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Hölder continuous solutions for integro-differential equations and maximal regularity

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Abstract

We characterize existence and uniqueness of solutions for a linear integro-differential equation in Hölder spaces. Our method is based on operator-valued Fourier multipliers. The solutions we consider may be unbounded. Concrete equations of the type we study arise in the modeling of heat conduction in materials with memory.

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1. Introduction

The aim of this paper is to establish a characterization of maximal relularity for the linear integro-differential equation

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$$\frac{d}{dt}\left(b_0 u(t) + \int_{-\infty}^{t} \beta(t-s)u(s)\,ds\right) + a_\infty u(t) = c_0 Au(t) - \int_{-\infty}^{t} \gamma(t-s)Au(s)\,ds + f(t), \quad (1.1)$$

where $t \in \mathbb{R}$ and $u(t)$ takes its values in a general Banach space X. Here $A : D(A) \subset X \to X$ is a (not necessarily densely defined) closed linear operator, $b_0, c_0, a_\infty \in \mathbb{R}$ and β and γ are real-valued functions defined on $[0, \infty)$.

As is well known, maximal regularity is very useful for treating semilinear and quasilinear problems and results in this direction have been studied extensively in recent years (see, for example, [1,7,20]).

Equation (1.1) is the abstract linear version of the following nonlinear partial integro-differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(b_0 u(t,x) + \int_{-\infty}^{t} \beta(t-s)u(s,x)\,ds\right) + a_\infty u(t,x) \\
&\quad = c_0 \Delta u(t,x) - \int_{-\infty}^{t} \gamma(t-s)\Delta u(s,x)\,ds + G\big(x, u(t,x), \nabla u(t,x)\big) + f(t,x), \\
&\quad t \in \mathbb{R},\ x \in \Omega, \qquad (1.2)
\end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the N-dimensional Laplace operator and ∇ is the gradient operator. The function $u(t,x)$ represents the temperature of the point $x \in \overline{\Omega}$ at time $t \in \mathbb{R}$, $f(t,x)$ is the heat supply, b_0 and c_0 (called respectively heat capacity and thermal conductivity constants) are positive. The relaxation functions β and γ are usually taken as

$$\beta(t) = \sum_{i=1}^{n} \beta_i e^{-b_i t}, \qquad \gamma(t) = \sum_{j=1}^{m} \gamma_j e^{-c_j t} \quad (1.3)$$

with $\beta_i, b_i, \gamma_j, c_j > 0$, where $n, m \in \mathbb{N}$.

Equation (1.2) under various boundary conditions on the operator Δ has been studied in [21] and arises in the study of heat flow in materials of the so-called fading memory type. The corresponding linear model has been discussed in [6,19], whereas nonlinear versions have been studied in [8,9,11,12,18] among others.

In case $\gamma(\cdot) \equiv 0$, global existence and Hölder regularity of the solution of the semilinear problem associated with (1.2) was studied by Sforza [26]. In [10], Clément and Prüss studied the problem in the space $L^p(\mathbb{R}; L^q(\Omega))$. In the above mentioned works, the authors prove global existence of bounded solutions using maximal regularity results for the linearized problem and a priori estimates for the solution of (1.1).

In case $\beta(\cdot) \equiv 0$ and $a_\infty = 0$, bounded solutions for the linear problem (1.1) were studied by Prüss [22] by making use of the theory of resolvent families. He assumes that A is the generator of an analytic semigroup and uses it in the construction of the resolvent. On the other hand, periodic solutions for the same linear problem were first considered by Da Prato and Lunardi [13] (see also the recent papers [16,17]).

Maximal regularity results for the linear problem (1.1) has been studied by Clément and Da Prato [6]. They also showed that if γ is bounded and the first moment of γ exists, then the linear problem (1.1) with $\beta = 0$ is essentially equivalent to the same problem with $\beta \neq 0$. Similar maximal regularity results for bounded solutions were obtained by Amann [2]. Our method differs from the above mentioned papers and allows us to obtain results for solutions which may be unbounded. In fact, we consider continuous functions and not only bounded functions in the Hölder space.

We will study directly the full problem (1.1) in Hölder spaces by a method based on operator-valued Fourier multiplier theorems, which was initiated by Weis in [28] (see also [3,27]) in the investigation of maximal regularity for abstract differential equations. The specific operator-valued Fourier multiplier theorem which we use are those established by Arendt, Batty and Bu in [3].

In contrast to all the above papers dealing with this subject, it is remarkable that well-posedness, in the sense that there exists a (unique) classical solution of (1.1) with maximal regularity, can be characterized completely in terms of the resolvent of A without any restriction on the Banach space X. Denote $R(\lambda, A)$ the resolvent of A at λ for $\lambda \in \rho(A)$. We show in Theorem 3.8 that problem (1.1) is C^α-well posed if and only if

$$b(\eta) := \frac{i\eta(b_0 + \tilde{\beta}(\eta)) + a_\infty}{c_0 - \tilde{\gamma}(\eta)} \in \rho(A) \quad \text{for all } \eta \in \mathbb{R} \quad \text{and}$$

$$\sup_{\eta\in\mathbb{R}} \left\| \frac{i\eta}{c_0 - \tilde{\gamma}(\eta)} R\big(b(\eta), A\big) \right\| < \infty,$$

where $\tilde{\eta}$ and $\tilde{\gamma}$ represent the Fourier transforms of η and γ, respectively (more precisely of their extensions to $\mathbb{R}$ by setting them equal to 0 on $(-\infty, 0)$).

Among the conditions that we impose on β and γ is one of k-regularity. This concept differs from the one used by Prüss [23, Definition 3.3]. Furthermore, we do not make in this paper any parabolicity assumption on the operator, not even that A generates a semigroup. In fact, we give examples showing that the condition that A be the generator of a semigroup is not necessary.

After some preliminaries on Fourier multipliers in Section 2, we prove our main result in Section 3. Our characterization of maximal regularity of problem (1.1) is new and extends some recent results of well-posedness of linear differential equations by Arendt, Batty and Bu [3, Section 6]. An important tool used to prove the uniqueness of the solution is the Carleman transform (see [4] and references therein). Some aspects of this transform needed for our proof are established in Appendix A at the end of this paper.

2. Preliminaries

Let X, Y be Banach spaces and let $0 < \alpha < 1$. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the spaces

$$\dot{C}^\alpha(\mathbb{R}, X) = \big\{ f : \mathbb{R} \to X\colon\ f(0) = 0,\ \|f\|_\alpha < \infty \big\}$$

normed by

$$\|f\|_\alpha = \sup_{t\neq s} \frac{\|f(t) - f(s)\|}{|t-s|^\alpha}.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all C^∞-functions in $\Omega \subseteq \mathbb{R}$ having compact support in Ω.

We denote by $\mathcal{F}f$ or $\tilde{f}$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)\,dt$$

$(s \in \mathbb{R},\ f \in L^1(\mathbb{R}; X))$.

Definition 2.1. Let $M : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha$-multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)\,ds = \int_{\mathbb{R}} \big(\mathcal{F}(\phi \cdot M)\big)(s) f(s)\,ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}, X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)\,dt \in \mathcal{B}(X, Y)$. Note that L is well defined, linear and continuous (cf. [3, Definition 5.2]).

Define the space $C^\alpha(\mathbb{R}, X)$ as the set

$$C^\alpha(\mathbb{R}, X) = \big\{ f : \mathbb{R} \to X:\ \|f\|_{C^\alpha} < \infty \big\}$$

with the norm

$$\|f\|_{C^\alpha} = \|f\|_\alpha + \big\|f(0)\big\|.$$

Let $C^{\alpha+1}(\mathbb{R}, X)$ be the Banach space of all $u \in C^1(\mathbb{R}, X)$ such that $u' \in C^\alpha(\mathbb{R}, X)$, equipped with the norm

$$\|u\|_{C^{\alpha+1}} = \|u'\|_{C^\alpha} + \big\|u(0)\big\|.$$

Observe from Definition 2.1 and the relation

$$\int_{\mathbb{R}} \big(\mathcal{F}(\phi M)\big)(s)\,ds = 2\pi(\phi M)(0) = 0,$$

that for $f \in C^\alpha(\mathbb{R}, X)$ we have $Lf \in C^\alpha(\mathbb{R}, X)$. Moreover, if $f \in C^\alpha(\mathbb{R}, X)$ is bounded then Lf is bounded as well (see [3, Remark 6.3]).

Remark 2.2. The test function space $C_c^\infty(\Omega)$ in Definition 2.1 can be replaced by the space $C_c^1(\Omega)$ of all C^1-functions in Ω having compact support in Ω. It follows from the fact that if $\varphi \in C_c^1(\Omega)$ then $\rho_n * \varphi \in C_c^\infty(\Omega)$ where ρ_n denotes a sequence of mollifying functions, and $\rho_n * f \to f$ in $L^1(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$ (see, e.g., [5, Théorème IV.22]).

The following multiplier theorem is due to Arendt, Batty and Bu [3, Theorem 5.3].

Theorem 2.3. *Let $M \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$ be such that*

$$\sup_{t \neq 0} \| M(t) \| + \sup_{t \neq 0} \| t M'(t) \| + \sup_{t \neq 0} \| t^2 M''(t) \| < \infty. \tag{2.2}$$

Then M is a $\dot{C}^\alpha$-multiplier.

Remark 2.4. If X is B-convex, in particular, if X is a *UMD* space, Theorem 2.3 remains valid if condition (2.2) is replaced by the following weaker condition

$$\sup_{t \neq 0} \| M(t) \| + \sup_{t \neq 0} \| t M'(t) \| < \infty, \tag{2.3}$$

where $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$ (cf. [3, Remark 5.5]).

Let $0 < \alpha < 1$. We denote by $L^1(\mathbb{R}_+, t^\alpha\, dt) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$ the set of all $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\int_0^\infty |a(t)| t^\alpha \, dt < \infty. \tag{2.4}$$

Observe that as consequence such an a is always in $L^1(\mathbb{R}_+)$.

Given $v \in C^\alpha(\mathbb{R}, X)$ $(0 < \alpha < 1)$ and $a \in L^1(\mathbb{R}_+, t^\alpha\, dt)$, we write

$$(a * v)(t) = \int_0^\infty a(s) v(t-s)\, ds = \int_{-\infty}^t a(t-s) v(s)\, ds. \tag{2.5}$$

From (2.4) the above integral is well defined. Moreover, it follows from the definition that

$$\text{if } v \in C^\alpha(\mathbb{R}, X) \quad \text{then} \quad a * v \in C^\alpha(\mathbb{R}, X) \quad \text{and} \quad \|a * v\|_\alpha \leqslant \|a\|_1 \|v\|_\alpha. \tag{2.6}$$

The Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ is denoted by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \operatorname{Re} \lambda > \omega,$$

whenever the integral is absolutely convergent for $\operatorname{Re} \lambda > \omega$. The relation between the Laplace transform of $f \in L^1(\mathbb{R}, X)$, $f(t) \equiv 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

For $u \in L^1(\mathbb{R}, X)$ of subexponential growth and denoting by $\hat{u}$ the Carleman transform of u (see Appendix A or [4, Chapter 4], or [23]) we have

$$\lim_{\sigma \to 0^+} \big(\hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho) \big) = \tilde{u}(\rho).$$

This follows from the definition of the Carleman transform and the dominated convergence theorem (see, e.g., [23, p. 19]).

For $\sigma > 0$ we define $L_\sigma u$ by

$$(L_\sigma u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho), \quad \rho \in \mathbb{R}. \tag{2.7}$$

Some properties of L_σ are collected in Appendix A.

3. A characterization of maximal regularity in Hölder spaces

In this section we characterize C^α-well-posedness.

Given $f \in C^\alpha(\mathbb{R}, X)$, we consider in this section the linear problem

$$\frac{d}{dt}\left(b_0 u(t) + \int_{-\infty}^{t} \beta(t-s)u(s)\,ds\right) + a_\infty u(t) = c_0 Au(t) - \int_{-\infty}^{t} \gamma(t-s)Au(s)\,ds + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A is a closed linear operator in X and $\beta, \gamma \in L^1(\mathbb{R}_+, t^\alpha\,dt) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

We denote by $[D(A)]$ the domain of A considered as a Banach space with the graph norm.

Definition 3.1. We say that (3.1) is C^α-well posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ such that (3.1) is satisfied.

Maximal regularity results for bounded solutions in Hölder spaces for (3.1) appear first to have been studied by Da Prato and Lunardi [13,14]. They studied the problem with periodic boundary conditions on the interval in case $\beta(\cdot) \equiv 0$ and $a_\infty = 0$. Some of these results were extended by Prüss [23]. In particular, [23, Section 12.5] gives necessary conditions for maximal regularity in Besov spaces which include the scale of Hölder spaces.

For the nonperiodic case, maximal regularity in Hölder spaces was proved by Clément and Da Prato [6]. First studies for the semilinear problem in a C^α-setting are due to Sforza [26]. She studied however (3.1) only in case $\gamma(\cdot) \equiv 0$.

In the construction of the solution of Eq. (3.1), we need the following lemma, which is a reformulation of the convolution theorem.

Lemma 3.2. *Let $0 < \alpha < 1$. Assume that $a \in L^1(\mathbb{R}_+, t^\alpha\,dt) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $u, v \in C^\alpha(\mathbb{R}, X)$. The following assertions are equivalent*:

(i) $a * v = u$.

(ii) $$\int_{\mathbb{R}} v(s)\mathcal{F}(\tilde{a} \cdot \phi)(s)\,ds = \int_{\mathbb{R}} u(s)\mathcal{F}(\phi)(s)\,ds \quad \textit{for all } \phi \in C_c^\infty(\mathbb{R}).$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be given. Extending a to $\mathbb{R}$ by setting $a(s) = 0$ for $s < 0$, we observe that

$$\begin{aligned}\mathcal{F}(\tilde{a}\cdot\phi)(-s) &= \int_{\mathbb{R}} e^{ist}\tilde{a}(t)\phi(t)\,dt = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ist}e^{-it\sigma}a(\sigma)\phi(t)\,dt\,d\sigma \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{it(s-\sigma)}\phi(t)\,dt\right)a(\sigma)\,d\sigma = \int_{\mathbb{R}}\mathcal{F}(\phi_-)(s-\sigma)a(\sigma)\,d\sigma \\ &= \big(a*\mathcal{F}(\phi_-)\big)(s),\end{aligned}$$

where $\phi_-(s) := \phi(-s)$. Then, since $(\mathcal{F}\phi)(-s) = \mathcal{F}(\phi_-)(s)$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}}(a*v)(\sigma)\mathcal{F}(\phi)(\sigma)\,d\sigma \\ &\quad= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} a(\sigma-s)v(s)\,ds\right)\mathcal{F}(\phi)(\sigma)\,d\sigma = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} a(s-\sigma)v(-s)\,ds\right)\mathcal{F}(\phi)(-\sigma)\,d\sigma \\ &\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} a(s-\sigma)\mathcal{F}(\phi)(-\sigma)v(-s)\,ds\,d\sigma = \int_{\mathbb{R}}\int_{\mathbb{R}} a(s-\sigma)\mathcal{F}(\phi_-)(\sigma)v(-s)\,d\sigma\,ds \\ &\quad= \int_{\mathbb{R}}\big(a*\mathcal{F}(\phi_-)\big)(s)v(-s)\,ds = \int_{\mathbb{R}} v(-s)\mathcal{F}\big(\tilde{a}(\cdot)\phi(\cdot)\big)(-s)\,ds = \int_{\mathbb{R}} v(s)\mathcal{F}\big(\tilde{a}(\cdot)\phi(\cdot)\big)(s)\,ds.\end{aligned}$$

Hence, (ii) follows immediately from (i). Conversely, assuming that (ii) is true, we obtain

$$\int_{\mathbb{R}}(a*v)(\sigma)\mathcal{F}(\phi)(\sigma)\,d\sigma = \int_{\mathbb{R}} u(s)\mathcal{F}(\phi)(s)\,ds$$

for all $\phi \in C_c^\infty(\mathbb{R})$, and then (i) follows. □

Remark 3.3.

(a) Recall from [3, Lemma 5.1] that $f \in C^\alpha(\mathbb{R}, X)$ is constant if and only if

$$\int_{\mathbb{R}} f(s)(\mathcal{F}\phi)(s)\,ds = 0 \quad \text{for all } \phi \in C_c^\infty\big(\mathbb{R}\setminus\{0\}\big).$$

Concerning Lemma 3.2, it follows from [4, Theorems 4.8.2 and 4.8.1] that if condition (ii) is satisfied for all $\phi \in C_c^\infty(\mathbb{R}\setminus\{0\})$ then $a*v-u$ is a polynomial. But then since for all t, we have $\|(a*v)(t)\| \leqslant C\|a\|_1(1+|t|^\alpha)$ and $\|u(t)\| \leqslant C(1+|t|^\alpha)$ with $0<\alpha<1$, we conclude that $a*v-u$ is constant.

(b) As in Remark 2.2 we observe that we can replace in (ii) the test function space $C_c^\infty(\mathbb{R})$ by $C_c^1(\mathbb{R})$.

In the next proposition, as usual we denote by $\rho(T)$, $R(\lambda, T)$ the resolvent set and resolvent of the operator T, respectively.

Observe that if $c_0 - \tilde{\gamma}(s) \neq 0$ for all $s \in \mathbb{R}$ then $\{\frac{1}{c_0 - \tilde{\gamma}(s)}\}$ is bounded. This follows from the Riemann–Lebesgue theorem. The last condition is satisfied if

$$\gamma \text{ is non-negative} \quad \text{and} \quad c_0 - \int_0^\infty \gamma(s)\,ds > 0. \tag{3.2}$$

This is usually the case in practical problems (see, e.g., [19] or [21] for a physical motivation). Remark that if γ is not necessarily non-negative, then condition $c_0 - \tilde{\gamma}(s) \neq 0$ is satisfied if instead of (3.2) we have

$$c_0 - \int_0^\infty |\gamma(s)|\,ds > 0. \tag{3.3}$$

The following result is related to [23, Proposition 11.5].

Proposition 3.4. *Suppose* $\gamma, \beta \in L^1(\mathbb{R}_+, t^\alpha\,dt) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *and* $c_0 - \tilde{\gamma}(s) \neq 0$ *for all* $s \in \mathbb{R}$. *Assume that* (3.1) *is* C^α*-well posed. Then*

$$\text{(i)} \quad b(\eta) := \frac{i\eta(b_0 + \tilde{\beta}(\eta)) + a_\infty}{c_0 - \tilde{\gamma}(\eta)} \in \rho(A) \quad \textit{for all } \eta \in \mathbb{R}, \quad \textit{and}$$

$$\text{(ii)} \quad \sup_{\eta \in \mathbb{R}} \left\| \frac{i\eta}{c_0 - \tilde{\gamma}(\eta)} R\big(b(\eta), A\big) \right\| < \infty.$$

Proof. Denote by $L : C^\alpha(\mathbb{R}, X) \to C^{1+\alpha}(\mathbb{R}, X)$ the bounded operator which associates to each $f \in C^\alpha(\mathbb{R}, X)$ the unique solution u of (3.1). Let $\eta \in \mathbb{R}$. Let $x \in D(A)$ be such that $Ax - b(\eta)x = 0$. Define $u(t) = e^{i\eta t}x$. Then it is not difficult to see that u is a solution of (3.1) with $f \equiv 0$. Hence, by uniqueness, $x = 0$.

Let $y \in X$ and define $f(t) = e^{i\eta t}y$. Let $u = Lf$. For fixed $s \in \mathbb{R}$ we define

$$v_1(t) = u(t+s) \quad \text{and} \quad v_2(t) = e^{i\eta s}u(t).$$

Then is easy to check that v_1 and v_2 are both solutions of (3.1) with f replaced by $e^{i\eta s}f$. By uniqueness, $u(t+s) = e^{i\eta s}u(t)$ for all $t, s \in \mathbb{R}$. In particular, it follows that $u(s) = e^{i\eta s}u(0)$ for all $s \in \mathbb{R}$. Let $x = u(0) \in D(A)$. Replacing $u(t) = e^{i\eta t}x$ in (3.1) we obtain

$$\big[i\eta\big(b_0 + \tilde{\beta}(\eta)\big) + a_\infty\big]u(t) = \big(c_0 - \tilde{\gamma}(\eta)\big)Au(t) + e^{i\eta t}y.$$

Taking $t = 0$ we conclude that $(b(\eta) - A)$ is bijective and

$$u(t) = \frac{1}{c_0 - \tilde{\gamma}(\eta)} R\big(b(\eta), A\big)e^{i\eta t}y.$$

Define $e_\eta(t) = e^{i\eta t}$ and $(e_\eta \otimes y)(t) = e_\eta(t)y$. We have the identity $\|e_\eta \otimes x\|_\alpha = K_\alpha|\eta|^\alpha\|x\|$ where $K_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$ (see [3, Section 3]). Hence

$$K_\alpha|\eta|^\alpha\left\|\frac{i\eta}{c_0-\tilde\gamma(\eta)}R\big(b(\eta),A\big)y\right\| = \left\|e_\eta\otimes\frac{i\eta}{c_0-\tilde\gamma(\eta)}R\big(b(\eta),A\big)y\right\|_\alpha = \|u'\|_\alpha$$
$$\leqslant \|u\|_{1+\alpha} = \|Lf\|_{1+\alpha} \leqslant \|L\|\|f\|_\alpha$$
$$\leqslant \|L\|\big(\|f\|_\alpha + \big\|f(0)\big\|\big) = \|L\|\big(\|e_\eta\otimes y\|_\alpha + \|y\|\big)$$
$$\leqslant \|L\|\big(K_\alpha|\eta|^\alpha + 1\big)\|y\|.$$

Therefore, for $\epsilon > 0$ we have

$$\sup_{|\eta|>\epsilon}\left\|\frac{i\eta}{c_0-\tilde\gamma(\eta)}R\big(b(\eta),A\big)y\right\| \leqslant \|L\|\sup_{|\eta|>\epsilon}\left(1+\frac{1}{K_\alpha|\eta|^\alpha}\right)\|y\| < \infty.$$

On the other hand, since $\{\frac{1}{c_0-\tilde\gamma(s)}\}$ is bounded and $\eta \to i\eta R(b(\eta),A)$ is continuous at $\eta = 0$, we obtain (ii) and the proof is complete. □

Definition 3.5. A function $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is called k-regular on $\mathbb{R}$ if for all $0 \leqslant n \leqslant k$ and $a \in L^1(\mathbb{R}_+, t^n\,dt)$ there is a constant $c > 0$ such that

$$\big|s^n\big[\tilde a(s)\big]^{(n)}\big| \leqslant c \tag{3.4}$$

for all $s \in \mathbb{R}$.

Remark 3.6.

(i) It follows from Theorem 2.3 that if a is 2-regular on $\mathbb{R}$, then $\tilde a(s)$ is a $\dot C^\alpha$-multiplier.
(ii) Assume a is 2-regular on $\mathbb{R}$ and $\tilde a(s) \neq c_0$ for all $s \in \mathbb{R}\setminus\{0\}$. Then the identities

$$s\big[\big(c_0-\tilde a(s)\big)^{-1}\big]' = \big(c_0-\tilde a(s)\big)^{-2}s\big[\tilde a(s)\big]' \quad \text{and}$$
$$s^2\big[\big(c_0-\tilde a(s)\big)^{-1}\big]'' = 2\big(c_0-\tilde a(s)\big)^{-3}\big\{s\big[\tilde a(s)\big]'\big\}^2 + \big(c_0-\tilde a(s)\big)^{-2}s^2\big[\tilde a(s)\big]''$$

show that $(c_0-\tilde a(s))^{-1}$ satisfies condition (2.2) and hence is a $\dot C^\alpha$-multiplier.

In what follows, we denote by *id* the function: $s \to is$ for all $s \in \mathbb{R}$. We also use the notation

$$b(s) := \frac{is(b_0+\tilde\beta(s)) + a_\infty}{c_0-\tilde\gamma(s)} \quad \text{and}$$
$$M(s) := \frac{1}{c_0-\tilde\gamma(s)}R\big(b(s),A\big) \tag{3.5}$$

for all $s \in \mathbb{R}$.

Lemma 3.7. *Assume that γ, β are 2-regular on $\mathbb{R}$ and $c_0 - \tilde\gamma(s) \neq 0$ for all $s \in \mathbb{R}$, and $\{b(s)\}_{s\in\mathbb{R}} \subseteq \rho(A)$. If*

$$\sup_{s\in\mathbb{R}}\big\|sM(s)\big\| < \infty,$$

then $id \cdot M$ and $\tilde{\beta} \cdot id \cdot M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X)$. Moreover M and $\tilde{\gamma} \cdot M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X, D(A))$.

Proof. Define

$$N(s) = \frac{Q(s)}{c_0 - \tilde{\gamma}(s)},$$

where $Q(s) := isR(b(s), A)$. Using Remark 3.6, it is easy to check that $(c_0 - \tilde{\gamma})^{-1}$ satisfies condition (2.2) of Theorem 2.3, and hence N is a $\dot{C}^\alpha$-multiplier if and only if Q is a $\dot{C}^\alpha$-multiplier.

We claim that Q satisfies the conditions of Theorem 2.3. In fact, note first that $[\tilde{\gamma}(s)]'$ is bounded since $\gamma \in L^1(\mathbb{R}, t\,dt)$ and hence from

$$\begin{aligned} b'(s) &= \big(c_0 - \tilde{\gamma}(s)\big)^{-1}\big(ib_0 + i\tilde{\beta}(s) + is\big[\tilde{\beta}(s)\big]'\big) \\ &\quad + \big(c_0 - \tilde{\gamma}(s)\big)^{-2}\big(is\big[\tilde{\gamma}(s)\big]'b_0 + \tilde{\beta}(s)is\big[\tilde{\gamma}(s)\big]' + a_\infty\big[\tilde{\gamma}(s)\big]'\big) \end{aligned} \tag{3.6}$$

and 1-regularity on $\mathbb{R}$ of γ and β, it follows that $b'(s)$ is bounded. Analogously, note that $s[\tilde{\gamma}(s)]''$ is bounded (by 2-regularity), and hence the identity

$$\begin{aligned} sb''(s) &= \big(c_0 - \tilde{\gamma}(s)\big)^{-1}\big\{2is\big[\tilde{\beta}(s)\big]' + is^2\big[\tilde{\beta}(s)\big]''\big\} \\ &\quad + \big(c_0 - \tilde{\gamma}(s)\big)^{-2}\big\{2ib_0 s\big[\tilde{\gamma}(s)\big]' + 2is\big[\tilde{\gamma}(s)\big]'\tilde{\beta}(s) + 2is\big[\tilde{\beta}(s)\big]'s\big[\tilde{\gamma}(s)\big]' \\ &\quad + ib_0 s^2\big[\tilde{\gamma}(s)\big]'' + i\tilde{\beta}(s)s^2\big[\tilde{\gamma}(s)\big]'' + a_\infty s\big[\tilde{\gamma}(s)\big]''\big\} \\ &\quad + 2\big(c_0 - \tilde{\gamma}(s)\big)^{-3}\big\{ib_0\big(s\big[\tilde{\gamma}(s)\big]'\big)^2 + i\tilde{\beta}(s)\big(s\big[\tilde{\gamma}(s)\big]'\big)^2 + a_\infty s\big[\tilde{\gamma}(s)\big]'\big[\tilde{\gamma}(s)\big]'\big\} \end{aligned}$$

shows that $sb''(s)$ is bounded.

It follows from the above and the identities

$$Q(s) = \big(c_0 - \tilde{\gamma}(s)\big)isM(s), \qquad sQ'(s) = Q(s) + i\big[Q(s)\big]^2 b'(s),$$

that $Q(s)$ and $sQ'(s)$ are bounded. Moreover, we have

$$s^2 Q''(s) = 2ib'(s)Q(s)^2 + isb''(s)Q(s)^2 - 2\big[b'(s)\big]^2 Q(s)^3,$$

and hence $s^2Q''(s)$ is also bounded. This proves the claim. We conclude that $N = id \cdot M$ is a $\dot{C}^\alpha$-multiplier.

We claim that M is a $\dot{C}^\alpha$-multiplier. In fact, note that by hypothesis $\sup_{|s|>\epsilon} \|M(s)\| < \infty$ for each $\epsilon > 0$, and the function $s \to M(s)$ is continuous at $t = 0$ since $\tilde{\gamma}(0) \neq c_0$. Hence $M(s)$ is bounded. Moreover,

$$sM'(s) = \frac{[\tilde{\gamma}(s)]'}{c_0 - \tilde{\gamma}(s)} sM(s) - sM(s)M(s)\big(c_0 - \tilde{\gamma}(s)\big)b'(s),$$

where from (3.6) we see that $(c_0 - \tilde{\gamma}(s))b'(s)$ is bounded. Since $\{\frac{1}{c_0 - \tilde{\gamma}(s)}\}$ is bounded, we conclude that $sM'(s)$ is bounded. Finally, taking into account Remark 3.6(ii) and the following calculations:

$$\begin{aligned}
M''(s) &= \big[\tilde{\gamma}'(s)\big(c_0-\tilde{\gamma}(s)\big)^{-1}M(s) - M(s)^2\big(c_0-\tilde{\gamma}(s)\big)b'(s)\big]' \\
&= \tilde{\gamma}''(s)\big(c_0-\tilde{\gamma}(s)\big)^{-1}M(s) + \big[\big\{\big(c_0-\tilde{\gamma}(s)\big)^{-1}\big\}'M(s) + \big(c_0-\tilde{\gamma}(s)\big)^{-1}M'(s)\big] \\
&\quad - 2M(s)M'(s)\big(c_0-\tilde{\gamma}(s)\big)b'(s) - M(s)^2\big[\big(c_0-\tilde{\gamma}(s)\big)b''(s) - \tilde{\gamma}'(s)b'(s)\big] \\
&= \tilde{\gamma}''(s)\big(c_0-\tilde{\gamma}(s)\big)^{-1}M(s) + \big(\tilde{\gamma}'(s)\big)^2\big(c_0-\tilde{\gamma}(s)\big)^{-2}M(s) \\
&\quad + \tilde{\gamma}'(s)\big(c_0-\tilde{\gamma}(s)\big)^{-1}\left[\frac{\tilde{\gamma}(s)}{c_0-\tilde{\gamma}(s)} - b'(s)\big(c_0-\tilde{\gamma}(s)\big)M(s)^2\right] \\
&\quad + 2M(s)\big(b'(s)\big)^2\big(c_0-\tilde{\gamma}(s)\big)^2M(s)^2 - 2M(s)b'(s)M(s)\tilde{\gamma}'(s) \\
&\quad - \big[\big(c_0-\tilde{\gamma}(s)\big)b''(s) - \tilde{\gamma}'(s)b'(s)\big]M(s)^2 \\
&= \big(c_0-\tilde{\gamma}(s)\big)\left[\frac{\tilde{\gamma}''(s)}{(c_0-\tilde{\gamma}(s))^2} + \frac{2\tilde{\gamma}'(s)}{(c_0-\tilde{\gamma}(s))^3}\right]M(s) \\
&\quad - \big[b'(s)\tilde{\gamma}'(s) + \big(c_0-\tilde{\gamma}(s)\big)b''(s)\big]M(s)^2 + 2\big(b'(s)\big)^2\big(c_0-\tilde{\gamma}(s)\big)^2M(s)^3,
\end{aligned}$$

we see that $s^2M''(s)$ is bounded and thus, the claim is proved.

Finally, to show that $\tilde{\beta}\cdot id\cdot M$ and $\tilde{\gamma}\cdot M$ are $\dot{C}^\alpha$-multipliers, we note that from 2-regularity, $\tilde{\beta}(s)$ and $\tilde{\gamma}(s)$ satisfy condition (2.2) of Theorem 2.3. But since $id\cdot M$ and M also satisfy these conditions, it is easy to verify that $\tilde{\beta}\cdot id\cdot M$ and $\tilde{\gamma}\cdot M$ satisfy (2.2).

Next, the identity

$$AM(s) = b(s)M(s) - \frac{1}{c_0-\tilde{\gamma}(s)}I = \frac{1}{c_0-\tilde{\gamma}(s)}\big[b_0isM(s) + \tilde{\beta}(s)isM(s) + a_\infty M(s) - I\big]$$

shows that $M(s)$ is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$. It follows that $\tilde{\gamma}\cdot M$ is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, D(A))$ and the lemma is proved. □

The following theorem, which is our main result, shows that the converse of Proposition 3.4 is valid under certain mild conditions on the kernels β and γ.

Theorem 3.8. *Let A be a closed linear operator defined on a Banach space X and $b_0, c_0, a_\infty \in \mathbb{R}$. Suppose $\gamma, \beta \in L^1_{loc}(\mathbb{R}_+)$ are 2-regular on $\mathbb{R}$ and there exists $\epsilon > 0$ such that*

$$\sup_{0\leqslant\sigma<\epsilon,\, \eta\in\mathbb{R}}\left|\frac{1}{c_0-\hat{\gamma}(\sigma+i\eta)}\right| < \infty. \tag{3.7}$$

Then the following assertions are equivalent:

(i) *Equation* (3.1) *is C^α-well posed.*
(ii) $b(\eta) := \frac{i\eta(b_0+\tilde{\beta}(\eta))+a_\infty}{c_0-\tilde{\gamma}(\eta)} \in \rho(A)$ *for all $\eta\in\mathbb{R}$ and*

$$\sup_{\eta\in\mathbb{R}}\left\|\frac{i\eta}{c_0-\tilde{\gamma}(\eta)}R\big(b(\eta), A\big)\right\| < \infty.$$

Proof. Note that 2-regularity implies that $\gamma, \beta \in L(\mathbb{R}_+, t^\alpha\, dt) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Therefore the implication (i) ⇒ (ii) follows by Proposition 3.4. We now prove the converse implication.

Let $f \in C^\alpha(\mathbb{R}, X)$. By Lemma 3.7 there exist $u_1, u_4 \in C^\alpha(\mathbb{R}, D(A))$ and $u_2, u_3 \in C^\alpha(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} u_1(s)(\mathcal{F}\phi_1)(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 \cdot M)(s) f(s)\, ds, \tag{3.8}$$

$$\int_{\mathbb{R}} u_2(s)(\mathcal{F}\phi_2)(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot id \cdot M)(s) f(s)\, ds, \tag{3.9}$$

$$\int_{\mathbb{R}} u_3(s)(\mathcal{F}\phi_3)(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\phi_3 \cdot \tilde{\beta} \cdot id \cdot M)(s) f(s)\, ds \quad \text{and} \tag{3.10}$$

$$\int_{\mathbb{R}} u_4(s)(\mathcal{F}\phi_4)(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\phi_4 \cdot \tilde{\gamma} \cdot M)(s) f(s)\, ds \tag{3.11}$$

for all $\phi_i \in C^1_c(\mathbb{R} \setminus \{0\})$ $(i = 1, 2, 3, 4)$ (cf. Remark 2.2). Choosing $\phi_1 = id \cdot \phi_2$ in (3.8), it follows from [3, Lemma 6.2] that $u_1 \in C^{1+\alpha}(\mathbb{R}, X)$ and

$$u_1' = u_2 + y_1 \tag{3.12}$$

for some $y_1 \in X$. Since $\beta \in L^1(\mathbb{R}_+, t\, dt)$ we can choose $\phi_2 = \phi_3 \cdot \tilde{\beta}$ in (3.9) and it follows from (3.10) that

$$\int_{\mathbb{R}} u_3(s)(\mathcal{F}\phi_3)(s) = \int_{\mathbb{R}} u_2(s)\mathcal{F}(\phi_3 \cdot \tilde{\beta})(s)\, ds$$

for all $\phi_3 \in C^1_c(\mathbb{R} \setminus \{0\})$. Hence by Lemma 3.2 and the remark following that lemma, we obtain $u_2 * \beta = u_3 + y_2$ for some $y_2 \in X$. Since $\beta \in L^1(\mathbb{R}_+)$, we conclude that there exists $y_4 \in X$ such that

$$u_3 = \left(u_1' - y_1\right) * \beta - y_2 = u_1' * \beta - \tilde{\beta}(0) y_1 - y_2 = u_1' * \beta - y_4. \tag{3.13}$$

Choosing $\phi_1 = \tilde{\gamma} \cdot \phi_4$ in (3.8) (since $\gamma \in L^1(\mathbb{R}_+, t\, dt)$), it follows from (3.11) that

$$\int_{\mathbb{R}} u_4(s)(\mathcal{F}\phi_4)(s)\, ds = \int_{\mathbb{R}} u_1(s)\mathcal{F}(\phi_4 \cdot \tilde{\gamma})(s)\, ds.$$

Hence by Lemma 3.2 and Remark 3.3, we obtain

$$u_1 * \gamma - u_4 = y_3 \tag{3.14}$$

for some $y_3 \in X$.

Define $M(s) = \frac{1}{c_0 - \tilde{\gamma}(s)} R(b(s), A)$ as in Lemma 3.7. Then making use of the resolvent equation, we obtain the identity

$$b_0 isM(s) + is\tilde{\beta}(s)M(s) + a_\infty M(s) = c_0 AM(s) - \tilde{\gamma}(s)AM(s) + I. \tag{3.15}$$

We multiply by ϕ take Fourier transforms and then integrate over $\mathbb{R}$ after taking the values at $f(s)$, we obtain

$$b_0 \int_{\mathbb{R}} \mathcal{F}(\phi \cdot id \cdot M)(s) f(s)\, ds + \int_{\mathbb{R}} \mathcal{F}(\phi \cdot \tilde{\beta} \cdot id \cdot M)(s) f(s)\, ds + a_\infty \int_{\mathbb{R}} \mathcal{F}(\phi \cdot M)(s) f(s)\, ds$$
$$= c_0 \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot M)(s) f(s)\, ds - \int_{\mathbb{R}} A\mathcal{F}(\phi \cdot \tilde{\gamma} \cdot M)(s) f(s)\, ds + \int_{\mathbb{R}} \mathcal{F}(\phi)(s) f(s)\, ds$$

for all $\phi \in C_c^1(\mathbb{R} \setminus \{0\})$. Using (3.8)–(3.11) we obtain from the above identity

$$b_0 \int_{\mathbb{R}} u_2(s)(\mathcal{F}\phi)(s)\, ds + \int_{\mathbb{R}} u_3(s)(\mathcal{F}\phi)(s)\, ds + a_\infty \int_{\mathbb{R}} u_1(s)(\mathcal{F}\phi)(s)\, ds$$
$$= c_0 \int_{\mathbb{R}} Au_1(s)(\mathcal{F}\phi)(s)\, ds - \int_{\mathbb{R}} Au_4(s)(\mathcal{F}\phi)(s)\, ds + \int_{\mathbb{R}} f(s)(\mathcal{F}\phi)(s)\, ds$$

for all $\phi \in C_c^1(\mathbb{R} \setminus \{0\})$. Using (3.12)–(3.14) in the above identity we conclude that

$$b_0 \int_{\mathbb{R}} u_1'(s)(\mathcal{F}\phi)(s)\, ds + \int_{\mathbb{R}} \big(u_1' * \beta\big)(s)(\mathcal{F}\phi)(s)\, ds + a_\infty \int_{\mathbb{R}} u_1(s)(\mathcal{F}\phi)(s)\, ds$$
$$= c_0 \int_{\mathbb{R}} Au_1(s)(\mathcal{F}\phi)(s)\, ds - \int_{\mathbb{R}} A(u_1 * \gamma)(s)(\mathcal{F}\phi)(s)\, ds + \int_{\mathbb{R}} f(s)(\mathcal{F}\phi)(s)\, ds$$

for all $\phi \in C_c^1(\mathbb{R} \setminus \{0\})$. By Remark 3.3 this implies that there exist $z \in X$ such that

$$b_0 u_1'(s) + \big(\beta * u_1'\big)(s) + a_\infty u_1(s) = c_0 Au_1(s) - (\gamma * Au_1)(s) + f(s) + z, \quad s \in \mathbb{R}.$$

Since $\tilde{\gamma}(0) \neq c_0$ and $b(0) \in \rho(A)$ we can define

$$u(t) = u_1(t) - \frac{1}{c_0 - \tilde{\gamma}(0)} \left(\frac{a_\infty}{c_0 - \tilde{\gamma}(0)} - A \right)^{-1} z =: u_1(t) - w. \tag{3.16}$$

Observe that $(c_0 - \tilde{\gamma}(0))Aw = a_\infty w - z$. Then

$$\begin{aligned} &c_0 Au(t) - (\gamma * Au)(t) + f(t) \\ &\quad = c_0 Au_1(t) - (\gamma * Au_1)(t) - \big(c_0 - \tilde{\gamma}(0)\big)Aw + f(t) \\ &\quad = b_0 u_1'(t) + \big(\beta * u_1'\big)(t) + a_\infty u_1(t) - \big(c_0 - \tilde{\gamma}(0)\big)Aw - z = b_0 u'(t) + (\beta * u')(t) + a_\infty u(t). \end{aligned}$$

This shows that u solves (3.1).

On the other hand, we observe from (3.12) that $u' \in C^\alpha(\mathbb{R}, X)$. Also, from Lemma 3.7 we have AM is $(\dot{C})^\alpha$-multiplier. Then there is $u_5 \in C^\alpha(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} u_5(s)(\mathcal{F}\phi_5)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_5 \cdot AM)(s) f(s)\,ds$$

for all $\phi_5 \in C_c^1(\mathbb{R} \setminus \{0\})$. Using Eq. (3.8) and the closedness of A we obtain from the above

$$\int_{\mathbb{R}} Au_1(s)(\mathcal{F}\phi_5)(s)\,ds = \int_{\mathbb{R}} A\mathcal{F}(\phi_5 \cdot M)(s) f(s)\,ds = \int_{\mathbb{R}} u_5(s)(\mathcal{F}\phi_5)(s)\,ds.$$

We conclude that there exists $y_5 \in X$ satisfying $Au_1 = u_5 + y_5$, proving that $Au \in C^\alpha(\mathbb{R}, X)$. Finally, from (2.6) it follows that $\gamma * Au$ and $\beta * u'$ also belong to $C^\alpha(\mathbb{R}, X)$.

In order to prove uniqueness, suppose that

$$c_0 Au(t) - (\gamma * Au)(t) = b_0 u'(t) + (\beta * u')(t) + a_\infty u(t), \tag{3.17}$$

where $u \in C^\alpha(\mathbb{R}, [D(A)]) \cap C^{1+\alpha}(\mathbb{R}, X)$. Thus, $u, u', Au \in C^\alpha(\mathbb{R}, X)$ and by (2.6), $\gamma * Au$, $\beta * u' \in C^\alpha(\mathbb{R}, X)$.

Let $\sigma > 0$ and consider the operator L_σ defined in (2.7) through the Carleman transform and apply it to (3.17). Then by Proposition A.2, we get

$$\begin{aligned} &-\big(c_0 - \hat{\gamma}(\sigma + i\rho)\big)\big[b_\sigma(\rho) - A\big](L_\sigma u)(\rho) \\ &\quad = 2\sigma \hat{u}(-\sigma + i\rho)\big(b_0 + \hat{\beta}(\sigma + i\rho)\big) + G_\gamma^{Au}(\sigma, \rho) + G_\beta^{u'}(\sigma, \rho) =: H_{\gamma,\beta}(\sigma, \rho), \end{aligned}$$

where

$$b_\sigma(\rho) := \frac{(\sigma + i\rho)(b_0 + \hat{\beta}(\sigma + i\rho)) + a_\infty}{c_0 - \hat{\gamma}(\sigma + i\rho)}$$

and $G_\gamma^{Au}(\sigma, \rho)$, $G_\beta^{u'}(\sigma, \rho)$ are given by (A.5).

Observe that $b_0(\rho) = b(\rho)$ as given by Eq. (3.5). Recall that by hypothesis $b(\rho) \in \rho(A)$ for all $\rho \in \mathbb{R}$. Therefore we have

$$\big(b_\sigma(\rho) - b(\rho)\big)\big(b(\rho) - A\big)^{-1}(L_\sigma u)(\rho) + (L_\sigma u)(\rho) = \frac{1}{\hat{\gamma}(\sigma + i\rho) - c_0}\big(b(\rho) - A\big)^{-1} H_{\gamma,\beta}(\sigma, \rho).$$

Let $\phi \in C_c^\infty(\mathbb{R})$. Multiplying by ϕ and integrating over $\mathbb{R}$ we obtain

$$\int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho)\,d\rho = \int_{\mathbb{R}} N_\sigma(\rho) H_{\gamma,\beta}(\sigma, \rho)\,d\rho - \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho)\,d\rho, \tag{3.18}$$

where

$$M_\sigma(\rho) := \phi(\rho)\big(b_\sigma(\rho) - b(\rho)\big)\big(b(\rho) - A\big)^{-1} \quad \text{and}$$
$$N_\sigma(\rho) := \frac{1}{\hat{\gamma}(\sigma + i\rho) - c_0}\phi(\rho)\big(b(\rho) - A\big)^{-1}.$$

Observe that $M_\sigma(\rho)$ and $N_\sigma(\rho)$ are both in $C_c^2(\mathbb{R}, \mathcal{B}(X))$ by the 2-regularity of β and γ. By Lemmas A.4 and A.5 we have that the right-hand side in Eq. (3.18) converges to zero. It follows from Proposition A.2(i) that

$$\int_{\mathbb{R}} u(\rho)(\mathcal{F}\phi)(\rho)\,d\rho = \lim_{\sigma\to 0}\int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho)\,d\rho = 0 \tag{3.19}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Therefore $u \equiv 0$. □

We remark that assumption (3.7) is satisfied if γ verifies Eq. (3.3). The conditions we put on β and γ are obviously fulfilled for kernels given by (1.3). The case where $a_\infty = 0$ and $\beta = \gamma \equiv 0$ was worked out by Arendt, Batty and Bu (see [3, Theorem 6.1]).

Corollary 3.9. *In the context of Theorem* 3.8*, if condition* (ii) *is valid, we have* u', Au, $\beta * u'$, $\gamma * Au \in C^\alpha(\mathbb{R}, X)$. *Moreover, there exists a constant* $C > 0$ *independent of* $f \in C^\alpha(\mathbb{R}, X)$ *such that*

$$\|u'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)} + \|\beta * u'\|_{C^\alpha(\mathbb{R},X)} + \|\gamma * Au\|_{C^\alpha(\mathbb{R},X)} \leqslant C\|f\|_{C^\alpha(\mathbb{R},X)}.$$

Proof. The first statement follows from the proof of Theorem 3.8. The second statement is a consequence of the closed graph theorem. □

Periodic solutions of (1.1) were studied in [16,17]. We note here that if f is periodic, then by uniqueness, the solution u is periodic as well. In fact, one can easily check that the conditions imposed on the kernels in [16,17] are implied by those we consider here.

Example 3.10. Let $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\}\colon\ 0 \leqslant \arg(\lambda) < \theta\}$ be a sector in the complex plane. Suppose A is a linear operator such that $\Sigma_\theta \subset \rho(A)$ and the estimate

$$\big\|\lambda R(\lambda, A)\big\| \leqslant M, \quad \lambda \in \Sigma_\theta, \tag{3.20}$$

is satisfied. Then in order to apply the theorem one needs only verify that $b(\eta) \in \Sigma_\theta$. Suppose, for instance, that A generates a bounded analytic semigroup (which need not be strongly continuous). Then (3.20) is satisfied on the sector Σ_θ with $\frac{\pi}{2} \leqslant \theta < \pi$ and since by the Riemann–Lebesgue lemma,

$$\lim_{|\eta|\to\infty} \frac{i\eta}{i\eta(b_0 + \tilde{\beta}(\eta)) + a_\infty} = b_0^{-1},$$

we obtain

$$\sup_{\eta\in\mathbb{R}}\left\|\frac{i\eta}{c_0-\tilde{\gamma}(\eta)}R\big(b(\eta),A\big)\right\|<\infty,$$

provided that $b(\eta)\in\Sigma_\theta\cup\{0\}$ and $b_0\neq 0$.

The situation where A generates an analytic semigroup has been studied by many authors. We mention [6,10,14,19,22] among others.

Example 3.11. Suppose A generates a bounded analytic semigroup on X and $0\in\rho(A)$. Then

$$\sup_{\eta\in\mathbb{R}}\big\|i\eta R(i\eta,A)\big\|<\infty. \tag{3.21}$$

This means that problem (1.1) for $\beta=\gamma\equiv 0$ and $a_\infty=0$, $b_0=c_0=1$ is C^α-well posed for both A and $-A$. Well posedness in the case $\beta=\gamma\equiv 0$ and $a_\infty=0$ was considered by Arendt, Batty and Bu (see [3, Theorem 6.1]).

Unless A is bounded, $-A$ is not a generator of a semigroup (compare [4, Corollary 3.7.18]). We observe that if B generates a bounded group of operators on X and $0\in\rho(B)$, then (3.21) is satisfied for $A=iB$. This follows from the Hille–Yosida theorem. In particular, this is the case if A is an invertible self-adjoint operator on a Hilbert space.

This example can be adapted to situations where β and γ are not necessarily identically zero.

Example 3.12. This example and the next one exhibits A and γ such that the conditions of Theorem 3.8 are satisfied but A is not necessarily the generator of a C_0-semigroup.

Let $X=l^2(\mathbb{N})$ and consider the operator A given by

$$(Au)_n=nu_n,\qquad D(A)=\big\{(u_n)\in l^2(\mathbb{N}):\ (n\cdot u_n)\in l^2(\mathbb{N})\big\}.$$

For the other parameters of the problem, we take $\beta\equiv 0$, $c_0=a_\infty=0$ and $b_0=-1$. It is clear that A does not generate a C_0-semigroup since $\sigma(A)=\{n:\ n\in\mathbb{N}\}$ is not contained in any left half plane. Define $\gamma(t)=e^{-\alpha t}$, $\alpha>0$. Clearly the sequence $\tilde{\gamma}(\eta)=\frac{1}{i\eta+\alpha}$ is 2-regular on $\mathbb{R}$ and $b(\eta)=-\eta^2+i\eta\alpha\in\rho(A)$ for all $\eta\in\mathbb{R}$. Moreover, for each $x=(x_n)\in l^2(\mathbb{N})$ we have

$$\begin{aligned}\left\|\frac{i\eta}{\tilde{\gamma}(\eta)}\big(b(\eta)I-A\big)^{-1}x\right\|^2&=\big\|i\eta(i\eta+\alpha)\big(i\eta(i\eta+\alpha)-A\big)^{-1}x\big\|^2=\sum_{n=1}^{\infty}\left|\frac{\eta^2-i\eta\alpha}{-\eta^2-i\eta\alpha-n}x_n\right|^2\\&=\sum_{n=1}^{\infty}\frac{\eta^4+\eta^2\alpha^2}{(\eta^2+n)^2+\eta^2\alpha^2}|x_n|^2\leqslant\sum_{n=1}^{\infty}\frac{\eta^4+\eta^2\alpha^2}{(\eta^2+n)^2}|x_n|^2.\end{aligned}$$

Since $\alpha>0$, we obtain $\frac{\eta^4+\eta^2\alpha^2}{\eta^4+2\eta^2n+n^2}\leqslant 1+\frac{\alpha^2}{2}$ for all $n\in\mathbb{N}$. Hence

$$\left\|\frac{i\eta}{\gamma(\eta)}\big(b(\eta)I-A\big)^{-1}x\right\|^2\leqslant\left(1+\frac{\alpha^2}{2}\right)\sum_{n=1}^{\infty}|x_n|^2=\left(1+\frac{\alpha^2}{2}\right)\|x\|^2$$

for all $\eta \in \mathbb{R}$. Then, the hypotheses of Theorem 3.8 are satisfied and we conclude that the problem is C^α-well posed.

Example 3.13. Suppose A is a self-adjoint operator on a Hilbert space. Observe that we can include here the case where A is not necessarily the generator of a C_0 semigroup choosing $\sigma(A)$ appropriately.

(i) We take $\gamma \equiv 0$, $\beta(t) = \beta_0 e^{-\nu t},\ t \geqslant 0$ where $\nu > 0$ and $\beta_0 \in \mathbb{R}$. Suppose $c_0 \neq 0$, $b_0 \neq 0$ and $a_\infty \in \mathbb{R}$. Then $\tilde{\beta}(\eta) = \frac{\beta_0}{\nu + i\eta},\ \eta \in \mathbb{R}$, and

$$b(\eta) = c_0^{-1}\left[i\eta\left(b_0 + \frac{\beta_0}{\nu + i\eta}\right) + a_\infty\right] = c_0^{-1}\left[a_\infty + \frac{\beta_0\eta^2}{\nu^2+\eta^2} + i\left(b_0\eta + \frac{\beta_0\nu\eta}{\nu^2+\eta^2}\right)\right].$$

It is clear that $\sup_{|\eta|\geqslant\varepsilon} \frac{|b(\eta)|}{|\Im(b(\eta))|} < \infty$ for each $\varepsilon > 0$. We make the additional assumptions that $b_0\beta_0 > 0$ (this ensures that $\Im(b(\eta)) \neq 0$ for all $\eta \in \mathbb{R} \setminus \{0\}$) and $b(0) = c_0^{-1} a_\infty \in \rho(A)$. Using the well-known estimate $\|(\lambda - A)^{-1}\| \leqslant |\Im(\lambda)|^{-1}$ for $\Im(\lambda) \neq 0$ (see, e.g., [25, Chapter X]), we see that Theorem 3.8 applies.

(ii) We take $\beta \equiv 0$ and $\gamma(t) = \gamma_0 e^{-\alpha t},\ t \geqslant 0$ so that $\tilde{\gamma}(\eta) = \frac{\gamma_0}{\alpha + i\eta},\ \eta \in \mathbb{R}$. With these choices, we have:

$$\begin{aligned} b(\eta) &= \frac{i\eta b_0 + a_\infty}{c_0 - \tilde{\gamma}(\eta)} = \frac{(\alpha + i\eta)(i\eta b_0 + a_\infty)}{c_0(\alpha + i\eta) - \gamma_0} \\ &= \frac{[(\alpha a_\infty - \eta^2 b_0) + i\eta(\alpha b_0 + a_\infty)][(c_0\alpha - \gamma_0) - ic_0\eta]}{(c_0\alpha - \gamma_0)^2 + c_0^2\eta^2}. \end{aligned}$$

Observe that

$$\left|\Im\big(b(\eta)\big)\right| = \frac{|\eta||(c_0\alpha - \gamma_0)(\alpha b_0 + a_\infty) - c_0(\alpha a_\infty - \eta^2 b_0)|}{(c_0\alpha - \gamma_0)^2 + c_0^2\eta^2}.$$

We make the following additional assumptions: $\alpha b_0 + a_\infty > 0$, $a_\infty < 0$, $b_0 > 0$, $c_0 > 0$, $c_0\alpha - \gamma_0 > 0$ and $b(0) = \frac{\alpha a_\infty}{c_0\alpha - \gamma_0} \in \rho(A)$.

Under these conditions, and in view of the inequality $\|(\lambda - A)^{-1}\| \leqslant |\Im(\lambda)|^{-1}$ for $\Im(\lambda) \neq 0$, it is readily seen that Theorem 3.8 applies.

As examples of operators fitting into this example, we may take: Let $X = L^2[0,1]$, $\theta \in \mathbb{R}$ and consider the operator A_θ defined by $A_\theta v(x) = iv'(x)$ with domain $D(A_\theta) = \{v \in AC[0,1],\ v(0) = e^{i\theta}v(1)\}$. Then A_θ is a self-adjoint operator with compact resolvent. In fact, the family $(A_\theta)_{\theta\in\mathbb{R}}$ constitutes the set of all self-adjoint extensions of the operator $\mathcal{A}$ given by $\mathcal{A}v(x) = iv'(x)$ with domain $D(\mathcal{A}) = \{v \in C^\infty[0,1],\ v(0) = v(1) = 0\}$ (see, e.g., [24, Chapter VIII] and [25, Chapter X]). Direct computation shows that the eigenvalues of A_θ are $\lambda_n = \theta + 2\pi n,\ n \in \mathbb{Z}$. The corresponding eigenvectors are the functions $v_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$.

This will correspond, in case (ii), to the boundary value problem

$$\begin{cases} b_0 u_t(t,x) + a_\infty u(t,x) = i c_0 u_x(t,x) - \int_{-\infty}^{t} \gamma(t-s) i u_x(s,x)\,ds + f(t,x), \\ \quad x \in [0,1],\ t \in \mathbb{R}, \\ u(t,0) = e^{i\theta} u(t,1), \quad t \in \mathbb{R}. \end{cases}$$

We summarize the discussion in the last example in the following proposition.

Proposition 3.14. *Suppose β and γ satisfy the assumptions of Theorem* 3.8*. Let B be the generator of a bounded strongly continuous group on the Banach space X, and set $A = iB$. Then if*

$$\Im\big(b(\eta)\big) \neq 0 \quad \textit{for all } \eta \in \mathbb{R} \quad \textit{and} \quad \sup_{\eta \in \mathbb{R}} |\eta| \big|\Im\big(b(\eta)\big)\big|^{-1} < \infty, \tag{3.22}$$

then problem (1.1) *is C^α-well posed.*

If instead of (3.22) *we assume that*

$$\Im\big(b(\eta)\big) \neq 0 \quad \textit{for all } \eta \in \mathbb{R}\setminus\{0\}, \quad \textit{and}$$
$$\sup_{\eta \in \mathbb{R}\setminus\{0\}} |\eta| \big|\Im\big(b(\eta)\big)\big|^{-1} < \infty \quad \textit{and} \quad b(0) \in \rho(A), \tag{3.23}$$

then problem (1.1) *is C^α-well posed.*

Proof. Since B generates a bounded group, there exists $M > 0$ such that $\|R(\lambda, B)\| + \|R(\lambda, -B)\| \leqslant M/|\Re\lambda|$ for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \neq 0$. Therefore the proposition follows from $\|R(b(\eta), iB)\| = \|R(-ib(\eta), B)\| \leqslant M/|\Re(-ib(\eta))| = M/|\Im(b(\eta))|$ for $\eta \in \mathbb{R}$.

For the second statement, we have $\|\eta R(b(\eta), A)\| \leqslant |\eta||\Im(b(\eta))|^{-1}$, $\eta \neq 0$. With the additional assumption that $b(0) \in \rho(A)$, this immediately leads to condition (ii) of Theorem 3.8. □

Remark 3.15. As an immediate application of Theorem 3.8 one can deal with semilinear problems of the following type

$$\frac{d}{dt}\left(b_0 u(t) + \int_{-\infty}^{t} \beta(t-s) u(s)\,ds\right) + a_\infty u(t)$$
$$= c_0 A u(t) - \int_{-\infty}^{t} \gamma(t-s) A u(s)\,ds + G\big(u(t)\big) + F(t). \tag{3.24}$$

We fix $w \in C^\alpha(\mathbb{R}, X)$, and suppose that $G(w(t)) \in C^\alpha(\mathbb{R}, X)$. Then, inserting w in the nonlinear term of (3.24) we arrive at the linear problem

$$L\big(u(t)\big) = G\big(w(t)\big) + F(t), \tag{3.25}$$

where

$$L\big(u(t)\big) := \frac{d}{dt}\left(b_0 u(t) + \int_{-\infty}^{t} \beta(t-s)u(s)\,ds\right) + a_\infty u(t) - c_0 Au(t) + \int_{-\infty}^{t} \gamma(t-s)Au(s)\,ds. \tag{3.26}$$

Suppose $F \in C^\alpha(\mathbb{R}, X)$. Under the hypothesis of Theorem 3.8 we obtain that there is a unique function

$$u(w) \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha\big(X, \big[D(A)\big]\big)$$

such that (3.25) is satisfied. Since $[D(A)] \hookrightarrow X$, it follows that

$$u(w) \in C^\alpha(\mathbb{R}, X).$$

Thus the fixed point map

$$w \to u(w) \tag{3.27}$$

is well defined in the Banach space $C^\alpha(\mathbb{R}, X)$. It can be shown that under appropriate conditions on the nonlinearity G, the above map possesses a fixed point which is obviously a solution of the semilinear problem (3.24). We observe that this type of approach has been recently carried out in [15] for some concrete problems arising in mathematical biology.

Appendix A. Remarks on Fourier, Carleman and Laplace transforms

In the Appendix we present some technical results which are used in Section 3.

We continue to use the symbol $\hat{f}(\lambda)$ for the Carleman transform:

$$\hat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)\,dt, & \operatorname{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)\,dt, & \operatorname{Re}\lambda < 0, \end{cases}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ is of subexponential growth; by this we mean

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|}\big\|f(t)\big\|\,dt < \infty \quad \text{for each } \epsilon > 0.$$

Lemma A.1. *If $\dot{u} \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ is of subexponential growth, then*

$$\hat{\dot{u}}(\lambda) = \lambda\hat{u}(\lambda) - u(0), \quad \operatorname{Re}\lambda \neq 0. \tag{A.1}$$

Moreover, if $a \in L^1(\mathbb{R}_+, t^\alpha\,dt)$ and $v \in C^\alpha(\mathbb{R}, X)$ then

$$\widehat{a * v}(\lambda) = \begin{cases} \hat{a}(\lambda)\hat{v}(\lambda) + \int_0^\infty a(s)\big(\int_0^s e^{-\lambda t} v(t-s)\,dt\big)\,ds, & \operatorname{Re}(\lambda) > 0, \\ -\int_{-\infty}^0 \big(\int_s^0 a(t-s)e^{-\lambda t}\,dt\big)v(s)\,ds, & \operatorname{Re}(\lambda) < 0. \end{cases} \tag{A.2}$$

The first formula is well known and the second is obtained by direct computation. We omit the details.

Recall from Section 2 that for $\sigma > 0$ we define $L_\sigma u$ by

$$(L_\sigma u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho), \quad \rho \in \mathbb{R}.$$

We denote by $\mathcal{S}(\mathbb{R})$ the Schwarz space of smooth rapidly decreasing functions on $\mathbb{R}$.

Statements (ii)–(iv) of the following proposition on the Carleman transform may be of independent interest.

Proposition A.2.

(i) *If* $v \in L^1(\mathbb{R}, (1+|t|)^{-k}\,dt, X)$ *for some* $k \in \mathbb{N} \cup \{0\}$ *then*

$$\lim_{\sigma \to 0} \int_{\mathbb{R}} (L_\sigma v)(\rho)\phi(\rho)\,d\rho = \int_{\mathbb{R}} v(\rho)(\mathcal{F}\phi)(\rho)\,d\rho$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

(ii) *If* $v \in C^\alpha(\mathbb{R}, X)$ *then*

$$\lim_{\sigma \to 0^+} \sigma \int_{\mathbb{R}} \hat{v}(-\sigma + i\rho)\phi(\rho)\,d\rho = 0$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

(iii) *If* $v \in C^{1+\alpha}(\mathbb{R}, X)$ *then*

$$\big(L_\sigma(v')\big)(\rho) = (\sigma + i\rho)\big(L_\sigma(v)\big)(\rho) + 2\sigma\hat{v}(-\sigma + i\rho). \tag{A.3}$$

(iv) *If* $v \in C^\alpha(\mathbb{R}, X)$ $(0 < \alpha < 1)$ *and* $a \in L^1(\mathbb{R}_+, t^\alpha\,dt)$, *then*

$$\big(L_\sigma(a * v)\big)(\rho) = \hat{a}(\sigma + i\rho)(L_\sigma v)(\rho) + G_a^v(\sigma, \rho) \tag{A.4}$$

with $\lim_{\sigma \to 0} \int_{\mathbb{R}} G_a^v(\sigma, \rho)\phi(\rho)\,d\rho = 0$ *for all* $\phi \in \mathcal{S}(\mathbb{R})$.

Proof. The first assertion is established in [4, p. 318]. We prove (ii). Using the definition of the Fourier transform and applying Fubini's theorem we have

$$\sigma \int_{\mathbb{R}} \hat{v}(-\sigma + i\rho)\phi(\rho)\,d\rho = \int_{\mathbb{R}} \sigma \left(\int_{-\infty}^{0} e^{(\sigma - i\rho)s} v(s)\,ds \right) \phi(\rho)\,d\rho$$

$$= \int_{-\infty}^{0} \int_{\mathbb{R}} \sigma e^{(\sigma - i\rho)s} \phi(\rho)\,d\rho\, v(s)\,ds = \int_{-\infty}^{0} \sigma e^{\sigma s} \tilde{\phi}(s) v(s)\,ds.$$

Since $v \in C^\alpha(\mathbb{R}, X)$ there exists a constant $c > 0$ such that

$$\big\| e^{\sigma s} \tilde{\phi}(s) v(s) \big\| \leqslant c\big(1 + |s|^\alpha\big)\big|\tilde{\phi}(s)\big|, \quad s \leqslant 0.$$

Because $\tilde{\phi} \in \mathcal{S}(\mathbb{R})$, the function $s \mapsto (1 + |s|^{\alpha})|\tilde{\phi}(s)|$ belongs to $L^1(\mathbb{R}_+)$. The assertion follows.

Assertion (iii) follows immediately from (A.1) and the definition. To finish the proof, we prove (iv). Using Lemma A.1 we have

$$\begin{aligned}
\widehat{(a*v)}(\sigma+i\rho) &= \hat{a}(\sigma+i\rho)\hat{v}(\sigma+i\rho) + \int_0^\infty a(\tau)\left(\int_0^\tau e^{-(\sigma+i\rho)t} v(t-\tau)\,dt\right)d\tau \\
&= \hat{a}(\sigma+i\rho)\hat{v}(\sigma+i\rho) + \int_0^\infty a(\tau)\left(\int_{-\tau}^0 e^{-(\sigma+i\rho)(t+\tau)} v(t)\,dt\right)d\tau \\
&= \hat{a}(\sigma+i\rho)\hat{v}(\sigma+i\rho) + \int_{-\infty}^0 \left(\int_{-s}^\infty a(\tau)e^{-(\sigma+i\rho)\tau}\,d\tau\right)e^{-(\sigma+i\rho)s} v(s)\,ds.
\end{aligned}$$

Similarly

$$\begin{aligned}
\widehat{(a*v)}(-\sigma+i\rho) &= -\int_{-\infty}^0 \left(\int_s^0 e^{(\sigma-i\rho)t} a(t-s)\,dt\right)v(s)\,ds \\
&= -\int_{-\infty}^0 \left(\int_s^0 e^{(\sigma-i\rho)(t-s)} a(t-s)\,dt\right)e^{(\sigma-i\rho)s} v(s)\,ds \\
&= -\int_{-\infty}^0 \left(\int_0^{-s} e^{(\sigma-i\rho)\tau} a(\tau)\,d\tau\right)e^{(\sigma-i\rho)s} v(s)\,ds \\
&= -\int_{-\infty}^0 \left(\int_0^\infty e^{-(\sigma+i\rho)\tau} a(\tau)\,d\tau + \int_0^{-s} e^{(\sigma-i\rho)\tau} a(\tau)\,d\tau\right)e^{(\sigma-i\rho)s} v(s)\,ds \\
&\quad + \int_{-\infty}^0 \left(\int_0^\infty e^{-(\sigma+i\rho)\tau} a(\tau)\,d\tau\right)e^{(\sigma-i\rho)s} v(s)\,ds \\
&= \hat{a}(\sigma+i\rho)\hat{v}(-\sigma+i\rho) - \int_{-\infty}^0 \left(\int_0^{-s} e^{(\sigma-i\rho)(s+\tau)} a(\tau)\,d\tau\right)v(s)\,ds \\
&\quad + \int_{-\infty}^0 \left(\int_0^\infty e^{-(\sigma+i\rho)\tau} a(\tau)\,d\tau\right)e^{(\sigma-i\rho)s} v(s)\,ds.
\end{aligned}$$

From this and the definition of L_σ, Eq. (A.4) follows with

$$\begin{aligned} G_a^v(\sigma,\rho) &:= \int_{-\infty}^{0}\left(\int_{-s}^{\infty} a(\tau)e^{-(\sigma+i\rho)\tau}\,d\tau\right)e^{-(\sigma+i\rho)s}v(s)\,ds \\ &\quad + \int_{-\infty}^{0}\left(\int_{0}^{-s} e^{(\sigma-i\rho)(s+\tau)}a(\tau)\,d\tau\right)v(s)\,ds \\ &\quad - \int_{-\infty}^{0}\left(\int_{0}^{\infty} e^{-(\sigma+i\rho)\tau}a(\tau)\,d\tau\right)e^{(\sigma-i\rho)s}v(s)\,ds \\ &=: I_1(\sigma,\rho)+I_2(\sigma,\rho)-I_3(\sigma,\rho). \end{aligned} \tag{A.5}$$

When it is clear from the context which function v is being used we shall write simply $G_a(\sigma,\rho)$ for $G_a^v(\sigma,\rho)$.

We now prove that $\lim_{\sigma\to 0}\int_{\mathbb{R}} G_a^v(\sigma,\rho)\phi(\rho)\,d\rho = 0$ for all $\phi\in\mathcal{S}(\mathbb{R})$. We consider each term separately. Let $\phi\in\mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} \int_{\mathbb{R}} I_1(\sigma,\rho)\phi(\rho)\,d\rho &= \int_{\mathbb{R}}\left[\int_{-\infty}^{0}\left(\int_{-s}^{\infty} e^{-\sigma\tau}a(\tau)\,d\tau\right)e^{-\sigma s}v(s)\,ds\right]e^{-i\rho(\tau+s)}\phi(\rho)\,d\rho \\ &= \int_{0}^{\infty}\left[\int_{-\tau}^{0}\left(\int_{\mathbb{R}} e^{-i\rho(\tau+s)}\phi(\rho)\,d\rho\right)v(s)e^{-\sigma s}\,ds\right]e^{-\sigma\tau}a(\tau)\,d\tau \\ &= \int_{0}^{\infty}\left[\int_{-\tau}^{0}\tilde{\phi}(\tau+s)e^{-\sigma s}v(s)\,ds\right]e^{-\sigma\tau}a(\tau)\,d\tau. \end{aligned}$$

Since $-\tau<s<0$ in the second integral, we obtain the estimate

$$\begin{aligned} \left\|\int_{-\tau}^{0} e^{-\sigma s}\tilde{\phi}(\tau+s)v(s)\,ds e^{-\sigma\tau}a(\tau)\right\| &\leqslant \int_{-\tau}^{0} e^{\sigma\tau}\left|\tilde{\phi}(\tau+s)\right|\left\|v(s)\right\|\,ds e^{-\sigma\tau}\left|a(\tau)\right| \\ &\leqslant \int_{\mathbb{R}}\left|\tilde{\phi}(\tau+s)\right|\left\|v(s)\right\|\,ds\left|a(\tau)\right| \\ &= \int_{\mathbb{R}}\left|\tilde{\phi}(s)\right|\left\|v(s-\tau)\right\|\,ds\left|a(\tau)\right|. \end{aligned}$$

Using the fact that $\tilde{\phi}\in\mathcal{S}(\mathbb{R})$, $v\in C^\alpha(\mathbb{R},X)$ and $a\in L^1(\mathbb{R}_+,t^\alpha\,dt)\cap L^1_{\mathrm{loc}}(\mathbb{R}_+)$ we easily verify that the last expression is in $L^1(\mathbb{R}_+)$. Analogously, we have

$$\int_{\mathbb{R}} I_2(\sigma,\rho)\phi(\rho)\,d\rho = \int_{\mathbb{R}} \left[\int_{-\infty}^{0} \left(\int_{0}^{-s} e^{\sigma\tau} a(\tau)\,d\tau \right) e^{\sigma s} v(s)\,ds \right] e^{-i\rho(\tau+s)}\phi(\rho)\,d\rho$$

$$= \int_{0}^{\infty} \left[\int_{-\infty}^{-\tau} \left(\int_{\mathbb{R}} e^{-i\rho(\tau+s)}\phi(\rho)\,d\rho \right) v(s) e^{\sigma s}\,ds \right] e^{\sigma\tau} a(\tau)\,d\tau$$

$$= \int_{0}^{\infty} \left[\int_{-\infty}^{-\tau} \tilde{\phi}(\tau+s) e^{\sigma s} v(s)\,ds \right] e^{\sigma\tau} a(\tau)\,d\tau.$$

In the same way as above we obtain the estimate

$$\left\| \int_{-\infty}^{-\tau} \tilde{\phi}(\tau+s) e^{\sigma s} v(s)\,ds e^{\sigma\tau} a(\tau) \right\| \leqslant \int_{-\infty}^{\tau} e^{-\sigma\tau} \left|\tilde{\phi}(\tau+s)\right| \left\|v(s)\right\| ds e^{\sigma\tau} \left|a(\tau)\right|$$

$$\leqslant \int_{\mathbb{R}} \left|\tilde{\phi}(s)\right| \left\|v(s-\tau)\right\| ds \left|a(\tau)\right|.$$

For the last term, using Fubini's theorem as above, we obtain

$$\int_{\mathbb{R}} I_3(\sigma,\rho)\phi(\rho)\,d\rho = \int_{0}^{\infty} \left[\int_{-\infty}^{0} \tilde{\phi}(\tau+s) e^{\sigma s} v(s)\,ds \right] e^{-\sigma\tau} a(\tau)\,d\tau,$$

from which the estimate

$$\left\| \int_{-\infty}^{0} \tilde{\phi}(\tau+s) e^{\sigma s} v(s)\,ds e^{-\sigma\tau} a(\tau) \right\| \leqslant \int_{\mathbb{R}} \left|\tilde{\phi}(s)\right| \left\|v(s-\tau)\right\| ds \left|a(\tau)\right|$$

follows. Applying the dominated convergence theorem, we obtain for all $\phi \in \mathcal{S}(\mathbb{R})$

$$\lim_{\sigma\to 0} \int_{\mathbb{R}} G_a^v(\sigma,\rho)\phi(\rho)\,d\rho = \int_{0}^{\infty} \int_{-\tau}^{0} \tilde{\phi}(\tau+s) v(s) a(\tau)\,ds\,d\tau + \int_{0}^{\infty} \int_{-\infty}^{-\tau} \tilde{\phi}(\tau+s) v(s) a(\tau)\,ds\,d\tau$$

$$- \int_{0}^{\infty} \int_{-\infty}^{0} \tilde{\phi}(\tau+s) v(s) a(\tau)\,ds\,d\tau = 0. \qquad \square$$

The following lemma gives an estimate on Fourier multipliers with compact support, needed for the proof of (3.19).

Lemma A.3. *Let $M \in C_c^2(\mathbb{R}, \mathcal{B}(X))$ and $u \in C^\alpha(\mathbb{R}, X)$. The Fourier transform $\mathcal{F}M$ of M satisfies*

$$\int_{\mathbb{R}} \|(\mathcal{F}M)(s)\| \|u(t-s)\| \, ds \leqslant c\big(\|M\|_{L^1} + \|M''\|_{L^1}\big)\big(1 + |t|^\alpha\big) \tag{A.6}$$

for some constant c depending only on α and u.

Proof. From the definition of $\mathcal{F}M$ it follows that $s^2(\mathcal{F}M)(s) = -(\mathcal{F}M'')(s)$. We have

$$\begin{aligned}\int_{\mathbb{R}} \big(1+s^2\big)\|(\mathcal{F}M)(s)\| \frac{\|u(t-s)\|}{1+s^2} \, ds &\leqslant \int_{\mathbb{R}} \big(\|M\|_{L^1} + \|M''\|_{L^1}\big) \frac{|t-s|^\alpha}{1+s^2} \, ds \\ &= K_M\left(\int_{|s|\leqslant|t|} \frac{2^\alpha |t|^\alpha}{1+s^2} \, ds + \int_{|s|>|t|} \frac{2^\alpha |s|^\alpha}{1+s^2} \, ds \right),\end{aligned}$$

where $K_M := \|M\|_{L^1} + \|M''\|_{L^1}$. This proves the lemma. □

In the following two lemmas we prove the convergence assertions used in the proof of the uniqueness part of Theorem 3.8. Recall from Section 3 that

$$M_\sigma(\rho) := \phi(\rho)\big(b_\sigma(\rho) - b(\rho)\big)\big(b(\rho) - A\big)^{-1}. \tag{A.7}$$

Lemma A.4. *Let M_σ be given by* (A.7) *with $\phi \in C_c^\infty(\mathbb{R})$ and $u \in C^\alpha(\mathbb{R}, X)$. Under the assumptions of Theorem* 3.8 *we have*

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho) \, d\rho = 0.$$

Proof. Since $(L_\sigma u)(\rho) = \int_{\mathbb{R}} e^{-\sigma|t|} e^{-i\rho t} u(t) \, dt$ we apply Fubini's theorem to obtain

$$\int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho) \, d\rho = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\rho t} M_\sigma(\rho) \, d\rho \, e^{-\sigma|t|} u(t) \, dt = \int_{\mathbb{R}} (\mathcal{F}M_\sigma)(t) e^{-\sigma|t|} u(t) \, dt.$$

It follows from Lemma A.3 that

$$\left\| \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho) \, d\rho \right\| \leqslant \int_{\mathbb{R}} \|(\mathcal{F}M_\sigma)(t)\| \|u(t)\| \, dt \leqslant 2c\big(\|M_\sigma\|_{L^1} + \|M_\sigma''\|_{L^1}\big).$$

It is easy to check that $\|M_\sigma\|_{L^1} + \|M_\sigma''\|_{L^1} \to 0$ as $t \downarrow 0$. □

Recall that for $\phi \in C_c^\infty(\mathbb{R})$, we define

$$N_\sigma(\rho) = \frac{1}{\hat{\gamma}(\sigma + i\rho) - c_0} \phi(\rho)\big(b(\rho) - A\big)^{-1}. \tag{A.8}$$

Lemma A.5. *Let N_σ be given by* (A.8) *with $\phi \in C_c^\infty(\mathbb{R})$ and $u', Au \in C^\alpha(\mathbb{R}, X)$. Under the assumptions of Theorem* 3.8 *we have*

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) H_{\gamma,\beta}(\sigma, \rho)\, d\rho = 0,$$

where $H_{\gamma,\beta}(\sigma, \rho)$ is given by

$$H_{\gamma,\beta}(\sigma, \rho) = 2\sigma \hat{u}(-\sigma + i\rho)\big(b_0 + \hat{\beta}(\sigma + i\rho)\big) + G_\gamma^{Au}(\sigma, \rho) + G_\beta^{u'}(\sigma, \rho).$$

Proof. We first show that $\lim_{\sigma\downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) G_\beta^{u'}(\sigma, \rho)\, d\rho = 0$.

Set $v = u'$. From (A.5) we have

$$\int_{\mathbb{R}} N_\sigma(\rho) G_\beta^{v}(\sigma, \rho)\, d\rho = \int_{\mathbb{R}} N_\sigma(\rho)\big(I_1(\sigma, \rho) + I_2(\sigma, \rho)\big)\, d\rho - \int_{\mathbb{R}} N_\sigma(\rho) I_3(\sigma, \rho)\, d\rho.$$

Note that by definition and using Fubini's theorem we have

$$\begin{aligned}
\int_{\mathbb{R}} N_\sigma(\rho) I_1(\sigma, \rho)\, d\rho &= \int_{\mathbb{R}} \int_{-\infty}^{0} \int_{-s}^{\infty} e^{-\sigma\tau} a(\tau) e^{-i\rho(\tau+s)} N_\sigma(\rho) v(s) e^{-\sigma s}\, d\tau\, ds\, d\rho \\
&= \int_0^\infty \left[\int_{-\tau}^{0} \left(\int_{\mathbb{R}} e^{-i\rho(\tau+s)} N_\sigma(\rho)\, d\rho \right) e^{-\sigma s} v(s)\, ds \right] e^{-\sigma\tau} a(\tau)\, d\tau \\
&= \int_0^\infty \left[\int_{-\tau}^{0} (\mathcal{F} N_\sigma)(\tau + s) v(s) e^{-\sigma s}\, ds \right] e^{-\sigma\tau} a(\tau)\, d\tau.
\end{aligned}$$

Also from Lemma A.3, we have the estimate

$$\left\| \int_{-\tau}^{0} \big[(\mathcal{F} N_\sigma)(\tau + s) v(s) e^{-\sigma s}\, ds\big] e^{-\sigma\tau} a(\tau) \right\| \leqslant c\big(\|N_\sigma\|_{L^1} + \|N_\sigma''\|_{L^1}\big)\big(1 + |\tau|^\alpha\big)\big|a(\tau)\big|.$$

Since by hypothesis $\sup_{\eta\in\mathbb{R}} \|\frac{i\eta}{c_0 - \tilde{\gamma}(\eta)} R(b(\eta), A)\| < \infty$, and β and γ are 2-regular, we verify that

$$\sup_{0 \leqslant \sigma \leqslant \epsilon} \big(\|N_\sigma\|_{L^1} + \|N_\sigma''\|_{L^1}\big) < \infty. \tag{A.9}$$

Proceeding in the same way we obtain similar estimates for the integral corresponding to I_2 and I_3. Applying the dominated convergence theorem, we have

$$\lim_{\sigma\to 0}\int_{\mathbb{R}} N_\sigma(\rho)G_\beta^v(\sigma,\rho)\,d\rho = \int_0^\infty\int_{-\tau}^0 (\mathcal{F}N_0)(\tau+s)v(s)a(\tau)\,ds\,d\tau + \int_0^\infty\int_{-\infty}^{-\tau} (\mathcal{F}N_0)(\tau+s)v(s)a(\tau)\,ds\,d\tau - \int_0^\infty\int_{-\infty}^{0} (\mathcal{F}N_0)(\tau+s)v(s)a(\tau)\,ds\,d\tau = 0.$$

Similarly, setting $v = Au$, the above arguments show that

$$\lim_{\sigma\downarrow 0}\int_{\mathbb{R}} N_\sigma(\rho)G_\gamma^{Au}(\sigma,\rho)\,d\rho = 0$$

since the conditions satisfied by β are also satisfied by γ.

It remains to show that $\lim_{\sigma\downarrow 0}\int_{\mathbb{R}} 2\sigma N_\sigma(\rho)\hat{u}(-\sigma+i\rho)(b_0+\hat{\beta}(\sigma+i\rho))\,d\rho = 0$.

Let $N_\sigma^\beta(\rho) := (b_0+\hat{\beta}(\sigma+i\rho))N_\sigma(\rho)$. By estimate (A.9) and the fact that $\hat{\beta}$ is 2-regular, we deduce that

$$\sup_{0\leqslant\sigma\leqslant\epsilon}\left(\left\|N_\sigma^\beta\right\|_{L^1} + \left\|\left(N_\sigma^\beta\right)''\right\|_{L^1}\right) < \infty.$$

On the other hand, we have

$$\int_{\mathbb{R}} N_\sigma^\beta(\rho)\hat{u}(-\sigma+i\rho)\,d\rho = -\int_{\mathbb{R}}\int_{-\infty}^0 e^{(\sigma-i\rho)t}N_\sigma^\beta(\rho)u(t)\,dt\,d\rho = -\int_{-\infty}^0 \left(\mathcal{F}N_\sigma^\beta\right)(t)e^{\sigma t}u(t)\,dt.$$

Then by Lemma A.3 we have for $0\leqslant\sigma\leqslant\epsilon$,

$$\left\|\int_{\mathbb{R}} N_\sigma^\beta(\rho)\hat{u}(-\sigma+i\rho)\,d\rho\right\| \leqslant 2c\sigma \sup_{0\leqslant\sigma\leqslant\epsilon}\left(\left\|N_\sigma^\beta\right\|_{L^1} + \left\|\left(N_\sigma^\beta\right)''\right\|_{L^1}\right).$$

This completes the proof. □

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