MILD WELL-POSEDNESS OF ABSTRACT DIFFERENTIAL EQUATIONS

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In Memory of Gunter Lumer

Abstract. We obtain spectral conditions that characterize mild well-posed inhomogeneous differential equations in a general Banach space X. L^p periodic solutions of first and second order equations are considered. The results are expressed in terms of operator-valued Fourier multipliers. Our approach provides a unified framework for various notions of strong and mild solutions. Applications to semilinear equations of second order in Hilbert spaces are given.

1. Introduction

Operator-valued Fourier multipliers and their applications to differential equations have received much attention recently. Among the many papers on the subject, we mention Arendt-Bu [5], Weis [18] and Denk-Hieber-Prüss [10]. Mild solutions of abstract differential equations are of great importance and are connected to operator semigroups and cosine functions for first and second order problems respectively (see e.g. the monograph [3]). It was discovered recently that for strong solutions of the first order problem, well-posedness did not require that the operator involved be the generator of a semigroup. In Arendt-Bu [5], a very simple and elegant characterization of strong well-posedness was established for periodic solutions. However the problem of characterizing mild well-posedness was left open, except when the operator A generates C_0-continuous semigroup. See the remark after Proposition 3.4. in [5].

The main objective of this paper is to establish a characterization of mild well-posedness for periodic solutions of differential equations of first and second order. We work with a different definition of mild solution and show that, for the first order Cauchy problem, it coincides with the one adopted by Arendt and Bu [5] in case A generates a strongly continuous semigroup. Actually, the definition of mild solutions that we adopt is inspired by Staffans [15] where he worked with a first order equation in Hilbert space.

Let A be a closed and densely defined operator in a Banach space X. We consider the inhomogeneous problem with periodic boundary conditions

$$P_{per}(f)\left\{\begin{array}{rcll} u'(t) &=& Au(t)+f(t), & t\in[0,2\pi],\\ u(0) &=& u(2\pi). & \end{array}\right.$$

where $f \in L^p((0,2\pi);X)$, $1 \le p < \infty$. A *strong* L^p-solution of $P_{per}(f)$ is a function $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ such that $P_{per}(f)$ is satisfied t-a.e. Assuming

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that X is a UMD space, Arendt and Bu [5] (see also Arendt [2]) have characterized strong L^p-well posedness of the periodic problem $P_{per}(f)$ in terms of the R-boundedness of the set $\{k(ikI - A)^{-1} : k \in \mathbb{Z}\}$.

Let $1 \leq p < \infty$. We will prove that $P_{per}(f)$ is $(W^{1,p}, L^p)$ mildly well-posed (see Definition 3.1) if and only if $i\mathbb{Z} \subset \rho(A)$ and $((ikI - A)^{-1})_{k\in\mathbb{Z}}$ is an L^p-multiplier. In the case of the Cauchy problem of second order, we introduce two new notions of mild solutions and this allows us to distinguish between having $((-k^2I-A)^{-1})_{k\in\mathbb{Z}}$ and $(ik(-k^2I-A)^{-1})_{k\in\mathbb{Z}}$ as L^p-multipliers. The latter gives a more transparent description of the concept of C^1-mild solution of the second order problem (see [8], [13] and [14]).

The interest in using Fourier multipliers comes from the fact that sufficient conditions for operator valued Fourier multipliers have been established recently (see [5], [10], [18] and [12]).

The paper is organized as follows. In section 2, we give some preliminaries on operator valued Fourier multipliers and strong well posedness of $P_{per}(f)$. In section 3, we establish a characterization of mild well-posedness of $P_{per}(f)$ and its connection to strongly continuous semigroups. Section 4 is concerned with the second order problem. There, we present two notions of mild well-posedness and characterize them through Fourier multipliers. Furthermore, we examine the situation when A is the generator of a strongly continuous cosine function on X. In Section 5, we present a unified approach to mild well posedness for the first and second order problems in UMD Banach spaces using Hardy-Sobolev spaces. Finally, in section 6, an application to semilinear equations in Hilbert spaces is considered.

2. Preliminaries

Let X be a Banach space. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If Y is another Banach space, we write $\mathcal{L}(X, Y)$ for the space of bounded linear operators from X to Y. By $\rho(A)$ we denote the resolvent set of the operator A, and we write $R(\lambda, A) = (\lambda - A)^{-1}$ when $\lambda \in \rho(A)$.

For $f \in L^1((0, 2\pi); X)$ denote by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$. The Fourier coefficients determine the function f; i.e., $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ a.e.

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ with their periodic extensions to $\mathbb{R}$. Thus, throughout, we consider the space $L^p_{2\pi}(\mathbb{R}; X)$ (which is also denoted by $L^p((0, 2\pi); X)$, $1 \leq p \leq \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

We recall the notion of operator-valued Fourier multiplier.

Definition 2.1. *Let $1 \leq p < \infty$. A sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p-multiplier if, for each $f \in L^p((0, 2\pi); X)$ there exists a function $g \in L^p((0, 2\pi); X)$ such that*

$$M_k\hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

If a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ is an L^p-multiplier, then there exists a unique bounded operator $\mathcal{M} : L^p((0,2\pi);X) \to L^p((0,2\pi);X)$ such that

$$\widehat{(\mathcal{M}f)}(k) = M_k \hat{f}(k),$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$.

Recall that a family $\mathbf{T} \subset \mathcal{L}(X,Y)$ is called R-bounded if there is a constant $C \geq 0$ such that

$$||\sum_{j=1}^{n} r_j \otimes T_j x_j||_{L^p(0,1;Y)} \leq C_p||\sum_{j=1}^{n} r_j \otimes x_j||_{L^p(0,1;X)} \tag{2.1}$$

for all $T_1,...,T_n \in \mathbf{T}$, $x_1,...,x_n \in X$ and $n \in \mathbb{N}$, for some $p \in [1, \infty)$. More information on $R-$boundedness and its relationship to L^p multipliers can be found in the reference [5], [10], [18]. If X is isomorphic to a Hilbert space, then, $R-$boundedness in $\mathcal{L}(X)$ is equivalent to boundedness. On the other hand, in any Banach space, $R-$boundedness is a necessary condition for L^p multipliers (see [5, Proposition 1.11, Proposition 1.13 and Proposition 1.17]).

We say that problem $P_{per}(f)$ is *strongly* L^p *well-posed* if for each $f \in L^p((0,2\pi);X)$ there exists a unique strong L^p-solution of $P_{per}(f)$.

In [5, Theorem 2.3] the following remarkable result was established: if X is a UMD space and $1 < p < \infty$ then the following assertions are equivalent:

(i) $P_{per}(f)$ is strongly L^p-well-posed.

(ii) $i\mathbb{Z} \subset \rho(A)$ and $(kR(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.

(iii) $i\mathbb{Z} \subset \rho(A)$ and $(kR(ik,A))_{k\in\mathbb{Z}}$ is $R-$bounded.

The equivalence $(i) \Leftrightarrow (ii)$ is valid in any Banach space and for $p = 1$ as well.

The concept of mild solution studied in [5, Section 3] is the following. Let $f \in L^1((0,2\pi);X)$. A function $u \in C([0,2\pi];X)$ is called a *mild solution* of the problem $P_{per}(f)$ if $u(0) = u(2\pi)$ and

$$\begin{cases} \displaystyle\int_0^t u(s)ds \in D(A), \quad \text{and} \\ \displaystyle u(t) - u(0) = A\int_0^t u(s)ds + \int_0^t f(s)ds \end{cases} \tag{2.2}$$

for all $t \in [0,2\pi]$. It is clear that every strong L^p-solution is a mild solution.

We say that problem $P_{per}(f)$ is L^p *mildly well-posed* if for each $f \in L^p((0,2\pi);X)$ there exists a unique mild solution of $P_{per}(f)$.

Now recall from [5, Proposition 3.4] that if $\overline{D(A)} = X$, and the problem $P_{per}(f)$ is L^p mildly well-posed then we have that $i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. In the following section we will use the above condition to characterize mild well posedness of $P_{per}(f)$, adopting a different notion of mild solution.

3. Mild-well posedness and L^p-multipliers

Let A be a closed operator in X with domain $D(A)$ and $1 \le p < \infty$. Define the operator $\mathcal{A}$ on $L^p((0,2\pi);X)$ by $D(\mathcal{A}) = W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and

$$\mathcal{A}u = u' - Au.$$

Here $W^{1,p}((0,2\pi);X)$ is the vector valued Sobolev space. When considering the space $L^p((0,2\pi);D(A))$, we equip $D(A)$ with the graph norm. We now define the notion of mild solution that we will use.

Definition 3.1. *We say that the problem $P_{per}(f)$ is $(W^{1,p},L^p)$ mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself as well as $W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself and which satisfies*

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the $(W^{1,p},L^p)$ mild solution of $P_{per}(f)$ and $\mathcal{B}$ the solution operator.

Clearly, the solution operator $\mathcal{B}$ above is unique, if it exists at all. The above notion of well-posedness is suggested by the paper Staffans [15] in case where $p = 2$ and X is a Hilbert space.

Our first main result in this paper characterizes $(W^{1,p},L^p)$ mildly well-posedness in terms of operator-valued L^p-multipliers in Banach spaces.

Theorem 3.2. *Let A be closed linear operator and assume $\overline{D(A)} = X$. Let $1 \le p < \infty$. Then the following assertions are equivalent:*

(i) $P_{per}(f)$ is $(W^{1,p},L^p)$ mildly well-posed.

(ii) $i\mathbb{Z} \subset \rho(A)$ and $(R(ik,A))_{k\in\mathbb{Z}}$is an L^p-multiplier.

Proof. $(ii) \to (i)$. Let $\mathcal{B}$ be the operator which maps $f \in L^p((0,2\pi);X)$ into the function $u \in L^p((0,2\pi);X)$ whose k^{th} Fourier coefficient is $R(ik,A)\hat{f}(k)$, i.e.

$$\widehat{(\mathcal{B}f)}(k) = R(ik,A)\hat{f}(k) = \hat{u}(k), \tag{3.1}$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$. By the remark following Definition 2.1, $\mathcal{B}$ is a bounded linear operator on $L^p((0,2\pi);X)$. Let $g \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and set $h = \mathcal{B}g$. Then,

$$ik\hat{h}(k) = R(ik,A)ik\hat{g}(k) = R(ik,A)\hat{g'}(k), \tag{3.2}$$

for all $k \in \mathbb{Z}$. Since $g' \in L^p((0,2\pi);X)$, there exists $w \in L^p((0,2\pi);X)$ such that

$$\hat{w}(k) = R(ik,A)\hat{g'}(k) \tag{3.3}$$

for all $k \in \mathbb{Z}$. Hence from (3.2), (3.3) and [5, Lemma 2.1] we obtain $h \in W^{1,p}((0,2\pi);X)$. Note that $\hat{h}(k) \in D(A)$, $k \in \mathbb{Z}$ since $\hat{h}(k) = R(ik,A)\hat{g}(k)$.

From (3.2), it follows that

$$A\hat{h}(k) = AR(ik,A)\hat{g}(k) = R(ik,A)\hat{g'}(k) - \hat{g}(k) \tag{3.4}$$

for all $k \in \mathbb{Z}$. Hence from (3.3), [5, Lemma 3.1] and the closedness of A, we conclude that $h(t) \in D(A)$ and $Ah(t) = w(t) - g(t)$ for almost all $t \in [0, 2\pi]$. We have proved that $\mathcal{B}$ that maps $W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ into itself. Continuity of $\mathcal{B}$ follows from the Closed Graph Theorem since the space $W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ embeds continuously into $L^p((0, 2\pi); X)$.

Finally, for $u \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ we have

$$\widehat{(\mathcal{A}u)}(k) = (ikI - A)\hat{u}(k), \tag{3.5}$$

for all $k \in \mathbb{Z}$. Hence from (3.1) and [5, Lemma 3.1] we obtain $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$.

$(i) \to (ii)$. Let $x \in X$ and $x_n \in D(A)$ such that $x_n \to x$. Fix $k \in \mathbb{Z}$ and let $f_n(t) = e^{ikt}x_n$ for all $n \in \mathbb{N}$ and $f_0(t) = e^{ikt}x$. Note that $\hat{f}_n(k) = x_n$ and $\hat{f}_n(j) = 0$ for $j \neq k$. Clearly $f_n \to f_0$ in the L^p-norm. Let $u_n = \mathcal{B}f_n$. Then we have

$$ik\hat{u}_n(k) - A\hat{u}_n(k) = \widehat{(\mathcal{A}u_n)}(k) = (\widehat{\mathcal{A}\mathcal{B}f_n})(k) = \hat{f}_n(k) = x_n.$$

Since $\mathcal{B}$ is bounded on $L^p((0, 2\pi); X)$, $u_n \to u_0 := \mathcal{B}f_0$ in the L^p-norm, we conclude that $\hat{u}_n(k) \to \hat{u}_0(k)$, and

$$ik\hat{u}_0(k) - A\hat{u}_0(k) = x.$$

Hence, for all $k \in \mathbb{Z}$, $(ikI - A)$ is surjective.

Let $x \in D(A)$ be such that $(ikI - A)x = 0$, for $k \in \mathbb{Z}$ fixed. Define $u(t) = e^{ikt}x$. Then, clearly, $u \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ and $u'(t) - Au(t) = \mathcal{A}u = 0$. Hence

$$u = \mathcal{B}\mathcal{A}u = 0,$$

and therefore $x = 0$. Since A is closed, we have proved that $i\mathbb{Z} \subset \rho(A)$.

Next we show that $(R(ik, A))_{k\in\mathbb{Z}}$is an L^p-multiplier. Let $f \in L^p((0, 2\pi); X)$. We observe that since $\overline{D(A)} = X$ and $1 \leq p < \infty$, the space $W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ is dense in $L^p((0, 2\pi); X)$. Hence there exists a sequence $f_n \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ such that $f_n \to f$ in the L^p-norm. Define

$$g_n = \mathcal{B}f_n, \quad n \in \mathbb{N}.$$

Then $g_n \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ and

$$g_n' - Ag_n = \mathcal{A}g_n = \mathcal{A}\mathcal{B}f_n = f_n, \quad n \in \mathbb{N}.$$

Taking Fourier coefficients, and using the fact that $i\mathbb{Z} \subset \rho(A)$, we obtain from the above

$$\hat{g}_n(k) = (ikI - A)^{-1}\hat{f}_n(k) \tag{3.6}$$

for all $k \in \mathbb{Z}$. Next, we note that $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p((0, 2\pi); X)$. By continuity of $\mathcal{B}$, there exists $g \in L^p((0, 2\pi); X)$ such that $g_n \to g$ in the L^p-norm. From this and using Hölder's inequality we deduce that $\hat{g}_n(k) \to \hat{g}(k)$ and, analogously, $\hat{f}_n(k) \to \hat{f}(k)$. Therefore we conclude from (3.6) that $\hat{g}(k) = (ikI - A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The claim is proved.

■

When X is a Hilbert space, the result was obtained by Staffans for $p = 2$. Even in this case, he could not obtain the full range $1 \leq p < \infty$ since his proof relied on Plancherel's theorem which is only valid when $X = H$ is a Hilbert space and $p = 2$.

Indeed, in the case of a Hilbert space, and for $1 < p < \infty$, a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(H)$ is an L^p-multiplier if

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty. \tag{3.7}$$

However, if in addition $p = 2$, then as a consequence of Plancherel's theorem,

$$\sup_{k\in\mathbb{Z}}(\|M_k\|) < \infty \tag{3.8}$$

is a necessary and sufficient condition for $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(H)$ to be a multiplier.

In a general Banach space, even finite dimensional, this is no longer the case. In [5, Theorem 1.3] (see also [2]), it is shown that for UMD spaces, $R-$boundedness of the sequences $(M_k)_{k\in\mathbb{Z}}$ and $(k(M_{k+1} - M_k))_{k\in\mathbb{Z}}$ is sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be an L^p-multiplier for $1 < p < \infty$. In the case of Hilbert spaces, the sufficiency of condition (3.7) is much older (see e.g. [6, Theorem 6.1.6, p. 135]). It is known that in a Banach space X, if condition (3.7) always implies that $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p multiplier for $1 < p < \infty$, then X is isomorphic to a Hilbert space (see [5, Section 1]).

If follows from the proof that the concept of mild solution considered here is related to the one studied by Da Prato and Grisvard in [9]. In that paper, they call *strict solutions* ("solutions strictes") what we call strong solutions and they term *strong solutions* ("solutions fortes") what corresponds to our $(W^{1,p}, L^p)$ mild solutions. In a sense, the present concept of mild solutions seems more natural. They appear as strong limits (in L^p) of strong solutions. Such solutions are important in the analysis of nonlinear problems.

It should also be noted that the solution $u(\cdot)$ in Theorem 3.2 depends continuously on the function f. Specifically, there exists a positive constant C such that

$$\|u\|_{L^p((0,2\pi);X)} \leq C\|f\|_{L^p((0,2\pi);X)},\ f \in L^p((0, 2\pi); X). \tag{3.9}$$

This is clear from the proof and is otherwise a consequence of Definition 3.1.

As direct consequence of [5, Proposition 3.4] we obtain the following result.

Corollary 3.3. *Let X be a Banach space and assume $\overline{D(A)} = X$. If $P_{per}(f)$ is L^p mildly well-posed then $P_{per}(f)$ is $(W^{1,p}, L^p)$ mildly well-posed.*

Using Theorem 3.2 and [5, Theorem 3.6] we obtain the following consequence in case A generates a C_0-semigroup.

Corollary 3.4. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and let $1 \leq p < \infty$. Then the following are equivalent.*

(i) $P_{per}(f)$ is L^p mildly well-posed.
(ii) $P_{per}(f)$ is $(W^{1,p}, L^p)$ mildly well-posed.
(iii) $i\mathbb{Z} \subset \rho(A)$ and $(R(ik, A))_{k\in\mathbb{Z}}$is an L^p-multiplier.
(iv) $1 \in \rho(T(2\pi))$.

In is important to note that the mild solutions provided by this corollary are continuous.

Remark 3.5. We observe that according [5, Proposition 1.11] condition (ii) in Theorem 3.2 implies that

(iii) $i\mathbb{Z} \subset \rho(A)$ and $(R(ik, A))_{k\in\mathbb{Z}}$ is R-bounded.

However the converse is false. It was shown in [5, Example 3.7]

4. Mild Solutions for Second Order Equations

This section is concerned with second order inhomogeneous problems of the form

$$(4.1)\qquad P^2_{per}(f)\begin{cases} u''(t) &= Au(t)+f(t), \qquad 0\le t\le 2\pi\\ u(0) &= u(2\pi),\\ u'(0) &= u'(2\pi),\end{cases}$$

in the space $L^p_{2\pi}(\mathbb{R};X), 1\le p<\infty$.

A *strong* L^p-solution of $P^2_{per}(f)$ is a function $u\in W^{2,p}((0,2\pi);X)\cap L^p((0,2\pi);D(A))$ such that $P^2_{per}(f)$ is satisfied t-a.e.

We say that problem $P^2_{per}(f)$ is *strongly* L^p *well-posed* if for each $f\in L^p((0,2\pi);X)$ there exists a unique strong L^p-solution of $P^2_{per}(f)$.

We define the operator $\mathcal{A}$ on $L^p((0,2\pi);X)$ by $D(\mathcal{A})=W^{2,p}((0,2\pi);X)\cap L^p((0,2\pi);D(A))$ and

$$\mathcal{A}u=u''-Au \quad \text{for} \quad u\in D(\mathcal{A}).$$

Mild solutions of second order problems have been studied in the paper [13] (see also [8] and [14]). There, two notions of mild solutions where considered. These notions, roughly speaking, correspond to integrating equation (4.1) once and twice respectively. Here, we introduce two new notions of mild solutions for (4.1) and establish characterizations which differentiate between the corresponding well-posedness in terms of Fourier multipliers even in case of Hilbert spaces. We show that when A generates a strongly continuous cosine function, then the notions of mild solutions introduced here coincide with those studied in [13].

Definition 4.1. *We say that the problem $P^2_{per}(f)$ is $(W^{2,p},L^p)$ mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself as well as $W^{2,p}((0,2\pi);X)\cap L^p((0,2\pi);D(A))$ into itself and which satisfies*

$$\mathcal{A}\mathcal{B}u=\mathcal{B}\mathcal{A}u=u$$

for all $u\in W^{2,p}((0,2\pi);X)\cap L^p((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the mild solution of order 2 (or $(W^{2,p},L^p)$ mild solution) of $P^2_{per}(f)$ and $\mathcal{B}$ the solution operator.

Proceeding as in the previous section, one obtains the following analog of Theorem 3.2.

Theorem 4.2. *Let A be closed and assume $\overline{D(A)}=X$. Let $1\le p<\infty$. Then the following assertions are equivalent:*

(i) $P^2_{per}(f)$ is $(W^{2,p},L^p)$ mildly well-posed.

(ii) $\{-k^2,\, k\in\mathbb{Z}\}\subset\rho(A)$ and $(R(-k^2,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Proof. $(ii)\to(i)$. For each $f\in L^p((0,2\pi);X)$, let $\mathcal{B}$ be the operator which maps f into the function $u\in L^p((0,2\pi);X)$ whose k^{th} Fourier coefficient is $R(-k^2,A)\hat{f}(k)$, i.e.

$$(4.2)\qquad \widehat{(\mathcal{B}f)}(k)=R(-k^2,A)\hat{f}(k)=\hat{u}(k),$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$. Clearly, $\mathcal{B}$ is a bounded linear operator on $L^p((0,2\pi);X)$. Let $g \in W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and set $h = \mathcal{B}g$. Then,

$$-k^2\hat{h}(k) = R(-k^2,A)(ik)^2\hat{g}(k) = R(-k^2,A)\hat{g''}(k), \tag{4.3}$$

for all $k \in \mathbb{Z}$. Since $g'' \in L^p((0,2\pi);X)$, there exists $w \in L^p((0,2\pi);X)$ such that

$$\hat{w}(k) = R(-k^2,A)\hat{g''}(k) \tag{4.4}$$

for all $k \in \mathbb{Z}$. Hence from (4.2), (4.3) and [5, Lemma 2.1] we obtain $h \in W^{2,p}((0,2\pi);X)$.

Since $h = \mathcal{B}g$, from (4.2) and (4.3) it follows that

$$A\hat{h}(k) = AR(-k^2,A)\hat{g}(k) = -k^2R(-k^2,A)\hat{g}(k) - \hat{g}(k) = R(-k^2,A)\hat{g''}(k) - \hat{g}(k) \tag{4.5}$$

for all $k \in \mathbb{Z}$. Hence from (4.4), [5, Lemma 3.1] and the closedness of A, we conclude that $h(t) \in D(A)$ and $Ah(t) = w(t) - g(t)$ for almost all $t \in [0,2\pi]$. We have proved that $\mathcal{B}$ that maps $W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself. Continuity of $\mathcal{B}$ follows from the Closed Graph Theorem since the space $W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ embeds continuously into $L^p((0,2\pi);X)$.

Finally, for $u \in W^{2,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ we have

$$\widehat{(\mathcal{A}u)}(k) = (-k^2I - A)\hat{u}(k), \tag{4.6}$$

for all $k \in \mathbb{Z}$. Hence from (4.2) and [5, Lemma 3.1] we obtain $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$.

$(i) \to (ii)$. We shall only give a sketch of the proof since it is analogous to the proof of the corresponding implication in Theorem 3.2. For $x \in X, k \in \mathbb{Z}$, fixed, we let $x_n \to x$ where $x_n \in D(A)$, $n \in \mathbb{N} \cup \{0\}$. Set $f_n(t) = e^{ikt}x_n$ and $f_0(t) = e^{ikt}x$. One first establishes that $\{-k^2,\ k \in \mathbb{Z}\} \subset \rho(A)$ and then using an approximation procedure, one proves that $(R(-k^2,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Note that $\widehat{\mathcal{B}f}(k) = R(-k^2,A)\hat{f}(k)$, $k \in \mathbb{Z}$.

■

Suppose A generates a strongly continuous cosine function $C(t)$ and denote by $S(t)$ the associate sine function. In what follows, we shall make use of the set

$$E = \{x \in X : t \to C(t)x \text{ is once continuously differentiable }\},$$

which under the norm $||x||_E = ||x|| + \sup_{0\le t\le 1} ||AS(t)x||$ is a Banach space (cf. [11] and [3, Section 3.14]).

Observe that if $(x,y) \in D(A) \times E$ and f is continuously differentiable on $[0,\ 2\pi]$, then the formula

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds, \tag{4.7}$$

defines a strong (classical) solution of (4.1) (see e.g. Travis and Webb [16, Proposition 2.4] or [3, Chapter 3] or [11]).

Using [13, Theorem 4.6] one immediately has the following corollary to Theorem 4.2.

Corollary 4.3. *Let A be the generator of a strongly continuous cosine function $C(t)$ and denote by $S(t)$ the associated sine function. For $1 \le p < \infty$ the following are equivalent:*

(i) $P^2_{per}(f)$ is $(W^{2,p}, L^p)$ mildly well-posed.

(ii) For any $f \in L^p_{2\pi}(\mathbb{R}; X)$ *there exists a unique* $(x, y) \in X \times X$ *such that* u *given by (4.7) is differentiable at* $t = 0$ *and* 2π*-periodic, i.e.* $u(0) = u(2\pi)$ *and* $u'(0) = u'(2\pi)$.

(iii) $S(2\pi) \in \mathcal{B}(X, E)$ *is invertible.*

In the context of Hilbert spaces, using [13, Corollary 4.7] we have the following.

Corollary 4.4. *Let* H *be a Hilbert space and let* A *be the generator of a strongly continuous cosine family* $C(t)$*. For* $1 \leq p < \infty$ *the following are equivalent:*
(i) $P^2_{per}(f)$ *is* $(W^{2,p}, L^p)$ *mildly well-posed.*

(ii) $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} ||R(-k^2; A)|| < \infty$.

We introduce the following definition of mild solution to equation (4.1).

Definition 4.5. *We say that the problem* $P^2_{per}(f)$ *is* $(W^{2,p}, W^{1,p})$ *mildly well-posed if there exists a linear operator* $\mathcal{B}$ *that maps* $L^p((0, 2\pi); X)$ *continuously into itself with range in* $W^{1,p}((0, 2\pi); X)$*, as well as* $W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ *into itself and which satisfies*

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$*. In this case the function* $\mathcal{B}f$ *is called the mild solution of order 1 (or* $(W^{2,p}, W^{1,p})$ *mild solution) of* $P^2_{per}(f)$ *and* $\mathcal{B}$ *the solution operator.*

Observe that this new notion of mild solutions is stronger than the previous one, namely the $(W^{2,p}, L^p)$ mild solution. This will be apparent in what follows.

When X and Y are Banach spaces, we write $X \hookrightarrow Y$ to indicate that X is continuously embedded into Y. The assertions contained in the following lemma are well-known.

Lemma 4.6. *Let* X*,* Y *and* Z *be Banach spaces such that* $Y \hookrightarrow Z$*. Then the following hold:*
(i) If the linear operator $T : X \longrightarrow Y$ *is continuous, then* $T : X \longrightarrow Z$ *is continuous.*

(ii) If the linear operator $T : X \longrightarrow Z$ *is continuous and* $T(X) \subset Y$*, then* $T : X \longrightarrow Y$ *is continuous.*

Proof. (i) follows by direct verification while (ii) is an immediate consequence of the Closed Graph Theorem. ■

In view of the lemma, in Definition 4.5, we can instead require that the solution operator $\mathcal{B}$ map $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$ continuously. One obtains the following result which, together with the above theorems, recognizes the multipliers establishing the differences between strong solutions, mild solutions of order one and mild solutions of order two.

Theorem 4.7. *Let A be closed and assume $\overline{D(A)} = X$. Let $1 < p < \infty$. Then the following assertions are equivalent:*

(i) $P^2_{per}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well-posed.

(ii) $\{-k^2,\ k \in \mathbb{Z}\} \subset \rho(A)$ and $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Proof. $(ii) \to (i)$. Observe that if $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier, then so is $(R(-k^2, A))_{k\in\mathbb{Z}}$. Again from the proof of Theorem 4.2, we have that from (ii) it follows that we can construct a solution operator $\mathcal{B}$. It remains to show that $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$. Since $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier, for any $f \in L^p((0, 2\pi); X)$, we can find a function $w \in L^p((0, 2\pi); X)$ such that $ikR(-k^2, A)\hat{f}(k) = \hat{w}(k),\ k \in \mathbb{Z}$. Recall that $\widehat{\mathcal{B}f}(k) = R(-k^2, A)\hat{f}(k),\ k \in \mathbb{Z}$. Hence, $ik\widehat{\mathcal{B}f}(k) = \hat{w}(k),\ k \in \mathbb{Z}$. Application of [3, Lemma 2.2] yields that $\mathcal{B}f \in W^{1,p}((0, 2\pi); X)$.

$(i) \to (ii)$. From the definition of well-posedness and Theorem 4.2, we see that (i) implies that $\{-k^2,\ k \in \mathbb{Z}\} \subset \rho(A)$ and $(R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. We have to show that $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p((0, 2\pi); X)$. Since $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$ and there exists $g \in W^{1,p}((0, 2\pi); X)$ such that $\widehat{\mathcal{B}f}(k) = \hat{g}(k) = R(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$, it follows from [5, Lemma 2.1], Definition 2.1 and the relation $\hat{g'}(k) = ik\hat{g}(k) = ikR(-k^2, A)\hat{f}(k)$, $k \in \mathbb{Z}$ that $(ikR(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.

∎

Remark 4.8. Observe that we have the following string of implications

Strongly L^p well-posed $\Longrightarrow (W^{2,p}, W^{1,p})$ mildly well-posed $\Longrightarrow (W^{2,p}, L^p)$ mildly well-posed.

Finally, from [13, Theorem 5.3 and Corollary 5.4] we obtain the following corollaries.

Corollary 4.9. *Let A be the generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ and let $1 \le p < \infty$. Then the following assertions are equivalent:*
(i) $P^2_{per}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well-posed.
(ii) For any $f \in L^p(0, 2\pi; X)$ there exists a unique $(x, y) \in E \times X$ such that u given by (4.7) is of class C^1 and 2π-periodic, i.e. $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.
(iii) $I - C(2\pi) \in \mathcal{B}(X; X)$ is invertible.

In the context of Hilbert spaces, we have:

Corollary 4.10. *Let H be a Hilbert space and A the generator of a strongly continuous cosine family $C(t)$ and let $1 \le p < \infty$. Then the following are equivalent:*
(i) $P^2_{per}(f)$ is $(W^{2,p}, W^{1,p})$ mildly well-posed.
(ii) $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{Z}} ||kR(-k^2; A)|| < \infty$.

Of course these two assertions are equivalent to assertion (ii) of the previous corollary.

5. Fractional differentiation and well-posedness

Let us consider the first order problem $P_{per}(f)$. We note from an examination of the proof of Theorem 3.2 that if in the definition of well-posedness (Definition 3.1) we further require that $\mathcal{B}$ map $L^p((0, 2\pi); X)$ into $W^{1,p}((0, 2\pi); X)$, one can show that this is equivalent to say that $\{ik\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(ikR(ik, A))_{k\in\mathbb{Z}}$ is an L^p multiplier. This shows that strong well-posedness (see Section 1 and [5, Theorem 2.3]) fits well into our framework. More precisely we have:

Theorem 5.1. *Let A be a closed and densely defined linear operator on X and let $1 \le p < \infty$. The following assertions are equivalent:*

(i) Problem $P_{per}(f)$ is $(W^{1,p}, L^p)$ mildly well-posed and the solution operator $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ continuously into itself with range in $W^{1,p}((0, 2\pi); X)$

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(ikR(ik, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Remark 5.2. Observe that if $P_{per}(f)$ is strongly L^p well-posed then condition (i) is satisfied. The converse is valid in UMD spaces by [5, Theorem 2.3].

Likewise, for the second order problem $P^2_{per}(f)$, we have the following proposition (Compare [5, Theorem 6.1] and [13, Theorem 2.1 (with $\alpha = 0$)]). See section 4 to recall the definition of strongly L^p well-posedness for problem $P^2_{per}(f)$.

Theorem 5.3. *Let A be a closed and densely defined linear operator on X and let $1 \le p < \infty$. The following assertions are equivalent:*

(i) Problem $P^2_{per}(f)$ is $(W^{2,p}, L^p)$ mildly well-posed and the solution operator $\mathcal{B}$ maps $L^p((0, 2\pi); X)$ continuously into itself with range in $W^{2,p}((0, 2\pi); X)$

(ii) $\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(k^2R(-k^2, A))_{k\in\mathbb{Z}}$ is an L^p multiplier.

Remark 5.4. Note that if $P^2_{per}(f)$ is strongly L^p well-posed then condition (i) is satisfied. The converse is valid in UMD spaces, which follows by the proof of [5, Theorem 6.1] (see also [13, Theorem 2.11]).

In UMD spaces, if $1 < p < \infty$, the multiplier conditions (ii) in Theorem 5.1 and Theorem 5.3 are equivalent respectively to the $R-$boundedness of $(ikR(ik, A))_{k\in\mathbb{Z}}$ and $(k^2R(-k^2, A))_{k\in\mathbb{Z}}$ (see [5]).

Comparing with [5] and [13], the difference is that here, we require the domain of A to be dense in X (see however Remark 5.8 below). And here we employ different proofs.

The above suggests that one can consider a one parameter family of concepts of well-posedness. In what follows, we shall restrict ourselves to the case of UMD spaces. So, let X be a UMD space. For $1 < p < \infty$ and $0 \le \alpha$, we define the space $H^{\alpha,p}((0, 2\pi); X)$ as:

$$H^{\alpha,p}((0, 2\pi); X) = \{f \in L^p((0, 2\pi); X),\ \exists g \in L^p((0, 2\pi); X) \text{ such that } \hat{g}(k) = |k|^\alpha \hat{f}(k),\ k \in \mathbb{Z}\}.$$

In the case of Hilbert spaces, this situation was studied by O. Staffans [15]. We note due to the UMD property (more precisely the continuity of the Hilbert transform on $L^p(0, 2\pi); X)$, we have

$$W^{m,p}((0, 2\pi); X) = H^{m,p}((0, 2\pi); X), \quad \text{for } 1 < p < \infty \text{ and } \quad m \in \mathbb{N} \cup \{0\} \tag{5.1}$$

(see for example [17, Chapter III], [1] and for the relationship with intermediate spaces, see [7, Chapter IV, especially Section 4.4, p.272]). Now we give the definition of (α, p) well-posedness for $P_{per}(f)$.

Definition 5.5. *We say that the problem $P_{per}(f)$ is (α, p) mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0, 2\pi); X)$ continuously into itself with range in $H^{\alpha,p}((0,\, 2\pi);\, X)$, as well as $W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ into itself and which satisfies*

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in W^{1,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$.

Then we have the following.

Theorem 5.6. *Let X be a UMD space and $0 \leq \alpha \leq 1$. Let A be closed linear operator and assume $\overline{D(A)} = X$ and $1 \leq p < \infty$. Then the following assertions are equivalent:*

(i) $P_{per}(f)$ is (α, p) mildly well-posed.

(ii) $i\mathbb{Z} \subset \rho(A)$ and $(|k|^{\alpha} R(ik, A))_{k\in\mathbb{Z}}$is an L^p-multiplier.

Proof. The proof is a modification of the proof of Theorem 3.2 and we omit it. ∎

In a similar manner, we can deal with the second order problem $P^2_{per}(f)$. For the definition of (α, p) *mild well-posedness*, we now modify Definition 4.5 (or Definition 4.1 for that matter) to require that $\mathcal{B}$ map $L^p((0,\, 2\pi); X)$ into $H^{2\alpha,p}((0,\, 2\pi); X)$ for $0 \leq \alpha \leq 1$.

The result is the following theorem.

Theorem 5.7. *Let X be a UMD space and $0 \leq \alpha \leq 1$. Let A be closed linear operator and assume $\overline{D(A)} = X$ and $1 \leq p < \infty$. Then the following assertions are equivalent:*

(i) $P^2_{per}(f)$ is (α, p) mildly well-posed.

(ii) $\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $(|k|^{2\alpha} R(-k^2, A))_{k\in\mathbb{Z}}$is an L^p-multiplier.

In UMD spaces, the case $\alpha = 1$ and $1 < p < \infty$ in Theorem 5.6 is Theorem 5.1. The reason is the continuity of the Hilbert transform on $L^p((0,\, 2\pi); X)$. Clearly, Theorem 5.7 with $\alpha = 1$ corresponds to Theorem 5.3. On the other hand, if $\alpha = 1/2$ in Theorem 5.7, then we see that Theorem 5.7 corresponds to Theorem 4.7.

Corresponding results may be stated in general Banach spaces. However, given the recently proved theorems on operator valued L^p multipliers in UMD spaces (see [5], [10] and [18]), it seems reasonable to single out this family of spaces. Observe that the spaces $H^{\alpha,p}((0,\, 2\pi);\, X)$ were used in [5, Section 4] in conjunction with Sobolev embedding theorems to obtain continuity of mild solutions, the latter however being defined differently than ours.

For $\alpha = 0$, the first order problem admits continuous mild solutions if we assume that A generates a strongly continuous semigroup. In the case of the second order problem, continuous mild solutions are obtained under the condition that A be the generator of a strongly continuous cosine function. We refer to [5] and [13] respectively.

Remark 5.8. Suppose X is a reflexive Banach space. Let A be a closed linear operator with domain and range in X. Then, as is well known, if $\{(\lambda_n)\} \subset \rho(A)$, $\lim_{n\to\infty} |\lambda_n| = \infty$ and $(\lambda_n R(\lambda_n, A))$ is bounded, then $\overline{D(A)} = X$. Therefore, when the condition (ii) in Theorem 5.1 or Theorem 5.3 is satisfied in a reflexive Banach space (in particular a UMD space), the closed operator A is automatically densely defined.

In order to justify the reasonableness of the restriction on α (i.e. $0 \le \alpha \le 1$) in the previous theorems, we establish the following proposition. It is probably well known but we do not have a ready reference.

Proposition 5.9. *Let X be a Banach space ($X \neq \{0\}$) and $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ and $\lim_{n\to\infty} |\lambda_n| = \infty$. Then for every $\varepsilon > 0$, $(|\lambda_n|^{1+\varepsilon} R(\lambda_n, A))$ is unbounded.*

Proof. Suppose to the contrary that $(|\lambda_n|^{1+\varepsilon} R(\lambda_n, A))$ is bounded, that is, there exists $M > 0$ such that $|\lambda_n|^{1+\varepsilon}\|R(\lambda_n, A))\| \le M$, $n \in \mathbb{N}$. Let $x \in D(A)$. Then there exist $\mu \in \rho(A)$ and $y \in X$ such that $x = R(\mu, A)y$. Clearly we may assume that $|\mu| < |\lambda_n|$, $n \in \mathbb{N}$.

Using the resolvent equation, we have

$$\begin{aligned} |\lambda_n|^{1+\varepsilon} R(\lambda_n, A)x &= |\lambda_n|^{1+\varepsilon} R(\lambda_n, A)R(\mu, A)y \\ &= \frac{|\lambda_n|^{1+\varepsilon}}{\mu - \lambda_n} (R(\lambda_n, A)y - R(\mu, A)y) \end{aligned}$$

It follows that $\frac{|\lambda_n|^{1+\varepsilon}}{|\mu-\lambda_n|}\|R(\lambda_n, A)y - R(\mu, A)y\| \le M\|x\|$ and thus

$$\frac{|\lambda_n|^{1+\varepsilon}}{|\mu - \lambda_n|}\|R(\mu, A)y\| \le M\|x\| + \frac{|\lambda_n|^{1+\varepsilon}}{|\mu - \lambda_n|}\|R(\lambda_n, A)y\| \le M(\|x\| + \frac{\|y\|}{|\mu - \lambda_n|})$$

for all $n \in \mathbb{N}$. Obviously, since $\lim_{n\to\infty} |\lambda_n| = \infty$, this is only possible if $R(\mu, A)y = 0$ and thus $y = 0$,that is $x = 0$. ■

In the light of this proposition, we see that the range of the parameter α in the last two theorems is the right one. Moreover, in view of the fact that every Fourier multiplier is bounded, we can say that the condition (ii) of Theorem 5.1 or Theorem 5.3 are the strongest possible.

6. Application to semi-linear equations in Hilbert spaces

Let X be a Hilbert space and denote by

$$Z := W^{2,p}((0, 2\pi); X) \cap L^p((0, 2\pi); D(A)).$$

In this section we consider the semilinear problem of second order

$$u''(t) = Au(t) + f(t, u(t)), \quad t \in [0, 2\pi], \tag{6.1}$$

where f is a continuous mapping of $L^p((0, 2\pi); X)$ into itself.

We say that a closed linear operator A belongs to the *class* $\mathcal{K}^2(X)$ if

$$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A) \text{ and } \sup_{k\in\mathbb{Z}} ||k^2 R(-k^2; A)|| < \infty. \tag{6.2}$$

Define the Nemytskii's superposition operator $\mathcal{N} : Z \to L^p((0, 2\pi); X)$ given by $\mathcal{N}(v)(t) = f(t, v(t))$ and the bounded linear operator

$$\mathcal{B} := \mathcal{A}^{-1} : L^p((0, 2\pi); X) \to Z$$

by $\mathcal{B}(g) = u$ where u is the unique solution of the linear problem

$$u''(t) = Au(t) + g(t).$$

Then, in order to obtain *strong solutions for (6.1)*, i.e. $u \in Z$ such that (6.1) is satisfied, we have to show that the operator $\mathcal{H} : Z \to Z$ defined by $\mathcal{H} = \mathcal{B}\mathcal{N}$ has a fixed point.

For example, if we assume that $\mathcal{B}$ is a compact operator, and we suppose that for some $M > 0$,

$$\sup_{\|u\|\le M} \|\mathcal{N}(u)\|_{L^p((0,2\pi);X)} \le M/\|\mathcal{B}\|, \tag{6.3}$$

then one may apply Schauder's fixed point theorem to $\mathcal{H}$ in the ball $\{u \in L^p((0, 2\pi); X) : \|u\| \le M\}$ to get existence of a strong solution for (6.1). This way one obtains the existence of solutions on $[0, 2\pi]$. More precisely, by applying the preceding argument, one proves the following result in Hilbert spaces.

Theorem 6.1. *Let H be a Hilbert space, and suppose $A \in \mathcal{K}^2(H)$. Assume that the unit ball of $D(A)$ is compact in H. Let f be given such that (6.3) is satisfied. Then the equation (6.1) has a strong solution, with $\|u\|_{L^2((0,2\pi);H)} \le M$.*

Proof. Since $A \in \mathcal{K}^2(H)$, for each $K \in \mathbb{Z}$ we can define operators $\mathcal{B}_K : L^2((0, 2\pi); H) \to L^2((0, 2\pi); H)$ by

$$(\mathcal{B}_K g)(t) = \sum_{k=-K}^{K} R(-k^2, A)\hat{g}(k)e^{ikt}. \tag{6.4}$$

Since the unit ball of $D(A)$ is compact in H, for each K, the operator $\mathcal{B}_K$ is a finite sum of compact operators, hence compact. Now, because of (6.2), as $K \to \infty$, $\mathcal{B}_K$ converges in norm to $\mathcal{B}$, so $\mathcal{B}$ is compact. The conclusion of the theorem is achieved by applying Schauder's fixed point theorem to the equation $u = \mathcal{B}f(u)$ in $\{u \in Z : \|u\| \le M\}$.

■

Remark 6.2. Note that if $P^2_{per}(f)$ is strongly L^p well-posed then $A \in \mathcal{K}^2(H)$. Indeed, we have by (ii) in Theorem 5.3 that $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$. Moreover, by Remark 5.4 and the comments following Definition 2.1, we know that

$$\sup_{k\in\mathbb{Z}} \|k^2 R(-k^2, A)\| < \infty. \tag{6.5}$$

On the other hand, we also obtain $A \in \mathcal{K}^2(H)$ under the weaker condition (i) in Theorem 5.3.

We can also obtain mild solutions for the semilinear problem (6.1) by relying instead on Corollary 4.10. Here, we take

$$Z = W^{1,p}((0, 2\pi); X).$$

We say that a closed linear operator A belongs to the *class* $\mathcal{K}^1(X)$ if A is the generator of a strongly continuous cosine family $C(t)$ on X and satisfies

$$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A) \text{ and } \sup_{k\in\mathbb{Z}} ||kR(-k^2; A)|| < \infty.$$

If A belongs to the class $\mathcal{K}^1(X)$ then, by Corollary 4.10, there exists a bounded linear operator

$$\mathcal{B} : L^p((0, 2\pi); X) \to Z.$$

We say that $u \in Z$ is a $(W^{2,p}, W^{1,p})$ *mild solution for (6.1)* if u is a fixed point of the equation

$$u = \mathcal{B}f(u)$$

With the same arguments as above, we arrive at:

Theorem 6.3. *Let H be a Hilbert space, and $A \in \mathcal{K}^1(H)$. Assume that the unit ball of $D(A)$ is compact in H. Let f be given such that (6.3) is satisfied. Then equation (6.1) has a $(W^{2,p}, W^{1,p})$ mild solution, with $\|u\|_{L^2((0,2\pi);H)} \leq M$.*

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