A CHARACTERIZATION OF PERIODIC SOLUTIONS FOR TIME-FRACTIONAL DIFFERENTIAL EQUATIONS IN UMD SPACES AND APPLICATIONS

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Abstract. We study the fractional differential equation $(*)$ $D^\alpha u(t)+BD^\beta u(t)+Au(t) = f(t)$, $0 \leq t \leq 2\pi$ $(0 < \beta < \alpha)$ in periodic Lebesgue spaces $L^p(0, 2\pi; X)$ where X is a Banach space. Using functional calculus and operator valued Fourier multiplier theorems, we characterize, in UMD spaces, the well posedness of $(*)$ in terms of R- boundedness of the sets $\{(ik)^\alpha((ik)^\alpha + (ik)^\beta B + A)^{-1}\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta B((ik)^\alpha+(ik)^\beta B+A)^{-1}\}_{k\in\mathbf{Z}}$. Applications to the fractional Cauchy problems with periodic boundary condition, which includes the time diffusion and fractional wave equations, as well as a abstract version of the Basset-Boussinesq-Oseen equation, are treated.

1. Introduction

Derivatives and integrals of fractional order [28, 33] have found many applications in recent studies. The interest in fractional analysis has been growing during the past few years. Fractional analysis has numerous applications: hydrology [8], physics [5, 35], kinetic theories [38], stochastic processes and many others (see [27]).

The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Grünwald, Letnikov and Riemann. For many decades afterwards, the theory of fractional derivatives was developed primarily as a theoretical field of mathematics (see [28, 29, 30] and [31] for general surveys). In the past few decades, many authors have pointed out that fractional order models are more appropriate than integer-order models for various real materials. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models in which such effects are, in fact, neglected.

In this paper we study *periodic* solutions for the equation

$$D^\alpha u(t) + BD^\beta u(t) + Au(t) = f(t),\ 0 \leq t \leq 2\pi, \tag{1.1}$$

where A and B are closed linear operators defined on a complex Banach space X with domains $D(A)$ and $D(B)$ respectively, $0 \leq \beta < \alpha \leq 2$. Time fractional equations, in particular diffusion and wave equations (i.e. (1.1) with $B = 0$ and $A = -\Delta$ where Δ is the Laplace operator in the $n-$dimensional space), have recently been treated by some authors (cf. [16, 17, 23, 34, 37]). However, to the knowledge of the authors, time fractional evolution equations with periodic boundary conditions have not been studied so far. One of the main difficulties is to determine the right definition of fractional derivative to be used in this case. In order to study periodic solutions, we consider here the Liouville-Grünwald fractional derivative, introduced in [11] in the scalar case (see Section 2 for a precise definition in the vector-valued case).

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An equation of the form (1.1) is referred to as the *composite fractional relaxation-oscillation equation.* We refer to [15, p.253] for a detailed explanation of different fractional differential equations that are special cases of (1.1).

Equation (1.1) can be regarded, when $\alpha = 2$ and $0 \le \beta \le 1$, as an interpolation between the complete second order equation

$$u''(t) + Bu'(t) + Au(t) = f(t), \tag{1.2}$$

and the classical (perturbed) abstract Cauchy problem of the second order

$$u''(t) + (A + B)u(t) = f(t). \tag{1.3}$$

Varying continuously the index of the fractional derivative β within the real interval $[0, 1]$ one can examine, which solutions survive when switching from the description (1.2) to (1.3).

In the special case where $B \equiv 0$ and $1 \le \alpha \le 2$, equation (1.1) interpolates between the abstract Cauchy problems of first and second order respectively. For example, when $-A = \Delta$, the Laplace operator on $L^p(\mathbf{R}^N)$, equation (1.1) interpolates between the heat and wave equations.

In this paper we are interested in the problem of L^p- *well posedness* (or maximal regularity) on the interval $[0, 2\pi]$ for the general fractional differential equation (1.1) with different periodic boundary conditions depending on the values of α and β. They are:

$$\begin{cases} u(0) = u(2\pi) \quad if \quad 0 < \beta < \alpha \le 1, \\ \\ D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi) \quad and \quad u(0) = u(2\pi) \quad if \qquad 0 < \beta \le 1 < \alpha \le 2, \\ \\ D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi),\ D^{\beta-1}u(0) = D^{\beta-1}u(2\pi),\ u(0) = u(2\pi)\, if\ 1 < \beta < \alpha \le 2, \end{cases} \tag{1.4}$$

In the case $\alpha = 2$ and $\beta = 1$, L^p- well posedness for the initial value problem corresponding to (1.1) on $[0, 2\pi]$ has been characterized in [12] in terms of Fourier multipliers. There, it was proved that if $B = A^\epsilon$ and $\epsilon \in (0, 1/2)$ then we cannot expect L^p-well posedness in general, even if A is a selfadjoint operator on a Hilbert space. On the other hand, in [19], well posedness for the periodic problem in case $B = A$ was characterized in terms of spectral properties of the data in the spaces Lebesgue $L^p(0, 2\pi; X)$, $1 \le p < \infty$ as well as in the periodic Besov spaces $B^s_{p,q}(0, 2\pi; X)$, $1 \le p, q \le \infty$. Note that in the latter framework, the case $p = q = \infty$ yields the Hölder-Zygmund spaces $\mathcal{C}^s$ for $s > 0$.

In this article, we apply the method of operator-valued Fourier multipliers to obtain a characterization of existence, uniqueness and well posedness for (1.1) in Lebesgue spaces. The corresponding techniques have been developed only recently and have proved very efficient for establishing well posedness results for evolution equations, both in the periodic and the initial value cases. Some of the contributions are: [1, 2, 4, 3, 13] and [36].

The plan of the paper is the following: in the second section, we present preliminary material on the Liouville-Grünwald fractional differentiation and integration and the concept of well-posedness in the case of Lebesgue spaces. In Section 3, we give the main abstract result of this paper. There, we are able to characterize solely in terms of spectral properties of A and B the well posedness of the problem (1.1) in $L^p(0, 2\pi; X)$, $1 < p < \infty$. In Section 4, we assume that A is "sectorial", and this allows us to give a very general result for the periodic fractional Cauchy problem, i.e. equation (1.1) in case $B = 0$, which seems to be new even in the scalar case. In section 5, we discuss the case $B = A^{1/2}$ recovering results similar to those of [12].

2. Preliminaries

In the paper [11], Butzer and Westphal introduced the fractional derivative directly as a limit of a fractional difference quotient. In the case of periodic functions, it enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as the connection with the classical Weyl fractional derivative (see [33]).

Let X, Y be complex Banach spaces. We denote by $\mathcal{B}(X,Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1} = (\lambda - A)^{-1}$.

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbf{R}$. Thus, throughout, we consider the space $L^p_{2\pi}(\mathbf{R}; X)$ (which is also denoted by $L^p(0, 2\pi; X)$, $1 \le p \le \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

Given $f \in L^p(0, 2\pi; X)$, $(1 \le p < \infty)$ the Riemann difference

$$\Delta_t^\alpha f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj) \tag{2.1}$$

(where $\binom{\alpha}{j} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}$) exists almost everywhere and

$$\|\Delta_t^\alpha f\|_{L^p(0,2\pi;X)} \le \sum_{j=0}^{\infty} |\binom{\alpha}{j}| \|f\|_{L^p(0,2\pi;X)} = O(1) \tag{2.2}$$

since $\binom{\alpha}{j} = O(j^{-j-1})$ as $j \to \infty$.

The following definition is the direct extension of [11, Definition 2.1] to the vector-valued case.

Definition 2.1. Let X be a complex Banach space, $\alpha > 0$ and $1 \le p < \infty$. If for $f \in L^p(0, 2\pi; X)$ there exists $g \in L^p(0, 2\pi; X)$ such that $\lim_{t\to 0^+} t^{-\alpha} \Delta_t^\alpha f = g$ in the $L^p(0, 2\pi; X)$norm, then g is called the α^{th} Liouville-Grünwald derivative of f in the mean of order p. We denote $g = D^\alpha f$.

Example 2.2. *The α^{th} fractional derivative of e^{iax} for any real a is given by $(ia)^\alpha e^{iax}$. In particular, $D^\alpha \sin x = \sin(x + \frac{\pi}{2}\alpha)$ and $D^\alpha \cos x = \cos(x + \frac{\pi}{2}\alpha)$.*

Now we recall that the Fourier series of $f \in L^p(0, 2\pi; X)(1 \le p < \infty)$ is defined for $k \in \mathbf{Z}$ by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt,$$

where for $k \in \mathbf{Z}$, $e_k(t) = e^{ikt}, t \in \mathbf{R}$. We also have the following properties.

Proposition 2.3. *For $f \in L^p(0, 2\pi; X)$, $1 \le p < \infty$, α, $\beta > 0$ we have*
(i) If $D^\alpha f \in L^p(0, 2\pi; X)$, then $D^\beta f \in L^p(0, 2\pi; X)$ for all $0 < \beta < \alpha$,
(ii) $D^\alpha D^\beta f = D^{\alpha+\beta} f$ whenever one of the two sides is well defined.

The proof is the same as in the scalar case, which is given in [11, Proposition 4.1]. In what follows we denote by $H^{\alpha,p}(0, 2\pi; X)$ the vector-valued function space $H^{\alpha,p}(0, 2\pi; X) := \{u \in L^p(0, 2\pi; X)$: there exists $v \in L^p(0, 2\pi; X)$ such that $\hat{v}(k) = (ik)^\alpha \hat{u}(k)$ for all $k \in \mathbf{Z}$ $\}$. By [11, Theorem 4.1] we also have

$$H^{\alpha,p}(0, 2\pi; X) = \{u \in L^p(0, 2\pi; X) : D^\alpha u \in L^p(0, 2\pi; X)\}.$$

Note that if $1 < \alpha \le 2$ then for $u \in H^{\alpha,p}(0, 2\pi; X)$, it follows that $u(0) = u(2\pi), D^{\alpha-1}(0) = D^{\alpha-1}(2\pi)$.

Let Φ_α be the function defined by

$$\Phi_\alpha(t) = \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{e^{ikt}}{(ik)^\alpha}, \quad t \in \mathbf{R} \setminus 2\pi\mathbf{Z}, \quad \alpha > 0 \tag{2.3}$$

where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i\alpha}{2}\mathrm{sgn} k}$. Note that $\Phi_\alpha \in L^1(0, 2\pi)$ (see [40] for more details) and hence for $u \in L^p(0, 2\pi; X)$,
$1 \le p < \infty$ and $\alpha > 0$, we can define

$$I^\alpha u(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t-s)\Phi_\alpha(s)ds. \tag{2.4}$$

The following lemma is essentially contained in [11, Theorem 4.1].

Lemma 2.4. *Let $1 \le p < \infty$ and let $u, w \in L^p(0, 2\pi; X)$. The following statements are equivalent*

(i) $\int_0^{2\pi} w(t)dt = 0$ *and there exists $x \in X$ such that*

$$u(t) = x + \frac{1}{2\pi}\int_0^{2\pi} w(t-s)\Phi_\alpha(s)ds \ a.e.on \ [0, 2\pi], \tag{2.5}$$

(ii) $\hat{w}(k) = (ik)^\alpha \hat{u}(k)$.

We will need the following definition of operator-valued Fourier multipliers.

Definition 2.5. For $1 \le p \le \infty$, $\alpha \ge 0$ we say that a sequence $\{M_k\}_{k\in\mathbf{Z}} \subset \mathcal{B}(X, Y)$ is an $(L^p, H^{\alpha,p})$-multiplier, if for each $f \in L^p_{2\pi}(\mathbf{R}; X)$ there exists $u \in H^{\alpha,p}(\mathbf{R}; Y)$ such that

$$\hat{u}(k) = M_k\hat{f}(k) \text{ for all } k \in \mathbf{Z}.$$

In particular, in case $\alpha = 0$ the definition coincides with the one contained in [2, Proposition 1.1]. The proof of the following lemma is similar to that of [2, Lemma 2.2] taking into account Lemma 2.4 above.

Lemma 2.6. *Let $1 \le p < \infty$, $\alpha > 0$ and $(M_k)_{k\in\mathbf{Z}} \subset \mathcal{B}(X)$. The following assertions are equivalent*

(i) $(M_k)_{k\in\mathbf{Z}}$ is an $(L^p, H^{\alpha,p})$-multiplier;

(ii) $((ik)^\alpha M_k)_{k\in\mathbf{Z}}$ is an (L^p, L^p)- multiplier.

Remark 2.7. Observe that if $0 \le \beta < \alpha$ then from the relation

$$(ik)^\beta M_k = (ik)^{\beta-\alpha}(ik)^\alpha M_k, \ k \in \mathbf{Z} \setminus \{0\},$$

it follows that $((ik)^\beta M_k)_{k\in\mathbf{Z}}$ is an (L^p, L^p)- multiplier whenever $((ik)^\alpha M_k)_{k\in\mathbf{Z}}$ is an (L^p, L^p)- multiplier.

Next, we define the notion of definition of the fractional differential equation and the associated concept of well-posedness.

Definition 2.8. Let $1 \le p < \infty$. A function u is called a strong L^p-solution of (1.1) if $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ such that $D^\beta u \in L^p(0, 2\pi; D(B))$ and equation (1.1) holds for almost all $t \in [0, 2\pi]$.

Definition 2.9. Let $1 \leq p < \infty$. We say that the problem (1.1) is strongly L^p-well posed (or has maximal regularity) if for every $f \in L^p(0, 2\pi; X)$ there exists a unique strong L^p-solution of (1.1).

The concept of maximal regularity has received much attention in recent years. It is connected to the question of closedness of the sum of two closed operators. It has proven very efficient in the treatment of non linear problems in partial differential equations (see for example [12]).

3. A characterization of strong L^p well posedness

In this section, we characterize well-posedness of the fractional differential equation (1.1). The theorems are in terms of operator-valued Fourier multipliers. The concept of $R-$boundedness is crucial in this context since we are concerned with the L^p spaces. Recent results on this concept as well as applications to concrete problems in the theory of partial differential equations can be found in [13]. Concrete examples of equations of mathematical physics are considered in [12].

Theorem 3.1. *Let X be a complex Banach space and $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed operators such that $D(A) \subset D(B)$. The following assertions are equivalent for $p \in [1, \infty)$:*
(i) The problem (1.1) is strongly L^p-well posed.
(ii) $H(ik) := ((ik)^\alpha + (ik)^\beta B + A)^{-1}$ exists in $\mathcal{B}(X)$ for each $k \in \mathbf{Z}$ and the sequences $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$ are (L^p, L^p)-multipliers.

Proof. $(i) \Longrightarrow (ii)$. Let $k \in \mathbf{Z}$ and $y \in X$. Define $f = e_k \otimes y$. Observe that $\hat{f}(k) = y$ and $\hat{f}(j) = 0$ for $j \neq k$. There exists a unique $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ such that $D^\beta u \in L^p(0, 2\pi; D(B))$ and $D^\alpha u(t) + BD^\beta u(t) + Au(t) = f(t)$ holds for almost all $t \in [0, 2\pi]$. Taking Fourier series on both sides and using the fact that A and B are closed, we obtain that $\hat{u}(k) \in D(A) \cap D(B) = D(A)$ and

$$(ik)^\alpha \hat{u}(k) + (ik)^\beta B\hat{u}(k) + A\hat{u}(k) = \hat{f}(k) = y.$$

Thus, the operator $((ik)^\alpha + (ik)^\beta B + A)$ is surjective.

Let $x \in D(A)$. If $((ik)^\alpha + (ik)^\beta B + A)x = 0$ then $u(t) = e^{ikt}x$ defines a periodic solution of equation (1.1) with $f \equiv 0$ (cf. Example 2.2). Hence $u = 0$ by the assumption of uniqueness and thus $x = 0$.

It remains to verify that $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$ are (L^p, L^p)-multipliers. Let $f \in L^p(0, 2\pi; X)$. By hypothesis, we deduce as above that $\hat{u}(k) \in D(A)$ and $((ik)^\alpha + (ik)^\beta B + A)\hat{u}(k) = \hat{f}(k)$. Then

$$\widehat{D^\alpha u}(k) = (ik)^\alpha \hat{u}(k) = (ik)^\alpha H(ik)\hat{f}(k).$$

From this, it follows that $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ is an (L^p, L^p)-multiplier. Similarly, $\widehat{D^\beta u}(k) \in D(B)$ and

$$\widehat{BD^\beta u}(k) = B\widehat{D^\beta u}(k) = (ik)^\beta B\hat{u}(k) = (ik)^\beta BH(ik)\hat{f}(k).$$

Therefore $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$is an (L^p, L^p)-multiplier.

$(ii) \Longrightarrow (i)$. Let $f \in L^p(0, 2\pi; X)$. By hypothesis there exists $v \in L^p(0, 2\pi; X)$ such that

$$\hat{v}(k) = (ik)^\alpha H(ik)\hat{f}(k), k \in \mathbf{Z}, \tag{3.1}$$

that is $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ is an (L^p, L^p)-multiplier. By Lemma 2.6 this is equivalent to say that $\{H(ik)\}_{k\in\mathbf{Z}}$
is an $(L^p, H^{\alpha,p})$-multiplier and hence we conclude that there exists $u \in H^{\alpha,p}(0, 2\pi; X)$ such that

$$\hat{u}(k) = H(ik)\hat{f}(k). \tag{3.2}$$

In particular, $\hat{u}(k) \in D(A) \subset D(B)$ for all $k \in \mathbf{Z}$ and $(ik)^\alpha \hat{u}(k) = \hat{v}(k)$. Therefore $u \in H^{\alpha,p}(0, 2\pi; X)$ and $D^\alpha u = v$.

Analogously, there exists $w \in L^p(0, 2\pi; X)$ such that

$$\hat{w}(k) = (ik)^\beta BH(ik)\hat{f}(k) = B(ik)^\beta \hat{u}(k), \tag{3.3}$$

Hence, by the closedness of B we obtain $D^\beta u \in L^p(0, 2\pi; D(B))$ and $BD^\beta u = w$.

From the identity

$$I = (ik)^\alpha H(ik) + (ik)^\beta BH(ik) + AH(ik), \tag{3.4}$$

we deduce that $\{AH(ik)\}_{k\in\mathbf{Z}}$ is an (L^p, L^p)-multiplier. Then there exists a function $z \in L^p(0, 2\pi; X)$ such that

$$\hat{z}(k) = AH(ik)\hat{f}(k) = A\hat{u}(k), k \in \mathbf{Z}. \tag{3.5}$$

Hence, by the closedness of A we obtain $u \in L^p(0, 2\pi; D(A))$ and $Au = z$. Using these relations and taking into account (3.4) we conclude that u is a strong L^p-solution of (1.1). We now prove uniqueness. If u is such that $D^\alpha u(t) + BD^\beta u(t) + Au(t) = 0$ then on the Fourier series side we get $\hat{u}(k) \in D(A)$ and $((ik)^\alpha + (ik)^\beta B + A)\hat{u}(k) = 0$. Using (3.4) we obtain that $\hat{u}(k) = 0$ for all $k \in \mathbf{Z}$ and thus $u = 0$.

□

Remark 3.2. In the case $B = A$ in the above theorem, thanks to the identity (3.4) we observe that (ii) can be reduced to requiring only that $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ be an (L^p, L^p)-multiplier. In fact, given $f \in L^p(0, 2\pi)$ we have that $u(t) = \dfrac{1}{\Gamma(\beta)}\displaystyle\int_{-\infty}^{t} (t-s)^{\beta-1} e^{-(t-s)} f(s)ds$ satisfies $\hat{u}(k) = \frac{1}{(ik)^{\beta+1}}\hat{f}(k)$, and this shows that $\frac{1}{(ik)^{\beta+1}}$ is an (L^p, L^p)-multiplier. In this case, we have

$$AH(ik) = \frac{1}{(ik)^{\beta+1}} - \frac{1}{(ik)^{\beta+1}}(ik)^\alpha H(ik)$$

is an (L^p, L^p)-multiplier.

Concrete criteria on Fourier multipliers can be found in the reference [13]. For $j \in \mathbf{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = \text{sgn}(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

Definition 3.3. A family $\mathbf{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there is a constant $C \geq 0$ such that

$$||\sum_{j=1}^{n} r_j \otimes T_j x_j||_{L^p(0,1;Y)} \leq C||\sum_{j=1}^{n} r_j \otimes x_j||_{L^p(0,1;X)} \tag{3.6}$$

for all $T_1, ..., T_n \in \mathbf{T},\ x_1, ..., x_n \in X$ and $n \in \mathbf{N}$, for some $p \in [1, \infty)$.

If (3.6) holds for some $p \in [1, \infty)$, then it holds for all $p \in [1, \infty)$. The best constant C in (3.6) is denoted by $R_p(T)$ or simply $R(T)$ if one does not wish to specify the dependence on p.

We recall that those Banach spaces X for which the Hilbert transform defined initially defined on $\mathcal{D}(\mathbf{R}, X)$ (the space of infinitely differentiable X-valued functions with compact support) by

$$(Hf)(t) = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \frac{1}{\pi}\int_{\epsilon\leq|s|\leq R} \frac{f(t-s)}{s}ds$$

extends to bounded on $L^p(\mathbf{R}, X)$ for some (and hence all) $p \in (1, \infty)$ are called UMD-spaces. The limit in the above formula is to be understood in the L^p sense. An alternative definition can be given using Fourier series may be found in [10]. The latter paper contains other characterizations of the UMD property, notably the one involving martingale differences in Banach spaces. More information on UMD spaces can be found in Amann [1] and Denk, Hieber and Prüss [13].

Theorem 3.4. *Let X be a UMD space. The following assertions are equivalent*

(i) $H(ik)$ exists in $\mathcal{B}(X)$ for each $k \in \mathbf{Z}$ and the sequences $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$ are (L^p, L^p)-multipliers.

(ii) $H(ik)$ exists in $\mathcal{B}(X)$ for each $k \in \mathbf{Z}$ and the sequences $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$ are R-bounded.

Proof. $(i) \implies (ii)$ follows by [2, Proposition 1.11].

$(ii) \implies (i)$ We define $M_k = (ik)^\alpha H(ik) = (I + c_k B + d_k A)^{-1}$ where $c_k = \frac{1}{(ik)^{\alpha-\beta}}$ and $d_k = \frac{1}{(ik)^\alpha}$, $k \neq 0$. Observe that for $\gamma > 0$ we have that $(i(k+1))^\gamma - (ik)^\gamma$ can be estimated by $(ik)^{\gamma-1}$ uniformly in k according to the definition of $(ik)^\gamma$ and the mean value theorem. Then, using the fact that

$$\begin{aligned} k(M_{k+1} - M_k) &= k((I + c_{k+1}B + d_{k+1}A)^{-1} - (I + c_k B + d_k A)^{-1}) \\ &= kH(i(k+1))[(I + c_k B + d_k A) - (I + c_{k+1}B + d_{k+1}A)]H(ik) \\ &= kH(i(k+1))[(c_k - c_{k+1})B + (d_k - d_{k+1})A]H(ik) \end{aligned}$$

and taking into account the identity (3.4) we obtain

$$\begin{aligned} k(M_{k+1} - M_k) &= H(i(k+1))k(c_k - c_{k+1})BH(ik) \\ &+ H(i(k+1))k(d_k - d_{k+1}) \\ &- H(i(k+1))k(d_k - d_{k+1})(ik)^\alpha H(ik) \\ &- H(i(k+1))k(d_k - d_{k+1})(ik)^\beta BH(ik). \end{aligned}$$

This proves that $k(M_{k+1} - M_k)$ is R-bounded. The conclusion follows by the operator-valued Marcinkiewicz multiplier theorem (see [2, Theorem 1.3]).

□

We observe that as in the case of Theorem 3.1 if $A = B$ then it is sufficient to assume in Theorem 3.4 (i) that $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ is a (L^p, L^p)-multiplier and in (ii) that $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ is R-bounded.

A combination of Theorem 3.4 and Theorem 3.1 yields the following result.

Corollary 3.5. *Let X be a UMD space, $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed operators such that $D(A) \subset D(B)$. The following assertions are equivalent for $p \in [1, \infty)$:*

(i) The problem (1.1) is strongly L^p-well posed.

(ii) $H(ik) := ((ik)^\alpha + (ik)^\beta B + A)^{-1}$ exists in $\mathcal{B}(X)$ for each $k \in \mathbf{Z}$ and the sequences $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ and $\{(ik)^\beta BH(ik)\}_{k\in\mathbf{Z}}$ are R-bounded.

Remark 3.6. In case $\alpha = 1$ or $\alpha = 2$ and $B \equiv 0$ the characterization of strongly L^p-well posedness in terms of R-boundedness of the set $\{(ik)^\alpha H(ik)\}_{k\in\mathbf{Z}}$ was first proved in [2, Theorem 2.3 and Theorem 6.1]. The case $\alpha = 2, \beta = 1$ and $B = A$, was proved in [19, Theorem 2.11].

Given $p \in (1, \infty)$, we define the maximal regularity space

$$MR_{\alpha,p}(X) = \{u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A)) : D^{\beta u} \in L^p(0, 2\pi; D(B))\}$$

with norm given by

$$||u||_{MR_{\alpha,p}(X)} = ||u||_{H^{\alpha,p}(0,2\pi;X)} + ||u||_{L^p(0,2\pi;D(A))} + ||D^\beta u||_{L^p(0,2\pi;D(B))}$$

We point out that if the problem (1.1) is strongly L^p-well posed then the operator

$$L : L^p(0, 2\pi; X) \to MR_{\alpha,p}(X)$$

which assigns to each f the unique solution $u := Lf$ of the problem (1.1), is a bounded linear operator. This is an easy consequence of the closedness of the operators A and B and the closed graph theorem. Hence, there exists a constant $M > 0$ such that for every $f \in L^p(0, 2\pi; X)$ we have

$$||u||_{H^{\alpha,p}(0,2\pi;X)} + ||u||_{L^p(0,2\pi;D(A))} + ||D^\beta u||_{L^p(0,2\pi;D(B))} \leq M||f||_{L^p(0,2\pi;X)}.$$

4. Well posedness for the periodic fractional Cauchy problem

Based on the abstract results of the first part, we give in this section a result to widely solve the fractional Cauchy problem with periodic boundary conditions, i.e. the equation

$$(4.1) \qquad P^\alpha_{per} \begin{cases} D^\alpha_t u(x,t) + A^\gamma_x u(x,t) = f(x,t), \quad t \in (0, 2\pi), \ \gamma > 0, \ 0 < \alpha \leq 2. \\ \\ u(x,0) - u(x,2\pi) = 0, \quad max\{0, \alpha - 1\}(u'(x,0) - u'(x,2\pi)) = 0. \end{cases}$$

We remark that the Cauchy problem with initial conditions and Caputo or Riemann-Louiville fractional derivative, has been studied in the literature in recent years. cf. [14], [20] and references therein.

We begin with some preliminaries on sectorial operators. Let $\Sigma_\phi \subset \mathbf{C}$ denote the open sector

$$\Sigma_\phi = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg \lambda| < \phi\}.$$

We denote by

$$\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbf{C} \text{ holomorphic}\}.$$

and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbf{C} \text{ holomorphic and bounded}\}.$$

$\mathcal{H}^\infty(\Sigma_\phi)$ is equipped with the norm

$$||f||^\phi_\infty = \sup_{|\arg \lambda| < \phi} |f(\lambda)|.$$

We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows

$$\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \{f \in \mathcal{H}(\Sigma_\phi) : ||f||^\phi_{\alpha,\beta} < \infty\},$$

where

$$||f||^\phi_{\alpha,\beta} = \sup_{|\lambda|\leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\geq 1} |\lambda^{-\beta} f(\lambda)|.$$

Definition 4.1. A closed linear operator A in X is called sectorial if the following conditions hold:
(i) $\overline{D(A)} = X, \overline{R(A)} = X, (-\infty, 0) \subset \rho(A)$;
(ii)$||t(t+A)^{-1}|| \leq M$ for all $t > 0$ and some $M > 0$.

A is called R-sectorial if the set $\{t(t+A)^{-1}\}_{t>0}$ is R-bounded.

The class of sectorial (resp. R-sectorial) operators in X will be denoted by $\mathcal{S}(X)$ (resp. $\mathcal{RS}(X)$. Set $\mathcal{R}_A(\phi) = R(\lambda(\lambda+A)^{-1} : |\arg \lambda| \leq \phi\}$.

If $A \in \mathcal{S}(X)$ then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and

$$\sup_{|\arg \lambda| < \phi} ||\lambda(\lambda+A)^{-1}|| < \infty.$$

We denote the *spectral angle* of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \quad \sup_{\lambda \in \Sigma_{\pi-\phi}} ||\lambda(\lambda+A)^{-1}|| < \infty\}$$

Definition 4.2. A sectorial operator A is said to admit a bounded $\mathcal{H}^\infty-$calculus if there are $\phi > \phi_A$ and a constant $K_\phi > 0$ such that

$$||f(A)|| \leq K_\phi ||f||_\infty^\phi \text{ for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{4.2}$$

The class of sectorial operators A which admit a bounded $\mathcal{H}^\infty-$calculus is denoted by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty-$angle is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : (4.2) \text{ holds }\}$$

When $A \in \mathcal{H}^\infty(X)$ we say that A *admits an R-bounded* $\mathcal{H}^\infty-$*calculus* if the set

$$\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta), ||f||_\infty^\theta \leq 1\}$$

is R-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The corresponding angle is defined in an obvious way and denoted by $\theta_A^{R\infty}$.

Remark 4.3. If A is a sectorial operator on a Hilbert space, Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$, Sobolev spaces $W^{s,p}(\Omega)$, $1 < p < \infty$, $s \in \mathbf{R}$ or Besov spaces $B^s_{p,q}(\Omega), 1 < p, q < \infty$, $s \in \mathbf{R}$ and A admits a bounded $\mathcal{H}^\infty$ calculus of angle β, then A already admits and $\mathcal{RH}^\infty$ calculus on the same angle β on each of the above described spaces (see Kalton and Weis [18]). More generally, this property is true whenever X is a UMD space with the so called property (α) (see [18]).

Example 4.4. *Well known examples for general classes of closed linear operators with a bounded $\mathcal{H}^\infty$ calculus are: normal sectorial operators in a Hilbert space; m-accretive operators in a Hilbert space; generators of bounded C_0-groups on L^p-spaces and negative generators of positive contraction semigroups on L^p-spaces.*

The following result will be useful in establishing L^p well posedness for the equation (4.1). It can be found in [13, Proposition 4.10].

Proposition 4.5. *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R\infty}$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda \in \Lambda}$ is R-bounded.*

The main result of this section is the following.

Theorem 4.6. *Assume that X is a UMD-space, $1 < p < \infty$, $0 < \alpha \leq 2$ and suppose that $A_x \in \mathcal{RH}^\infty(X)$ with angle $\theta_{A_x}^{R_\infty} \in (0, \frac{\pi}{\gamma}(1 - \frac{\alpha}{2}))$ and $0 \in \rho(A_x)$. Then problem P_{per}^α is strongly L^p-well posed.*

Proof. Since $0 < \theta_{A_x}^{R_\infty} < \frac{\pi}{\gamma}(1 - \frac{\alpha}{2})$ there exists $\beta > \theta_{A_x}^{R_\infty}$ such that $\beta < \frac{\pi}{\gamma}(1 - \frac{\alpha}{2})$. For each $z \in \Sigma_\beta$ and $k \in \mathbf{Z}, k \neq 0$, define $F(ik, z) = (ik)^\alpha((ik)^\alpha + z^\gamma)^{-1}$. Note that the fraction $\frac{z^\gamma}{(ik)^\alpha}$ belong to the sector $\Sigma_{\beta\gamma + \frac{\pi\alpha}{2}}$ where $\beta\gamma + \frac{\alpha\pi}{2} < \pi$. Hence the distance from the sector $\Sigma_{\beta\gamma + \frac{\pi\alpha}{2}}$ to -1 is always positive. In consequence, there exists a constant $M > 0$ independent of $k \in \mathbf{Z}$ and $z \in \Sigma_\beta$ such that

$$|F(ik, z)| = |\frac{1}{1 + \frac{z^\gamma}{(ik)^\alpha}}| \leq M.$$

Hence, Proposition 4.5 shows that the set

$$\{F(ik, A_x)\}_{k \in \mathbf{Z} \setminus \{0\}}$$

is R-bounded. In particular, and since A_x is invertible, the operators $H(ik) := ((ik)^\alpha + A_x^\gamma)^{-1}$ exist for all $k \in \mathbf{Z}$. It then follows that $H(ik)$ exist in $\mathcal{B}(X)$ for all $k \in \mathbf{Z}$ and the sequence

$$\{(ik)^\alpha((ik)^\alpha + A_x^\gamma)^{-1}\}_{k \in \mathbf{Z}}$$

is R-bounded. By Theorem 3.4 and Theorem 3.1 with $B \equiv 0$, we conclude that the problem P_{per}^α is strongly L^p-well posed. □

Remark 4.7. We note that, in case $\gamma = 1$, the same type of conditions of the above theorem leads with the L^p-maximal regularity of the equation $D^\alpha u(t) + Au(t) = f(t)$ with Riemann-Liouville derivative of order $\alpha \in (0, 2)$ and appropriate, non periodic, initial conditions (see [6, Proposition 4.10, p.59]).

Corollary 4.8. *Let $1 < p < \infty$. Suppose that A generates a bounded group on a UMD space X such that $0 \in \rho(A)$ and*

$$0 < \alpha < 1 + |2n - \beta|, \quad 2n - 1 < \beta < 2n + 1, \quad n \in \mathbf{N}.$$

Then for every $f \in L^p(0, 2\pi; X)$ there exists a unique $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ such that

$$(4.3) \qquad P_{per}^\alpha \begin{cases} D^\alpha u(t) + (-1)^{n+1}(-A)^\beta u(t) = f(t), \quad t \in (0, 2\pi). \\ \\ u(0) - u(2\pi) = 0, \quad max\{0, \alpha - 1\}(u'(0) - u'(2\pi)) = 0. \end{cases}$$

holds for almost all $t \in [0, 2\pi]$.

Proof. Since A generates a bounded group, it follows from [9, Corollary 2.12] that $(-1)^{n+1}(-A)^\beta$ generates an analytic semigroup of angle $(1 - |2n - \beta|)\frac{\pi}{2}$. The result follows by Remark 4.3 and Theorem 4.6 with $\gamma = 1$. □

Let us recall that a linear operator A defined on X is called *non-negative* if $(-\infty, 0) \subset \rho(A)$ and there exists $M > 0$ such that

$$(4.4) \qquad \|\lambda R(\lambda, A)\| \leq M, \qquad \lambda < 0.$$

A closed linear operator A is said to be *positive* if it is non-negative and if, in addition, $0 \in \rho(A)$. For further information on positive operators we refer to the monograph by Martinez and Sanz [25].

Since each self-adjoint, positive operator admits a bounded $\mathcal{RH}^\infty$ calculus of angle 0, we get the following result.

Corollary 4.9. *Let A be a selfadjoint, positive operator defined on a Hilbert space H and $0 < \alpha \leq 2$. Then for every $f \in L^p(0, 2\pi; X)$ there exists a unique $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ such that P_{per} holds for almost all $t \in [0, 2\pi]$.*

Zaslavsky [38] introduced in 1994 the fractional kinetic equation

$$D^\beta g(t,x) = Lg(t,x) + p_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \quad 0 < \beta < 1, \tag{4.5}$$

for Hamiltonian chaos, where $0 < \beta < 1$, $p_0(x)$ is an arbitrary initial condition. The existence of a solution for (4.5) was proved in [7, Theorem 3.1] when L is the generator of a uniformly bounded continuous semigroup on a Banach space X. As a concrete application of Theorem 4.6 we prove the following result.

Corollary 4.10. *Let L be the generator of a bounded C_0 semigroup on $L^q(\mathbf{R}), 1 < q < \infty$, such that $L \in \mathcal{RH}^\infty(X)$ with angle $\theta_L^{R_\infty} \in (0, \pi(1-\frac{\alpha}{2}))$ and $0 \in \rho(L)$. If $p_0 \in L^q(\mathbf{R})$ then for all $\alpha < \frac{1}{p}$ there exists a unique $g \in H^{\alpha,p}(0, 2\pi; L^q(\mathbf{R})) \cap L^p(0, 2\pi; D(L))$ which satisfies the periodic fractional equation*

$$\begin{cases} D^\alpha g(t,x) = Lg(t,x) + p_0(x)\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} & 0 < \alpha < 1,\ 0 \leq t \leq 2\pi; \\ \\ g(0,x) = g(2\pi, x), \end{cases} \tag{4.6}$$

for almost all $t \in (0, 2\pi)$.

Proof. A direct computation shows that for $f(t) := p_0(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ we have

$$\int_0^{2\pi} ||f(t)||_{L^q}^p dt = ||p_0||_{L^q}^p (\Gamma(1-\alpha))^{-p}[\frac{t^{1-\alpha p}}{1-\alpha p}]_0^{2\pi} = ||p_0||_{L^q}^p (\Gamma(1-\alpha))^{-p}\frac{(2\pi)^{1-\alpha p}}{1-\alpha p}.$$

Consequently, $f \in L^p(0, 2\pi; L^q(\mathbf{R}))$ if and only if $\alpha < \frac{1}{p}$. Then the result is a consequence of Theorem 4.6.

□

5. Well posedness of an abstract BBO equation

Let $0 < \alpha \leq 4$, $a > 0$. In this section we consider the equation

$$D^\alpha u(t) + aA^{1/2}D^{\alpha/2}u(t) + Au(t) = f(t),\ 0 \leq t \leq 2\pi, \tag{5.1}$$

with appropriate periodic boundary conditions, and where A is a sectorial operator on X. The case $\alpha = 2$ corresponds to a abstract wave equation studied in [12] where the initial value problem is studied.

Take $X = \mathbf{C}$ and let $A = m$ be a positive real number. When $\alpha = 1$ and $a = \frac{1}{\sqrt{m}}$, equation (5.1) is an abstract version of the Basset-Boussinesq-Oseen (BBO) equation:

$$u'(t) + D^{1/2}u(t) + mu(t) = f(t),\ 0 \leq t \leq 2\pi. \tag{5.2}$$

The BBO equation expresses the Lagrangian acceleration of a spherical particle in an unsteady flow as the sum of the viscous, gravitational/buoyancy, virtual mass, and Basset forces acting respectively on the particle (see [26] and [24]). Before we proceed to the analysis of this equation, it seems appropriate to make a remark on the role of the fractional derivative in this case. From a physical point of view, the order α in the fractional derivative D^α expresses memory or history effects. So, as $\alpha \to 1$ the fractional derivative $D^\alpha \to D^1$ and any memory effect is lost. On the other hand, as $\alpha \to 0$, we have $D^\alpha \to I$ and the memory effect is maximum. With this analogy, the BBO equation accounts for a perfectly symmetrical viscoelastic memory, half-way between an elastic $\alpha = 0$ and a viscous memory $\alpha = 1$ (cf. [21]).

For every $\lambda \in \mathbf{C},\ z \in \mathbf{C} \setminus (-\infty, 0]$ we define

$$F^1(\lambda, z) = \lambda^\alpha(\lambda^\alpha + a\lambda^{\alpha/2}z^{1/2} + z)^{-1} \text{ and } F^2(\lambda, z) = \lambda^{\alpha/2}z^{1/2}(\lambda^\alpha + a\lambda^{\alpha/2}z^{1/2} + z)^{-1}.$$

We need the following Lemma.

Lemma 5.1. *Let $a > 0$, $\theta > 0$ and $0 < \alpha \le 2$ be such that one of the following conditions is satisfied*

(i) $a \ge 2$ and either $\theta_A^{R_\infty} \in (0, \pi)$ in case $0 < \alpha \le 2$ or $\theta_A^{R_\infty} \in (0, 2\pi(1 - \frac{\alpha}{4}))$ in case $2 < \alpha \le 4$.

(ii) $a \in (0, 2)$ and either $\theta \in (0, \pi - 2\varphi)$ in case $0 < \alpha \le 2$ or $\theta_A^{R_\infty} \in (0, 2\pi(1 - \frac{\alpha}{4}) - 2\varphi)$ in case $2 < \alpha \le 4$, where in each case $\varphi = \arctan \frac{\sqrt{4-a^2}}{a}$.

Then there are constants $\beta > \theta$, and $M \ge 0$ such that $F^j(ik, \cdot) \in \mathcal{H}^\infty(\Sigma_\beta)$ for all $k \in \mathbf{Z}$ and

$$\sup_{k \in \mathbf{Z}} ||F^j(ik, \cdot)||_{\mathcal{H}^\infty(\Sigma_\beta)} \le M$$

Proof. We analyze each case separately.

Case 1. $a \ge 2$ and $\theta \in (0, \pi)$ in case $0 < \alpha \le 2$, (respectively $\theta_A^{R_\infty} \in (0, 2\pi(1 - \frac{\alpha}{4}))$ in case $2 < \alpha \le 4$).

Define $c_1 = \frac{a}{2} + \frac{\sqrt{a^2-4}}{2}$ and $c_2 = \frac{a}{2} - \frac{\sqrt{a^2-4}}{2}$. Note that c_1 and c_2 are strictly positive real numbers and

$$(ik)^\alpha + a(ik)^{\alpha/2}z^{1/2} + z = (ik)^\alpha c_1c_2(\frac{1}{c_1} + \frac{z^{1/2}}{(ik)^{\alpha/2}})(\frac{1}{c_2} + \frac{z^{1/2}}{(ik)^{\alpha/2}}),\ k \ne 0.$$

Since $\theta < \pi < 2\pi(1 - \frac{\alpha}{4})$ for $0 < \alpha \le 2$ (resp. $\theta < 2\pi(1 - \frac{\alpha}{4})$ for $2 < \alpha \le 4$) there exists $\beta > \theta$ such that

$$\frac{\beta}{2} < \pi(1 - \frac{\alpha}{4}). \tag{5.3}$$

Note that for every $z \in \Sigma_\beta$ the fraction $\frac{z^{1/2}}{(ik)^{\alpha/2}}$, $k \ne 0$, belong to the sector $\Sigma_{\frac{\beta}{2} + \frac{\delta\alpha}{2} + \frac{\alpha\pi}{4}}$, which does not contain the semi-axis $(-\infty, 0]$ due to (5.3). As a consequence, the distance from $-\frac{1}{c_1}$ and $-\frac{1}{c_2}$ to the sector $\Sigma_{\frac{\beta}{2} + \frac{\alpha\pi}{4}}$ is positive and we conclude that there exists a constant $M_1 \ge 0$ independent of $k \in \mathbf{Z}$ and $z \in \Sigma_\beta$ such that for $k \in \mathbf{Z} \setminus \{0\}$,

$$|F^1(ik, z)| = \frac{1}{c_1c_2|\frac{1}{c_1} + \frac{z^{1/2}}{(ik)^{\alpha/2}}||\frac{1}{c_2} + \frac{z^{1/2}}{(ik)^{\alpha/2}}|} \le M_1. \tag{5.4}$$

Analogously, for $k \in \mathbf{Z} \setminus \{0\}$, we write

$$(ik)^\alpha + a(ik)^{\alpha/2}z^{1/2} + z = z^{1/2}(ik)^{\alpha/2}c_1(\frac{1}{c_1} + \frac{z^{1/2}}{(ik)^{\alpha/2}})(c_2 + \frac{(ik)^{\alpha/2}}{z^{1/2}}).$$

and observe that the fraction $\frac{(ik)^{\alpha/2}}{z^{1/2}}$ also belong to the sector $\Sigma_{\frac{\beta}{2}+\frac{\alpha\pi}{4}}$. It follows that there exists a constant $M_2 \geq 0$ independent of (ik) and z such that for $k \in \mathbf{Z} \setminus \{0\}$,

$$|F^2(ik,z)| = \frac{1}{c_1|\frac{1}{c_1} + \frac{z^{1/2}}{(ik)^{\alpha/2}}||c_2 + \frac{(ik)^{\alpha/2}}{z^{\alpha/2}}|} \leq M_2. \tag{5.5}$$

Case 2. $0 < a < 2$ and $\theta \in (0, \pi - 2\varphi)$ in case $0 < \alpha \leq 2$, (respectively $\theta_A^{R_\infty} \in (0, 2\pi(1-\frac{\alpha}{4}) - 2\varphi)$ in case $2 < \alpha \leq 4$).

Define $c_1 = e^{i\varphi}$and $c_2 = e^{-i\varphi}$, where $\varphi = \arctan \frac{\sqrt{4-a^2}}{a}$. Since $\theta < \pi - 2\varphi < 2\pi(1-\frac{\alpha}{4}) - 2\varphi$ for $0 < \alpha \leq 2$, there exists $\beta > \theta$ such that

$$\frac{\beta}{2} + \varphi < \pi(1 - \frac{\alpha}{4}). \tag{5.6}$$

Let $k \in \mathbf{Z}$ and $z \in \Sigma_\beta$. The fraction $\frac{z^{1/2}}{(ik)^{\alpha/2}}$ belong to the sector $\Sigma_{\frac{\beta}{2}+\frac{\alpha\pi}{4}}$, which now, because of (5.6), does not contain the sector $\Sigma_{\pi-\varphi}$. Since $-c_2 = -e^{-i\varphi} = -\frac{1}{c_1}$ and $-\frac{1}{c_2} = -e^{i\varphi}$ are precisely in the sector $\Sigma_{\pi-\varphi}$, we conclude that the distance from $-c_2, -\frac{1}{c_1}$ and $-\frac{1}{c_2}$ to the sector $\Sigma_{\pi-\varphi}$ is positive and hence there exists a constant $M > 0$ independent of $k \in \mathbf{Z}$ and $z \in \Sigma_\beta$ such that (5.4) and (5.5) also hold in this case.

□

The following is the main result of this section.

Theorem 5.2. *Assume that X is a UMD-space, $1 < p < \infty$ and suppose that $A \in \mathcal{RH}^\infty(X)$ with angle $\theta_A^{R_\infty}$. Let $f \in L^p(0, 2\pi; X)$. Suppose one of the following conditions is satisfied*

(i) $a \geq 2$ and either $\theta_A^{R_\infty} \in (0, \pi)$ in case $0 < \alpha \leq 2$ or $\theta_A^{R_\infty} \in (0, 2\pi(1-\frac{\alpha}{4}))$ in case $2 < \alpha \leq 4$; or

(ii) $a \in (0, 2)$ and either $\theta_A^{R_\infty} \in (0, \pi - 2\varphi)$ in case $0 < \alpha \leq 2$ or $\theta_A^{R_\infty} \in (0, 2\pi(1-\frac{\alpha}{4}) - 2\varphi)$ in case $2 < \alpha \leq 4$, where $\varphi = \arctan \frac{\sqrt{4-a^2}}{a}$,

then there exists a unique function $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ with $D^{\alpha/2}u \in L^p(0, 2\pi; D(A^{1/2}))$ which solves (5.1) for almost all $t \in [0, 2\pi]$.

Proof. As a consequence of Lemma 5.1 and Proposition 4.5, the sets

$$\{F^j(ik, A)\}_{k \in \mathbf{Z}} \quad j = 1, 2$$

are R-bounded. In particular, the operators $H(ik) := ((ik)^\alpha + a(ik)^{\alpha/2}A^{1/2} + A)^{-1}$ exists for all $k \in \mathbf{Z}$. It then follows that $H(ik)$ exists in $\mathcal{B}(X)$ for all $k \in \mathbf{Z}$ and the operator sequences

$$\{(ik)^\alpha H(ik)\}_{k \in \mathbf{Z}} \text{ and } \{(ik)^{\alpha/2} A^{1/2} H(ik)\}_{k \in \mathbf{Z}}$$

are R-bounded. By Theorem 3.4 and Theorem 3.1 we conclude that problem (5.1) is L^p-well posed.

□

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