tag{14}$$

with $g(s) = e^{(2(c/a)-a(4\sin^2(\theta/2))^\alpha)s}$. Applying Lemma 2.3 we get that

$$\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]g(s)ds, \quad t\ge 0, \tag{15}$$

corresponds to the inverse Laplace transform of

$$\begin{aligned}\frac{1}{\lambda}\hat{g}\left(\lambda+\frac{c^2}{\lambda a^2}\right) &= \frac{1}{\lambda}\frac{1}{\lambda+\frac{c^2}{\lambda a^2}-(2(c/a)-a(4\sin^2(\theta/2))^\alpha)}\\ &= \frac{a^2}{\lambda^2a^2+c^2-2c\lambda a+a^3\lambda(4\sin^2(\theta/2))^\alpha}\\ &= \frac{a^2}{(\lambda a-c)^2+a^3\lambda(4\sin^2(\theta/2))^\alpha}.\end{aligned} \tag{16}$$

Since $a_\theta = (4\sin^2(\theta/2))^\alpha$, we have $(a\lambda - c)^2 + a^3 a_\theta\lambda = a^2(\lambda - r_1(\theta)(\lambda - r_2(\theta))$ where

$$r_1(\theta) = \frac{c}{a}-\frac{aa_\theta}{2}+\frac{1}{2}\sqrt{a_\theta(a^2a_\theta-4c)} \quad \text{and} \quad r_2(\theta) = \frac{c}{a}-\frac{aa_\theta}{2}-\frac{1}{2}\sqrt{a_\theta(a^2a_\theta-4c)}.$$

Using [10, Chapter V, 5.2 (5)] to calculate the inverse Laplace transform of (16), we obtain

$$\int_0^t J_0[2(c/a)\sqrt{(t-s)s}]g(s)ds = \frac{1}{r_1(\theta)-r_2(\theta)}(e^{r_1(\theta)t}-e^{r_2(\theta)t}), \quad t\ge 0. \tag{17}$$

Substituting (17) into (14), and since by hypothesis $\frac{a^2 4^{\alpha-1}}{c} \leq 1$ implies $a^2 a_\theta - 4c \leq 0$, we obtain:

$$
\begin{aligned}
S_t^\alpha(n) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-\frac{a a_\theta}{2}t}\left(e^{\frac{1}{2}it\sqrt{a_\theta(4c-a^2a_\theta)}} - e^{-\frac{1}{2}it\sqrt{a_\theta(4c-a^2a_\theta)}}\right)}{i\sqrt{a_\theta(4c-a^2a_\theta)}} e^{-in\theta}d\theta \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{\frac{-a a_\theta t}{2}}\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}} e^{-in\theta}d\theta \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} s(\theta,t)\cos(n\theta)d\theta, \quad t \geq 0, \quad n \in \mathbb{Z},
\end{aligned}
\tag{18}
$$

where we have used that the function $\theta \to s(\theta,t) := \frac{e^{\frac{-a a_\theta t}{2}}\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}$ is even. This proves (iii).

For (iv), we choose $\{\varphi(n)\}_{n\in\mathbb{Z}} = \{\delta_{0,n}\}_{n\in\mathbb{Z}} \in \ell_p(\mathbb{Z})$. Then, by (12) we obtain $\mathcal{S}_t\delta_{0,n} = S_t^\alpha(n)$ for each n and the conclusion follows from part (ii).

□

Remark 3.2. *We note that formula* (13) *shows that* $S_t^\alpha(n)$ *corresponds to the Fourier coefficients, or inverse discrete-time Fourier transform, of the term* $\frac{e^{\frac{-a a_\theta t}{2}}\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}$.

For all $t \geq 0, n \in \mathbb{Z}$, we define

$$
D_t^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{-a a_\theta t}{2}}\left(\cos(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)}) - \frac{a a_\theta \sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}\right)\cos(n\theta)d\theta.
\tag{19}
$$

Note that $D_t^\alpha(n) := \frac{\partial S_t^\alpha(n)}{\partial t}$, and in particular, $D_0^\alpha(n) = \delta_{0,n},\ n \in \mathbb{Z}$. We also have

$$
\begin{aligned}
\frac{\partial D_t^\alpha(n)}{\partial t} =& -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a a_\theta}{2} e^{\frac{-a a_\theta t}{2}} \times \\
& \times \left(\cos(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)}) - \frac{a a_\theta \sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}\right)\cos(n\theta)d\theta \\
& - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{-a a_\theta t}{2}} \times \\
& \times \left(\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})\frac{1}{2}\sqrt{a_\theta(4c-a^2a_\theta)} + \frac{a a_\theta}{2}\cos(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})\right)\cos(n\theta)d\theta,
\end{aligned}
\tag{20}
$$

and from here we deduce using formula (6) from Theorem 2.2 that $\frac{\partial D_t^\alpha(n)}{\partial t} = -aK^\alpha(n)$ at $t = 0$.

Our main result is stated as follows:

Theorem 3.3. *Let $\varphi, \psi \in \ell_1(\mathbb{Z})$ and $f : \mathbb{Z} \times \mathbb{R}_+ \longrightarrow \mathbb{C}$ be continuous in the second variable and such that $\sup_{t\geq 0} \|f(\cdot,t)\|_{\ell_1} < \infty$. Suppose that for $c > 0$ and $a \geq 0$ we have*

$$4^{\alpha-1}a^2 \leq c.$$

Then, the unique solution bounded on $\mathbb{Z} \times [0,T]$ for each $T > 0$ of (9) is given by

$$u(n,t) = D_t^\alpha * \varphi(n) + S_t^\alpha * (\psi + a(-\Delta_d)^\alpha \varphi)(n) + \int_0^t S_{t-s}^\alpha * f(n,s)ds, \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{21}$$

Proof. The case $a = 0$ is well-known and, as we said in the introduction, the formula (21) coincides in such case. Suppose $a \neq 0$. Denote by $A := -(-\Delta_d)^\alpha$. By [18, Theorem 1.3] the operator A is bounded in $\ell_p(\mathbb{Z})$ for all $1 \leq p \leq \infty$ and $\sigma(A) = [-4^\alpha, 0]$. We can apply Laplace transform to (9) obtaining

$$\lambda^2 \hat{u}(n,\lambda) - \lambda\varphi(n) - \psi(n) - aA(\lambda\hat{u}(n,\lambda) - \varphi(n)) - cA\hat{u}(n,\lambda) = \hat{f}(n,\lambda), \quad n \in \mathbb{Z}. \tag{22}$$

We then have:

$$(\lambda^2 - aA\lambda - cA)\hat{u}(n,\lambda) = \lambda\varphi(n) + (\psi(n) - aA\varphi(n)) + \hat{f}(n,\lambda), \quad n \in \mathbb{Z}.$$

Observe that for any $z \in \mathbb{C}$ we have the identity $Re(\frac{z^2}{az+c}) = Re(z)^3(a+c) + Im(z)^2(aRez - c)$. Therefore for any $\lambda \in \mathbb{C}$ with $Re(\lambda) > \frac{c}{a}$ we have that $Re(\frac{\lambda^2}{a\lambda+c}) > 0$ and hence $\frac{\lambda^2}{a\lambda+c} \in \rho(A)$, the resolvent set of A. If we denote

$$R(\lambda) = (\lambda^2 - aA\lambda - cA)^{-1} = \frac{1}{(a\lambda + c)} \left(\frac{\lambda^2}{a\lambda + c} - A \right)^{-1}, \quad Re(\lambda) > \frac{c}{a},$$

we get:

$$\hat{u}(n,\lambda) = \lambda R(\lambda)\varphi(n) + R(\lambda)(\psi(n) - aA\varphi(n)) + R(\lambda)\hat{f}(n,\lambda), \quad n \in \mathbb{Z}. \tag{23}$$

Let now prove that $P(t) \equiv \mathcal{S}_t^\alpha$ defined in (10) satisfies $\hat{P}(\lambda) = R(\lambda)$. Indeed,

$$\begin{aligned} \hat{P}(\lambda) &= \int_0^\infty e^{-(\lambda+(c/a))t} \int_0^t J_0[2(c/a)\sqrt{(t-s)s}]e^{2(c/a)s}e^{asA}dsdt \\ &= \int_0^\infty e^{-(\lambda+(c/a))t} \int_0^t J_0[2(c/a)\sqrt{(t-s)s}]G(s)dsdt, \end{aligned} \tag{24}$$

where $G(s) := e^{(2(c/a)+aA)s} \in \mathcal{B}(\ell_p(\mathbb{Z}))$ for all $1 \leq p \leq \infty$. From Lemma 2.3, expression (24) equals to

$$\frac{1}{\lambda + (c/a)} \hat{G} \left(\lambda + (c/a) + \frac{(c/a)^2}{(\lambda + (c/a))} \right). \tag{25}$$

Since $\hat{G}(\mu) = (\mu - 2(c/a) - aA)^{-1}$ exists for all $\mu \in \mathbb{C}$ with $Re(\mu) > \frac{2c}{a}$, a computation shows that:

$$\frac{1}{\lambda + (c/a)} \hat{G} \left(\lambda + (c/a) + \frac{(c/a)^2}{(\lambda + (c/a))} \right) = \frac{1}{a\lambda + c} \left(\frac{\lambda^2}{a\lambda + c} - A \right)^{-1} = R(\lambda), \tag{26}$$

for all $\lambda \in \mathbb{C}$ with $Re(\lambda)$ sufficiently large, and the claim is proved.

Since clearly $t \to P(t)\varphi$ is differentiable for each $\varphi \in \ell_p(\mathbb{Z})$, $1 \leq p \leq \infty$, and $P(0) = 0$, applying now inverse Laplace transform to (23) we obtain:

$$u(n,t) = P'(t)\varphi(n) + P(t)(\psi - aA\varphi)(n) + \int_0^t P(t-s)f(n,s)ds, \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{27}$$

is a solution of the equation in (9).

Note that a computation using the property $D_0^\alpha(n) = \delta_{0,n},\ n \in \mathbb{Z}$ together with Theorem 3.1 part (iii), shows that $P'(0) = I$ and then:

$$u(n,0) = P'(0)\varphi(n) + P(0)(\psi(n) - aA\varphi(n)) = \varphi(n), \quad n \in \mathbb{Z}. \tag{28}$$

On the other hand, differentiating in (27) and evaluating in $t = 0$ we get:

$$\frac{\partial u(n,0)}{\partial t} = P''(0)\varphi(n) + P'(0)(\psi(n) - aA\varphi(n)) + P(0)f(n,t) = P''(0)\varphi(n) + \psi(n) - aA\varphi(n). \tag{29}$$

where, using the fact that $\frac{\partial D_t^\alpha(n)}{\partial t} = -aK^\alpha(n)$ at $t = 0$, we obtain from Theorem 2.2 that

$$P''(0)\varphi(n) = -a(-\Delta_d)^\alpha\varphi(n) = aA\varphi(n), \quad n \in \mathbb{Z}. \tag{30}$$

Finally, replacing in (29) we arrive to $\frac{\partial u(n,0)}{\partial t} = \psi(n)$. This proves that $u(n,t)$ is a solution of the initial value problem (9). In order to see that such solution is bounded on $\mathbb{Z} \times [0,T]$ for each $T > 0$ we first note that the estimate $|\sin(x)/x| \leq 1$, $x \in \mathbb{R}$, applied to formula (13) shows that $|S_t^\alpha(n)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}d\theta = t$ for each $t \geq 0$, and analogously from the formula (19) we obtain the estimate $|D_t^\alpha(n)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 + |a||a_\theta|\frac{t}{2})d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 + |a|4^\alpha\frac{t}{2})d\theta = 1 + |a|2^{2\alpha-1}t$ for all $t \geq 0$. Then, Young's convolution inequality implies

$$\sup_{n\in\mathbb{Z}}|(D_t^\alpha * \varphi)(n)| = \|D_t^\alpha * \varphi\|_{\ell_\infty} \leq \|D_t^\alpha\|_{\ell_\infty}\|\varphi\|_{\ell_1} = \sup_{n\in\mathbb{Z}}|D_t^\alpha(n)|\|\varphi\|_{\ell_1} \leq (1 + |a|2^{2\alpha-1}t)\|\varphi\|_{\ell_1}.$$

Therefore, we conclude that $\sup_{t\in[0,T]}\sup_{n\in\mathbb{Z}}|(D_t^\alpha * \varphi)(n)| \leq (1 + |a|2^{2\alpha-1}T)\|\varphi\|_{\ell_1}$ for each $T > 0$. Analogously, since $(-\Delta_d)^\alpha$ is well defined as a bounded operator in $\ell_1(\mathbb{Z})$ we obtain the estimate

$$\sup_{t\in[0,T]}\sup_{n\in\mathbb{Z}}|(S_t^\alpha * (\psi - aA\varphi))(n)| \leq T(\|\psi\|_{\ell_1} + |a|\|(-\Delta_d)^\alpha\|\|\varphi\|_{\ell_1}),$$

for each $T > 0$. We also have

$$\begin{aligned}\sup_{n\in\mathbb{Z}}\left|\int_0^t S_{t-s}^\alpha * f(n,s)ds\right| &\leq \int_0^t \sup_{n\in\mathbb{Z}}|S_{t-s}^\alpha * f(n,s)|ds \leq \int_0^t (t-s)\sup_{\tau\geq 0}\|f(\cdot,\tau)\|_{\ell_1}ds \\ &= \sup_{\tau\geq 0}\|f(\cdot,\tau)\|_{\ell_1}\int_0^t sds = \sup_{\tau\geq 0}\|f(\cdot,\tau)\|_{\ell_1}\frac{t^2}{2}.\end{aligned}$$

Hence, by hypothesis, $\sup_{t\in[0,T]}\sup_{n\in\mathbb{Z}}\left|\int_0^t S_{t-s}^\alpha * f(n,s)ds\right| \leq \sup_{\tau\geq 0}\|f(\cdot,\tau)\|_{\ell_1}\frac{T^2}{2} < \infty$ for each $T > 0$. It proves that the solution $u(n,t)$ is bounded in $\mathbb{Z} \times [0,T]$ for each $T > 0$.

Finally, assume that (9) has two bounded solutions u_1 and u_2 with the same initial data φ, ψ, and set $v := u_1 - u_2$. Then v is a solution of the following initial value problem

$$\begin{cases} \dfrac{\partial^2 v}{dt^2}(n,t) &= -a(-\Delta_d)^\alpha \dfrac{\partial v}{dt}(n,t) - c(-\Delta_d)^\alpha v(n,t), \quad t \geq 0, \quad n \in \mathbb{Z}, \\ v(n,0) &= 0, \quad n \in \mathbb{Z}, \\ v'(n,0) &= 0, \quad n \in \mathbb{Z}. \end{cases}$$

Integrating two times we obtain the equivalent abstract Volterra equation

$$w(t) = \int_0^t k(t-s)Aw(s)ds, \quad w(t) \in \ell_\infty(\mathbb{Z}), \quad t \geq 0,$$

where $w(t)(n) := v(n,t)$ and $k(t) := a + ct$. Since $k(t)$ is a creep function [22, Chapter I, Definition 4.4] and A generates a cosine family on $\ell_\infty(\mathbb{Z})$ because A is bounded in such space, we deduce that the above abstract Volterra equation admits a unique resolvent [22, Theorem 4.3 and Corollary 1.1] and hence has zero as its unique solution. Therefore, we have that $v \equiv 0$ and hence $u_1 \equiv u_2$. It proves the uniqueness and the theorem.

□

Remark 3.4. *Looking at the formula* (19)*, we note that the solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due to the factor* $e^{\frac{-aa_\theta t}{2}}$. *This behavior is consistent in the case* $a = 0$ *because in such case, this factor disappears and the integrand in formula* (19) *entirely coincides with formula* (4).

Remark 3.5. *It should be noted that recently, by results from Ikehata-Todorova-Yordanov and Ikehata-Onodera* [9]*, we know that the asymptotic profile of the solution to the equation*

$$u_{tt}(x,t) - \Delta u(x,t) - \Delta u_t(x,t) = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{31}$$

is the so-called diffusion wave, that is,

$$\hat{u}(\xi,t) \sim P_1 e^{-t|\xi|^2/2} \frac{\sin(t|\xi|)}{|\xi|} \quad (t \to \infty), \tag{32}$$

where $\hat{u}(t,\xi)$ *represents the partial Fourier transform of the solution* $u(t,x)$ *with respect to the* x*-variable. In contrast, Theorem 3.3 and Theorem 3.1 show that the profile of the solution of equation* (1) *with* $f \equiv 0$, $\varphi \equiv 0$ *and* $\psi(n) = \delta_{0,n}$

$$u_{tt}(n,t) + a(-\Delta_d)^\alpha u(n,t) + c(-\Delta_d)^\alpha u_t(n,t) = 0, \quad t \geq 0, \quad n \in \mathbb{Z},$$

has the form

$$\tilde{u}(\theta,t) = e^{\frac{-aa_\theta t}{2}} \frac{\sin(\frac{t}{2}\sqrt{a_\theta(4c - a^2 a_\theta)})}{\sqrt{a_\theta(4c - a^2 a_\theta)}},$$

which corresponds to the Fourier coefficients of the fundamental solution $u(n,t)$ *with respect to the first variable. In particular, when* $c = a = 1$ *and* $\alpha = 1$ *we find* $a_\theta = 2 - 2\cos\theta =$

$(2\sin(\theta/2)^2$ *and hence* $\sqrt{a_\theta(4c-a^2a_\theta)} = 2|\sin(\theta)|$. *Therefore*

$$\tilde{u}(\theta,t) = e^{-t(2\sin(\theta/2)^2/2}\frac{\sin(t|\sin(\theta)|}{2|\sin(\theta)|},$$

which is comparable to (32). *An analogous similarity can be found in the reference* [9, Theorem 1] *between the asymptotic profile of* (31) *with two nonzero initial conditions and the Fourier coefficients of the fundamental solution* (19).

Our next result provides qualitative properties of solutions to equation (9).

Theorem 3.6. *Let* $\varphi, \psi \in \ell_1(\mathbb{Z})$ *and* $f : \mathbb{Z}\times\mathbb{R}_+ \longrightarrow \mathbb{C}$ *be continuous in the second variable and such that* $\sup_{t\geq 0}\|f(\cdot,t)\|_{\ell_1} < \infty$. *Suppose that* $c > 0$, $a \in \mathbb{R}$ *and* $4^{\alpha-1}a^2 \leq c$. *If* $f(\cdot,t)$ *is discrete almost periodic in the first variable, for each* $t \geq 0$, *then the unique solution* $u(\cdot,t)$ *of* (9) *given by* (21) *is also discrete almost periodic in the first variable.*

Proof. In order to prove that $u(n,t)$ (9) is discrete almost periodic, it is sufficient to show that $D_t^\alpha * \varphi(n)$, $S_t^\alpha * \psi(n)$, $S_t^\alpha(-\Delta_d)^\alpha\varphi(n)$ and $\int_0^t S_{t-s}^\alpha * f(n,s)$ are discrete almost periodic.

Indeed, let show that $D_t^\alpha * \varphi(n)$ is discrete almost periodic. According to Theorem 2.5, since $\varphi \in \ell_1(\mathbb{Z})$ it is sufficient to prove that $D_t^\alpha(n)$ is an almost periodic function in $n \in \mathbb{Z}$. If we denote $g_\theta(n) := \cos(n\theta)$ the sequence $\{g_\theta(n)\}_{n\in\mathbb{Z}}$ is almost periodic in $n \in \mathbb{Z}$ for every $\theta \in \mathbb{R}$ as shown in [3]. Thus, for every $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that any set consisting of $N(\epsilon)$ consecutive integers contains at least one integer p such that

$$\sup_{n\in\mathbb{Z}}|g_\theta(n+p) - g_\theta(n)| < \epsilon.$$

On the other hand, since by hypothesis $4^{\alpha-1}a^2 \leq c$, then $4c - a^2a_\theta \geq 0$ and the function $h : \theta \to \frac{\sin(\frac{t}{2}\sqrt{a_\theta(4c-a^2a_\theta)})}{\sqrt{a_\theta(4c-a^2a_\theta)}}$ is bounded for all $\theta \in (-\pi,\pi)$. Thus, there exists $K > 0$ such that $|h(\theta)| < K$ for all $\theta \in (-\pi,\pi)$. It is also clear that $|a_\theta| = |(4\sin^2(\theta/2))^\alpha| \leq 4^\alpha$ for all $\theta \in (-\pi,\pi)$. Consequently, we have

$$|D_t^\alpha(n+p) - D_t^\alpha(n)| \;\; \leq \tfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{\frac{-aa_\theta t}{2}}(1+4^\alpha a)K|g_\theta(n+p)-g_\theta(n)|d\theta.$$

Hence, we obtain

$$\sup_{n\in\mathbb{Z}}|D_t^\alpha(n+p) - D_t^\alpha(n)| < \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} e^{\frac{-aa_\theta t}{2}}(1+4^\alpha a)K d\theta$$

and the conclusion holds. The discrete almost periodicity of $S_t^\alpha * \psi(n)$, $S_t^\alpha * (-\Delta_d)^\alpha\varphi(n)$ follow similarly taking into account Theorem 3.1 formula (13).

Let now show that $\int_0^t S_{t-s}^\alpha * f(n,s)$ is discrete almost periodic. Applying Theorem 2.5 it follows that the function

$$F(n,t,s) := S_{t-s}^\alpha * f(n,s)$$

is discrete almost periodic. Thus, given $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that any set consisting of $N(\epsilon)$ consecutive integers contains at least one integer p such that

$$|F(n+p,t,s) - F(n,t,s)| < \epsilon, \quad n \in \mathbb{Z}$$

for each $n \in \mathbb{Z}$ and every $t, s \in \mathbb{R}^+$. If we denote $\xi(n,t) := \int_0^t F(n,t,s)ds$ it follows immediately that $|\xi(n+p,t) - \xi(n,t)| < \epsilon t, n \in \mathbb{Z}$ which finishes the proof.

□

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