ALGEBRA HOMOMORPHISMS DEFINED VIA CONVOLUTED SEMIGROUPS AND COSINE FUNCTIONS

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Abstract. Transform methods are used to establish algebra homomorphisms related to convoluted semigroups and convoluted cosine functions. Such families are now basic in the study of the abstract Cauchy problem. The framework they provide is flexible enough to encompass most of the concepts used up to now to treat Cauchy problems of the first and second order in general Banach spaces. Starting with the study of the classical Laplace convolution and a cosine convolution, along with associated dual transforms, natural algebra homomorphisms are introduced which capture the convoluted semigroup and cosine function properties. These correspond to extensions of the Cauchy functional equation for semigroups and the abstract d'Alembert equation for the case of cosine operator functions. The algebra homomorphisms obtained provide a way to prove Hille-Yosida type generation theorems for the operator families under consideration.

1. introduction

Abstract Cauchy problems not well posed in classical sense (that is, those that cannot be handled by the Hille-Yosida-Phillips-Feller-Miyadera theorem) and their applications to partial differential equations, notably equations of mathematical physics, have received much attention lately. Historically, J.L. Lions ([21]) was the first to introduce a concept, namely distribution semigroups to deal with such problems. Subsequently, Da Prato [10] introduced the notion of regularized semigroup, now known as $C-$semigroup. Ultradistribution semigroups were studied by Chazarain ([5]). The results of Chazarain are recorded in the monograph [22] by Lions and Magenes (the French edition appeared in 1970). Hyperfunction solutions have been treated by Ouchi in [29].

More recently, the notion of integrated semigroup appeared as a very flexible tool for solving Cauchy problems. They are in correspondence with exponential distribution semigroups in the sense of Lions (see [1]). Convoluted semigroups were introduced by Cioranescu [8] to include the correspondence between the theory of ultradistribution semigroups and Beal's constructive approach to the abstract Cauchy problem of first order. The monographs [2] and [23] contain the theories of these families and their inter-relationships as well as applications to partial differential equations.

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In the present paper we establish algebra homomorphism on certain convolution algebras denoted $(\mathcal{T}_k(\tau), *)$ and $(\mathcal{T}_k(\tau), *_c)$ using convoluted semigroups and convoluted cosine functions. We note that these algebra homomorphisms include the Laplace transform and the cosine Fourier transform. Such homomorphism shed a new light on the fundamental equations governing the families under consideration. Previous results in this direction were obtained by Galé and Miana in [12] and Miana [27] for integrated semigroups and integrated cosine families respectively. See also [24] and [25]. As direct consequences they defined fractional powers for the generator of temperated integrated semigroups [24, Section 5] and subordinated holomorphic semigroups with respect to integrated groups, cf. [12, Theorem 4.5]. Previous work on concrete integrated semigroups have been studied in [14], [15], [28]. See also [2] and references therein.

In this setting, algebra homomorphisms from $L^1(\mathbb{R}^+)$ into $\mathcal{B}(X)$ is equivalent to the Hille-Yosida theorem on the generation of uniformly bounded C_0- semigroups, see [6] and [7]. Our results have as a consequence Hille-Yosida type theorems (see Theorem 5.8 and Theorem 6.8) which includes the cases of integrated semigroups and integrated cosine functions.

We begin with a detailed study of two convolution products and the dual operations. The first is the Laplace convolution:

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds, \ t \geq 0$$

where f, g are locally integrable functions on $\mathbb{R}_+$. For the second transform, we first consider the following:

$$(f \circ g)(t) = \int_t^\infty f(s-t)g(s)ds, \ t \geq 0.$$

The cosine convolution product is then given by:

$$f *_c g = \frac{1}{2}(f * g + f \circ g + g \circ f).$$

A striking fact is that one can immediately relate the two convolution products (Laplace convolution and cosine convolution) with the fundamental equations defining convoluted semigroups and convoluted cosine functions. In fact, one of the main consequences of our results in this paper indicates that given a densely defined closed operator $A : D(A) \subset X \to X$ and $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in \operatorname{supp}(k)$, $\operatorname{abs}(|k|) < \infty$ and $\beta > \max\{\operatorname{abs}(|k|), 0\}$, the following assertions are equivalent:

(i) The integral equation

$$u(t) = A\int_0^t u(s)ds + \int_0^t k(s)xds, \quad t \geq 0,$$

has a unique exponentially bounded solution u_x for all $x \in X$.

(ii) $\{z \in \mathbb{C} : \Re z > \beta\} \subset \rho(A)$ and there exists a bounded algebra homomorphism $\Theta_k : (\mathcal{T}_k(e_{-\beta}), *) \to \mathcal{B}(X)$ such that $\Theta_k(e_\lambda) = (\lambda - A)^{-1}$, where $e_\lambda(t) = e^{-\lambda t}$.

(iii) A generates a k-convoluted semigroup $(S_k(t))_{t\geq 0}$ of bounded linear operators such that $||S_k(t)|| \leq Me^{\beta t}$,

(iv) There exists an exponentially bounded and strongly continuous family $(S_k(t))_{t\geq 0}$ such that $S_k(0) = 0$ and

$$S_k(t)S_k(s) = \int_t^{t+s} k(t+s-u)S_k(u)du - \int_0^s k(t+s-u)S_k(u)du,$$

for $t, s \geq 0$. Moreover $Ax = \lim_{t\to 0^+} \dfrac{S_k(t)x - x}{\int_0^t k(s)ds}$ for all $x \in D(A)$.

Under similar hypothesis, the following assertions are equivalent in the case of convoluted cosine functions.

(i) The integral equation

$$v(t) = A\int_0^t (t-s)v(s)ds + \int_0^t k(s)xds, \quad t \geq 0,$$

has a unique exponentially bounded solution v_x for all $x \in X$.

(ii) $\{z^2 \in \mathbb{C} : \Re z > \beta\} \subset \rho(A)$ and there exists a bounded algebra homomorphism $\Psi_k : (\mathcal{T}_k(e_{-\beta}), *_c) \to \mathcal{B}(X)$ such that $\Psi_k(e_\lambda) = \lambda(\lambda^2 - A)^{-1}$.

(iii) A generates a convoluted cosine family $(C_k(t))_{t\geq 0}$ of bounded linear operators such that $||C_k(t)|| \leq Me^{\beta t}$,

(iv) There exists an exponentially bounded and strongly continuous family $(C_k(t))_{t\geq 0}$ such that $C_k(0) = 0$ and

$$Q_k(t)C_k(s) + C_k(t)Q_k(s) = \int_t^{t+s} k(t+s-u)C_k(u)du - \int_0^s k(t+s-u)C_k(u)du,$$

for $t, s \geq 0$, where $Q_k(t) = \int_0^t C_k(s)ds$. Moreover $Ax = \lim_{t\to 0^+} \dfrac{C_k(t)x - x}{\int_0^t (t-s)k(s)ds}$ for all $x \in D(A)$.

This paper is organized as follows: In sections 2 and 3, by means of the dual map of the Tichmarsh-Foiaş convolution operator, we define an space $\mathcal{D}_k$ with norms associated to the convolution transforms $*$ and $*_c$, and a map W_k from $\mathcal{D}_k$ to the set of test functions on $[0, \infty)$ such that $W_k(k \circ f) = f$ for all $f \in \mathcal{D}_k$ (Definition 2.7). In the particular case of $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(\alpha > 0, t > 0)$ the map W_α is the Weyl fractional derivative of order α. We next show that the space $\mathcal{D}_k$ is closed under the convolution products $*, \circ, *_c$ (Theorem 2.10). In section 4, we use Laplace transform to establish an equivalence between pseudo resolvents and functional equations (Theorem 4.2) which facilitates the comprehension of the striking functional equations defining k-convoluted semigroups and cosine functions. In section 5 we give a new characterization of k-convoluted semigroups in terms of a bounded algebra homomorphism Θ_k defined by means of the map W_k introduced in section 2 (Theorem 5.7). In the limit (semigroup) case $k(t) = \delta_0$ (the Dirac measure), we get the Laplace transform associated to the C_0-semigroup $(S(t))_{t\geq 0}$, given by $\Theta_{\delta_0}(f) = \int_0^\infty f(t)S(t)dt$. In case $k(t) = \frac{t^{n-1}}{n!}$ we obtain $\Theta_k(f) = (-1)^n \int_0^\infty f^{(n)}(t)S_n(t)dt$ where $(S_n(t))_{t\geq 0}$ is the n-times integrated semigroup, studied in [2, p.235], [3, Theorem 4.4] and [32, Theorem 4.13]. The case of α-times integrated semigroups $(S_\alpha(t))_{t\geq 0}$, i.e. $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, gives $\Theta_k(f) = \int_0^\infty W_\alpha f(t)S_\alpha(t)dt$ and was considered in [24, Theorem 3.1]. In section 6 we present analogous results for the k-convoluted cosine functions, taking in account that we must consider the convolution $*_c$ instead of the convolution $*$. For α-times integrated cosine functions, similar results have been proved in [25] and [27].

2. Inversion of convolution transforms

Throughout this paper we denote by X a complex Banach space, and $\mathcal{B}(X)$ the Banach algebra of bounded linear operators on X. For a closed linear operator A on X, we denote by $D(A), Ker(A), \rho(A), R(A)$ its domain, kernel, range and resolvent set, respectively.

Let $k \in L^1_{loc}(\mathbb{R}_+)$. The objective of this section is to define a space $\mathcal{D}_k$, which will be closed under different convolution products, and then to define associated maps W_k related to the inversion of convolution transforms.

Let $\mathcal{D}_+$ be the set of $\mathcal{C}^\infty$ functions of compact support on $[0,\infty)$. We denote by $L^1(\mathbb{R}^+)$ the set of Lebesgue integrable functions, i.e., f is a measurable function and

$$\|f\|_1 := \int_0^\infty |f(t)|dt < \infty,$$

and $L^1_\tau(\mathbb{R}^+)$ the set of functions f such that $f\tau \in L^1(\mathbb{R}^+)$ where $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is a measurable function, and the norm $\|f\|_\tau := \|f\tau\|_1$. The space $L^1[0,a]$ is given by the integrable functions on the interval $[0,a]$.

Analogously we denote by $L^1_{loc}(\mathbb{R}^+)$ the set of locally integrable functions. Given $k, g \in L^1(\mathbb{R}^+)$ the (finite or Laplace) convolution product is defined by

$$(k * g)(t) := \int_0^t k(t-s)g(s)ds, \quad t \geq 0.$$

It is well known that the convolution product $*$ is commutative and associative and satisfies $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. We will consider a dual convolution product on $L^1(\mathbb{R}^+)$ defined as follows: Given $k, g \in L^1(\mathbb{R}^+)$, we define

$$(f \circ g)(t) := \int_t^\infty f(s-t)g(s)ds, \qquad t \geq 0. \tag{2.1}$$

Moreover, the cosine convolution product $f *_c g$ is defined by

$$f *_c g := \frac{1}{2}(f * g + f \circ g + g \circ f), \qquad f, g \in L^1(\mathbb{R}^+) \tag{2.2}$$

(see [26] and [27]).

It is easy to check that $\|f \circ g\|_1 \leq \|f\|_1 \|g\|_1$ and hence $k \circ g \in L^1(\mathbb{R}^+)$. Notice that in general the convolution product $\circ$ is non commutative,

$$e_\lambda \circ e_\mu = \frac{1}{\mu + \lambda} e_\mu \neq e_\mu \circ e_\lambda, \quad \lambda, \mu \in \mathbb{C}^+$$

where $e_\lambda(t) := e^{-\lambda t}$ for $\lambda \in \mathbb{C}^+, t \in \mathbb{R}^+$; and non associative, $e_\lambda \circ (e_\mu \circ e_\theta) \neq (e_\lambda \circ e_\mu) \circ e_\theta$, with $\lambda, \mu, \theta \in \mathbb{C}^+$. Observe that $(k \circ g)(t) = \int_t^\infty k(s-t)g(s)ds = \int_0^\infty k(s)g(s+t)dt$. From this, it follows that if supp$(g) \subset [0,\, a]$, then supp$(k \circ g) \subset [0,\, a]$. We write by supp$(h)$ the usual support of a function h defined in $\mathbb{R}^+$. More generally, we have the following result which can be proved easily using Fubini's theorem, see [26, Proposition 2.1, Theorem 3.1. (iii)]

Proposition 2.1. *Given $f, g \in \mathcal{D}_+$ and $k \in L^1_{loc}(\mathbb{R}^+)$, then $f \circ g, k \circ g, \in \mathcal{D}_+$ and*

$$k \circ (f \circ g) = f \circ (k \circ g).$$

In the case that $k, f, g \in L^1(\mathbb{R}^+)$, we have that

$$k \circ (f \circ g) = (k * f) \circ g = (f * k) \circ g = f \circ (k \circ g).$$

In what follows we consider Dirac measures $(\delta_t)_{t\geq 0}$ defined by $\langle \delta_t, f\rangle := f(t)$ for $f\in\mathcal{D}_+$ and $t\geq 0$. Given $f\in L^1_{loc}(\mathbb{R}^+)$, we write $\delta_t\circ f(s)=\chi_{[t,\infty)}(s)f(s-t)$, $\delta_t\circ f(s)=f(s+t)$ and $f\circ\delta_t(s)=\chi_{[0,t]}(s)f(t-s)$ for $t,s\geq 0$, see more details in [26].

Proposition 2.2. *Take $f,k\in L^1_{loc}(\mathbb{R}^+)$ and we define f_t,k_t by $f_t(s):=\chi_{[0,t]}(s)f(t-s)$ for $t,s\geq 0$. Then $(f_t)_{t\geq 0}\subset L^1(\mathbb{R}^+)$, $\|f_t\|_1=\int_0^t|f(s)|ds$,*

$$f_t * k_s=\int_t^{t+s}k(t+s-u)f_u du-\int_0^s k(t+s-u)f_u du,\qquad t,s\geq 0; \tag{2.3}$$

$$f_t\circ k_s=\int_0^s f(t-s+u)k_u du,\qquad t\geq s\geq 0; \tag{2.4}$$

$$f_s\circ k_t=\int_{t-s}^t f(u-t+s)k_u du,\qquad t\geq s\geq 0. \tag{2.5}$$

Proof. Since $f\in L^1_{loc}(\mathbb{R}^+)$, then $f_t\in L^1(\mathbb{R}^+)$ and $\|f_t\|_1=\int_0^t|f(s)|ds$ for $t\geq 0$. Note that $f_t=f\circ\delta_t$ and

$$f_t * k_s=(f\circ\delta_t)*k_s=f\circ(\delta_t*k_s)-(\delta_t\circ f)\circ k_s,\qquad t,s\geq 0,$$

see a similar proof in [26, Theorem 4.1]. Since $\delta_t*k_s=\chi_{[t,\infty)}k_{s+t}$ we have that

$$\left(f\circ(\delta_t*k_s)\right)(x)=\int_x^{t+s}f(u-x)\chi_{[t,\infty)}(u)k(t+s-u)du=\int_t^{t+s}k(t+s-u)f_u(x)du,$$

when $x\leq t+s$. On the other hand,

$$(\delta_t\circ f)\circ k_s(x)=\int_x^s f(t+u-x)k(s-u)du=\int_x^s k(t+s-y)f(y-x)dy=\int_0^s k(t+s-u)f_u(x)du,$$

when $x\leq s$. Then we conclude the equality (2.3).

Now we take $t\geq s\geq 0$. Then we have that

$$f_t\circ k_s=(f\circ\delta_t)\circ k_s=\delta_t\circ(f*k_s)-(\delta_t\circ f)*k_s,$$

see again [26, Theorem 4.1]. Take $x\geq 0$, and

$$\begin{aligned} f_t\circ k_s(x) &= \int_0^{t+x}f(t+x-y)k_s(y)dy-\int_0^x k(s-x+y)f(y+t)dy\\ &= \int_x^{s+x}f(t+r-s)k(r-x)dr-\int_s^{s+x}k(r-x)f(r-s+t)dr\\ &= \int_x^s f(t+r-s)k(r-x)dr=\int_0^s f(t-s+u)k_u(x)du. \end{aligned}$$

The equality (2.4) is proven.

We consider $t\geq s\geq 0$ and $x\geq 0$. We get that

$$\begin{aligned} f_s\circ k_t(x) &= \left(\delta_s\circ(f*k_t)(x)-(\delta_s\circ f)*k_t\right)(x)\\ &= \int_0^{s+x}f(s+x-y)k(t-y)dy-\int_0^x f(s+x-y)k(t-y)dy\\ &= \int_x^{s+x}f(s+x-y)k(t-y)dy=\int_{t-s}^t f(r+s-t)k(r-x)dr \end{aligned}$$

$$= \int_{t-s}^{t} f(u-t+s)k_u(x)du.$$

The proof of equality (2.5) is finished. □

Given $a > 0$ and $k \in L^1_{loc}(\mathbb{R}^+)$ the convolution operator $T_k : L^1[0,a] \to L^1[0,a]$ defined by $T_k(f) := k * f$ was considered by E. C. Titchmarsh [30] and C. Foiaş [11].

Theorem 2.3. *(Titchmarsh-Foiaş theorem) Let $a > 0$ and $k \in L^1_{loc}(\mathbb{R}^+)$. Consider the convolution operator $T_k : f \mapsto k * f$ as an operator from $L^1[0,a]$ into $L^1[0,a]$. Then the following statements are equivalent.*

(i) *T_k has dense range;*
(ii) *T_k is injective;*
(iii) *$0 \in supp(k)$.*

For a proof see [4, Theorem 2.1]. Given $a > 0$ and $k \in L^1_{loc}(\mathbb{R}^+)$, we define the operator $T'_k : L^1[0,a] \to L^1[0,a]$ by

$$T'_k(f) := k \circ f, \qquad f \in L^1[0,a]. \tag{2.8}$$

Note that T_k and T'_k are dual in the sense that

$$\int_0^a g(t)T'_k(f)(t)dt = \int_0^a f(t)T'_k(g)(t)dt \tag{2.9}$$

for $f, g \in C[0,a]$, the set of continuous functions on the interval $[0,a]$. Our next result is a dual of the Titchmarsh-Foiaş theorem.

Theorem 2.4. *Let $a > 0$, $k \in L^1_{loc}(\mathbb{R}^+)$ and the operators $T_k, T'_k : L^1[0,a] \to L^1[0,a]$ defined as above. Then the following statements are equivalent.*

(i) *$0 \in supp(k)$;*
(ii) *T_k has dense range;*
(iii) *T_k is injective;*
(iv) *T'_k has dense range;*
(v) *T'_k is injective.*

Proof. We show that (ii) ⇒ (iv) ⇒(iii) and (ii)⇒ (v) ⇒ (i) and we apply the Titchmarsh-Foiaş theorem to complete the proof.

(ii) ⇒ (iv) Take $(h_n) \subset C[0,a]$ such that $h_n \geq 0$, $\int_0^a h_n(t)dt = 1$ and $\mathrm{supp}(h_n) \subset [0, \frac{1}{n}]$. It is easy to check that $h_n \circ g \to g$ in $L^1[0,a]$ for $g \in L^1[0,a]$. Since T_k has dense range, there exists $(f_n) \subset L^1[0,a]$ such that $k * f_n - h_n \to 0$ in $L^1[0,a]$. Take $g \in L^1[0,a]$ and we apply Proposition 2.1 to show that

$$k \circ (f_n \circ g) - g = (k * f_n) \circ g - g = ((k * f_n) - h_n) \circ g + h_n \circ g - g \to 0$$

in $L^1[0,a]$.

(iv) ⇒ (iii) Since T'_k has dense range, there exist $(g_n) \subset L^1[0,a]$ such that $k \circ g_n \to \chi_{(0,a)}$ on $L^1[0,a]$. Suppose that $T_k(f) = 0$ for some $f \in L^1[0,a]$. We apply Proposition 2.1 to get

$$0 = (k * f) \circ g_n = f \circ (k \circ g_n) \to f \circ \chi_{(0,a)}$$

on $L^1[0,a]$. Hence $\int_t^a f(s-t)ds = 0$ for $t \geq 0$, and we conclude that $f = 0$.

(ii) $\Rightarrow$ (v) Since T_k has dense range, there exist $(h_n) \subset L^1[0,a]$ such that $k * h_n \to \chi_{(0,a)}$ on $L^1[0,a]$. Suppose that $T'_k(f) = 0$ for some $f \in L^1[0,a]$. We apply Proposition 2.1 to get

$$0 = g_n \circ (k \circ f) = (k * g_n) \circ f \to \chi_{(0,a)} \circ f$$

on $L^1[0,a]$. Hence $\int_t^a f(s)ds = 0$ for $t \geq 0$, and we conclude that $f = 0$.

(v) $\Rightarrow$ (i) Suppose that $0 \notin \mathrm{supp}(k)$. Then there exists $\varepsilon > 0$ such that $k = 0$ on $[0,\varepsilon]$. Take $0 \neq \phi \in \mathcal{D}_+$ with $supp(\phi) \subset [0,\varepsilon]$. Note that $T'_k(\phi) = 0$ due to

$$kT'_k(\phi)(t) = k \circ \phi(t) = \int_t^\varepsilon k(s-t)\phi(s)ds = 0,$$

for $t \leq \varepsilon$. Since T'_k is injective we conclude that $\phi = 0$ and obtain a contradiction. □

Some properties of the operator T'_k are given in the next Theorem.

Theorem 2.5. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ be such that $0 \in supp(k)$.*

(i) *Then $T'_k : \mathcal{D}_+ \to \mathcal{D}_+$ is a injective, linear and continuous homomorphism such that*

$$T'_k(f \circ g) = f \circ T'_k(g), \qquad f, g \in \mathcal{D}_+.$$

(ii) *Take $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function and $C > 0$ such that $\int_0^t |k(s)|ds \leq C\tau(t)$ for $t \geq 0$. Then the map T'_k extends to a linear and continuous map from $L^1_\tau(\mathbb{R}^+)$ to $L^1(\mathbb{R}^+)$, which we denote again by $T'_k : L^1_\tau(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$ such that $\|T'_k\| \leq C$.*

Proof. Part (i) is an easy consequence from Proposition 2.1 and Theorem 2.4. To show (ii), take $f \in L^1_\tau(\mathbb{R}^+)$ and apply Fubini theorem to obtain that

$$\|T'_k(f)\|_1 = \int_0^\infty |\int_t^\infty k(s-t)f(s)ds|dt \leq \int_0^\infty \int_0^s |k(s-t)|dt\,|f(s)|ds \leq \|f\,\tau\|_1,$$

and we conclude the proof. □

Remark 2.6. Notice that, in general $T_k(\mathcal{D}_+) \not\subset \mathcal{D}_+$ for $k \in L^1_{loc}(\mathbb{R}^+)$. An easy example is provided by $k = \chi_{(0,\infty)}$.

With the above ingredients, we are able to present the following definition.

Definition 2.7. *Take $k \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(k)$. We define the space $\mathcal{D}_k$ by $\mathcal{D}_k := T'_k(\mathcal{D}_+)$ and the map $W_k : \mathcal{D}_k \to \mathcal{D}_+$, by*

$$f(t) =: T'_k(W_k f)(t) = \int_t^\infty k(s-t)W_k f(s)ds, \qquad f \in \mathcal{D}_k, \tag{2.10}$$

for $t \geq 0$.

Note that $\mathcal{D}_k$ is a subspace and W_k is a right-inverse of T'_k on $\mathcal{D}_k$. Also, observe that the functions $W_k f$ may be defined in a larger space than $\mathcal{D}_k$.

In some cases, it is possible to calculate explicitly $W_k f$. The following example will be very useful in the forthcoming sections: Take $k \in L^1_{loc}(\mathbb{R}^+)$ such that $\mathrm{abs}(k) < \infty$, where

$$\mathrm{abs}(k) := \inf\{\Re\lambda \ ; \ \text{ exists } \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda s}k(s)ds =: \hat{k}(\lambda)\},$$

see definition and properties in [2, Section 1.4]. If $\hat{k}(z) \neq 0$, for $\Re z > \mathrm{abs}(k)$ then we have

$$W_k(e_z) = \frac{1}{\hat{k}(z)} e_z. \tag{2.11}$$

where $e_z(t) := e^{-zt}$ for $t \geq 0$. Observe that we use the fact that T_k' is injective.

The following Lemma and Proposition, provide more information about the map W_k and the space $\mathcal{D}_k$.

Lemma 2.8. *Take $k, l \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(k) \cap supp(l)$.*

(i) *Take $f \in \mathcal{D}_k$. Then $f^{(n)} \in \mathcal{D}_k$ and $W_k(f^{(n)}) = (W_k f)^{(n)}$ for $n \geq 1$.*

(ii) *Then $0 \in supp(k * l)$, $\mathcal{D}_{k*l} \subset \mathcal{D}_k \cap \mathcal{D}_l$ and*

$$W_k f = l \circ W_{k*l} f, \quad f \in \mathcal{D}_{k*l}.$$

Proof. It is enough to show the equality (i) for $n = 1$. Note that

$$f(t) = \int_0^\infty k(s) W_k f(t+s) ds,$$

and then

$$f'(t) = \int_0^\infty k(s)(W_k f)'(t+s) ds = \int_t^\infty k(s-t)(W_k f)'(s) ds,$$

for $t \geq 0$.

By Titchmarsh's theorem (see, for example [4, Corollary 1.7]), $0 \in \text{supp}(k * l)$. Take $f \in \mathcal{D}_{k*l}$. Then

$$f = (k * l) \circ W_{k*l} f = k \circ (l \circ W_{k*l} f),$$

by Proposition 2.1. Hence $f \in \mathcal{D}_k$ and $W_k f = l \circ W_{k*l} f$ and we have proved (ii). $\square$

Proposition 2.9. *Take $z \in \mathbb{C}$, $e_z(t) := e^{-zt}$ for $t \geq 0$, $k \in L^1_{loc}(\mathbb{R}^+)$ such that $0 \in supp(k)$ and $k_z(t) := e^{zt} k(t)$ for $t \geq 0$. Then $k_z \in L^1_{loc}(\mathbb{R}^+)$ and $0 \in supp(k_z)$. Morever $\mathcal{D}_{k_z} = \{e_{-z} f \mid f \in \mathcal{D}_k\}$ and*

$$W_{k_z} f = e_z W_k(e_{-z} f), \qquad f \in \mathcal{D}_k.$$

Proof. Take $f \in \mathcal{D}_{k_z}$. Then there exists $W_{k_z} f \in \mathcal{D}_{k_z}$ such that

$$f(t) = \int_t^\infty F(s-t) e^{z(s-t)} W_{k_z} f(s) ds, \qquad t \geq 0.$$

Then we write

$$e^{zt} f(t) = \int_t^\infty F(s-t) e^{zs} W_{k_z} f(s) ds, \qquad t \geq 0,$$

and $e_{-z} f \in \mathcal{D}_k$ and $W_k(e_{-z} f) = e_{-z} W_{k_z} f$. Similarly we take $e_{-z} f$ with $t \in \mathcal{D}_k$ and we conclude that $e_{-z} f \in \mathcal{D}_{k_z}$ with $W_{k_z} f = e_z W_k(e_{-z} f)$. $\square$

Now we present some examples of functions k, spaces $\mathcal{D}_k$ and maps W_k. A limit case is for $k = \delta_0$, the Dirac measure concentrated in $\{0\}$. Then $\mathcal{D}_k = \mathcal{D}_+$ and $W_k = I_{\mathcal{D}_+}$.

Examples

(1) Given $\alpha > 0$, take $k(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$. In this case $T_k' : \mathcal{D}_+ \to \mathcal{D}_+$ is the Weyl fractional integral of order α, $W_{-\alpha}$,

$$(T_k)' f(t) = W_{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s) ds, \qquad f \in \mathcal{D}_+,$$

for $t \geq 0$. The map $W_k : \mathcal{D}_+ \to \mathcal{D}_+$ is the Weyl fractional derivative of order α, W_α,

$$W_\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty (t-x)^{n-\alpha-1} f(t) dt, \qquad f \in \mathcal{D}_+,$$

for $n > \alpha$ see, for example [12, Section 0.2]. Note that in this case $\mathcal{D}_k = \mathcal{D}_+$.

(2) Given $\alpha > 0$ and $z \in \mathbb{C}$, take $k(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{zt}$. By Proposition 2.9, and Example (1), we have that $\mathcal{D}_k = \mathcal{D}_+$ and

$$W_k f = e_z W_\alpha(e_{-z} f), \qquad f \in \mathcal{D}_+.$$

In particular for $\alpha = 1$, and $k(t) = e^{zt}$, we get $W_k f = -(zf + f')$; for $\alpha = 2$, $k(t) = te^{zt}$, we get

$$W_k f(t) = z^2 f(t) + 2zf'(t) + f''(t), \qquad f \in \mathcal{D}_+,$$

for $t \geq 0$.

(3) Take $k \in L^2(\mathbb{R}^+) \subset L^1_{loc}(\mathbb{R}^+)$ with $0 \in \operatorname{supp}(k)$. We write $\overline{K}(t) := k(-t)\chi_{(-\infty,0]}(t)$, for $t \in \mathbb{R}$. Let $f \in \mathcal{D}_+$. Then

$$k \circ f(t) = \int_{-\infty}^{\infty} \overline{K}(t-s) f(s) ds, \qquad t \geq 0.$$

Now take $g \in \mathcal{D}_k$. Then there exists $f \in \mathcal{D}_+$ such that $k \circ f = g$. We apply Fourier transform, $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, to conclude that

$$\mathcal{F}(\overline{K})(t)\mathcal{F}(f)(t) = \mathcal{F}(g)(t), \qquad t \in \mathbb{R}.$$

Then $\{t \in \mathbb{R} \,|\mathcal{F}(\overline{K})(t) = 0\} \subset \{t \in \mathbb{R} \,|\mathcal{F}(g)(t) = 0\}$. We conclude that there exists $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in \operatorname{supp}(k)$ such that $\mathcal{D}_k \neq \mathcal{D}_+$.

Our next result extends [12, Proposition 1.2] and [25, Proposition 1]. In particular, it shows that the space $\mathcal{D}_k$ is closed under the convolution products $*$, $\circ$ and $*_c$.

Theorem 2.10. *Take $f, g \in \mathcal{D}_k$. Then $f * g, f \circ g, f *_c g \in \mathcal{D}_k$ and*

$$\begin{aligned}
W_k(f * g)(s) &= \int_0^s W_k g(r) \int_{s-r}^s k(t+r-s) W_k f(t) dt\, dr \\
&\quad - \int_s^\infty W_k g(r) \int_s^\infty k(t+r-s) W_k f(t) dt\, dr, \\
W_k(f \circ g) &= f \circ W_k g, \\
W_k(f *_c g) &= \frac{1}{2} \left(W_k(f * g) + f \circ W_k g + g \circ W_k f \right).
\end{aligned}$$

Proof. Take $f, g \in \mathcal{D}_k$ and denote by h the right side member of the equality $W_k(f * g)$. Then

$$\begin{aligned}
h(s) &= \left(\int_0^s \int_{s-r}^\infty - \int_0^\infty \int_s^\infty \right) k(t+r-s) W_k f(t) dt\, W_k g(r) dr \\
&= (W_k g * f)(s) - \int_0^\infty \int_0^\infty k(t+r) W_k f(t+s) dt\, W_k g(r) dr,
\end{aligned}$$

for $s > 0$ and h belongs to $\mathcal{D}_+$. On the other hand, if $x > 0$,

$$\begin{aligned}
\int_x^\infty k(s-x)(W_k g * f)(s) ds &= \int_x^\infty k(s-x) \int_0^s W_k g(s-r) f(r) dr\, ds \\
&= g * f(x) + \int_x^\infty \int_r^\infty k(s-x) W_k g(s-r) f(r) ds\, dr \\
&= g * f(x) + \int_0^\infty \int_0^\infty \int_0^\infty W_k g(s) k(s+r) W_k f(x+t+r) k(t) dt\, dr\, ds.
\end{aligned}$$

where we apply Fubbini theorem and definition of $W_k f$. In conclusion, we have

$$\begin{aligned}
&\int_x^\infty k(s-x)h(s)ds = (g*f)(x)\\
&\quad + \int_0^\infty \int_0^\infty \int_0^\infty W_k g(s) k(s+r) W_k f(x+t+r) k(t) dt\, dr\, ds\\
&\quad - \int_x^\infty k(s-x) \int_0^\infty \int_0^\infty k(t+r) W_k f(t+s) dt\, W_k g(r) dr\, ds\\
= &\quad (g*f)(x) + \int_0^\infty \int_0^\infty \int_0^\infty W_k g(s) k(s+r) W_k f(x+t+r) k(t) dt\, dr\, ds\\
&- \int_0^\infty \int_0^\infty \int_0^\infty W_k g(r) k(t+r) W_k f(x+s+t) k(s) ds\, dt\, dr = (g*f)(x),
\end{aligned}$$

for every $x > 0$, as desired.

To check $W_k(f \circ g)$, we apply Proposition 2.1 to obtain

$$k \circ (f \circ W_k g) = f \circ (k \circ W_k g) = f \circ g.$$

We conclude that $W_k(f \circ g) = f \circ W_k g$. Since W_k is a linear map we get $W_k(f *_c g)$. $\square$

3. Banach algebras for convolution products

Let $k \in L^1_{loc}(\mathbb{R}^+)$. The main aim of this section is to define an algebraic norm on $\mathcal{D}_k$ for the convolution product $*$ and $*_c$ (Theorem 3.5). This will be done with the help of certain classes of weight functions that we state in the following.

Definition 3.1. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ and $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ be a locally integrable function.*

(i) *We say that $\tau \in A_k$, if there exists $C > 0$ such that*

$$\int_0^s \tau(u)|k(r+s-u)|du + \int_r^{r+s} \tau(u)|k(r+s-u)|du \le C\tau(r)\tau(s), \qquad 0 \le s \le r.$$

(ii) *We say that $\tau \in B_k$ if $\tau \in A_k$, is a non decreasing, continuous function and there exists $M > 0$ such that*

$$\int_0^t |k(s)|ds \le M\tau(t), \qquad t \ge 0.$$

Example 3.2. (1) For $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, examples of functions in A_k or B_k, are given in [12, p.15]:

(i) $\tau(t) := t^\beta(1 + t^\mu)$, with $0 \le \beta \le \alpha$ and $\beta + \mu \ge \alpha$ belongs to A_k;
(ii) $\tau(t) := t^\alpha \omega(t)$ whenever ω is a non decreasing continuous weight on $[0, \infty)$ belongs to B_k ;
(iii) $\tau(t) := t^\nu e^{\rho t}$, for $0 \le \nu \le \alpha$ and $\rho > 0$ belongs to A_k.

(2) For $k(t) = e^{-\alpha t}$ with $\alpha > 0$, then $\tau(t) := te^{\beta t}$ for $\beta \ge \alpha$ belongs to B_k.
(3) Take $k \in L^1_{loc}(\mathbb{R}^+)$ and τ a non decreasing function such that $\tau(2t) \le C\tau(t)$ and $\int_0^t |k(s)|ds \le C\tau(t)$ for $t \ge 0$. Then $\tau \in B_k$, in particular if $\int_0^{2t} |k(s)|ds \le C \int_0^t |k(s)|ds$ then $\tau(t) := \int_0^t |k(s)|ds$ belongs to B_k.
(4) Take $k \in L^1_{loc}(\mathbb{R}^+)$ such that $\mathrm{abs}(|k|) < \infty$. Then $e_{-\beta} \in B_k$ for $\beta > \max\{\mathrm{abs}(|k|), 0\}$.

The next lemma is an extension of the integrated semigroup case obtained in [25].

Lemma 3.3. *Take $k \in L^1_{loc}(\mathbb{R}^+)$ and $\tau \in B_k$. There exists $C > 0$ such that*

(i) $\displaystyle\int_{t-s}^{t} |k(r-t+s)|\tau(r)dr \le C\tau(t)\tau(s)$,

(ii) $\displaystyle\int_{0}^{s} |k(r+t-s)|\tau_\alpha(r)dr \le C\tau(t)\tau(s)$,

for $0 \le s \le t$.

Proof. As τ is a non decreasing continuous function, we get that

$$\int_{t-s}^{t} |k(r-t+s)|\tau(r)dr \le \tau(t)\int_{t-s}^{t} |k(r-t+s)|dr = \tau(t)\int_0^s |k(u)|du \le C\tau(t)\tau(s).$$

The part (ii) is proven in a similar way. □

The next results extends [12, Proposition 1.4].

Theorem 3.4. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$ and $\tau \in A_k$. Then the integral*

$$\|f\|_{k,\tau} := \int_0^\infty |W_k f(t)|\tau(t)dt, \qquad f \in \mathcal{D}_k,$$

defines an algebra norm on $\mathcal{D}_k$ for the convolution product $*$. We denote by $\mathcal{T}_k(\tau)$ the Banach space obtained as the completion of $\mathcal{D}_k$ in the norm $\| \quad \|_{k,\tau}$.

Proof. It is straightforward to check that $\|\cdot\|_{k,\tau}$ is a norm on $\mathcal{D}_k$. We apply Theorem 2.10 to get

$$\begin{aligned}\|f*g\|_{k,\tau} \;&\le\; \int_0^\infty\int_0^t |W_k g(r)|\int_{t-r}^t k(s+r-t)|W_k f(s)|ds\,dr\tau(t)dt\\ &+\; \int_0^{+\infty}\int_t^\infty |W_k g(r)|\int_t^\infty k(s+r-t)|W_k f(s)|ds\,dr\tau(t)dt,\end{aligned}$$

for all $f, g \in \mathcal{D}_k$. Using Fubini theorem and simplifying we get

$$\|f*g\|_{k,\tau} \le \int_0^\infty |W_k g(r)|\int_0^\infty |W_k f(s)|\int_{I(r,s)} k(s+r-t)\tau(t)dt\,ds\,dr,$$

where $I(r,s) = [0, \min(r,s)] \cup [\max(r,s), r+s]$. Since $\tau \in A_k$, we obtain

$$\|f*g\|_{k,\tau} \le C\left(\int_0^\infty |W_k g(r)|\tau(r)dr\right)\left(\int_0^\infty |W_k f(s)|\tau(s)ds\right)$$

and the result follows. □

The next theorem extends [25, Theorem 3].

Theorem 3.5. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$ and $\tau \in B_k$. Then the integral*

$$\|f\|_{k,\tau} = \int_0^\infty |W_k f(t)|\tau(t)dt, \qquad f \in \mathcal{D}_k,$$

defines an algebra norm on $\mathcal{D}_k$ for the convolution product $*_c$. Moreover, if we denote by $\mathcal{T}_k(\tau)$ the Banach space obtained as the completion of $\mathcal{D}_k$ in the norm $\| \quad \|_{k,\tau}$ then $\mathcal{T}_k(\tau) \hookrightarrow L^1(\mathbb{R}^+)$.

Proof. First we check that $\|f\circ g\|_{k,\tau}\le C\|f\|_{k,\tau}\,\|g\|_{k,\tau}$, for $f,g\in\mathcal{D}_k$. By Theorem 2.10, we have that

$$\begin{aligned}\|f\circ g\|_{k,\tau} &\le \int_0^\infty \tau(t)\int_t^\infty\int_{s-t}^\infty |k(u-s+t)||W_kf(u)|du|W_kg(s)|dsdt\\ &\le \int_0^\infty |W_kg(s)|\int_0^s |W_kf(u)|\int_{s-u}^s |k(u-s+t)|\tau(t)dtduds\\ &\quad+\int_0^\infty |W_kg(s)|\int_s^\infty |W_kf(u)|\int_0^s |k(u-s+t)|\tau(t)dtduds\\ &\le C\|f\|_{k,\tau}\|g\|_{k,\tau}.\end{aligned}$$

where we have applied Lemma 3.3. As $f*_c g=\frac{1}{2}(f*g+f\circ g+g\circ f)$, we apply Theorem 3.4 to conclude that there exists $C>0$ such that

$$\|f*_c g\|_{k,\tau}\le C\|f\|_{k,\tau}\|g\|_{k,\tau},\qquad f,g\in\mathcal{D}_k.$$

Take $f\in\mathcal{D}_k$ and

$$\|f\|_1=\int_0^\infty\Big|\int_t^\infty k(s-t)W_kf(s)ds\Big|dt\le\int_0^\infty |W_kf(s)|\int_0^s|k(s-t)|dsdt\le C\|f\|_{k,\tau},$$

and we conclude that $\mathcal{T}_k(\tau)\hookrightarrow L^1(\mathbb{R}^+)$. ☐

Our next result, identify the elements of the space $\mathcal{T}_k(\tau)$ in case $\tau\in B_k$.

Theorem 3.6. *Take $k\in L^1_{loc}(\mathbb{R}^+)$ with $0\in supp(k)$ and $\tau\in B_k$. Then $f\in\mathcal{T}_k(\tau)$ if and only if there exists $g\in L^1_\tau(\mathbb{R}^+)$ such that $T'_k(g)=f$. Following the usual conventions, we write $W_kf=g$.*

Proof. Take $f\in\mathcal{T}_k(\tau)\hookrightarrow L^1(\mathbb{R}^+)$. Then there exists $(f_n)_{n\in\mathbb{N}}\subset\mathcal{D}_k$ such that $f_n\to f$ in $\mathcal{T}_k(\tau)$. Note that there exists a sequence $(g_n)_{n\in\mathbb{N}}\subset\mathcal{D}_+$ such that $k\circ f_n=g_n$ for $n\in\mathbb{N}$ and $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1_\tau(\mathbb{R}^+)$. Then there exists $g\in L^1_\tau(\mathbb{R}^+)$ such that $g_n\to g$ in $L^1_\tau(\mathbb{R}^+)$. By Theorem 2.5 (ii), $f_n=k\circ g_n\to k\circ g$ in $L^1(\mathbb{R}^+)$. We conclude that $k\circ g=f$.

Let $g\in L^1_\tau(\mathbb{R}^+)$ such that $T'_k(g)=f$. Then there exists $(g_n)_{n\in\mathbb{N}}\subset\mathcal{D}_+$ such that $g_n\to g$ in $L^1_\tau(\mathbb{R}^+)$. By Theorem 2.5(i) and (ii), $f_n:=k\circ g_n\in\mathcal{D}_+$ and $f_n\to f$ in $L^1(\mathbb{R}^+)$. Moreover, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{T}_k(\tau)$ and then there exists limit. We conclude that $f_n\to f$ in $\mathcal{T}_k(\tau)$. ☐

The following theorem specializes in case $\tau=e_{-\beta}$ (see Example 3.2(4)).

Theorem 3.7. *Take $k\in L^1_{loc}(\mathbb{R}^+)$ such that $abs(|k|)<\infty$ and $\beta>\max\{abs(|k|),0\}$. We consider the Banach algebra $(\mathcal{T}_k(e_{-\beta}),*)$.*

(i) *Take $(K_t)_{t\ge0}\subset L^1(\mathbb{R}^+)$ defined by*

$$K_t(s):=\chi_{[0,t]}(s)\int_0^{t-s}k(u)du,\qquad t,s\ge0.\tag{3.1}$$

Then $W_kK_t=\chi_{[0,t]}$, $(K_t)_{t\ge0}\subset\mathcal{T}_k(e_{-\beta})$, $\widehat{K_{(\cdot)}(s)}(\lambda)=e_\lambda(s)\frac{1}{\lambda}\hat{k}(\lambda)$, for $\lambda>\beta$, and

$$K_t*K_s=\int_t^{t+s}K(t+s-u)K_udu-\int_0^sK(t+s-u)K_udu\tag{3.2}$$

$$(3.3)\qquad \begin{aligned} 2(K_t *_c K_s) &= \int_t^{t+s} K(t+s-u)K_u du - \int_0^s K(t+s-u)K_u du \\ &\quad + \int_0^s K(u+t-s)K_u du + \int_{t-s}^t K(u+s-t)K_u du. \end{aligned}$$

where $K(u) := \int_0^t k(u)du$, *for* $t \geq 0$.

(ii) *Take* $z \in \Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}$. *Then* $e_z \in \mathcal{T}_k(e_{-\beta})$,

$$\|e_z\|_{k,e_{-\beta}} = \frac{1}{|\hat{k}(z)|}\frac{1}{\Re z - \beta},$$

and the family $(e_z)_{z\in\Omega_k}$ *is a pseudo-resolvent in* $\mathcal{T}_k(e_{-\beta})$, *i.e.*,

$$e_z * e_\mu = \frac{1}{(\mu - z)}\left(e_z - e_\mu\right), \qquad z, \mu \in \Omega_{k,\beta}.$$

Proof. Take $t \geq 0$ and note that $\chi_{[0,t)} \in L^1_{e_{-\beta}}(\mathbb{R}^+)$, and

$$k \circ \chi_{[0,t)}(s) = \int_s^\infty k(u-s)\chi_{[0,t)}(u)du = \chi_{[0,t)}(s)\int_s^t k(u-s)du = \chi_{[0,t)}(s)\int_0^{t-s} k(u)du,$$

for $t, s \geq 0$. By Theorem 3.6, $K_t \in \mathcal{T}_k(e_{-\beta})$ and $W_k K_t = \chi_{[0,t)}$ for $t \geq 0$. Take $\lambda > \beta$,

$$\int_0^\infty e^{-\lambda t}K_t(s)dt = \int_s^\infty \int_0^{t-s} k(u)du e^{-\lambda t}dt = e^{-\lambda s}\int_0^\infty k(u)du\int_u^\infty e^{-\lambda t}dt = e_\lambda(s)\frac{1}{\lambda}\hat{k}(\lambda),$$

for $s \geq 0$. To conclude the equality (3.2), we apply the equality (2.3) to functions K and $(K_t)_{t\geq 0}$, defined by

$$K(t) := \int_0^t k(u)du, \qquad K_t(s) := \chi_{[0,t]}(s)K(t-s), \qquad t, s \geq 0.$$

To obtain, the equality (3.3), we apply equalities (2.3), (2.4), and (2.5) to functions K and $(K_t)_{t\geq 0}$ defined above.

Take $z \in \Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}$. By formula (2.11) and Theorem 3.6, functions $e_z \in \mathcal{T}_k(e_{-\beta})$ and

$$\|e_z\|_{k,e_{-\beta}} = \frac{1}{|\hat{k}(z)|}\frac{1}{\Re z - \beta}.$$

Since $\mathcal{T}_k(e_{-\beta}) \subset L^1(\mathbb{R}^+)$, the family $(e_z)_{z\in\Omega_k}$ is a pseudo-resolvent in $\mathcal{T}_k(e_{-\beta})$. $\square$

4. Laplace transforms and functional equations

Let X be a complex Banach space. If $\Omega \subset \mathbb{C}$ is an open set and $R : \Omega \to \mathcal{B}(X)$ is an analytic function obeying

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu), \quad \lambda, \mu \in \Omega,$$

then we call $R(\lambda)$ an (operator valued) *pseudo-resolvent*. We recall that given a pseudo resolvent $R : \Omega \to \mathcal{B}(X)$ then there is an operator A on X such that $R(\lambda) = (\lambda - A)^{-1}$ for all $\lambda \in \Omega$ if and only if $KerR(\lambda) = \{0\}$ (see [2, Proposition B.6]).

Let $S, \ T : \mathbb{R}^+ \to \mathcal{B}(X)$ be strongly continuous functions satisfying $||S(t)|| \leq Me^{\omega t}$ and $||T(t)|| \leq Me^{\omega t}$ $(t \geq 0)$ for some $\omega \in \mathbb{R}$, $M \geq 0$ (for simplicity we may assume the same constants). Denote $\hat{S}(\lambda)=\displaystyle\int_0^\infty e^{-\lambda t}S(t)dt$ the Laplace transform of S (respectively T). For $h\in\mathbb{R}$ we shall denote S^h the translation $S^h(u):=S(u+h)\chi_{[-h,+\infty)}(u)$ for $u\in\mathbb{R}$ and

$$(T*S)(t):=\int_0^t T(t-s)S(s)ds,\quad t>0,$$

the convolution product between T and S. The following lemma shows how the Laplace transform connects the product of two operator-valued functions, including translations.

Lemma 4.1. *Let $S,\quad T:[0,\infty)\to\mathcal{B}(X)$ be strongly continuous functions satisfying the assumptions above. For $\lambda>\mu>\omega$, the following identities are valid:*

$$\hat{S}(\lambda)\hat{T}(\mu)=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(t)T(s)dsdt \tag{4.1}$$

$$\frac{1}{\mu-\lambda}(\hat{S}(\lambda)-\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(t+s)dsdt \tag{4.2}$$

$$\frac{1}{\mu-\lambda}\hat{T}(\mu)[\hat{S}(\lambda)-\hat{S}(\mu)]=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S_t)(s)dsdt \tag{4.3}$$

Defining $S(t)=S(-t)$ for $t<0$ we have,

$$\frac{1}{\mu+\lambda}(\hat{S}(\lambda)+\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(s-t)dsdt,\quad \lambda+\mu>0. \tag{4.4}$$

and

$$\frac{1}{\mu+\lambda}\hat{T}(\mu)(\hat{S}(\lambda)+\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^{-t})(s)dsdt,\quad \lambda+\mu>0. \tag{4.5}$$

and defining $S(t):=-S(-t)$ for $t<0$ we obtain

$$\frac{-1}{\mu+\lambda}\hat{T}(\mu)(\hat{S}(\lambda)-\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^{-t})(s)dsdt,\quad \lambda+\mu>0. \tag{4.6}$$

Proof. Is clear that (4.1) holds. The proof of (4.2) is contained in [1, Proposition 2.2]. To prove (4.3) we observe first the following identities

$$\hat{S}^t(\mu)=e^{\mu t}\hat{S}(\mu)-e^{\mu t}\int_0^t e^{-\mu s}S(s)ds$$

and, for $\lambda>\mu$

$$\int_0^\infty e^{-(\lambda-\mu)t}\int_0^t e^{-\mu s}S(s)dsdt=\frac{1}{\lambda-\mu}\hat{S}(\lambda),$$

which can be proved by integration by parts. Then $\displaystyle\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^t)(s)dsdt=\int_0^\infty e^{-\lambda t}\hat{T}(\mu)\hat{S}^t(\mu)dt$

$$= \hat{T}(\mu)\int_0^\infty e^{-\lambda t}[e^{\mu t}\hat{S}(\mu) - e^{\mu t}\int_0^t e^{-\mu s}S(s)ds]dt$$
$$= \hat{T}(\mu)\hat{S}(\mu)\int_0^\infty e^{-(\lambda-\mu)t}dt - \hat{T}(\mu)\int_0^\infty e^{-(\lambda-\mu)t}\int_0^t e^{-\mu s}S(s)dsdt$$
$$= \frac{1}{\mu-\lambda}\hat{T}(\mu)[\hat{S}(\lambda) - \hat{S}(\mu)].$$

To show (4.4), note that by means of integration by parts we can prove the following identity

$$\int_0^\infty e^{-(\lambda+\mu)t}\int_0^t e^{\mu s}S(-s)dsdt = \frac{1}{\lambda+\mu}\hat{S}(\lambda),$$

and hence

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(s-t)dsdt &= \int_0^\infty e^{-(\lambda+\mu)t}\int_{-t}^\infty e^{-\mu s}S(s)dsdt \\ &= \int_0^\infty e^{-(\lambda+\mu)t}\int_0^\infty e^{-\mu s}S(s)dsdt \\ &+ \int_0^\infty e^{-(\lambda+\mu)t}\int_0^t e^{\mu s}S(-s)dsdt \\ &= \frac{1}{\mu+\lambda}(\hat{S}(\lambda)+\hat{S}(\mu)).\end{aligned}$$

We now show (4.5). Using the fact that $S(t)$ is even function, we have

$$\hat{S}^{-t}(\mu) = e^{-\mu t}\hat{S}(\mu) + e^{-\mu t}\int_0^t e^{\mu s}S(s)ds.$$

By means of integration by parts we can prove the following identity

$$\int_0^\infty e^{-(\lambda+\mu)t}\int_0^t e^{\mu s}S(-s)dsdt = \frac{1}{\lambda+\mu}\hat{S}(\lambda),$$

hence

$$\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^{-t})(s)dsdt = \int_0^\infty e^{-\lambda t}\hat{T}(\mu)\hat{S}^{-t}(\mu)dt$$

$$\begin{aligned}&= \hat{T}(\mu)\int_0^\infty e^{-\lambda t}[e^{-\mu t}\hat{S}(\mu) + e^{-\mu t}\int_0^t e^{\mu s}S(s)ds]dt \\ &= \hat{T}(\mu)\hat{S}(\mu)\int_0^\infty e^{-(\lambda+\mu)t}dt + \hat{T}(\mu)\int_0^\infty e^{-(\lambda+\mu)t}\int_0^t e^{\mu s}S(s)dsdt \\ &= \frac{1}{\mu+\lambda}\hat{T}(\mu)[\hat{S}(\lambda)+\hat{S}(\mu)]\end{aligned}$$

and we conclude the claim. The last identity is proved similarly. □

The main result of this section is the following theorem.

Theorem 4.2. *Let $S,\ T : \mathbb{R}^+ \to \mathcal{B}(X)$ be strongly continuous functions such that for some $M,\ \omega \in \mathbb{R}$, we have $||S(t)|| \leq Me^{\omega t}$ and $||T(t)|| \leq Me^{\omega t}$. Assume that $\hat{T}(\lambda)$ is invertible and let $R(\lambda)$ be defined by*

$$R(\lambda) = \hat{T}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} S(s) ds, \quad \lambda > \omega. \tag{4.7}$$

Then $(R(\lambda))_{\lambda>\omega}$ *is a pseudo resolvent if and only if*

$$S(t)S(s) = (T * S^t)(s) - (T^t * S)(s), \quad t, s \in \mathbb{R}. \tag{4.8}$$

Proof. Let $\lambda > \mu > \omega$. According the definition of pseudo resolvent, we obtain from (4.7) that $(R(\lambda))_{\lambda>\omega}$ is a pseudo resolvent if and only if

$$\hat{S}(\lambda)\hat{S}(\mu) = \frac{1}{\mu-\lambda}[\hat{T}(\mu)\hat{S}(\lambda)-\hat{T}(\lambda)\hat{S}(\mu)] = \frac{1}{\mu-\lambda}\hat{T}(\mu)[\hat{S}(\lambda)-\hat{S}(\mu)]+\frac{1}{\mu-\lambda}[\hat{T}(\mu)-\hat{T}(\lambda)]\hat{S}(\mu).$$

Now we can use (4.1) and (4.3) to obtain

$$\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s} S(s)S(t) ds dt = \int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}((T * S^t)(s) - (T^t * S)(s)) ds dt.$$

Then the claim follows from the uniqueness theorem for Laplace transforms (see [2, Theorem 1.7.3]). ☐

Remark 4.3. Note that condition (4.8) implies $S(t)S(s) = S(s)S(t)$ for all $s, t \geq 0$ and $S(t)S(0) = 0$ for all $t \geq 0$.

Let $k \in L^1_{loc}(\mathbb{R}^+)$ and $T(t)x := k(t)x$ for $x \in X$. We observe that the right hand side of (4.8) can be written as $(k * S)(t+s) - (k^s * S)(t) - (k^t * S)(s)$, which is equivalent to

$$S(t)S(s) = \int_t^{t+s} k(t+s-r)S(r)dr - \int_0^s k(t+s-r)S(r)dr.$$

This functional equation is the key to define k-convoluted semigroups, see Proposition 5.3 below.

In the particular case $k(t) = \frac{t^{n-1}}{n!}$, Theorem 4.2 appears in [1, Proposition 2.2 and Theorem 2.1], see also [2, Proposition 3.2.4]. More generally we can take $T(t)x = \frac{t^{n-1}Cx}{n!}$, where C is a regularizing operator (see [13, Definition 2.2]) and obtain the corresponding functional equation for the theory of regularized semigroups introduced by Da Prato [10]. This point of view of Theorem 4.2 have been applied in the D'Alembert functional equation for cosine functions (see [2, Proposition 3.14.4]) or in the function equation for n-times integrated cosine functions ([3]).

5. Algebra homomorphisms defined via k-convoluted semigroups

The definition of k-convoluted semigroups was introduced by the first time by Cioranescu [8]. The below definition, is adapted from the theory of C_0-semigroups as appears in [2, Theorem 3.1.12] and also used in [20] to define the concept of regularized families. The main purpose of this section is to show an algebra homomorphism using the k-convoluted families as integral kernel, see Theorem 5.5 below.

Definition 5.1. *Let A be a closed operator, $k \in L^1_{loc}(\mathbb{R}^+)$ and $(S_k(t))_{t\geq 0} \subset \mathcal{B}(X)$ a strongly continuous operator family. The family $(S_k(t))_{t\geq 0}$ is a k-convoluted semigroup (or k-semigroup in short) generated by A if $S_k(t)A \subset AS_k(t)$, $\int_0^t S_k(s)xds \in D(A)$ for $t \geq 0$ and $x \in X$ and*

$$A\int_0^t S_k(t)xdt = S_k(t)x - \int_0^t k(s)dsx, \quad x \in X, \tag{5.1}$$

for $t \geq 0$; in this case the operator A is called the generator of $(S_k(t))_{t\geq 0}$. We say that $(S_k(t))_{t\geq 0}$ is exponentially bounded if there exists $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S_k(t)\| \leq Ce^{\omega t}$ for $t \geq 0$; non degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

If A generates a k-convoluted semigroup $(S_k(t))_{t\geq 0}$, then $S_k(0) = 0$ and $S_k(t)x \in \overline{D(A)}$ for $t \geq 0$ and $x \in X$. See these definitions and properties, for example in [9], [16] and [18]. Notice that k-convoluted semigroups are a particular case of regularized resolvent families defined in [20].

In the exponentially bounded case, equivalent definitions can be given (see e.g. [16, Definition 3.5]) based on the Laplace transform of vector-valued functions. Note that Hille-Yosida type theorems for local convoluted semigroups are proved in [16], based on the study of the vector-valued local Laplace transform. Well-known examples of k-convoluted semigroups are α-times integrated semigroup where $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0$.

The next proposition shows how to generate new k-convoluted semigroups.

Proposition 5.2. *Let $k, l \in L^1_{loc}(\mathbb{R}^+)$ and $(S_k(t))_{t\geq 0}$ a k-convoluted semigroup generated by A. Then $((l * S_k)(t))_{t\geq 0}$ is a $k * l$-convoluted semigroup generated by A.*

Proof. It is straightforward to check $(l * S_k)(t)A \subset A(l * S_k)(t)$,

$$\int_0^t (l * S_k)(s)xds = (\chi_{(0,\infty)} * (l * S_k))(t) = (l * (\chi_{(0,\infty)} * S_k))(t) \in D(A)$$

for $t \geq 0$ and $x \in X$. Now we check the equality (5.1),

$$\begin{aligned} A\int_0^t (l * S_k)(t)xdt &= A(\chi_{(0,\infty)} * (l * S_k))(t)x = A(l * (\chi_{(0,\infty)} * S_k))(t)x \\ &= (l * (S_k - \chi_{(0,\infty)} * k))(t)x = (l * S_k)(t)x - (\chi_{(0,\infty)} * (l * k))(t)x, \end{aligned}$$

and we conclude the proof. □

The next characterization can be used as a definition to *local* k-convoluted semigroups. It has the advantage to offer an algebraic character which may be used in many different contexts.

Proposition 5.3. [18, Proposition 2.2] *Let $k \in L^1_{loc}(\mathbb{R}^+)$, A a closed linear operator and $(S_k(t))_{t\geq 0}$ a non-degenerate strongly continuous operator family. Then $(S_k(t))_{t\geq 0}$ is a k-convoluted semigroup generated by A if and only if $S_k(0) = 0$ and*

$$S_k(t)S_k(s) = (k * (S_k)^t)(s) - (k^t * S_k)(s), \tag{5.2}$$

for $t, s \geq 0$

Remark 5.4. In the case of exponentially bounded k-convoluted semigroups, the proof of the above proposition can be directly deduced from Theorem 4.2 and [20, Proposition 3.1].

In what follows, recall that $\mathcal{D}_k := T_k'(\mathcal{D}_+)$ where T_k' is defined by (2.8). Moreover, we will make extensively use of the maps $W_k : \mathcal{D}_k \to \mathcal{D}_+$ defined in (2.10). The following is the main result of this section.

Theorem 5.5. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, and $(S_k(t))_{t\geq 0}$ a non-degenerate k-convoluted semigroup generated by A. We define the map $\Theta_k : \mathcal{D}_k \to \mathcal{B}(X)$ by*

$$\Theta_k(f)x := \int_0^\infty W_k f(t) S_k(t) x dt, \qquad x \in X.$$

Then the following properties hold.

(i) *$\Theta_k(f * g) = \Theta_k(f)\Theta_k(g)$ for any $f, g \in \mathcal{D}_k$.*
(ii) *$\Theta_k(f)x \in D(A)$ and $A\Theta_k(f)x = -\Theta_k(f')x - f(0)x$ for any $f \in \mathcal{D}_k$ and $x \in X$.*
(iii) *If $x \in D(A)$ then the map $t \mapsto S(t)x$ is differentiable almost everywhere and*

$$\Theta_k(f')x = -\int_0^\infty W_k f(t) \frac{d}{dt} S_k(t) x dt, \qquad f \in \mathcal{D}_k.$$

(iv) *Take $l \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(l)$ and $((l * S_k)(t))_{t\geq 0}$ the $k * l$-convoluted semigroup given in Proposition 5.2. Then*

$$\Theta_{k*l}(f) = \Theta_k(f), \qquad f \in \mathcal{D}_{k*l}.$$

Proof. Since $W_k f \in \mathcal{D}_+$ then the expresion

$$\Theta_k(f)x = \int_0^\infty W_k f(t) S_k(t) x dt,$$

is well defined for $x \in X$ and $\Theta_k : \mathcal{D}_k \to \mathcal{B}(X)$ defines a linear map.

Take $f, g \in \mathcal{D}_k$ and then $f * g \in \mathcal{D}_k$ by Theorem 2.10. Now we show that $\Theta_k(f * g) = \Theta_k(f)\Theta_k(g)$. By Theorem 2.10 again, we have that

$$\begin{aligned} \Theta_k(f * g)x &= \int_0^\infty W_k(f * g)(t) S_\alpha(t) x dt \\ &= \int_0^\infty \int_0^t W_k g(r) \int_{t-r}^t k(s + r - t) W_k f(s) ds dr S_k(t) x dt \\ &\quad - \int_0^\infty \int_t^\infty W_k g(r) \int_t^\infty k(s + r - t) W_k f(s) ds dr S_\alpha(t) x dt. \end{aligned}$$

By Fubini theorem, we obtain the four integral expressions

$$\begin{aligned} \Theta_k(f * g)x &= \int_0^\infty W_k g(r) \int_0^r W_k f(s) \int_r^{s+r} k(s + r - t) S_k(t) x dt ds dr \\ &\quad + \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \int_s^{s+r} k(s + r - t) S_k(t) x dt ds dr \\ &\quad - \int_0^\infty W_k g(r) \int_0^r W_k f(s) \int_0^s k(s + r - t) S_k(t) x dt ds dr \\ &\quad - \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \int_0^r k(s + r - t) S_k(t) x dt ds dr \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty W_k g(r) \int_0^r W_k f(s) \left(\int_r^{s+r} - \int_0^s k(s+r-t) S_k(t) x dt \right) dsdr \\
&\quad + \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \left(\int_s^{s+r} - \int_0^r k(s+r-t) S_k(t) x dt \right) dsdr \\
&= \int_0^\infty W_k g(r) S_k(r) \int_0^r W_k f(s) S_k(s) x dsdr \\
&\quad + \int_0^\infty W_k g(r) S_k(r) \int_r^\infty W_k f(s) S_k(s) x dsdr = \Theta_k(g)\Theta_k(f),
\end{aligned}
$$

where we have applied the formula (5.2).

To show the part (ii), we consider $f \in \mathcal{D}_k$ and $x \in X$. We apply Lemma 2.8 and formula (5.1) to get

$$
\begin{aligned}
A\Theta_k(f)x &= -A \int_0^\infty (W_k f)'(t) \int_0^t S_k(s) x dsdt = -\int_0^\infty W_k f'(t) \left(S_k(t)x - \int_0^t k(s) dsx \right) dt \\
&= -\Theta_k(f')x - \int_0^\infty W_k f(t) k(t) dtx = -\Theta_k(f')x - f(0)x.
\end{aligned}
$$

To show the part (iii), take $x \in D(A)$ and by the formula (5.1), we obtain

$$
\frac{d}{dt} S_k(t)x = AS_k(t)x + k(t)x,
$$

almost everywhere and by the part (ii) we get that

$$
\int_0^\infty W_k f(t) \frac{d}{dt} S_k(t) x dt = A \int_0^\infty W_k f(t) S_k(t) x dt + \int_0^\infty W_k f(t) k(t) x dt = -\Theta_k(f')x.
$$

To finish the proof take $f \in \mathcal{D}_{k*l}$ and

$$
\Theta_{k*l}(f) = \int_0^\infty W_{k*l} f(t) (l * S_k)(t) x dt = \int_0^\infty (k \circ W_{k*l} f)(t) S_k(t) x dt = \Theta(f),
$$

where we apply Fubini theorem and Lemma 2.8. The proof is concluded. □

Next result extends some known results for n-times integrated semigroups (see [3, Theorem 4.4] and [32]) and α-times integrated semigroups, [24, Theorem 3.1].

Corollary 5.6. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $\tau \in A_k$ and $(S_k(t))_{t\geq 0}$ a non-degenerate k-convoluted semigroup such that $\|S(t)\| \leq C\tau(t)$ for $t \geq 0$. Then the map $\Theta_k : (\mathcal{T}_k(\tau), *)) \to \mathcal{B}(X)$ defined by*

$$
\Theta_k(f)x := \int_0^\infty W_k f(t) S_k(t) x dt, \qquad x \in X, f \in \mathcal{T}_k(\tau),
$$

is a bounded algebra homomorphism.

The following consequence is one of the main results of this paper. It shows an interesting interplay between different areas like functional equations, algebra and operator-valued analytic functions, in addition to the link with integral equations of Volterra type as stated in the introduction (cf. also [20]).

Theorem 5.7. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $abs(|k|) < \infty$ and $\beta > \max\{abs(|k|), 0\}$; $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ a strongly continuous family such that $\|S(t)\| \leq Ce^{\beta t}$ for $t \geq 0$. Define*

$$R(\lambda)x := \frac{1}{\hat{k}(\lambda)} \int_0^\infty e^{-\lambda s} S(s)xds, \qquad \lambda \in \Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}, \tag{5.3}$$

and the map $\Theta_k : (\mathcal{T}_k(e_{-\beta}), *) \to \mathcal{B}(X)$ by

$$\Theta_k(f) := \int_0^\infty W_k f(t) S(t)xdt, \qquad f \in \mathcal{D}_k,$$

for $x \in X$. Then the following conditions are equivalent.

(i) *The family $(S(t))_{t\geq 0}$ satisfies the equation (5.2).*
(ii) *The map Θ_k is a bounded algebra homomorphism.*
(iii) *The family $(R(\lambda))_{\lambda \in \Omega_{k,\beta}}$ is a pseudo-resolvent.*

Proof. The proof (i) $\Rightarrow$ (ii) follows the same lines that Theorem 5.5 (i).
To show (ii) $\Rightarrow$ (iii), take $(e_\lambda)_{\lambda \in \Omega_{k,\beta}} \subset \mathcal{T}_k(e_{-\beta})$. Note that $\Theta_k(e_\lambda) = R(\lambda)$ for $\lambda \in \Omega_{k,\beta}$ by (2.11). Since $(e_\lambda)_{\lambda \in \Omega_{k,\beta}}$ is a pseudo-resolvent in $\mathcal{T}_k(e_{-\beta})$ we conclude that $R(\lambda))_{\lambda \in \Omega_{k,\beta}}$ is a pseudo-resolvent.
The proof of (iii)$\Rightarrow$ (i) is included in Theorem 4.2, taking $T(t)x := k(t)x$ for $t \geq 0$. $\square$

Our last result in this section, show that the existence of bounded algebra homomorphisms implies the existence of a Lipschitz bounded convoluted semigroup or, equivalently, that Hille-Yosida type conditions are satisfied.

Theorem 5.8. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $abs(|k|) < \infty$, $\beta > \max\{abs(|k|), 0\}$, $\Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}$ and A a closed operator. Suppose that there exists an algebra bounded homomorphism $\Theta_k : (\mathcal{T}_k(e_{-\beta}), *) \to \mathcal{B}(X)$ such that $\Theta_k(e_\lambda) = (\lambda - A)^{-1}$ for $\lambda \in \Omega_{k,\beta}$. Then*

(i) *the operator A generates a K-convoluted semigroup, $(S_K(t))_{t\geq 0}$, such that*
$$\|S_K(t+h) - S_K(t)\| \leq Che^{\beta(t+h)}, \qquad t \geq 0,$$
where $K(t) := \int_0^t k(u)du$ for $t \geq 0$.
(ii) $\|\dfrac{d^n}{d\lambda^n}\left(\hat{k}(\lambda)R(\lambda, A)\right)\| \leq \dfrac{Mn!}{(\lambda - \beta)^{n+1}}$ *for $n \geq 0$, and $\lambda \in \Omega_{k,\beta} \cap \mathbb{R}^+$.*

Proof. We consider the family of functions $(K_t)_{t\geq 0}$ defined by (3.1). Note that $(K_t)_{t\geq 0} \subset \mathcal{T}_k(e_{-\beta})$ and $W_k K_t = \chi_{[0,t]}$ by Theorem 3.7 (i). We define $S_K(t) := \Theta_k(K_t)$ for $t \geq 0$. Since $(K_t)_{t\geq 0}$ satisfies the equation (3.2) then $(S_K(t))_{t\geq 0}$ satisfies (5.2). Note that

$$\|S_K(t+h) - S_K(t)\| \leq \|\Theta_k\| \, \|\chi_{[0,t+h]} - \chi_{[0,t]}\|_{k,e_{-\beta}} = \|\Theta_k\| \int_t^{t+h} e^{\beta s} ds \leq \|\Theta_k\| h e^{\beta(t+h)}.$$

To show that $(S_K(t))_{t\geq 0}$ is a K-convoluted semigroup generated by A, it is enough to show that

$$(\lambda - A)^{-1}x = \frac{1}{\hat{K}(\lambda)} \int_0^\infty e^{-\lambda t} S_K(t)xdt, \qquad \lambda \in \Omega_{k,\beta}, \quad x \in X,$$

see [18, Theorem 3.1 (ii)]. Since Θ_k is a continuous map,

$$\int_0^\infty e^{-\lambda t} S_K(t)xdt = \Theta_k\left(\int_0^\infty K_t e^{-\lambda t} dt\right)x = \hat{k}(\lambda)\frac{1}{\lambda}\Theta_k(e_\lambda)x = \hat{K}(\lambda)(\lambda - A)^{-1}x,$$

where we have applied Theorem 3.7 (i). We conclude the proof of the part (i). The part (ii) is a consequence of (i) and [18, Theorem 3.3 (i)]. □

Remark 5.9. In the case that A is a closed densely defined operator, Theorem 5.8 may be improved: in this case, the operator A generates a k-convoluted semigroup, $(S_k(t))_{t\geq 0}$, such that $\|S_k(t)\| \leq Ce^{\beta t}$ for $t \geq 0$, see similar ideas and proof for α-times integrated semigroup in [24, Theorem 4.9] and [25, Theorem 8].

6. Algebra homomorphisms defined via k-convoluted cosine functions

The notion of k-convoluted cosine functions appears implicitly introduced in [20], as a generalization of the concept of n-times integrated cosine functions given in [3, section 6].

Definition 6.1. *Let A be a closed operator, $k \in L^1_{loc}(\mathbb{R}^+)$ and $(C_k(t))_{t\geq 0} \subset \mathcal{B}(X)$ a strongly continuous operator family. The family $(C_k(t))_{t\geq 0}$ is a k-convoluted cosine function generated by A if $C_k(t)A \subset AC_k(t)$, $\int_0^t (t-s)C_k(s)xds \in D(A)$ for $t \geq 0$ and $x \in X$ and*

$$A\int_0^t (t-s)C_k(t)xdt = C_k(t)x - \int_0^t k(s)xds, \quad x \in X, \tag{6.1}$$

for $t \geq 0$; in this case the operator A is called the generator of $(C_k(t))_{t\geq 0}$.

If A generates a k-convoluted cosine function $(C_k(t))_{t\geq 0}$, then $C_k(0) = 0$. See other properties, for example in [19]. In [17, Definition 2.1] the k-convoluted cosine functions are given for exponentially functions in terms of the Laplace transform. α-Times integrated cosine functions are recovered taking $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. The following result appears in [19, Lemma 4.4]. It is observed also in [20, Remark 2.4].

Proposition 6.2. *Let $k, l \in L^1_{loc}(\mathbb{R}^+)$ and $(C_k(t))_{t\geq 0}$ a k-convoluted cosine function generated by A. Then $((l * C_k)(t))_{t\geq 0}$ is a $k * l$-convoluted cosine function generated by A.*

The proof of the following result, analog to Proposition 5.3, can be deduced from [31, Theorem 2.4] taking $C = I$ and replacing $t^{n-1}/(n-1)!$ by $k(t)$ in the proof of the cited Theorem.

Proposition 6.3. *Let $k \in L^1_{loc}(\mathbb{R}^+)$, A a closed linear operator and $(C_k(t))_{t\geq 0}$ a non-degenerate strongly continuous operator family. Then $(C_k(t))_{t\geq 0}$ is a k-convoluted cosine function generated by A if and only if $C_k(0) = 0$,*

$$\begin{aligned} 2C_k(t)C_k(s) &= \int_t^{s+t} k(s+t-r)C(r)dr - \int_0^s k(s+t-r)C(r)dr \\ &+ \int_{t-s}^t k(r-t+s)C(r)dr + \int_0^s k(r+t-s)C(r)dr, \end{aligned} \tag{6.2}$$

for $t \geq s \geq 0$; and the operator A satisfies

$$C_k(t)x - \int_0^t k(s)dsx = \int_0^t (t-s)C_k(s)yds, \quad \text{for all } t \geq 0.$$

where $x \in D(A)$ and $Ax = y$.

Remark 6.4. Defining $C_k(t) = C_k(-t)$ and $k(t) = -k(-t)$ for $t < 0$ we observe that (6.2) can be rewritten as

$$2C_k(t)C_k(s) = (k * (C_k)^t)(s) - (k^t * C_k)(s) + (k * (C_k)^{-t})(s) - (k^{-t} * C_k)(s), \quad t, s \in \mathbb{R}.$$

Assuming exponential boundedness of $C_k(t)$, we can give a direct proof of Proposition 6.3 based only in the formulas for the Laplace transformation given in section 4. In fact, we note (cf. [20, Proposition 3.1]) that $(C_k(t))_{t\geq 0}$ is a k-convoluted cosine function generated by A if and only if $(\lambda^2 - A)^{-1}$ exists in $\mathcal{B}(X)$ and

$$\lambda(\lambda^2 - A)^{-1}x = \frac{1}{\hat{k}(\lambda)} \int_0^\infty e^{-\lambda t} C_k(t)x dt, \quad x \in X. \tag{6.3}$$

Define $R(\lambda)x = \frac{1}{\hat{k}(\lambda)} \int_0^\infty e^{-\lambda t} C_k(t)x dt$. Hence if (6.3) holds, then $R(\sqrt{\lambda})/\sqrt{\lambda}$ is a pseudo-resolvent and, conversely, if $R(\sqrt{\lambda})/\sqrt{\lambda}$ is a pseudo-resolvent, then there exists an operator A such that (6.3) holds (since $C_k(t)$ is non-degenerate). Finally, we note that $R(\sqrt{\lambda})/\sqrt{\lambda}$ is a pseudo-resolvent if and only if

$$R(\lambda)R(\mu) = \frac{1}{\mu^2 - \lambda^2}[\mu R(\lambda) - \lambda R(\mu)]$$

which is equivalent to

$$\hat{C}_k(\lambda)\hat{C}_k(\mu) = \frac{1}{\mu^2 - \lambda^2}[\mu \hat{k}(\mu)\hat{C}_k(\lambda) - \lambda \hat{k}(\lambda)\hat{C}_k(\mu)].$$

From the identity $\frac{1}{\mu^2-\lambda^2} = \frac{1}{2\lambda}[\frac{1}{\mu-\lambda} - \frac{1}{\mu+\lambda}] = \frac{1}{2\mu}[\frac{1}{\mu-\lambda} + \frac{1}{\mu+\lambda}]$ we deduce that the above identity is the same as

$$\begin{aligned} \hat{C}_k(\lambda)\hat{C}_k(\mu) &= \tfrac{1}{2}\tfrac{1}{\mu-\lambda}\hat{k}(\mu)[\hat{C}_k(\lambda) - \hat{C}_k(\mu)] - \tfrac{1}{2}\tfrac{1}{\mu-\lambda}\hat{C}(\mu)[\hat{k}(\lambda) - \hat{k}(\mu)] \\ &+ \tfrac{1}{2}\tfrac{1}{\mu+\lambda}\hat{k}(\mu)[\hat{C}_k(\lambda) + \hat{C}_k(\mu)] + \tfrac{1}{2}\tfrac{1}{\mu+\lambda}\hat{C}(\mu)[\hat{k}(\lambda) - \hat{k}(\mu)]. \end{aligned}$$

Finally, that the above identity is equivalent to (6.2) can be deduced from formulas (4.3), (4.5) and (4.6).

Our following main result is the analog of Theorem 5.5 which was given for k-convoluted semigroups.

Theorem 6.5. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, and $(C_k(t))_{t\geq 0}$ a non-degenerate k-convoluted cosine function generated by A. We define the map $\Psi_k : \mathcal{D}_k \to \mathcal{B}(X)$ by*

$$\Psi_k(f)x := \int_0^\infty W_k f(t) C_k(t)x dt, \qquad x \in X.$$

Then the following properties hold.

(i) $\Psi_k(f *_c g) = \Psi_k(f)\Psi_k(g)$ *for any* $f, g \in \mathcal{D}_k$.
(ii) $\Psi_k(f)x \in D(A)$ *and* $A\Psi_k(f)x = \Psi_k(f'')x - f'(0)x$ *for any* $f \in \mathcal{D}_k$ *and* $x \in X$.
(iii) *Take $l \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(l)$ and $((l * C_k)(t))_{t\geq 0}$ the $k * l$-convoluted cosine function given in Proposition 6.3. Then*

$$\Psi_{k*l}(f) = \Psi_k(f), \qquad f \in \mathcal{D}_{k*l}.$$

Proof. Since $W_k f \in \mathcal{D}_+$ then the expression

$$\Psi_k(f)x = \int_0^\infty W_k f(t) C_k(t) x dt,$$

is well defined for $x \in X$ and $\Psi_k : \mathcal{D}_k \to \mathcal{B}(X)$ defines a linear map. By Theorem 2.10, we have that

$$\begin{aligned}\Psi_k(f *_c g)x &= \int_0^\infty W_k(f *_c g)(t) C_k(t) x dt \\ &= \frac{1}{2}\int_0^\infty \left(W_k(f * g) + f \circ W_k g + g \circ W_k f\right)(t) C_k(t) x dt,\end{aligned}$$

for $x \in X$. Using Theorem 2.10 again, we get

$$\begin{aligned}&\int_0^\infty W_k(f * g)(t) C_k(t) x dt \\ &\quad = \int_0^\infty W_k g(r) \int_0^r W_k f(s) \left(\int_r^{s+r} - \int_0^s k(s+r-t) C_k(t) x dt\right) ds dr \\ &\qquad + \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \left(\int_s^{s+r} - \int_0^r k(s+r-t) C_k(t) x dt\right) ds dr.\end{aligned}$$

Again, by Fubini Theorem

$$\int_0^\infty (f \circ W_k g)(t) C_k(t) x dt = \int_0^\infty W_k g(r) \int_0^r f(r-t) C_k(t) x dt dr$$

and using $f(r-t) = (k \circ W_k f)(r-t)$, we have that

$$\begin{aligned}\int_0^\infty (f \circ W_k g)(t) C_k(t) x dt &= \int_0^\infty W_k g(r) \int_0^r W_k f(s) \int_{r-s}^r k(s-r+t) C_k(t) x dt ds dr \\ &\quad + \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \int_0^r k(t+s-r) C_k(t) x dt ds dr.\end{aligned}$$

In the same way, we get also

$$\begin{aligned}\int_0^\infty (g \circ W_k f)(t) C_k(t) x dt &= \int_0^\infty W_k g(r) \int_r^\infty W_k f(s) \int_{s-r}^r k(s-r+t) C_k(t) x dt ds dr \\ &\quad + \int_0^\infty W_k g(r) \int_0^r W_k f(s) \int_0^r k(t+s-r) C_k(t) x dt ds dr.\end{aligned}$$

We join these six summands, apply the equality (6.2) and have that

$$\begin{aligned}\Psi_k(f *_c g)x &= \int_0^\infty W_k g(r) C_k(r) \int_0^r W_k f(s) C_k(s) x ds dr \\ &\quad + \int_0^\infty W_k g(r) C_k(r) \int_r^\infty W_+^k f(s) C_k(s) x ds dr = \Psi_k(g)\Psi_k(f)x,\end{aligned}$$

and the part (i) is shown.

To show (ii), we consider $f \in \mathcal{D}_k$ and $x \in X$. We apply Lemma 2.8 and formula (6.1) to get

$$\begin{aligned} A\Psi_k(f)x &= A\int_0^\infty (W_k f)''(t)\int_0^t (t-s)C_k(s)xdsdt \\ &= \int_0^\infty W_k f''(t)\left(C_k(t)x - \int_0^t k(s)dsx\right)dt \\ &= \Psi_k(f'')x - \int_0^\infty W_k f'(t)k(t)dtx = \Psi_k(f'')x - f'(0)x. \end{aligned}$$

The proof of (iii) follows the same lines that the proof of Theorem 5.5 (iv). $\square$

Now we consider the algebra $(\mathcal{T}_k(\tau), *_c)$ defined in Theorem 3.5 to obtain bounded algebra homomorphisms.

Corollary 6.6. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $\tau \in B_k$ and $(C_k(t))_{t\geq 0}$ a non-degenerate k-convoluted cosine function such that $\|C_k(t)\| \leq C\tau(t)$ for $t \geq 0$. Then the map $\Psi_k : (\mathcal{T}_k(\tau), *_c) \to \mathcal{B}(X)$ defined by*

$$\Psi_k(f)x := \int_0^\infty W_k f(t)C_k(t)xdt, \qquad x \in X, f \in \mathcal{T}_k(\tau),$$

is a bounded algebra homomorphism.

The next Theorem is the main result of this section. It extends [25, Theorem 4] and [27, Theorem 4.8]. Note the interesting fact that the convolution $*$ in Theorem 5.7 must be replaced now by the convolution $*_c$.

Theorem 6.7. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $abs(|k|) < \infty$ and $\beta > \max\{abs(|k|), 0\}$; $(C(t))_{t\geq 0} \subset \mathcal{B}(X)$ a strongly continuous family such that $\|C(t)\| \leq Ce^{\beta t}$ for $t \geq 0$. Define*

$$R(\lambda)x := \frac{1}{\hat{k}(\lambda)}\int_0^\infty e^{-\lambda s}C(s)xds, \qquad \lambda \in \Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}, \tag{6.4}$$

and the map $\Psi_k : (\mathcal{T}_k(e_{-\beta}), *_c) \to \mathcal{B}(X)$ by

$$\Psi_k(f) := \int_0^\infty W_k f(t)C(t)xdt, \qquad f \in \mathcal{D}_k,$$

for $x \in X$. Then the following conditions are equivalent.

(i) *The family $(C(t))_{t\geq 0}$ satisfies the equation (6.2).*
(ii) *The map Ψ_k is a bounded algebra homomorphism.*
(iii) *The family $(\frac{R(\sqrt{\lambda})}{\sqrt{\lambda}})_{\sqrt{\lambda}\in\Omega_{k,\beta}}$ is a pseudo-resolvent.*
(iv) *The family $(C(t))_{t\geq 0}$ satisfies the equation*

$$Q(t)C(s) + C(t)Q(s) = (k * Q^t)(s) - (k^t * Q)(s) \quad t, s > 0, \tag{6.5}$$

where $Q(t) := \int_0^t C(s)ds$ for $t \geq 0$.

Proof. The proof (i) $\Rightarrow$ (ii) follows the same lines that Theorem 6.5 (i).
To show (ii) $\Rightarrow$ (iii), take $(e_\lambda)_{\lambda\in\Omega_{k,\beta}} \subset \mathcal{T}_k(e_{-\beta})$. Note that $\Psi_k(e_\lambda) = R(\lambda)$ for $\lambda \in \Omega_{k,\beta}$ by (2.11). Since $(e_\lambda)_{\lambda\in\Omega_{k,\beta}}$ satisfies

$$e_\lambda *_c e_\nu = \frac{1}{\lambda^2 - \nu^2}\left(\lambda e_\nu - \nu e_\lambda\right), \tag{6.6}$$

in $(\mathcal{T}_k(e_{-\beta}), *_c)$ ([27, Formula (1.4)]), we conclude that $(\frac{R(\sqrt{\lambda})}{\sqrt{\lambda}})_{\sqrt{\lambda}\in\Omega_{k,\beta}}$ is a pseudo-resolvent.
(iii)$\Rightarrow$ (iv) Multiplying by $\frac{\mu+\lambda}{\lambda\mu}$, we observe that

$$\frac{1}{\mu^2-\lambda^2}[\mu R(\lambda) - \lambda R(\mu)] = R(\lambda)R(\mu)$$

is equivalent to

$$\frac{1}{\mu-\lambda}[\frac{R(\lambda)}{\lambda} - \frac{R(\mu)}{\mu}] = \frac{R(\lambda)}{\lambda}R(\mu) + R(\lambda)\frac{R(\mu)}{\mu}.$$

Using (6.4) and then multiplying by $\hat{k}(\lambda)\hat{k}(\mu)$ we get

$$\frac{1}{\mu-\lambda}[\hat{k}(\mu)\hat{Q}(\lambda) - \hat{k}(\lambda)\hat{Q}(\mu)] = \hat{Q}(\lambda)\hat{C}(\mu) + \hat{C}(\lambda)\hat{Q}(\mu).$$

We note that we can rewrite the above identity as

$$\hat{Q}(\lambda)\hat{C}(\mu) + \hat{C}(\lambda)\hat{Q}(\mu) = \frac{1}{\mu-\lambda}\hat{k}(\mu)[\hat{Q}(\lambda) - \hat{Q}(\mu)] - \frac{1}{\mu-\lambda}\hat{Q}(\mu)[\hat{k}(\lambda) - \hat{k}(\mu)]. \tag{6.7}$$

Hence, using formula (4.3) we obtain (6.5).
(iv)$\Rightarrow$ (i) We define $Q(-t) := -Q(t)$ and $k(-t) := -k(t)$ for $t < 0$. Then $C(t)$ is an even function for $t < 0$ and hence replacing t by $-t$ in (6.5) we obtain

$$-Q(t)C(s) + C(t)Q(s) = (k * Q^{-t})(s) - (k^{-t} * Q)(s), \tag{6.8}$$

for $t, s \in \mathbb{R}$. Summing (6.8) and (6.5) we get

$$2C(t)Q(s) = (k * Q^t)(s) - (k^t * Q)(s) + (k * Q^{-t})(s) - (k^{-t} * Q)(s).$$

Finally, differentiating with respect to s in the above identity, we obtain (i). $\square$

In the next statement, we show that the existence of bounded algebra homomorphisms give existence of a Lipschitz bounded convoluted cosine function which, in turn, is equivalent to Hille-Yosida type conditions due to the Arendt-Widder's Theorem (see [1] and [2]).

Theorem 6.8. *Let $k \in L^1_{loc}(\mathbb{R}^+)$ with $0 \in supp(k)$, $abs(|k|) < \infty$, $\beta > \max\{abs(|k|), 0\}$, $\Omega_{k,\beta} := \{\Re z > \beta \ : \ \hat{k}(z) \neq 0\}$ and A a closed operator. Suppose that there exists an algebra bounded homomorphisms $\Psi_k : (\mathcal{T}_k(e_{-\beta}), *_c) \to \mathcal{B}(X)$ such that $\Psi_k(e_\lambda) = \lambda(\lambda^2 - A)^{-1}$ for $\lambda^2 \in \Omega_{k,\beta}$. Then*

(i) *the operator A generates a K-convoluted cosine function, $(C_K(t))_{t\geq 0}$, such that*

$$\|C_K(t+h) - C_K(t)\| \leq Che^{\beta(t+h)}, \qquad t \geq 0,$$

where $K(t) := \int_0^t k(u)du$ for $t \geq 0$.

(ii) $\|\frac{d^n}{d\lambda^n}\left(\lambda\hat{k}(\lambda)R(\lambda^2, A)\right)\| \leq \frac{Mn!}{(\lambda-\beta)^{n+1}}$ *for $n \geq 0$, and $\lambda \in \Omega_{k,\beta} \cap \mathbb{R}^+$.*

Proof. We consider the family of functions $(K_t)_{t\geq 0}$ defined by (3.1). Note that $(K_t)_{t\geq 0} \subset \mathcal{T}_k(e_{-\beta})$ by Theorem 3.7 (i) and $W_k K_t = \chi_{[0,t]}$. We define $C_K(t) := \Psi_k(K_t)$ for $t \geq 0$. Since $(K_t)_{t\geq 0}$ satisfies the equation (3.3) then $(C_K(t))_{t\geq 0}$ satisfies the equation(6.2). As in proof of Theorem 5.8(ii), we show that

$$\|C_K(t+h) - C_K(t)\| \leq \|\Psi_k\| h e^{\beta(t+h)}.$$

To show that $(C_K(t))_{t\geq 0}$ is a K-convoluted semigroup generated by A, it is enough to show that

$$\lambda(\lambda^2 - A)^{-1}x = \frac{1}{\hat{K}(\lambda)} \int_0^\infty e^{-\lambda t} C_K(t) x dt, \qquad \lambda \in \Omega_{k,\beta}, \quad x \in X,$$

see [17, Theorem 2.3]. Since Ψ_k is a continuous map,

$$\int_0^\infty e^{-\lambda t} C_K(t) x dt = \Psi_k \left(\int_0^\infty K_t e^{-\lambda t} dt \right) x = \hat{k}(\lambda) \frac{1}{\lambda} \Psi_k(e_\lambda) x = \hat{K}(\lambda) \lambda (\lambda^2 - A)^{-1} x,$$

where we have applied Theorem 3.7 (i). We conclude the proof of the part (i). The part (ii) is a consequence of (i) and [17, Theorem 2.3]. □

Remark 6.9. In the case that A is a closed densely defined operator, Theorem 6.8 may be improved: in this case, the operator A generates a k-convoluted cosine function, $(C_k(t))_{t\geq 0}$, such that $\|C_k(t)\| \leq Ce^{\beta t}$ for $t \geq 0$, see similar ideas and proof for α-times cosine function in [25, Theorem 8].

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