PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS IN BANACH SPACES

VALENTIN KEYANTUO AND CARLOS LIZAMA

Abstract. We use operator-valued Fourier multiplier theorems to study second order differential equations in Banach spaces. We establish maximal regularity results in L^p and C^s for strong solutions of a complete second order equation.

In the second part, we study mild solutions for the second order problem. Two types of mild solutions are considered. When the operator A involved is the generator of a strongly continuous cosine function, we give characterizations in terms of Fourier multipliers and spectral properties of the cosine function. The results obtained are applied to elliptic partial differential operators.

1. Introduction

The main purpose of this paper is to prove existence and uniqueness of the periodic second order Cauchy problem

$$\begin{cases} u''(t) - aAu(t) - \alpha Au'(t) &= f(t), \qquad 0 \leq t \leq 2\pi \\ u(0) &= u(2\pi), \\ u'(0) &= u'(2\pi), \end{cases} \tag{1.1}$$

on a Banach space X. Here, A is a closed linear operator on X, $a, \alpha \in \mathbb{R}$, and $f \in L^p_{2\pi}(\mathbb{R}; X)$ or $f \in C^s_{2\pi}(\mathbb{R}; X)(0 < s < 1)$. Throughout, $L^p_{2\pi}(\mathbb{R}; X)$ (resp. $C^s_{2\pi}(\mathbb{R}; X)$) stands for the space of $2\pi-$periodic functions that are $p-$summable (resp. Hölder continuous of exponent s) on $[0, 2\pi]$. Special attention is paid to the case $\alpha = 0$. In this case we study mild solutions as well.

Problem (1.1) corresponds to a special case of the inhomogeneous complete second order Cauchy problem

$$u''(t) - Au(t) - Bu'(t) = f(t), \tag{1.2}$$

which has been extensively studied by semigroup methods since the pioneering paper of J.-L. Lions [26]. Other early contributions are due to Sobolevskii [34] where A and B can in fact depend on t, and Fattorini [22, Chapter VIII]. In these works, the emphasis is placed on reducing (1.2) to a first order system in a product space and then using semigroup theory. The theory of (1.2) is considerably more complicated

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than of its incomplete counterpart (i.e. with $B = 0$). Recently, Xiao and Liang [40] presented a unified treatment of the complete second order Cauchy problem with differential operators as coefficient operators in $L^p(\mathbb{R}^N), (1 \leq p \leq \infty)$ and other function spaces. Still another approach to problem (1.2) can be found in the paper [10] by Chill and Srivastava.

Motivation of our work relies in the application of maximal regularity results to the study of nonlinear problems. Semilinear problems of the form (1.1) with periodic conditions appears for example [1, Theorem 4.1], whereas a related quasilinear problem was treated by Nakao and Okochi [28, Section 5]. On the other hand, when $A = B = \Delta$ is the n-dimensional Laplacian and f depends on $u, u_{x_i}, u_{x_i x_j}, u_t, u'_{x_i}, u'_{x_i x_j}$ Ebihara [19] (see also [20])proved for equation (1.1) uniqueness of solutions in a certain class of functions and the global existence of a (classical) solution.

In this work, we are able to give necessary and sufficient conditions in order to obtain existence and uniqueness of periodic solutions for (1.1) in the spaces $L^p_{2\pi}(\mathbb{R}; X), 1 < p < \infty$ and $C^s_{2\pi}(\mathbb{R}; X), 0 < s < 1$ (solutions in the Besov spaces $B^s_{p,q}(0, 2\pi; X)$ are also considered). Namely, we obtain that when X has the UMD property, then for every $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists a unique strong L^p-solution of (1.1) if and only if $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

The concept of R-boundedness was introduced by Bourgain [8]. Since large classes of classical operators are R-bounded (cf. Girardi-Weis [24] and the recent memoir of Denk, Hieber and Prüss [16]), the assumptions in this approach are not too restrictive for applications.

In contrast to the above result, we show that additional restrictions on X or even R-boundedness are not needed in case of periodic Hölder spaces, namely; for every $f \in C^s_{2\pi}(\mathbb{R}; X), s \in (0, 1)$ there exists a unique strong $C^s_{2\pi}$-solution of (1.1) such that $u' \in C^s_{2\pi}(\mathbb{R}; D(A))$ and $u'', Au, Au' \in C^s_{2\pi}(\mathbb{R}; X)$ if and only if $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\sup_k ||\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}|| < \infty$. The situation is similar for Besov spaces.

In the case of mild solutions, we consider two notions of such solutions when the operator A is densely defined. We define and characterize, mild (resp. mild of class C^1) periodic solutions. Moreover, if A generates a cosine function $C(t)$ with $S(t)$ the associated sine family, then we are able to obtain the following characterization: for every $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists a unique $(x, y) \in X \times X$ (resp. $(x, y) \in E \times X$) such that u given by $u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds$ is 2π-periodic, i.e. $u(0) = u(2\pi)$ and $u'(0) = u(2\pi)$ if and only if $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $(R(-k^2, A))_{k\in\mathbb{Z}}$ (resp. $(kR(-k^2, A))_{k\in\mathbb{Z}}$) is an (L^p, L^p)-multiplier. Equivalently, if and only if $S(2\pi) \in \mathcal{B}(X, E)$ is invertible (resp. $1 \in \rho(C(2\pi))$). Here, E is the space of vectors $x \in X$ for which $C(t)x$ is continuously differentiable. These results are new and complete those of [11] and [32].

To achieve our goals, we make extensive use of recent results from the papers Arendt-Bu [6], [5] and Keyantuo-Lizama [25], and the methods are based on operator-valued Fourier multipliers theorems. In the non-periodic case, operator-valued Fourier multiplier theorems have been established by Amann [3], Weis [38], [39], Girardi-Weis [23], and Arendt-Batty-Bu [5].

The paper is organized as follows: In Section 2, strong L^p solutions of (1.1) are studied. Section 3 deals with Hölder continuous periodic solutions. When $\alpha = 0$ and $a = 1$ we study mild solutions and mild solutions of class C^1 respectively in Section 4 and Section 5. Particular attention is paid to the situation where A is the infinitesimal generator a strongly continuous cosine function. Throughout, we discuss examples involving the Laplace operator (as well as more general elliptic operators) with Dirichlet boundary conditions on an open subset Ω of $\mathbb{R}^N$.

2. Periodic solutions on $L^p(0, 2\pi; X)$

For a function $f \in L^1(0, 2\pi; X)$, we denote by $\hat{f}(k), k \in \mathbb{Z}$ the k-th Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt,$$

where for $k \in \mathbb{Z}$, $e_k(t) = e^{ikt}, t \in \mathbb{R}$.

Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$.

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, throughout, we consider the space $L^p_{2\pi}(\mathbb{R}; X)$ (which is also denoted by $L^p(0, 2\pi; X)$, $1 \leq p \leq \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

We begin with some preliminaries about operator-valued Fourier multipliers. More information may be found in Arendt-Bu [6] for the periodic case and Amann [2], Weis [38] for the non-periodic case.

Definition 2.1. *For $1 \leq p \leq \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an (L^p, L^p)-multiplier, if for each $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists $u \in L^p_{2\pi}(\mathbb{R}; Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

It follows from the uniqueness theorem of Fourier series that u is uniquely determined by f. Moreover, if $(M_k) \subset \mathcal{B}(X, Y)$ is an $(L^p(X), L^p(Y))-$multiplier and $(N_k) \subset \mathcal{B}(Y, Z)$ an $(L^p(Y), L^p(Z))-$multiplier, then it follows from the definition that $(N_k M_k)$ is an $(L^p(X), L^p(Z))-$multiplier.

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0,1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

Definition 2.2. *A family* $\mathbf{T} \subset \mathcal{B}(X,Y)$ *is called R-bounded if there is a constant* $C \geq 0$ *such that*

$$||\sum_{j=1}^{n} r_j \otimes T_j x_j||_{L^p(0,1;Y)} \leq C_p || \sum_{j=1}^{n} r_j \otimes x_j||_{L^p(0,1;X)} \tag{2.1}$$

for all $T_1, ..., T_n \in \mathbf{T},\ x_1, ..., x_n \in X$ *and* $n \in \mathbb{N}$, *for some* $p \in [1, \infty)$.

If (2.1) holds for some $p \in [1, \infty)$, then it holds for all $p \in [1, \infty)$. The best constant C in (2.1) is denoted by $R_p(T)$.

It follows readily from the definition that any R-bounded family is bounded. The converse of this assertion holds only in spaces which are isomorphic to Hilbert spaces (see [6], Proposition 1.13 and the comments preceding it).

Remark 2.3.

a) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{B}(X)$ be R-bounded sets, then $\mathbf{S} \cdot \mathbf{T} := \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R- bounded and

$$R_p(\mathbf{S} \cdot \mathbf{T}) \leq R_p(\mathbf{S}) \cdot R_p(\mathbf{T}).$$

b) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded whenever $\Omega \subset \mathbb{C}$ is bounded (I denotes the identity operator on X). This follows from Kahane's inequality (see [6, Lemma 1.7]). We shall use this remark frequently.

The following theorem, due to Arendt-Bu [6, Theorem 1.3], is the discrete analogue of the operator-valued version of Mikhlin's theorem. It is concerned with UMD spaces.

We recall that those Banach spaces X for which the Hilbert transform defined by

$$(Hf)(t) = \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \frac{1}{\pi} \int_{\epsilon \leq |s| \leq R} \frac{f(t-s)}{s} ds$$

is bounded on $L^p(\mathbb{R}, X)$ for some (and hence all) $p \in (1, \infty)$ are called UMD-spaces. The limit in the above formula is to be understood in the L^p sense. An alternative definition using Fourier series may be found in [9]. This paper contains other characterizations of the UMD property, notably the one involving martingale differences in Banach spaces.

Examples of UMD spaces include Hilbert spaces, Lebesgue spaces, as well as vector valued Lebesgue spaces $L^p(\Omega,\ \mu),\ 1 < p < \infty$, $L^p(\Omega,\ \mu;\ X),\ 1 < p < \infty$ when X is a UMD space and the Schatten-von Neumann classes $C_p(H),\ 1 < p < \infty$, of operators on a Hilbert space. Every UMD space is superreflexive, i.e. has an equivalent norm under which it is uniformly convex. The spaces $L^1(\Omega,\ \mu), L^\infty(\Omega,\ \mu)$ (if infinite dimensional) and $C^s([0,\ 2\pi];\ X)$ are not reflexive and therefore are not

UMD. More information on UMD spaces can be found in Amman [2], Bourgain [8], De Pagter-Witvliet [17] and Denk, Hieber and Prüss [16].

Theorem 2.4. *(Operator-valued Marcinkiewicz multiplier theorem)*
Let X, Y be UMD-spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier for $1 < p < \infty$.

Remark 2.5.

(1) If $X = Y$ is a UMD space and $M_k = m_k I$ with $m_k \in \mathbb{C}$, then the condition
$$\sup_k |m_k| + \sup_k |k(m_{k+1} - m_k)| < \infty$$
implies that the set $\{M_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. (see [6] or [2, Theorem 4.4.3]). This is the vector-valued extension of the original theorem of Marcinkiewicz. The result does not hold in general if the UMD condition is dropped.

(2) We note that in the case where $X = Y = L^p(0, 1)$, $1 < p < \infty$, which is a typical example of a UMD space, another sufficient condition for a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ to be a Fourier multiplier may be found in [18].

The following concept of k-regularity ($k = 1, 2$) introduced in [25] is the discrete analog for the notion of k-regularity related to Volterra integral equations, see [30, Chapter I, Section 3.2].

Definition 2.6. *A sequence $\{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ is called*
a) 1-regular if the sequence $\{\frac{k(a_{k+1} - a_k)}{a_k}\}_{k\in\mathbb{Z}}$ is bounded.
b) 2-regular if it is 1-regular and the sequence
$$\{\frac{k^2(a_{k+1} - 2a_k + a_{k-1})}{a_k}\}_{k\in\mathbb{Z}}$$
is bounded.

The following result on Fourier multipliers and resolvent operators is proved in [25, Proposition 2.8].

Proposition 2.7. *Let A be a closed linear operator defined on a UMD space X. Let $\{b_k\}_{k\in\mathbb{Z}} \in \mathbb{C}\backslash\{0\}$ be a 1-regular sequence such that $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent*

(i) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, $1 < p < \infty$.

(ii) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

In what follows, and for $n \in \mathbb{N}$, we denote by $H^{n,p}(0, 2\pi; X)$ the space of all $u \in L^p(0, 2\pi; X)$ for which there exists $v \in L^p(0, 2\pi; X)$ such that $\hat{v}(k) = (ik)^n \hat{u}(k)$

for all $k \in \mathbb{Z}$ (cf. [6, Section 2]). Note that for $u \in H^{2,p}(0, 2\pi; X)$, it follows that $u(0) = u(2\pi), u'(0) = u'(2\pi)$.

Remark 2.8.

We recall from [6, Lemma 2.2] that for $1 \leq p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$, we have that $(kM_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)- multiplier if and only if $(M_k)_{k\in\mathbb{Z}}$is an $(L^p, H^{1,p})$-multiplier. Hence, inductively, we get that, for all $n \in \mathbb{N}$, $(k^n M_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier if and only if $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{n,p})$-multiplier.
A similar remark holds for the Hölder spaces C^s (upon inspection of the proof of [6, Lemma 2.2]) and will be used in the next section.

We always assume that $a \neq 0$ but otherwise, do not put any condition on its sign.

Definition 2.9. *Let $1 < p < \infty$. A function u is called a strong L^p-solution of (1.1) if $u \in H^{2,p}(0, 2\pi; X) \cap H^{1,p}(0, 2\pi; D(A))$ and equation (1.1) holds for almost all $t \in [0, 2\pi]$.*

We have the following auxiliary result.

Lemma 2.10. *Consider the sequence $b_k = \frac{-k^2}{a+i\alpha k}$, $k \in \mathbb{Z}$. Then $(b_k)_{k\neq 0}$ is 2-regular.*

The proof follows by direct computation and we omit it.

The following is the main result of this section.

Theorem 2.11. *Let X be a UMD space and $A : D(A) \subset X \to X$ be a closed linear operator. The following assertions are equivalent for $1 < p < \infty$:*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}; X)$ there exists a unique strong L^p-solution of (1.1).

(ii) $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1})\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

(iii) $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. $(i) \implies (ii)$. We shall follow the same lines as the proof of [6, Theorem 2.3]. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f = e_k \otimes y$. There exists $u \in H^{2,p}(0, 2\pi; X)$ such that $u''(t) - aAu(t) - \alpha Au'(t) = f(t)$. Taking Fourier series on both sides we obtain that $\hat{u}(k) \in D(A)$ and $-k^2\hat{u}(k) - aA\hat{u}(k) - \alpha ikA\hat{u}(k) = \hat{f}(k) = y$. Thus, $(-k^2 I - (a + i\alpha k)A)$ is surjective. Let $x \in D(A)$. If $(-k^2 I - (a + i\alpha k)A)x = 0$, that is $Ax = \dfrac{-k^2}{a + i\alpha k}x$ then $u(t) = e^{ikt}x$ defines a periodic solution of $u''(t) - aAu(t) - \alpha Au'(t) = 0$. Hence $u = 0$ by the assumption of uniqueness and thus $x = 0$. Since A is closed, we conclude that $\{b_k := \frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subset \rho(A)$.

Next we show that $\{b_k(b_kI-A)^{-1}\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. Let $f \in L^p(0, 2\pi; X)$. By hypothesis, there exists a unique $u \in H^{2,p}(0, 2\pi; X)$ such that $u''(t) - aAu(t) - \alpha Au'(t) = f(t)$. Taking Fourier series, we deduce that $\hat{u}(k) \in D(A)$ and $(-k^2I - (a + i\alpha k)A)\hat{u}(k) = \hat{f}(k)$ or $(a + i\alpha k)(b_kI - A)\hat{u}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}$. Hence,

$$-k^2\hat{u}(k) = b_k(b_kI - A)^{-1}\hat{f}(k) \text{ for all } k \in \mathbb{Z}. \tag{2.2}$$

Note that by definition of $H^{2,p}(0, 2\pi; X)$, there exists $v \in L^p(0, 2\pi; X)$ such that $\hat{v}(k) = -k^2\hat{u}(k)$ for all $k \in \mathbb{Z}$. Hence, $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

$(ii) \implies (i)$. Let $f \in L^p(0, 2\pi; X)$. By hypothesis there exists $v \in L^p(0, 2\pi; X)$ such that

$$\hat{v}(k) = b_k(b_kI - A)^{-1}\hat{f}(k) = -k^2M_k\hat{f}(k), \tag{2.3}$$

where $M_k = \dfrac{1}{a + i\alpha k}(\dfrac{-k^2}{a + i\alpha k} - A)^{-1}$.

By Remark 2.8 , we obtain from (2.3) that $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{2,p})$-multiplier and hence we conclude that there exists $u \in H^{2,p}(0, 2\pi; X)$ such that

$$\hat{u}(k) = \frac{1}{a + i\alpha k}(b_kI - A)^{-1}\hat{f}(k). \tag{2.4}$$

In particular, $\hat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$.

Again by Remark 2.8, $(kM_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{1,p})$-multiplier and hence there exists $w \in H^{1,p}(0, 2\pi; X)$ such that $\hat{w}(k) = ik\hat{u}(k) =: \widehat{u'}(k)$.

Now, the identity $A(b_k - A)^{-1} = b_k(b_k - A)^{-1} - I$ shows that

$$\begin{aligned} A\hat{u}(k) &= \frac{1}{a + i\alpha k}A(b_kI - A)^{-1}\hat{f}(k) \\ &= \frac{1}{a + i\alpha k}b_k(b_kI - A)^{-1}\hat{f}(k) - \frac{1}{a + i\alpha k}\hat{f}(k) \\ &= \frac{1}{a + i\alpha k}\hat{v}(k) - \frac{1}{a + i\alpha k}\hat{f}(k), \end{aligned}$$

Since by Remark 2.5 $\frac{1}{a+i\alpha k}I$ is a Fourier multiplier, by Lemma 3.1 in [AB] we conclude that $u(t) \in D(A)$. Also we see that $Au \in L^p(0, 2\pi, X)$. Similarly, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} Ak\hat{u}(k) &= \frac{k}{a + i\alpha k}b_k(b_k - A)^{-1}\hat{f}(k) - \frac{k}{a + i\alpha k}\hat{f}(k) \\ &= \frac{k}{a + i\alpha k}\hat{v}(k) - \frac{k}{a + i\alpha k}\hat{f}(k) \end{aligned}$$

and arguing as above, we obtain $u'(t) \in D(A)$ and $Au' \in L^p_{2\pi}(\mathbb{R}; X)$.Also we note from (2.3) and (2.4) that $u'' := v \in L^p(0, 2\pi, X)$.

Finally, from (2.4) we have $(-k^2I - (a + i\alpha k)A)\hat{u}(k) = \hat{f}(k)$. It follows from the uniqueness theorem of Fourier coefficients that (1.1) holds for almost all $t \in [0, 2\pi]$. We have proved that u is a strong L^p-solution of (1.1). It remains to show uniqueness.

Let u be such that

$$u''(t) - aAu(t) - Au'(t) = 0,$$

then $\hat{u}(k) \in D(A)$ and $(-k^2 I - (a + i\alpha k)A)\hat{u}(k) = 0$. Since $\frac{-k^2}{a+i\alpha k} \in \rho(A)$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.

$(ii) \Leftrightarrow (iii)$. Follows from Proposition 2.7 and the observation that $\{\frac{-k^2}{a+i\alpha k}\}$ is 1-regular, according Definition 2.6 and Lemma 2.10.

□

Corollary 2.12. *Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a closed linear operator. Then, for $1 < p < \infty$, the following assertions are equivalent:*
(i) For every $f \in L^p_{2\pi}(\mathbb{R}; H)$ there exists a unique strong L^p-solution of (1.1).
(ii) $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_k ||\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}|| < \infty$.

Proof. This follows from Theorem 2.11, Proposition 2.7 and the fact that in the context of Hilbert spaces, R-boundedness and boundedness are identical concepts. This in turn follows from Plancherel's theorem and the fact that the Rademacher system $\{r_j(t)\}$ is an orthonormal family in $L^2(0, 1; \mathbb{C})$ (see Clément-de Pagter-Sukochev-Witvliet [13]).

□

The solution $u(\cdot)$ given by Theorem 2.11 actually satisfies the following maximal regularity property.

Corollary 2.13. *In the context of Theorem 2.11, if condition (ii) is satisfied we have: u'', Au, $Au' \in L^p_{2\pi}(\mathbb{R}; X), u' \in L^p_{2\pi}(\mathbb{R}; D(A))$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that the following inequality holds:*

$$||u''||_{L^p_{2\pi}(\mathbb{R};X)} + ||u'||_{L^p_{2\pi}(\mathbb{R};D(A))} + ||Au||_{L^p_{2\pi}(\mathbb{R};X)} + ||Au'||_{L^p_{2\pi}(\mathbb{R};X)} \le C||f||_{L^p_{2\pi}(\mathbb{R};X)} \tag{2.5}$$

Proof.
The first assertion follows from the proof of Theorem 2.11. The estimate (2.5) is a consequence of the Closed Graph Theorem.

□

Remark 2.14.

Fejer's Theorem (see [6, Proposition 1.1] or [4, Theorem 4.2.19]) can be used to construct the solution $u(\cdot)$ given by Theorem 2.11. More precisely, for $b_k :=$

$\frac{-k^2}{a+i\alpha k}$, $k \in \mathbb{Z}$ we have:

$$u(\cdot) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes b_k R(b_k, A)\hat{f}(k)$$

with convergence in $L^p(0, 2\pi; X)$. This remark will be used in Section 4 and Section 5 in the construction of mild solutions.

Example 2.15.

Let $A = d^2/dx^2$ defined in $L^2(0,\pi)$ with $D(A) = \{v \in H^{2,2}(0,\pi) : v(0) = v(\pi) = 0\}$. This is the Laplacian on the interval with Dirichlet boundary conditions. Then A is self-adjoint. This is a linearized version of the case considered by Nakao and Okochi [28]. As we will see below, the condition $a > 0$ used by these authors is not necessary when we treat with the linear case.

More generally, Let $\mathcal{A}$ be the Laplace operator given by $\mathcal{A}u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ on $L^2(\Omega)$ where Ω is a bounded open subset of $\mathbb{R}^N$. The precise definition of the operator A we have in mind is the following.

We define a form $a(.,.)$ with domain $D(a) = H_0^1(\Omega)$ on $L^2(\Omega)$ by:

$$a(u,v) = -\sum_{i,j=1}^{N} \int_\Omega D_i u(x)\overline{(D_j v)(x)}\, dx.$$

Let A be the closed operator on $L^2(\Omega)$ associated with the quadratic form $a(.,.)$. Then A is a realization of $\mathcal{A}$ with Dirichlet boundary conditions; i.e. $u = 0$ on the boundary $\partial\Omega$ of Ω in an appropriate sense. If Ω is smooth, e.g. $\partial\Omega$ is of class C^1 then $u = 0$ almost everywhere (for the surface measure) on $\partial\Omega$.

The above results apply to the operator A so defined.

The following corollary shows that the operator A considered here need not be a semigroup generator, much less a cosine function generator.

Corollary 2.16. *Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a self-adjoint operator such that $0 \in \rho(A)$. Assume that $\alpha \neq 0$. Then, for $1 < p < \infty$, we have:*

For every $f \in L^p_{2\pi}(\mathbb{R}; H)$ there exists a unique strong L^p-solution of (1.1).

Proof. Since $b_k = \frac{-k^2}{a+i\alpha k}$, $k \in \mathbb{Z}$, we have $\Re(b_k) = \frac{-k^2 a}{a^2+\alpha^2 k^2}$ and $\Im(b_k) = \frac{k^3\alpha}{a^2+\alpha^2 k^2}$. For a self-adjoint operator A it is well known that for $z \in \mathbb{C} \setminus \mathbb{R}$, $z \in \rho(A)$ and $\|(z-A)^{-1}\| \leq \frac{1}{|\Im(z)|}$. It follows that for $k \neq 0$,

$$\begin{aligned}\|b_k(b_k-A)^{-1}\|^2 &\le \frac{|b_k|^2}{|\Im(b_k)|^2} \le 1+\left(\frac{\Re(b_k)}{\Im(b_k)}\right)^2\\ &\le 1+\left(\frac{k^2a}{k^3\alpha}\right)^2 \le 1+\frac{a^2}{\alpha^2}.\end{aligned}$$

Therefore $\sup\{\|b_k(b_k-A)^{-1}\|,\ k\in\mathbb{Z}\} \le \|A^{-1}\| + \sqrt{1+\frac{a^2}{\alpha^2}}$ and Corollary 2.12 applies.

□

3. Periodic solutions on Hölder spaces

In the present section, we deal with with periodic solutions in Hölder and Besov spaces. The results apply to a large class of elliptic operators.

Let X be a Banach space. For $0<s<1$ we denote by $C^s(\mathbb{R};X)$ the space of all continuous functions $f:\mathbb{R}\to X$ such that

$$||f(t)-f(t')|| \le c|t-t'|^s \quad (t,\,t'\in\mathbb{R})$$

for some $c\ge 0$.

By $C_{2\pi}(\mathbb{R};X)$ we denote the space of all 2π-periodic continuous functions $f:\mathbb{R}\to X$. We let $C^s_{2\pi}(\mathbb{R};X)=C^s(\mathbb{R};X)\cap C_{2\pi}(\mathbb{R};X)$.

In the paper [5], Arendt and Bu showed that the analogue of Marcinkiewicz's operator-valued Fourier multiplier theorem on L^p holds for the Hölder space $C^s_{2\pi}(\mathbb{R};X)$ without restrictions on the space X. Moreover, the concept of $R-$boundedness is not used. In this section, we use the results of [5] to characterize maximal regularity of equation (1.1).

Definition 3.1. *Let X and Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$. We will say that $\{M_k\}_{k\in\mathbb{Z}}$ is a $C^s_{2\pi}$-multiplier, if for each $f\in C^s_{2\pi}(\mathbb{R};X)$ there exists $u\in C^s_{2\pi}(\mathbb{R};Y)$ such that*

$$\hat{u}(k)=M_k\hat{f}(k) \text{ for all } k\in\mathbb{Z}.$$

The following condition on sequences $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ was introduced in [5] to study Fourier multipliers for Hölder continuous functions. It is also used in the study of multipliers of Besov spaces of which the spaces $C^s(\mathbb{R};X)$ of $X-$valued Hölder continuous functions of exponent s are a special instance.

Definition 3.2. *We say that a sequence $\{M_k\}_{k\in\mathbb{Z}}\subset\mathcal{B}(X,Y)$ is M- bounded if*

$$\sup_{k\in\mathbb{Z}}||M_k||<\infty,\quad \sup_{k\in\mathbb{Z}}||k(M_{k+1}-M_k)||<\infty, \tag{3.1}$$

$$\sup_{k\in\mathbb{Z}}||k^2(M_{k+1}-2M_k+M_{k-1})||<\infty. \tag{3.2}$$

The following general multiplier theorem is due to Arendt-Bu [5, Theorem 3.4].

Theorem 3.3. *Let X and Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be an M-bounded sequence. Then $\{M_k\}_{k\in\mathbb{Z}}$ is a $C^s_{2\pi}$-multiplier.*

The following proposition is the analogue of Proposition 2.7 and corresponds to a particular case of [25, Proposition 3.4].

Proposition 3.4. *Let A be a closed linear operator defined on the Banach space X. Let $\{b_k\}_{k\in\mathbb{Z}} \in \mathbb{C}\backslash\{0\}$ be a 2-regular sequence such that $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent*

(i) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is a $C^s_{2\pi}$-multiplier.

(ii) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.

For $f \in C^s_{2\pi}(\mathbb{R};X)$ let

$$||f||_s = \sup\{\frac{||f(t)-f(t')||}{|t-t'|^s} : t,t' \in [0,2\pi], t \neq t'\} + \|f\|_\infty.$$

Then $C^s_{2\pi}(\mathbb{R};X)$ under the above norm is a Banach space. We define

$$C^{1+s}_{2\pi}(\mathbb{R};X) = \{f \in C^1(0,2\pi;X) : f \text{ is } 2\pi \text{ -periodic and } f' \in C^s_{2\pi}(\mathbb{R};X)\},$$

and

$$C^{2+s}_{2\pi}(\mathbb{R};X) = \{f \in C^2(0,2\pi;X) : f \text{ is } 2\pi \text{ -periodic and } f'' \in C^s_{2\pi}(\mathbb{R};X)\}.$$

Definition 3.5. *Let $0 < s < 1$. A function $u \in C^{2+s}_{2\pi}(\mathbb{R};X)$ is called a strong $C^s_{2\pi}$-solution of (1.1) if $u(t) \in D(A)$, $u'(t) \in D(A)$ and (1.1) holds for all $t \in [0,2\pi]$.*

The advantage in the following result, compared with Theorem 2.11, is that the multiplier theorems used do not require any restriction on the Banach space X, in contrast to the L^p results which depended upon the UMD property.

Theorem 3.6. *Let A be a closed linear operator defined on a Banach space X. The following assertions are equivalent:*

(i) $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_k ||\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}|| < \infty$.

(ii) For every $f \in C^s_{2\pi}(\mathbb{R};X)$ there exists a unique strong $C^s_{2\pi}$-solution of (1.1) such that $u'', Au, Au' \in C^s_{2\pi}(\mathbb{R};X)$, and $u' \in C^s_{2\pi}(\mathbb{R};D(A))$.

The proof follows the same lines as the proof of Theorem 2.11 in Section 2. We omit the details.

Example 3.7.

Let A the operator of multiplication by a real-valued function $m(x)$ on $L^p(\Omega,\,\mu),\ 1\leq p\leq\infty$. We assume that $0\in\rho(A)$ and $\alpha\neq 0$. This implies that $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}}\subseteq\rho(A)$. We show that the estimate $\sup_k||\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I-A)^{-1}||<\infty$ is satisfied. In fact, if $Af(x)=m(x)f(x),\ f\in L^p(\Omega,\,\mu)$ where $m(x)\in\mathbb{R}$, a.e. $x\in\Omega$, then we have

$$\begin{aligned}||(\frac{-k^2}{a+i\alpha k}I-A)^{-1}|| &= \text{ess}-\sup_{x\in\Omega}|(\frac{-k^2}{a+i\alpha k}-m(x))^{-1}|\\ &= \text{ess}-\sup_{x\in\Omega}|(\frac{-k^2(a-i\alpha k)}{a^2+\alpha^2k^2}-m(x))^{-1}|.\end{aligned}$$

But $|(\dfrac{-k^2(a-i\alpha k)}{a^2+\alpha^2k^2}-m(x))^{-1}|\leq\dfrac{a^2+\alpha^2k^2}{|\alpha k^3|}$ and thus,

$$\begin{aligned}\sup_{k\neq 0}\|\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I-A)^{-1}\| &\leq \sup_{k\neq 0}(\frac{k^2}{|a+i\alpha k|}\frac{a^2+\alpha^2k^2}{|\alpha k^3|})\\ &\leq \sup_{k\neq 0}\frac{\sqrt{a^2+\alpha^2k^2}}{|\alpha||k|}<\infty.\end{aligned}$$

For a more concrete example, let A be the operator considered at the end of Section 2, that is, the Laplace operator on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions defined through quadratic forms. Then one can define a family of closed operators A_p on the spaces $L^p(\Omega),\ 1\leq p<\infty$ consistent with A ($A=A_2$.) These operators generate analytic semigroups of angle $\frac{\pi}{2}$ on $L^p(\Omega)$. The same is true for the spaces $C_0(\Omega)$ provided that Ω be regular in the sense of Wiener (see for instance [4, Chapter 6]). We refer to Pazy [29] and Davies [15] for a complete presentation which includes more general elliptic operators. The above results apply to all these cases.

We recall the definition of periodic Besov spaces. Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$, $\mathcal{S}'$ be the space of all tempered distributions on $\mathbb{R}$ and $\mathcal{D}'(\mathbb{T};X)$ the space of $X-$valued $2\pi-$periodic distributions. Let $\Phi(\mathbb{R})$ be the set of all systems $\phi=\{\phi_j\}_{j\geq 0}\subset\mathcal{S}$ satisfying

$$\text{supp}(\phi_0)\subset[-2,2]$$
$$\text{supp}(\phi_j)\subset[-2^{j+1},-2^{j-1}]\cup[2^{j-1},2^{j+1}],\quad j\geq 1$$
$$\sum_{j\geq 0}\phi_j(t)=1,\quad t\in\mathbb{R}$$

and for $\alpha\in\mathbb{N}\cup\{0\}$, there exists $C_\alpha>0$ such that

$$\sup_{j\geq 0,x\in\mathbb{R}}2^{\alpha j}||\phi_j^{(\alpha)}(x)||\leq C_\alpha. \tag{3.3}$$

That such systems exist is a well known fact which is related to the Littlewood-Paley decomposition (see e.g. [2], [3], [4, Chapter 8], [7])).

Let $1 \le p, q \le \infty, s \in \mathbb{R}$ and $\phi = (\phi_j)_{j\ge 0} \in \Phi(\mathbb{R})$. The $X-$valued periodic Besov spaces are defined by

$$B^{s,\phi}_{p,q} = \{f \in \mathcal{D}'(\mathbb{T}; X) : ||f||_{B^{s,\phi}_{p,q}} = (\sum_{j\ge 0} 2^{sjq}||\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)||^q_p)^{1/q} < \infty\}.$$

The space $B^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and the norms $||\cdot||_{B^{s,\phi}_{p,q}}$ are equivalent. As a consequence, we will denote $||\cdot||_{B^{s,\phi}_{p,q}}$ simply by $||\cdot||_{B^s_{p,q}}$. We refer to the paper [7, section 1] for more details.

In the context of Besov spaces, using the above techniques, one can establish the following theorem.

Theorem 3.8. *Let $1 \le p, q \le \infty$ and $s > 0$. Let A be a closed linear operator defined on a Banach space X. The following assertions are equivalent:*

(i) $\{\frac{-k^2}{a+i\alpha k}\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_k \|\frac{-k^2}{a+i\alpha k}(\frac{-k^2}{a+i\alpha k}I - A)^{-1}\| < \infty$.

(ii) For every $f \in B^s_{p,q}(\mathbb{T}; X)$ there exists a unique strong $B^s_{p,q}$-solution of (1.1) such that u'', Au, $Au' \in B^s_{p,q}(\mathbb{T}; X)$, and $u' \in B^s_{p,q}(\mathbb{T}; D(A))$.

Here, the notion of $B^s_{p,q}$-solution is defined in the same way as L^p-solution and C^s-solution in Definition 2.9 and Definition 3.5 respectively. In this case, one uses the analogue of Theorem 3.3 (see [7, Theorem 4.5]), along with [25, Proposition 3.4].

We remark that the above example applies to this case as well. For more on vector-valued Besov spaces, we refer to Arendt Batty and Bu [7]. A result similar to Theorem 3.8 was proved in [25]. Operator-valued Fourier multipliers on Besov spaces built on $L^p(\mathbb{R}; X)$ were studied by Amann [3].

We now consider a more general class of examples which includes elliptic equations.

Let us first recall that a linear operator A defined on X is called *non-negative* if $(-\infty, 0) \subset \rho(A)$ and there exists $M > 0$ such that

$$\|\lambda R(\lambda, A)\| \le M, \qquad \lambda < 0. \tag{3.4}$$

A closed linear operator A is said to be *positive* if it is non-negative and if, in addition, $0 \in \rho(A)$. For further information on positive operators we refer to the recent monograph by Martinez-Sanz [27].

It is well known that the above estimate implies that there exists $\theta \in (0, \pi)$ such that $\Sigma_\theta := \{z \in \mathbb{C},\ |arg(z)| > \pi - \theta\} \subset \rho(A)$ and

$$\|\lambda R(\lambda, A)\| \le M,\ \lambda \in \Sigma_\theta. \tag{3.5}$$

If the operator $-A$ generates a bounded strongly continuous semigroup, then A is a non-negative operator and one can take in this case any θ such that $\theta > \frac{\pi}{2}$. In case $-A$ generates a bounded analytic semigroup, the above estimate holds on a (left) sector of angle larger than $\frac{\pi}{2}$ (thus $\theta < \frac{\pi}{2}$ is allowed).

We have the following:

Corollary 3.9. *Suppose that $a > 0$ and $0 < s < 1$ and let Let A be a closed linear operator defined on a Banach space X.*

(1) *Assume that $0 \in \rho(A)$ and $-A$ is the generator of a bounded analytic semigroup (which need not be of class C_0). Then, for every $f \in C^s_{2\pi}(\mathbb{R}; X)$ there exists a unique strong $C^s_{2\pi}$-solution of (1.1) such that $u' \in C^s_{2\pi}(\mathbb{R}; D(A))$ and u'', Au, $Au' \in C^s_{2\pi}(\mathbb{R}; X)$.*

(2) *Assume that $\alpha = 0$. Then the conclusion of* (1) *holds for any positive operator A.*

Proof. To prove (1), it is enough to observe that $\frac{-k^2}{a+i\alpha k} = \frac{-k^2 a}{a^2+\alpha^2k^2} + i\frac{\alpha k^3}{a^2+\alpha^2k^2}$ and that the real part is negative if $k \neq 0$. As for (2), we note that in that case, $b_k := \frac{-k^2}{a+i\alpha k} = \frac{-k^2}{a} < 0$ for $k \neq 0$ and thanks to (3.4) and the fact that $0 \in \rho(A)$, the estimate in Theorem 3.8 is satisfied.

□

This corollary applies to a large class of elliptic operators with bounded measurable coefficients in $L^p(\Omega)$.

Theorems of this type were obtained using the method of sums of operators. See the monographs [2], [30] by Prüss and Amann respectively. Our treatment here is particularly simple and does not involve the condition of bounded imaginary powers.

For the case $a < 0$, we have the following:

Corollary 3.10. *Assume that $a < 0$, $\alpha \neq 0$, $0 \in \rho(A)$, $0 < s < 1$ and A generates a bounded analytic semigroup of angle $\frac{\pi}{2}$. Then the conclusion of Corollary 3.9 holds.*

Proof. To see this observe that the real part of $\frac{-k^2}{a+i\alpha k} = \frac{-k^2 a}{a^2+\alpha^2k^2} + i\frac{\alpha k^3}{a^2+\alpha^2k^2}$ is positive if $k \neq 0$. Therefore estimate in Theorem 3.8, (i) is satisfied.

□

Similar results hold true in the context of Besov spaces. These results apply to general elliptic operators on the L^p-spaces (see [15]). For elliptic operators on spaces of continuous functions, we refer to [4, Chapter 6], [29, Section 7.3].

4. Mild periodic solutions: Case $\alpha = 0$

In this section we study mild solutions of the abstract Cauchy problem (1.1) in case $\alpha = 0$ and $a = 1$ that is,

$$u''(t) - Au(t) = f(t), \qquad 0 \leq t \leq 2\pi. \tag{4.1}$$

Strong solutions of problem (4.1) where characterized by Arendt and Bu (see [6, Section 6]) but the study of mild periodic solutions was left open. However, we note that mild solutions in the non-periodic case were studied recently by Schweiker [33] for the inhomogeneous problem with $f \in \mathbf{BUC}(\mathbb{R}, X)$. In this and the next section we complete the study initiated in [6].

We recall from [6, Theorem 3.6] that, if A is the generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ and $1 \leq p < \infty$ then the existence of a unique mild solution for the Cauchy problem

$$\begin{cases} u'(t) &= Au(t) + f(t), t \in (0, 2\pi) \\ u(0) &= u(2\pi) \end{cases}$$

is equivalent to the following: $i\mathbb{Z} \subset \rho(A)$ and $(R(ik, A))_{k\in\mathbb{Z}}$ is an (L^p, L^p) -multiplier. On the other hand, by a result of Prüss [31], this is also equivalent to have: $1 \in \rho(T(2\pi))$.

Formally, if we consider (4.1) with periodic boundary conditions and use the Fourier series, then we get

$$\hat{u}(k) = R(-k^2, A)\hat{f}(k).$$

By a result of Cioranescu-Lizama [11], it is known that if A is the generator of a cosine function $\{C(t)\}_{t\in\mathbb{R}}$ then the existence of a unique mild periodic solution of class C^1 for (4.1) is characterized by the condition $1 \in \rho(C(2\pi))$ which , in Hilbert spaces, is equivalent to the boundedness of the set

$$\{kR(-k^2, A)\}_{k\in\mathbb{Z}}.$$

On the other hand, if $S(t)$ denotes the associated sine function then Schüler [32] proved that existence of mild periodic solutions for (4.1) (not of class C^1) is characterized by the condition that $S(2\pi) : X \to E$ is invertible. Here E denotes the set $E = \{x \in X : t \to C(t)x$ is once continuously differentiable $\}$, which is a Banach space under an appropriate norm (cf. Fattorini [21]). Moreover, Schüler proved that the above description is equivalent, also in Hilbert spaces, to the boundedness of the set

$$\{R(-k^2, A)\}_{k\in\mathbb{Z}}.$$

The above results, give us to consider two notions of mild solutions for the problem (4.1) in order to obtain the corresponding analogues to the Arendt-Bu result. In order to do that, we slightly modify the notion of mild solutions to the case that A is not necessarily the generator of a cosine function, obtaining characterizations which are analogues to [6, Proposition 3.2].

We begin with our first definition of the notion of "2-times mild" solution.

Definition 4.1. *For given $f \in L^1_{loc}(\mathbb{R}; X)$, a function $u \in C(0, 2\pi; X)$ differentiable at $t = 0$ is called a mild solution of the problem (4.1) if*

$$(4.2)\qquad \begin{cases} \displaystyle\int_0^t (t-s)u(s)ds \in D(A), \\ u(t) = A\displaystyle\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds + u(0) + tu'(0), \end{cases}$$

for all $0 \le t \le 2\pi$.

We note that our definition is a natural extension of classical ones. In [4, p. 120 and p. 206] for example, the definition of mild solution for (4.1) uses only (4.2) alone, in the case where $f \equiv 0$. Here we consider the nonhomogeneous problem.

Lemma 4.2. *Let* $g_f(t) = \int_0^t (t-s)f(s)ds$. *Then the Fourier coefficients of* g_f *are given by:*

$$\hat{g}_f(k) = \frac{-1}{2\pi ik}g_f(2\pi) + \frac{1}{k^2}\hat{f}(0) - \frac{1}{k^2}\hat{f}(k),$$

for all $k \neq 0$.

Proof. Note that $g_f(0) = g'_f(0) = 0$. We have:

$$\begin{aligned} \hat{g}_f(k) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}g_f(t)dt \\ &= \frac{-1}{2\pi ik}g_f(2\pi) + \frac{1}{2\pi ik}\int_0^{2\pi} e^{-ikt}g'_f(t)dt \\ &= \frac{-1}{2\pi ik}g_f(2\pi) + \frac{1}{2\pi ik}[\frac{-1}{ik}\int_0^{2\pi} f(s)ds + \frac{1}{ik}\int_0^{2\pi} e^{-ikt}f(t)dt] \\ &= \frac{-1}{2\pi ik}g_f(2\pi) + \frac{1}{k^2}\hat{f}(0) - \frac{1}{k^2}\hat{f}(k). \end{aligned}$$

The proof is complete.

□

Remark 4.3.

Recall from Section 3 that we denote by $C_{2\pi}(\mathbb{R};X)$ the space of all continuous and 2π-periodic functions with $u(0) = u(2\pi)$. By differentiability of a function $u \in C_{2\pi}(\mathbb{R};X)$ at $t = 0$ or $t = 2\pi$ we mean that the following limits exist and are equal

$$\lim_{t\to 0+}\frac{u(t)-u(0)}{t} = \lim_{t\to 2\pi-}\frac{u(t)-u(2\pi)}{t-2\pi}.$$

We denote this by $u'(0) = u'(2\pi)$.

Theorem 4.4. *Let* $f \in L^1_{2\pi}(\mathbb{R};X)$ *and let* $u \in C_{2\pi}(\mathbb{R};X)$ *be differentiable at* $t = 0$. *Assume that* $\overline{D(A)} = X$. *Then* u *is a mild solution of problem (4.1) satisfying* $u'(0) = u'(2\pi)$ *if and only if*

$$\hat{u}(k) \in D(A) \text{ and } (-k^2 I - A)\hat{u}(k) = \hat{f}(k), \tag{4.3}$$

for all $k \in \mathbb{Z}$.

Proof. Assume that u is a mild solution. Letting $t = 2\pi$ in (4.2) we get that $\int_0^{2\pi}(2\pi - s)u(s)ds = g_u(2\pi) \in D(A)$ and $Ag_u(2\pi) + g_f(2\pi) = -2\pi u'(0)$.

Consider the functions $v(t) = \int_0^t (t-s)u(s)ds = g_u(t)$ and $w(t) = u(t) - g_f(t) - u(0) - tu'(0)$. Then by [6, Lemma 3.1] we obtain $\hat{g}_u(k) \in D(A)$ and $A\hat{g}_u(k) = \hat{w}(k)$.

Since u is differentiable at $t = 2\pi$, $u'(2\pi)$ exists and hence by (4.2), closedness of A and using the fact that $u'(0) = u'(2\pi)$ we obtain $\hat{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} u(s)ds \in D(A)$ and $A\hat{u}(0) = -\hat{f}(0)$.

Now, by Lemma 4.2 we have for $k \neq 0$, $\hat{g}_u(k) = \frac{-1}{2\pi ik}g_u(2\pi) + \frac{1}{k^2}\hat{u}(0) - \frac{1}{k^2}\hat{u}(k)$. Since $\hat{u}(0) \in D(A)$ we obtain $\hat{u}(k) \in D(A)$ and hence $A\hat{g}_u(k) = \frac{-1}{2\pi ik}Ag_u(2\pi) + \frac{1}{k^2}A\hat{u}(0) - \frac{1}{k^2}A\hat{u}(k)$.

On the other hand, $\hat{w}(k) = \hat{u}(k) - \hat{g}_f(k) + \frac{1}{ik}u'(0) = \hat{u}(k) + \frac{1}{2\pi ik}g_f(2\pi) - \frac{1}{k^2}\hat{f}(0) + \frac{1}{k^2}\hat{f}(k) + \frac{2\pi}{2\pi ik}u'(0)$. Therefore $\frac{-1}{k^2}A\hat{u}(k) = \hat{u}(k) + \frac{1}{k^2}\hat{f}(k)$, for all $k \neq 0$. This proves (4.3) for all $k \in \mathbb{Z}\backslash\{0\}$. Since we have already proved $-A\hat{u}(0) = \hat{f}(0)$ we obtain (4.3) for all $k \in \mathbb{Z}$.

Conversely, assume that (4.3) holds. Let $x^* \in D(A^*)$, where A^* is the adjoint of A. By [4, Proposition B.10], it suffices to show that

$$\int_0^t (t-s) < u(s), A^*x^* > ds =< u(t), x^* > - < u(0), x^* >$$
$$- < tu'(0), x^* > - \int_0^t (t-s) < f(s), x^* > ds.$$

Define $w(s) =< u(s), A^*x^* > + < f(s), x^* >$. Then by (4.3) we obtain

$$\hat{w}(k) =< \hat{u}(k), A^*x^* > + < \hat{f}(k), x^* >= -k^2 < \hat{u}(k), x^* >, \tag{4.4}$$

for all $k \in \mathbb{Z}$.

Define $h(t) = g_w(t) - < u(t), x^* > + < tu'(0), x^* >$. Then, by (4.4) we obtain $\hat{w}(0) = 0$ and hence $\hat{h}(k) = \frac{-1}{2\pi ik}g_w(2\pi) + \frac{1}{k^2}\hat{w}(0) - (\frac{1}{k^2}\hat{w}(k) + < \hat{u}(k), x^* >) - \frac{1}{ik} < u'(0), x^* >= \frac{-1}{2\pi ik}g_w(2\pi) - \frac{1}{ik} < u'(0), x^* >$, for all $k \neq 0$.

We conclude from the uniqueness that $h(t) - \frac{t}{2\pi}g_w(2\pi) - t < u'(0), x^* >$ is constant; i.e $h(t) = \frac{t}{2\pi}g_w(2\pi) + t < u'(0), x^* > - < u(0), x^* >$ since $g_w(0) = 0$. Hence

$$g_w(t) - < u(t), x^* >= \frac{t}{2\pi}g_w(2\pi) - < u(0), x^* > \tag{4.5}$$

for all $t \in [0, 2\pi]$. Since $u(t)$ is differentiable at $t = 0$ from (4.5)we obtain $g_w'(0) - < u'(0), x^* >= \frac{1}{2\pi} g_w(2\pi)$. But $g_w'(0) = 0$, and hence $g_w(2\pi) = -2\pi < u'(0), x^* >$. Therefore (4.5) together with the definition of w yield equation (4.2).

Since u is differentiable at $t = 2\pi$, from (4.5) we also obtain $g_w'(2\pi) - < u'(2\pi), x^* >= \frac{1}{2\pi} g_w(2\pi)$ which implies $\hat{u}(0) \in D(A)$ and $A \int_0^{2\pi} u(s)ds + \int_0^{2\pi} f(s)ds - u'(2\pi) = -u'(0)$. But, by (4.3) with $k = 0$ we have $A\hat{u}(0) + \hat{f}(0) = 0$. Therefore $u'(0) = u'(2\pi)$, and the theorem is proved.

□

In case A generates a strongly continuous cosine family, mild solutions can be described differently.

Recall that if A generates a cosine function $C(t)$ and $S(t) := \int_0^t C(s)ds$ is the associated sine function, then for $x \in X$, $\int_0^t S(s)xds \in D(A)$ and

$$A \int_0^t S(s)xds = C(t)x - x, \;\; t \geq 0. \tag{4.6}$$

Furthermore, recall that $C(t)$ and therefore $S(t)$ are exponentially bounded: there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| + \|S(t)\| \leq Me^{\omega t}, \;\; t \geq 0.$$

We shall also make use of the set

$$E = \{x \in X : t \to C(t)x \text{ is once continuously differentiable } \},$$

which under the norm $||x||_E = ||x|| + \sup_{0 \leq t \leq 1} ||AS(t)x||$ is a Banach space (cf. [22] and [4, Section 3.14]).

Observe that if $(x, y) \in D(A) \times E$ and f is continuously differentiable on $[0, 2\pi]$, then the formula

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds, \tag{4.7}$$

defines a classical solution of (4.1) (see e.g. Travis and Webb [35, Proposition 2.4]). If $(x, y) \in X \times X$ and $f \in L^1_{loc}(\mathbb{R}; X)$ we may consider (4.7) as providing a "mild" solution to (4.1) with $u(0) = x$ and $u'(0) = y$; this mild solution is of class C^1 if and only if $(x, y) \in E \times X$.

In the proof of Theorem 4.6 below, we will need to establish the following lemma.

Lemma 4.5. *Assume that A generates a strongly continuous cosine function $C(t)$ and let $f \in L^p(0, 2\pi; X)$, $1 \leq p < \infty$ or $f \in C(0, 2\pi; X)$. Then $\int_0^t (t-s) \int_0^s S(t-\sigma)f(\sigma)d\sigma ds \in D(A)$ and*

$$A \int_0^t (t-s) \int_0^s S(s-\sigma)f(\sigma)d\sigma ds = \int_0^t S(t-s)f(s)ds - \int_0^t (t-s)f(s)ds \tag{4.8}$$

for all $t \geq 0$.

Proof. To see this, we approximate f with a sequence of smooth functions, say , continuously differentiable, f_n with values in X. Then by [35, Proposition 2.4], we have $\int_0^t S(t-s)f_n(s)ds \in D(A)$ and

$$A\int_0^t S(t-s)f_n(s)ds = \frac{d^2}{dt^2}(\int_0^t S(t-s)f_n(s)ds) - f_n(t),\ 0 \le t \le 2\pi.$$

Integrating this relation, we obtain, using the fact that A is closed,

$$(4.9)\quad A\int_0^t (t-s)\int_0^s S(s-\sigma)f_n(\sigma)d\sigma ds = \int_0^t S(t-s)f_n(s)ds - \int_0^t (t-s)f_n(s)ds$$

for $0 \le t \le 2\pi$. Let $n,\ m \in \mathbb{N}$ and $t \ge 0$. For the two integrals appearing in the right hand side of (4.9) we have, with $t \ge 0$ fixed:

$$\|\int_0^t (t-s)(f_n(s)-f_m(s))ds\| \le \int_0^t (t-s)\|f_n(s)-f_m(s)\|ds$$

and

$$\|\int_0^t S(t-s)(f_n(s)-f_m(s))ds\| \le M\int_0^t (t-s)e^{\omega(t-s)}\|f_n(s)-f_m(s)\|ds.$$

For the integral in the left hand side we have the following estimate:

$$\begin{aligned}&\|\int_0^t (t-s)\int_0^s S(t-\sigma)(f_n(\sigma)-f_m(\sigma))ds\|\\ \le M&\int_0^t (t-s)\int_0^s e^{\omega(s-\sigma)}\|f_n(\sigma)-f_m(\sigma)\|d\sigma ds.\end{aligned}$$

Letting n go to infinity in (4.9) and appealing to the Closed Graph Theorem, we obtain the desired relation.

□

Theorem 4.6. *Let A be the generator of a strongly continuous cosine function $C(t)$ and denote by $S(t)$ the associated sine function. For $1 \le p < \infty$ the following are equivalent:*

(i) For any $f \in L^p_{2\pi}(\mathbb{R};X)$ there exists a unique $(x,y) \in X \times X$ such that u given by (4.7) is differentiable at $t=0$ and 2π-periodic, i.e. $u(0)=u(2\pi)$ and $u'(0)=u'(2\pi)$.

(ii)$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $(R(-k^2,A))_{k\in\mathbb{Z}}$ is an (L^p,L^p)-multiplier.

(iii) $S(2\pi) \in \mathcal{B}(X,E)$ is invertible.

Proof. $(i) \to (ii)$. By hypothesis the function u given by (4.7) is in $C_{2\pi}(\mathbb{R};X)$ and differentiable at $t=0$. In particular $u(0)=u(2\pi)=x$, $u'(0)$ exists and $u'(0)=u'(2\pi)=y$. Moreover, $\overline{D(A)}=X$ since A generates a cosine function on X.

From the properties of cosine functions, it is clear that $\int_0^t (t-s)u(s)ds \in D(A)$ and by Lemma 4.5 we obtain

$$\begin{aligned} A\int_0^t (t-s)u(s)ds &= A\int_0^t (t-s)C(s)xds + A\int_0^t (t-s)S(s)yds \\ &+ A\int_0^t (t-s)\int_0^s S(s-\sigma)f(\sigma)d\sigma ds \\ &= C(t)x - x + S(t)y - ty \\ &+ \int_0^t S(t-s)f(s)ds - \int_0^t (t-s)f(s)ds \\ &= u(t) - u(0) - tu'(0) - \int_0^t (t-s)f(s)ds \end{aligned}$$

This proves (4.2), that is u is a mild solution of the problem (4.1).

Since $u'(0) = u'(2\pi)$ by hypothesis, it follows from Theorem 4.4 that $\hat{u}(k) \in D(A)$ and $(-k^2 - A)\hat{u}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proceeding as in the proof of Theorem 2.11 we obtain $\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$. Let $f \in L^p(0, 2\pi; X)$. It follows that $\hat{u}(k) = R(-k^2; A)\hat{f}(k)$ for all $k \in \mathbb{Z}$. Now the claim follows from [6, Proposition 1.1].

$(ii) \to (i)$. Let $f \in L^p_{2\pi}(\mathbb{R}; X)$ and define $f_n = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k \otimes \hat{f}(k)$, and $u_n = \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k \otimes R(-k^2, A)\hat{f}(k)$. The hypothesis implies that $u = \lim_{n\to\infty} u_n$ exists in $L^p_{2\pi}(\mathbb{R}; X)$. Clearly the functions u_n satisfy $u_n''(t) = Au_n(t) + f_n(t)$ as strong solutions. Hence, by uniqueness of Fourier coefficients, $u_n(t)$ satisfies

$$u_n(t) = C(t)u_n(0) + S(t)u_n'(0) + \int_0^t S(t-s)f_n(s)ds$$

and

$$u_n'(t) = C'(t)u_n(0) + C(t)u_n'(0) + \int_0^t C(t-s)f_n(s)ds. \tag{4.10}$$

Since $u_n(0) = u_n(2\pi)$ and $u_n'(0) = u_n'(2\pi)$ we obtain

$$(I - C(2\pi))u_n(0) = S(2\pi)u_n'(0) + \int_0^{2\pi} S(2\pi - s)f_n(s)ds$$

and

$$(I - C(2\pi))u_n'(0) = C'(2\pi)u_n(0) + \int_0^{2\pi} C(2\pi - s)f_n(s)ds \tag{4.11}$$

respectively. Obviously $\int_0^t S(t-s)f_n(s)ds \to \int_0^t S(t-s)f(s)ds$ as $n \to \infty$ and $\int_0^t C(t-s)f_n(s)ds \to \int_0^t C(t-s)f(s)ds$ as $n \to \infty$, uniformly for $t \in [0, 2\pi]$. Using the arguments in [32, Theorem 3] we can deduce that $u_n(0)$ and $u_n'(0)$ converge to some x, y in X, respectively, as $n \to \infty$. More precisely, since by the hypothesis $0 \in \rho(A)$, we can define

$$\begin{aligned} u_n(0) &= (I - C(2\pi))u_n(0) + C(2\pi)u_n(0) \\ &= \frac{1}{2}\int_0^{2\pi} (S(2\pi - s) + S(s))f_n(s)ds + 2\int_0^{2\pi} C(2\pi - s)u_n(s)ds \\ &- \frac{1}{2}A^{-1}\int_0^{2\pi} (C(s) + C(2\pi - s))f_n(s)ds \\ &- \frac{1}{2}\int_0^{2\pi} (S(s) - S(2\pi - s))f_n(s)ds - \int_0^{2\pi}\int_0^t S(2\pi - s)f_n(s)ds, \end{aligned}$$

and

$$\begin{aligned} u_n'(0) &= -\frac{1}{2}S(2\pi)\int_0^{2\pi} (C(s) + C(2\pi - s))f_n(s)ds \\ &+ \frac{1}{2}\int_0^{2\pi} (S(2\pi - s) + S(s))f_n(s)ds \\ &+ \int_0^{2\pi} C(s)\int_0^s C(s - \tau)(f_n(\tau) + f_n(-\tau))d\tau ds \\ &- \int_0^{2\pi}\int_0^s C(\tau)f_n(\tau)d\tau ds. \end{aligned}$$

We conclude that u verifies the formula

$$u(t) = C(t)x + S(t)y + \int_0^t S(t - s)f(s)ds,$$

and satisfy $u(0) = u(2\pi)$. Also from (4.11) we get that $C'(2\pi)u_n(0)$ converge. Then by (4.10) we obtain that $u'(2\pi)$ exists. Hence u is differentiable at $t = 0$ and $u'(0) = u'(2\pi)$.

$(iii) \to (i)$. Let $f \in L^p(0, 2\pi; X)$. The hypothesis implies $0 \in \rho(A)$ by [32, Lemma 3]. Choose

$$x = -\frac{1}{2}S(2\pi)^{-1}A^{-1}\int_0^{2\pi} [C(s) + C(2\pi - s)]f(s)ds$$

and

$$y = \frac{1}{2}S(2\pi)^{-1}\int_0^{2\pi} [S(s) - S(2\pi - s)]f(s)ds.$$

Note that x and y are well defined as elements of X.

Let $u(t) := C(t)x + S(t)y + \int_0^t S(t - s)f(s)ds$. Then $u(0) = u(2\pi)$. Next, define $v(2\pi) = AS(2\pi)x + C(2\pi)y + \int_0^{2\pi} S(2\pi - s)f(s)ds$. Then $u'(2\pi) = v(2\pi)$ exists and, by periodicity of u, we conclude that $u'(0)$ exists and satisfy $u'(0) = u'(2\pi)$.

$(i) \to (iii)$. Follows from [32, Theorem 1].

□

An immediate consequence of Theorem 4.6 is the following result in Hilbert spaces (see also [32, Theorem 1 and Theorem 3]).

Corollary 4.7. *Let H be a Hilbert space and let A be the generator of a strongly continuous cosine family $C(t)$ and denote by $S(t)$ the associated sine function. For $1 \le p < \infty$ the following are equivalent:*

(i) For any $f \in L^p(0, 2\pi; H)$ there exists a unique $(x, y) \in H \times H$ such that u given by (4.7) is differentiable at $t = 0$ and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

(ii)$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $(R(-k^2, A))_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

(iii) $S(2\pi) \in \mathcal{B}(H, E)$ is invertible.

(iv) $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{Z}} ||R(-k^2; A)|| < \infty$.

5. Mild periodic solutions of class C^1

In the present section, we discuss another notion of mild solution for the undamped equation (that is, $\alpha = 0$). Since a strong solution must be twice differentiable (in the classical or Sobolev sense), it is natural to examine solutions which do not require this condition. We also obtain a characterization in the case where the operator A generates a cosine function on X.

Definition 5.1. *For $f \in L^1_{loc}(\mathbb{R}, X)$, a function $u \in C^1(0, 2\pi; X)$ is called a mild solution of class C^1 of problem (4.1) if*

$$\begin{cases} & \displaystyle\int_0^t u(s)ds \in D(A), \\ u'(t) = & A\displaystyle\int_0^t u(s)ds + \int_0^t f(s)ds + u'(0), \end{cases} \tag{5.1}$$

for all $0 \le t \le 2\pi$.

We establish the following.

Theorem 5.2. *Let $f \in L^1_{2\pi}(\mathbb{R}, X)$ and $u \in C^1(0, 2\pi; X)$ be such that $u(0) = u(2\pi)$. Assume that $\overline{D(A)} = X$. Then u is a mild solution of class C^1 for the problem (4.1) such that $u'(0) = u'(2\pi)$ if and only if*

$$\hat{u}(k) \in D(A) \text{ and } (-k^2 - A)\hat{u}(k) = \hat{f}(k), \tag{5.2}$$

for all $k \in \mathbb{Z}$.

Proof. Assume that u is a mild solution of class C^1. Letting $t = 2\pi$ in (5.1) obtain $\int_0^{2\pi} u(s)ds \in D(A)$ and

$$u'(2\pi) = A\int_0^{2\pi} u(s)ds + \int_0^{2\pi} f(s)ds + u'(0).$$

Hence $-A\hat{u}(0) = \hat{f}(0)$, proving (5.2) for $k = 0$.
As in the proof of Theorem 4.4 we obtain by (5.1) $\hat{u}(k) \in D(A)$ and
$ik\hat{u}(k) + \frac{1}{2\pi}(u(2\pi) - u(0)) = A[\frac{1}{ik}\hat{u}(k) - \frac{1}{ik}\hat{u}(0)] + \frac{1}{ik}\hat{f}(k) - \frac{1}{ik}\hat{f}(0)$
for all $k \neq 0$. Hence $-k^2\hat{u}(k) - A\hat{u}(k) = \hat{f}(k)$ which proves (5.2) for all $k \neq 0$.
Conversely, assume that (5.2) holds. Let $x^* \in D(A^*)$. It suffices to show that

$$\begin{aligned} < u'(t), x^* > &= \int_0^t < u(s), A^*x^* > ds + \int_0^t < f(s), x^* > ds \\ &+ < u'(0), x^* > . \end{aligned}$$

Define $w(s) =< u(s), A^*x^* > + < f(s), x^* >$. Then by (5.2) we have $\hat{w}(k) = -k^2 < \hat{u}(k), x^* >$. In particular $\hat{w}(0) = 0$. Next, also define $h(t) =< u'(t), x^* >$. Then $\hat{h}(k) =< ik\hat{u}(k), x^* >= \frac{1}{ik}\hat{w}(k) = \frac{1}{ik}\hat{w}(k) - \frac{1}{ik}\hat{w}(0)$, for all $k \neq 0$. It follows that $h(t) - \int_0^t w(s)ds$ is constant; i.e. $h(t) = \int_0^t w(s)ds+ < u'(0), x^* >$. This proves that u is mild solution. Letting $t = 2\pi$ we get $u'(2\pi) = A\hat{u}(0) + \hat{f}(0) + u'(0) = u'(0)$, proving the theorem.

□

Theorem 5.3. *Let A be the generator of a strongly continuous cosine family $C(t)$ and let $1 \leq p < \infty$. Then the following are equivalent:*

(i) For any $f \in L^p(0, 2\pi; X)$ there exists a unique $(x, y) \in E \times X$ such that u given by (4.7) is 2π-periodic, i.e. $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$.

(ii)$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $(kR(-k^2, A))_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

(iii) $I - C(2\pi) \in \mathcal{B}(X; X)$ is invertible.

Proof. *(i)* $\to$ *(ii)*. Since $x \in E$, the function u given by (4.7) is C^1, and $\int_0^t u(s)ds \in D(A)$. Also, $u'(t) = AS(t)x + C(t)y + \int_0^t C(t-s)f(s)ds$ and we have

$$\begin{aligned} A\int_0^t u(s)ds &= A\int_0^t C(s)xds + A\int_0^t S(s)yds + A\int_0^t\int_0^\sigma S(\sigma - s)f(s)d\sigma ds \\ &= AS(t)x + C(t)y - y + \int_0^t C(t-s)f(s)ds - \int_0^t f(s)ds \\ &= u'(t) - y - \int_0^t f(s)ds = u'(t) - u'(0) - \int_0^t f(s)ds. \end{aligned}$$

Hence, by Theorem 5.2, $\hat{u}(k) \in D(A)$ and $(-k^2 - A)\hat{u}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}$. As in the proof of Theorem 2.11 we obtain $\{-k^2\}_{k\in\mathbb{Z}} \subset \rho(A)$. Let $f \in L^p(0, 2\pi; X)$. We have $\hat{u}(k) = R(-k^2; A)\hat{f}(k)$ for all $k \in \mathbb{Z}$, and because u is of class C^1, also $\widehat{u'}(k) = ikR(-k^2; A)\hat{f}(k)$ for all $k \in \mathbb{Z}$, where $u' \in L^p(0, 2\pi; X)$. Hence $(kR(-k^2; A))_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

$(ii) \to (i)$. Let $f \in L^p(0, 2\pi; X)$ and define

$$f_n = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes \hat{f}(k), \; n \in \mathbb{N}.$$

By hypothesis and [6, Lemma 2.2], $(R(-k^2; A))_{k\in\mathbb{Z}}$ is an $(L^p, H^{1,p})$-multiplier. Define $u_n = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes R(-k^2, A)\hat{f}(k)$, $n \in \mathbb{N}$. The hypothesis implies that $u = \lim_{n\to\infty} u_n$ exists in $H^{1,p}(0, 2\pi; X)$. Hence $u' = \lim_{n\to\infty} u_n'$ exists in $L^p(0, 2\pi; X)$, where

$$u_n' = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes ikR(-k^2, A)\hat{f}(k).$$

Clearly the functions u_n are 2π periodic on $\mathbb{R}$, $u_n'(0) = u_n'(2\pi)$ and $u_n''(t) = Au_n(t) + f_n(t)$ for all $n \in \mathbb{Z}$. Hence, $u_n(t)$ satisfies

$$u_n(t) = C(t)u_n(0) + S(t)u_n'(0) + \int_0^t S(t-s)f_n(s)ds.$$

Obviously $\int_0^t S(t-s)f_n(s)ds \to \int_0^t S(t-s)f(s)ds$ as $n \to \infty$, uniformly for $t \in [0, 2\pi]$. As in the proof of Theorem 4.6 or, alternatively, using the arguments as in [11, Theorem 2] we can deduce that $u_n(0)$ and $u_n'(0)$ converge to some $x \in E$ and $y \in X$, respectively, as $n \to \infty$. More precisely to verify that $x \in E$, we make use of the identity

$$\begin{aligned} AS(t)u_n(0) &= -C(t)u_n'(0) - \frac{1}{t}S(t)u_n'(0) + \frac{2}{t}\int_0^t C(t-s)u_n'(s)ds \\ &- \frac{2}{t}\int_0^t C(t-s)\int_0^s C(s-\tau)f_n(\tau)d\tau ds. \end{aligned}$$

Observe that $\int_0^t C(t-s)u_n'(s)ds \to \int_0^t C(t-s)u'(s)ds$ as $n \to \infty$, uniformly for $t \in [0, 2\pi]$. Hence the term on the right side of the above equality converges uniformly for $t \in [0, 2\pi]$. It follows that $\{AS(t)u_n(0)\}$ converges as $n \to \infty$, uniformly for $t \in [0, 2\pi]$. Thus $\{u_n(0)\}$ is a Cauchy sequence in E endowed with its norm as defined in the previous section. This proves that $x \in E$.

We conclude that

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds,$$

satisfying $u(0) = u(2\pi)$. Moreover u is C^1 since $x \in E$. Hence $u'(0) = u'(2\pi)$. Uniqueness of the solution follows from (5.2).

$(iii) \to (i)$. Let $f \in L^p_{2\pi}(\mathbb{R}; X)$. Choose

$$x = (I - C(2\pi))^{-1}S(\pi)\int_0^\pi C(\pi - s)(f(s) + f(-s))ds$$

and

$$y = (I - C(2\pi))^{-1}AS(\pi)\int_0^{\pi} S(\pi - s)(f(s) - f(-s))ds.$$

Define $u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds$. Then $u(0) = u(2\pi)$. Since $x \in ImS(\pi)$ (and $ImS(\pi) \subseteq E$, see [35, Proposition 2.2]) it results that $(x, y) \in E \times X$. It follows that u is actually of class C^1 and $u'(0) = u'(2\pi)$.

$(i) \to (iii)$. Follows from [11, Theorem 1].

□

As a consequence of the foregoing theorem, we obtain the following characterization in Hilbert spaces (see also [11, Theorem 2]).

Corollary 5.4. *Let H be a Hilbert space and A the generator of a strongly continuous cosine family $C(t)$ and let $1 \le p < \infty$. Then the following are equivalent:*

(i) For any $f \in L^p(0, 2\pi; H)$ there exists a unique $(x, y) \in E \times H$ such that u given by (4.7) is of class C^1 and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

(ii)$\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $(kR(-k^2, A))_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

(iii) $1 \in \rho(C(2\pi))$

(iv) $\{-k^2 : k \in \mathbb{Z}\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{Z}} ||kR(-k^2; A)|| < \infty$.

Example 5.5.

Let Ω be the cube $\Omega = \{x = (x_1, x_2, ... x_N),\ 0 < x_j < L,\ 1 \le j \le N\} \subset \mathbb{R}^N$. Let A be the Laplacian in $L^2(\Omega)$ with Dirichlet boundary conditions as defined at the end of Section 2. Then A is self-adjoint with compact resolvent and the eigenvalues of A are given by: $\lambda_n = -\pi^2 L^{-2}(n_1^2 + n_2^2 + ... + n_N^2)$ where $n = (n_1, n_2, ..., n_N)$ and $n_j \in \mathbb{N}$. By the spectral theorem for self-adjoint operators, one sees that A is the generator of a cosine function $C(t)$ (see e.g. [4, Example 3.14.16]) Moreover, the eigenvalues of $C(t)$ are given by $\mu_n(t) = \cos(t\sqrt{-\lambda_n})$ for $n \in \mathbb{N}^N$. When $t = 2\pi$, the eigenvalues are $\mu_n(2\pi) = \cos(2\pi\sqrt{-\lambda_n})$. It follows that condition (iii) of Corollary 5.4 is satisfied if $1 \notin \overline{\{\cos(2\pi\sqrt{-\lambda_n}), n \in \mathbb{N}\}}$.

However, when $N = 1$, this condition is never satisfied. In fact, $\cos(2\pi\sqrt{-\lambda_n}) = \cos(\frac{2\pi^2}{L}n)$ and if $\frac{2\pi}{L} \in \mathbb{Q}$, say $\frac{2\pi}{L} = \frac{p}{q}$ then $1 \in \{\cos(\frac{p}{q}\pi n),\ n \in \mathbb{N}\}$. On the other hand, if $\frac{2\pi}{L} \notin \mathbb{Q}$, then the set $\{\cos(\frac{p}{q}\pi n),\ n \in \mathbb{N}\}$ is dense in $[-1, 1]$. This is easily seen because in this case $G := \{\frac{2\pi^2}{L}k + 2m\pi,\ k,\ m \in \mathbb{Z}\}$ is a dense subgroup of $\mathbb{R}$.

In case $N > 1$, observe that $\lambda_n = -\pi^2 L^{-2}(n_1^2 + n_2^2 + ... + n_N^2) = -\pi^2 L^{-2} n_1^2 (1 + \dfrac{n_2^2}{n_1^2} + ... + \dfrac{n_N^2}{n_1^2})$. From this we see that the above analysis applies to this situation as well.

As a result, condition (iii) of Corollary 5.4 is not satisfied by the operator A.

Example 5.6.

Let $\gamma \in \mathbb{R}\setminus\{0\}$, $\varepsilon > 0$ with $\varepsilon \notin \{k^2,\ k \in \mathbb{N}\}$ and consider on $L^2(\mathbb{R})$ the operator$A$ given by $A = \dfrac{\partial^2}{\partial x^2} + \gamma \dfrac{\partial}{\partial x} - \varepsilon$ with domain the Sobolev space $H^2(\mathbb{R})$. Taking Fourier series we obtain $\mathcal{F}(Af)(\xi) = (-\xi^2 + i\gamma\xi - \varepsilon)\mathcal{F}(f)(\xi)$ for all $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$. From this we see that A generates a cosine function. Actually, the corresponding operator generates a cosine function on $L^p(\mathbb{R})$, $1 \le p < \infty$. This follows from a perturbation argument (see [4, Corollary 3.14.13] or [22]). For $k \in \mathbb{Z}$, $|k(-k^2 + \xi^2 + \varepsilon - i\gamma\xi)^{-1}| = |\dfrac{k}{\sqrt{(k^2 - \xi^2 - \varepsilon)^2 + \gamma^2\xi^2}}|$. The operator A is invertible. Now consider, for k fixed, the function $\varphi(\xi) = \dfrac{k^2}{(k^2 - \xi^2 - \varepsilon)^2 + \gamma^2\xi^2}$. Then $\lim_{|\xi|\to\infty} \varphi(\xi) = 0$. The critical points of φ are $\xi = 0$ with $\varphi(0) = \frac{k^2}{(k^2-\varepsilon)^2}$ and ξ_0 with $\xi_0^2 = k^2 - \varepsilon - \dfrac{\gamma}{2}$ for which $\varphi(\xi_0) = \dfrac{k^2}{\frac{\gamma^2}{4} + \gamma^2(k^2 - \varepsilon - \frac{\gamma}{2})}$, in case $\xi_0^2 = k^2 - \varepsilon - \dfrac{\gamma}{2} \ge 0$. Using Plancherel's Theorem we obtain: $\sup_{k\neq 0} \|k(-k^2 - A)^{-1}\| \le M$. So that Corollary 5.4 and hence Corollary 4.7 both apply.

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University of Puerto Rico, Department of Mathematics, Faculty of Natural Sciences, PO Box 23355, PR - 00931 U.S.A.
E-mail address: keyantuo@uprrp.edu

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago-Chile.
E-mail address: clizama@lauca.usach.cl