Strong solutions to stochastic Volterra equations ⋆

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Abstract

In this paper stochastic Volterra equations admitting exponentially bounded resolvents are studied. After obtaining convergence of resolvents, some properties for stochastic convolutions are studied. Our main results provide sufficient conditions for strong solutions to stochastic Volterra equations.

Key words: Stochastic linear Volterra equation, resolvent, strong solution, stochastic convolution.
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1 Introduction

We deal with the following stochastic Volterra equation in a separable Hilbert space H

$$X(t) = X_0 + \int_0^t a(t-\tau)\, AX(\tau) d\tau + \int_0^t \Psi(\tau)\, dW(\tau)\ , \qquad t \geq 0\ , \tag{1}$$

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where $X_0 \in H$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and A is a closed unbounded linear operator in H with a dense domain $D(A)$. The domain $D(A)$ is equipped with the graph norm $|\cdot|_{D(A)}$ of A, i.e. $|h|_{D(A)} := (|h|_H^2 + |Ah|_H^2)^{1/2}$, where $|\cdot|_H$ denotes the norm in H.

In this work the equation (1) is driven by a cylindrical Wiener process W and Ψ is an appropriate process defined later. Let us emphasize that the results obtained for cylindrical Wiener process W are valid for classical (genuine) Wiener process, too.

Equation (1) arises, in the deterministic case, in a variety of applications as model problems. Well-known techniques like localization, perturbation, and coordinate transformation allow to transfer results for such problems to parabolic integro-differential equations on smooth domains, see [16, Chapter I, Section 5]. In these applications, the operator A typically is a differential operator acting in spatial variables, like the Laplacian, the Stokes operator, or the elasticity operator. The function a should be thought of as a kernel like $a(t) = e^{-\eta t} t^{\beta-1}/\Gamma(\beta); \eta \geq 0, \beta \in (0,2)$. Equation (1) is an abstract stochastic version of the mentioned deterministic model problems. The stochastic approach to integral equations has been recently used due to the fact that in applications the level of accuracy for a given deterministic model not always seem to be significantly changed with increasing model complexity. Instead, the stochastic approach provides better results. A typical example is the use of stochastic integral equations in rainfall-runoff models, see [9].

Our main results concerning (1), rely essentially on techniques using a strongly continuous family of operators $S(t), t \geq 0$, defined on the Hilbert space H and called the `resolvent` (precise definition will be given below). Hence, in what follows, we assume that the deterministic version of equation (1) is `well-posed`, that is, admits a resolvent $S(t), t \geq 0$. Our aim is to provide sufficient conditions to obtain a strong solution to the stochastic equation (1).

This paper is organized as follows. In section 2 we prove the main deterministic ingredient for our construction; this is an extension of results of [2] allowing here that the operator A in (1) will be the generator of a C_0-semigroup, not necessarily of contraction type. Section 3 contains the main definitions and concepts used in the paper. In Section 4 we compare mild and weak solutions while in the last section we provide sufficient condition for stochastic convolution to be a strong solution to the equation (1). We note that this is an improvement of the known results about existence of strong solutions for stochastic differential equations.

2 Convergence of resolvents

In this section we recall some definitions connected with the deterministic version of the equation (1), that is, the equation

$$u(t) = \int_0^t a(t-\tau)\, Au(\tau)d\tau + f(t), \quad t \geq 0, \tag{2}$$

in a Banach space B. In (2), the operator A and the kernel function a are the same as previously considered in the introduction and f is a B-valued function.

Problems of this type have attracted much interest during the last decades, due to their various applications in mathematical physics like viscoelasticity, thermodynamics, or electrodynamics with memory, cf. [16].

By $S(t),\ t \geq 0$, we denote the family of resolvent operators corresponding to the Volterra equation (2), if it exists, and defined as follows.

Definition 1 *(see, e.g. [16])*
A family $(S(t))_{t\geq 0}$ of bounded linear operators in B is called `resolvent` *for (2) if the following conditions are satisfied:*

(1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
(2) $S(t)$ commutes with the operator A, that is, $S(t)(D(A)) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
(3) the following `resolvent equation` *holds*

$$S(t)x = x + \int_0^t a(t-\tau)AS(\tau)xd\tau \tag{3}$$

for all $x \in D(A),\ t \geq 0$.

Necessary and sufficient conditions for the existence of the resolvent family have been studied in [16]. Let us emphasize that the resolvent $S(t),\ t \geq 0$, is determined by the operator A and the function a, so we also say that the pair (A, a) admits a resolvent family. Moreover, as a consequence of the strong continuity of $S(t)$ we have $\sup_{t\leq T} ||S(t)|| < +\infty$ for any $T \geq 0$.

We shall use the abbreviation $(a \star f)(t) = \int_0^t a(t-s)f(s)ds, \quad t \in [0, T]$, for the convolution of two functions.

Definition 2 *We say that function $a \in L^1(0, T)$ is* `completely positive` *on*

$[0,T]$ *if for any* $\mu \geq 0$*, the solutions of the convolution equations*

$$s(t) + \mu(a \star s)(t) = 1 \quad \textit{and} \quad r(t) + \mu(a \star r)(t) = a(t) \tag{4}$$

satisfy $s(t) \geq 0$ *and* $r(t) \geq 0$ *on* $[0,T]$.

Kernels with this property have been introduced by Clément and Nohel [2]. We note that the class of completely positive kernels appears naturally in the theory of viscoelasticity. Several properties and examples of such kernels appears in [16, Section 4.2].

Definition 3 *Suppose* $S(t),\ t \geq 0$*, is a resolvent for (2).* $S(t)$ *is called* `exponentially bounded` *if there are constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that*

$$||S(t)|| \leq M\, e^{\omega t}, \quad \textit{for all} \ \ t \geq 0.$$

(M, ω) *is called a* `type` *of* $S(t)$.

Let us note that in contrary to the case of semigroups, not every resolvent needs to be exponentially bounded even if the kernel function a belongs to $L^1(\mathbb{R}_+)$. The resolvent version of the Hille–Yosida theorem (see, e.g., [16, Theorem 1.3]) provides the class of equations that admit exponentially bounded resolvents. An important class of kernels providing such class of resolvents are $a(t) = t^{\beta-1}/\Gamma(\beta),\ \alpha \in (0,2)$ or the class of completely monotonic functions. For details, counterexamples and comments we refer to [5].

In this paper the results concerning convergence of resolvents for the equation (1) in a Banach space B contained in Theorems 4 and 5 play the key role. They are significant generalizations of the results of Clément and Nohel obtained in [2] for contraction semigroups.

Theorem 4 *Let* A *be the generator of a* C_0*-semigroup in* B *and suppose the kernel function* a *is completely positive. Then* (A, a) *admits an exponentially bounded resolvent* $S(t)$*. Moreover, there exist bounded operators* A_n *such that* (A_n, a) *admit resolvent families* $S_n(t)$ *satisfying* $||S_n(t)|| \leq Me^{w_0 t}$ $(M \geq 1,\ w_0 \geq 0)$ *for all* $t \geq 0$ *and*

$$S_n(t)x \to S(t)x \quad \textit{as} \quad n \to +\infty \tag{5}$$

for all $x \in B,\ t \geq 0$*. Additionally, the convergence is uniform in* t *on every compact subset of* $\mathbb{R}_+$.

Proof The first assertion follows directly from [15, Theorem 5] (see also [16, Theorem 4.2]). Since A generates a C_0-semigroup $T(t),\ t \geq 0$, the resolvent

set $\rho(A)$ of A contains the ray $[w, \infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \qquad \text{for } \lambda > w, \qquad k \in \mathbb{N}.$$

Define

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \qquad n > w \tag{6}$$

the *Yosida approximation* of A.

Then

$$\begin{aligned} ||e^{tA_n}|| &= e^{-nt}||e^{n^2 R(n,A)t}|| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k}{k!} ||R(n, A)^k|| \\ &\leq M e^{(-n + \frac{n^2}{n-w})t} = M e^{\frac{nwt}{n-w}}. \end{aligned}$$

Hence, for $n > 2w$ we obtain

$$||e^{A_n t}|| \leq M e^{2wt}. \tag{7}$$

Taking into account the above estimate and the complete positivity of the kernel function a, we can follow the same steps as in [15, Theorem 5] to obtain that there exist constants $M_1 > 0$ and $w_1 \in \mathbb{R}$ (independent of n, due to (7)) such that

$$||[H_n(\lambda)]^{(k)}|| \leq \frac{M_1}{(\lambda - w_1)^{k+1}} \qquad \text{for } \lambda > w_1,$$

where $H_n(\lambda) := (\lambda - \lambda \hat{a}(\lambda) A_n)^{-1}$. Here and in the sequel the hat indicates the Laplace transform. Hence, the generation theorem for resolvent families implies that for each $n > 2\omega$, the pair (A_n, a) admits resolvent family $S_n(t)$ such that

$$||S_n(t)|| \leq M_1 e^{w_1 t}. \tag{8}$$

In particular, the Laplace transform $\hat{S}_n(\lambda)$ exists and satisfies

$$\hat{S}_n(\lambda) = H_n(\lambda) = \int_0^{\infty} e^{-\lambda t} S_n(t) dt, \qquad \lambda > w_1.$$

Now recall from semigroup theory that for all μ sufficiently large we have

$$R(\mu, A_n) = \int_0^\infty e^{-\mu t}\, e^{A_n t}\, dt$$

as well as,

$$R(\mu, A) = \int_0^\infty e^{-\mu t}\, T(t)\, dt\,.$$

Since $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$, we deduce that for all λ sufficiently large, we have

$$H_n(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A_n) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda))t} e^{A_n t} dt\,,$$

and

$$H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R(\frac{1}{\hat{a}(\lambda)}, A) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda))t} T(t) dt\,.$$

Hence, from the identity

$$H_n(\lambda) - H(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)} [R(\frac{1}{\hat{a}(\lambda)}, A_n) - R(\frac{1}{\hat{a}(\lambda)}, A)]$$

and the fact that $R(\mu, A_n) \to R(\mu, A)$ as $n \to \infty$ for all μ sufficiently large (see, e.g. [14, Lemma 7.3], we obtain that

$$H_n(\lambda) \to H(\lambda) \quad \text{as } n \to \infty\,. \tag{9}$$

Finally, due to (8) and (9) we can use the Trotter-Kato theorem for resolvent families of operators (cf. [13, Theorem 2.1]) and the conclusion follows. ■

An analogous result like Theorem 4 holds in other cases.

Theorem 5 *Let A be the generator of a strongly continuous cosine family. Suppose any of the following:*

(i) *$a \in L^1_{\text{loc}}(\mathbb{R}_+)$ is completely positive;*
(ii) *the kernel fuction a is a creep function with a_1 log-convex;*
(iii) *$a = c \star c$ with some completely positive $c \in L^1_{\text{loc}}(\mathbb{R}_+)$.*

Then (A, a) admits an exponentially bounded resolvent $S(t)$. Moreover, there exist bounded operators A_n such that (A_n, a) admit resolvent families $S_n(t)$ satisfying $||S_n(t)|| \leq M e^{w_0 t}$ $(M \geq 1,\ w_0 \geq 0)$ for all $t \geq 0,\ n \in \mathbb{N}$, and

$$S_n(t)x \to S(t)x \quad \textit{as} \quad n \to +\infty$$

for all $x \in B$, $t \geq 0$. Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

The proof follows from [16, Theorem 4.3], where the definition of a creep function can be found, or [15, Theorem 6] and proof of Theorem 4. Therefore it is omitted.

Remark 6 Other examples of the convergence (5) for the resolvents are given, e.g., in [2] and [7]. In the first paper, the operator A generates a linear continuous contraction semigroup. In the second of the mentioned papers, A belongs to some subclass of sectorial operators and the kernel a is an absolutely continuous function fulfilling some technical assumptions.

Proposition 7 *Let A, A_n and $S_n(t)$ be given as in Theorem 4. Then $S_n(t)$ commutes with the operator A, for every n sufficiently large and $t \geq 0$.*

Proof For each n sufficiently large the bounded operators A_n admit a resolvent family $S_n(t)$, so by the complex inversion formula for the Laplace transform we have

$$S_n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} e^{\lambda t} H_n(\lambda) d\lambda$$

where Γ_n is a simple closed rectifiable curve surrounding the spectrum of A_n in the positive sense.

On the other hand, $H_n(\lambda) := (\lambda - \lambda \hat{a}(\lambda) A_n)$ where $A_n := nA(n - A)^{-1}$, so each A_n commutes with A on $D(A)$ and then each $H_n(\lambda)$ commutes with A, on $D(A)$, too.

Finally, because A is closed, and all the following integrals are convergent (exist) we have for all n sufficiently large and $x \in D(A)$

$$\begin{aligned} AS_n(t)x &= A \int_{\Gamma_n} e^{\lambda t} H_n(\lambda) x d\lambda = \int_{\Gamma_n} e^{\lambda t} A H_n(\lambda) x d\lambda \\ &= \int_{\Gamma_n} e^{\lambda t} H_n(\lambda) A x d\lambda = S_n(t) A x \,. \end{aligned}$$

■

3 Solutions and the stochastic convolution

Let H and U be two separable Hilbert spaces and $Q \in L(U)$ be a linear bounded symmetric nonnegative operator. A Wiener process W with covariance operator Q is defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We assume that the process W is a cylindrical one, that is, we do not assume that $\mathrm{Tr}Q < +\infty$. In this case, the process W has values in some superspace of U. Let us note that the results obtained in the paper for cylindrical Wiener process are valid in classical Wiener process, too. Namely, when $\mathrm{Tr}Q < +\infty$, we can take $U = H$ and $\Psi = I$.

This is apparently well-known that the construction of the stochastic integral with respect to cylindrical Wiener process requires some particular terms. We will need the subspace $U_0 := Q^{1/2}(U)$ of the space U, which endowed with the inner product $\langle u, v\rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$ forms a Hilbert space. Among others, an important role is played by the space of Hilbert-Schmidt operators. The set $L_2^0 := L_2(U_0, H)$ of all Hilbert-Schmidt operators from U_0 into H, equipped with the norm $|C|_{L_2(U_0,H)} := (\sum_{k=1}^{+\infty} |Cf_k|_H^2)^{1/2}$, where $\{f_k\}$ is an orthonormal basis of U_0, is a separable Hilbert space.

According to the theory of stochastic integral with respect to cylindrical Wiener process we have to assume that Ψ belongs to the class of measurable L_2^0-valued processes.

Let us introduce the norms

$$
\begin{aligned}
||\Psi||_t &:= \left\{ \mathbb{E} \left(\int_0^t |\Psi(\tau)|_{L_2^0}^2 \, d\tau \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \mathbb{E} \int_0^t \left[\mathrm{Tr}(\Psi(\tau)Q^{\frac{1}{2}})(\Psi(\tau)Q^{\frac{1}{2}})^* \right] d\tau \right\}^{\frac{1}{2}}, \quad t \in [0, T].
\end{aligned}
$$

By $\mathcal{N}^2(0, T; L_2^0)$ we denote a Hilbert space of all L_2^0-predictable processes Ψ such that $||\Psi||_T < +\infty$.

It is possible to consider a more general class of integrands, see, e.g. [12], but in our opinion it is not worthwhile to study the general case. That case produces a new level of difficulty additionally to problems related to long time memory of the system. So, we shall study the equation (1) under the below Probability Assumptions (abbr. (PA)):

(1) X_0 is an H-valued, $\mathcal{F}_0$-measurable random variable;
(2) $\Psi \in \mathcal{N}^2(0, T; L_2^0)$ and the interval $[0, T]$ is fixed.

Definition 8 *Assume that (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `strong solution` *to (1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0,T]$; for any $t \in [0,T]$*

$$\int_0^t |a(t-\tau)AX(\tau)|_H \, d\tau < +\infty, \quad P-a.s. \tag{10}$$

and for any $t \in [0,T]$ the equation (1) holds P-a.s.

Let A^* denote the adjoint of A with a dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$.

Definition 9 *Let (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `weak solution` *to (1), if $P(\int_0^t |a(t-\tau)X(\tau)|_H d\tau < +\infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds*

$$\langle X(t), \xi\rangle_H = \langle X_0, \xi\rangle_H + \langle \int_0^t a(t-\tau)X(\tau)\, d\tau, A^*\xi\rangle_H + \langle \int_0^t \Psi(\tau) dW(\tau), \xi\rangle_H, \ P-\text{a.s.}$$

Remark 10 This definition has sense for a cylindrical Wiener process because the scalar process $\langle \int_0^t \Psi(\tau) dW(\tau), \xi\rangle_H$, for $t \in [0,T]$, is well-defined.

Definition 11 *Assume that X_0 is an H-valued $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (1), if*

$$\mathbb{E}\left(\int_0^t |S(t-\tau)\Psi(\tau)|^2_{L_2^0} \, d\tau\right) < +\infty \quad \textit{for} \quad t \le T \tag{11}$$

and, for arbitrary $t \in [0,T]$,

$$X(t) = S(t)X_0 + \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad P-a.s. \tag{12}$$

where $S(t)$ is the resolvent for the equation (2), if it exists.

In some cases weak solutions of equation (1) coincides with mild solutions of (1), see e.g. [11]. In consequence, having results for the convolution on the right hand side of (12) we obtain results for weak solutions.

In the paper we will use the following well-known result.

Proposition 12 *(see, e.g. [4, Proposition 4.15])*
Assume that A is a closed linear unbounded operator with the dense domain $D(A) \subset H$ and $\Phi(t),\ t \in [0,T]$ is an $L_2(U_0,H)$-predictable process. If $\Phi(t)(U_0) \subset D(A)$, $P-a.s.$ for all $t \in [0,T]$ and

$$P\left(\int_0^T |\Phi(s)|^2_{L_2^0}\,ds < \infty\right) = 1, \quad P\left(\int_0^T |A\Phi(s)|^2_{L_2^0}\,ds < \infty\right) = 1,$$

then $$P\left(\int_0^T \Phi(s)\,dW(s) \in D(A)\right) = 1$$

and $$A\int_0^T \Phi(s)\,dW(s) = \int_0^T A\Phi(s)\,dW(s), \quad P-a.s.$$

In what follows we assume that (2) admits a resolvent family $S(t)$, $t \geq 0$. We introduce the stochastic convolution

$$W^{\Psi}(t) := \int_0^t S(t-\tau)\Psi(\tau)\,dW(\tau), \tag{13}$$

where Ψ belongs to the space $\mathcal{N}^2(0,T;L_2^0)$. Note that, because resolvent operators $S(t),\ t \geq 0$, are bounded, then $S(t-\cdot)\Psi(\cdot) \in \mathcal{N}^2(0,T;L_2^0)$, too.

Let us formulate some auxiliary results concerning the convolution $W^{\Psi}(t)$.

Proposition 13 *Assume that (2) admits resolvent operators $S(t),\ t \geq 0$. Then, for arbitrary process $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, the process $W^{\Psi}(t),\ t \geq 0$, given by (13) has a predictable version.*

Proposition 14 *Assume that $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Then the process $W^{\Psi}(t),\ t \geq 0$, defined by (13) has square integrable trajectories.*

For the proofs of Propositions 13 and 14 we refer to [11].

Proposition 15 *Let $a \in BV(\mathbb{R}_+)$ and suppose that (2) admits a resolvent family $S \in C^1(0,\infty;L(H))$. Let X be a predictable process with integrable trajectories. Assume that X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0,T]$ and (11) holds. If for any $t \in [0,T]$ and $\xi \in D(A^*)$*

$$\langle X(t),\xi\rangle_H = \langle X_0,\xi\rangle_H + \int_0^t \langle a(t-\tau)X(\tau), A^*\xi\rangle_H d\tau \tag{14}$$

$$+\int_0^t \langle \xi, \Psi(\tau) dW(\tau)\rangle_H, \quad P-a.s.,$$

then

$$X(t) = S(t)X_0 + \int_0^t S(t-\tau)\Psi(\tau)dW(\tau), \quad t \in [0,T]. \tag{15}$$

Remark 16 If (1) is parabolic and the kernel a is 3-monotone, understood in the sense defined by Prüss [16, Section 3], then $S \in C^1(0,\infty; L(H))$ and $a \in BV(\mathbb{R}_+)$ respectively.

Proposition 17 *Assume that A is a closed linear unbounded operator with the dense domain $D(A)$, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $S(t)$, $t \geq 0$, are resolvent operators for the equation (2). If $\Psi \in \mathcal{N}^2(0,T;L^0_2)$, then the stochastic convolution W^Ψ fulfills the equation (14) with $X_0 \equiv 0$.*

For the proofs of Proposition 15 and 17 we refer to [11].

Remark 18 Assume that $X_0 \in D(A)$ a.s. and, in addition, $X_0 \in Ker(A)$. Define $Y^\Psi(t) = W^\Psi(t) + X_0$ then $Y^\Psi(t)$ satisfies the equation (1). In fact, for any $t \in [0,T]$ and $\xi \in D(A^*)$ we have

$$\begin{aligned}
\langle Y^\Psi(t), \xi\rangle_H &= \langle W^\Psi(t), \xi\rangle_H + \langle X_0, \xi\rangle_H = \int_0^t \langle a(t-\tau)W^\Psi(\tau), A^*\xi\rangle_H d\tau + \langle X_0, \xi\rangle_H \\
&= \int_0^t \langle a(t-\tau)(Y^\Psi(\tau) - X_0), A^*\xi\rangle_H d\tau + \langle X_0, \xi\rangle_H \\
&= \int_0^t \langle a(t-\tau)Y^\Psi(\tau), A^*\xi\rangle_H d\tau - \int_0^t a(\tau)\langle X_0, A^*\xi\rangle_H d\tau + \langle X_0, \xi\rangle_H \\
&= \int_0^t \langle a(t-\tau)Y^\Psi(\tau), A^*\xi\rangle_H d\tau + \langle X_0, \xi\rangle_H.
\end{aligned}$$

In consequence, in what follows we restrict ourselves to the case $X_0 \equiv 0$.

Corollary 19 *Assume that A is a linear bounded operator in H, $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $S(t)$, $t \geq 0$, are resolvent operators for the equation (2). If Ψ belongs to $\mathcal{N}^2(0,T;L^0_2)$ then*

$$W^\Psi(t) = \int_0^t a(t-\tau)AW^\Psi(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau)\,. \tag{16}$$

Remark 20 The formula (16) says that the convolution W^Ψ is a strong solution to (1) with $X_0 \equiv 0$ if the operator A is bounded.

4 Strong solution

In this section we provide sufficient conditions under which the stochastic convolution $W^{\Psi}(t)$, $t \geq 0$, defined by (13) is a strong solution to the equation (1).

Theorem 21 *Suppose that assumptions of Theorem 4 or Theorem 5 hold. Then the equation (1) has a strong solution. Precisely, the convolution W^{Ψ} defined by (13) is the strong solution to (1) with $X_0 \equiv 0$.*

The proof of Theorem 21 bases on the following result.

Lemma 22 *Let A be a closed linear unbounded operator with the dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$. Suppose that assumptions of Theorem 4 or Theorem 5 hold. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then (16) holds.*

Proof Because formula (16) holds for any bounded operator, then it holds for the Yosida approximation A_n of the operator A, too, that is

$$W_n^{\Psi}(t) = \int_0^t a(t-\tau)A_n W_n^{\Psi}(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau),$$

where

$$W_n^{\Psi}(t) := \int_0^t S_n(t-\tau)\Psi(\tau)dW(\tau)$$

and

$$A_n W_n^{\Psi}(t) = A_n \int_0^t S_n(t-\tau)\Psi(\tau)dW(\tau).$$

Recall that by assumption $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because the operators $S_n(t)$ are deterministic and bounded for any $t \in [0,T]$, $n \in \mathbb{N}$, then the operators $S_n(t-\cdot)\Psi(\cdot)$ belong to $\mathcal{N}^2(0,T;L_2^0)$, too. In consequence, the difference

$$\Phi_n(t-\cdot) := S_n(t-\cdot)\Psi(\cdot) - S(t-\cdot)\Psi(\cdot) \tag{17}$$

belongs to $\mathcal{N}^2(0,T;L_2^0)$ for any $t \in [0,T]$ and $n \in \mathbb{N}$. This means that

$$\mathbb{E}\left(\int_0^t |\Phi_n(t-\tau)|_{L_2^0}^2 d\tau\right) < +\infty \tag{18}$$

for any $t \in [0,T]$.

Let us recall that the cylindrical Wiener process $W(t)$, $t \geq 0$, can be written in the form

$$W(t) = \sum_{j=1}^{+\infty} f_j \, \beta_j(t), \tag{19}$$

where $\{f_j\}$ is an orthonormal basis of U_0 and $\beta_j(t)$ are independent real Wiener processes. From (19) we have

$$\int_0^t \Phi_n(t-\tau)\, dW(\tau) = \sum_{j=1}^{+\infty} \int_0^t \Phi_n(t-\tau)\, f_j \, d\beta_j(\tau). \tag{20}$$

Then, from (18)

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{+\infty} |\Phi_n(t-\tau)\, f_j|_H^2\right) d\tau\right] < +\infty \tag{21}$$

for any $t \in [0,T]$. Next, from (20), properties of stochastic integral and (21) we obtain for any $t \in [0,T]$,

$$\mathbb{E}\left|\int_0^t \Phi_n(t-\tau)\, dW(\tau)\right|_H^2 = \mathbb{E}\left|\sum_{j=1}^{+\infty} \int_0^t \Phi_n(t-\tau)\, f_j \, d\beta_j(\tau)\right|_H^2 \leq$$
$$\mathbb{E}\left[\sum_{j=1}^{+\infty} \int_0^t |\Phi_n(t-\tau)\, f_j|_H^2 d\tau\right] \leq \mathbb{E}\left[\sum_{j=1}^{+\infty} \int_0^T |\Phi_n(T-\tau)\, f_j|_H^2 d\tau\right] < +\infty.$$

By Theorem 4 or 5, the convergence (5) of resolvent families is uniform in t on every compact subset of $\mathbb{R}_+$, particularly on the interval $[0,T]$. Now, we use (5) in the Hilbert space H, so (5) holds for every $x \in H$. Then, for any fixed j,

$$\int_0^T |[S_n(T-\tau) - S(T-\tau)]\, \Psi(\tau)\, f_j|_H^2 d\tau \tag{22}$$

tends to zero for $n \to +\infty$. Summing up our considerations, particularly using (21) and (22) we can write

$$\sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \Phi_n(t-\tau) dW(\tau)\right|_H^2 \equiv \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t [S_n(t-\tau) - S(t-\tau)]\Psi(\tau) dW(\tau)\right|_H^2 \leq$$

$$\leq \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^T |[S_n(T-\tau)-S(T-\tau)]\Psi(\tau)\, f_j|_H^2 d\tau\right] \to 0$$

as $n \to +\infty$.

Hence, by the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}\mathbb{E}\left|W_n^{\Psi}(t)-W^{\Psi}(t)\right|_H^2 = 0. \tag{23}$$

By assumption, $\Psi(\cdot,\cdot)(U_0)\subset D(A),\ P-a.s.$ Because $S(t)(D(A))\subset D(A)$, then $S(t-\tau)\Psi(\tau)(U_0)\subset D(A),\ P-a.s.$, for any $\tau\in[0,t],\ t\geq 0$. Hence, by Proposition 12, $P(W^{\Psi}(t)\in D(A))=1$.

For any $n\in\mathbb{N}$, $t\geq 0$, we have

$$|A_nW_n^{\Psi}(t)-AW^{\Psi}(t)|_H \leq N_{n,1}(t)+N_{n,2}(t),$$

where

$$\begin{aligned}
N_{n,1}(t) &:= |A_nW_n^{\Psi}(t)-A_nW^{\Psi}(t)|_H,\\
N_{n,2}(t) &:= |A_nW^{\Psi}(t)-AW^{\Psi}(t)|_H = |(A_n-A)W^{\Psi}(t)|_H\,.
\end{aligned}$$

Then

$$\begin{aligned}
|A_nW_n^{\Psi}(t)-AW^{\Psi}(t)|_H^2 &\leq N_{n,1}^2(t)+2N_{n,1}(t)N_{n,2}(t)+N_{n,2}^2(t)\\
&< 3[N_{n,1}^2(t)+N_{n,2}^2(t)].
\end{aligned} \tag{24}$$

Let us study the term $N_{n,1}(t)$. Note that the unbounded operator A generates a semigroup. Then we have for the Yosida approximation the following properties:

$$A_nx = J_nAx \quad \text{for any } x\in D(A), \quad \sup_n ||J_n|| < \infty \tag{25}$$

where $A_nx = nAR(n,A)x = AJ_nx$ for any $x\in H$, with $J_n := nR(n,A)$. Moreover (see [6, Chapter II, Lemma 3.4]):

$$\begin{aligned}
&\lim_{n\to\infty} J_nx = x \qquad \text{for any } x\in H,\\
&\lim_{n\to\infty} A_nx = Ax \qquad \text{for any } x\in D(A).
\end{aligned} \tag{26}$$

By Proposition 7, $AS_n(t)x = S_n(t)Ax$ for all $x\in D(A)$. So, by Propositions 7 and 12 and the closedness of A we can write

$$A_n W_n^{\Psi}(t) \equiv A_n \int_0^t S_n(t-\tau)\Psi(\tau)dW(\tau)$$
$$= J_n \int_0^t AS_n(t-\tau)\Psi(\tau)dW(\tau) = J_n \left[\int_0^t S_n(t-\tau)A\Psi(\tau)dW(\tau)\right].$$

Analogously,

$$A_n W^{\Psi}(t) = J_n \left[\int_0^t S(t-\tau)A\Psi(\tau)dW(\tau)\right].$$

By (25) we have

$$N_{n,1}(t) = |J_n \int_0^t [S_n(t-\tau) - S(t-\tau)]A\Psi(\tau)dW(\tau)|_H$$
$$\leq |\int_0^t [S_n(t-\tau) - S(t-\tau)]A\Psi(\tau)dW(\tau)|_H \ .$$

From assumptions, $A\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Then the term $[S_n(t-\tau) - S(t-\tau)]A\Psi(\tau)$ may be treated like the difference Φ_n defined by (17).

Hence, from (25) and (23), for the first term of the right hand side of (24) we have

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}(N_{n,1}^2(t)) = 0.$$

For the second term of (24), that is $N_{n,2}^2(t)$, we can follow the same steps as above for proving (23).

$$N_{n,2}(t) = |A_n W^{\Psi}(t) - AW^{\Psi}(t)|_H$$
$$\equiv \left| A_n \int_0^t S(t-\tau)\Psi(\tau)dW(\tau) - A\int_0^t S(t-\tau)\Psi(\tau)dW(\tau)\right|_H =$$
$$= \left|\int_0^t [A_n - A]S(t-\tau)\Psi(\tau)dW(\tau)\right|_H \ .$$

From assumptions, $\Psi, A\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because $A_n, S(t),\ t \geq 0$ are bounded, then $A_n S(t-\cdot)\Psi(\cdot) \in \mathcal{N}^2(0,T;L_2^0)$, too. Analogously, $AS(t-\cdot)\Psi(\cdot) = S(t-\cdot)A\Psi(\cdot) \in \mathcal{N}^2(0,T;L_2^0)$.

Let us note that the set of all Hilbert-Schmidt operators acting from one separable Hilbert space into another one, equipped with the operator norm

defined on page 8 is a separable Hilbert space. Particularly, sum of two Hilbert-Schmidt operators is a Hilbert-Schmidt operator, see e.g. [1]. Therefore, we can deduce that the operator $(A_n - A)\ S(t-\cdot)\Psi(\cdot) \in \mathcal{N}^2(0,T;L_2^0)$, for any $t \in [0,T]$. Hence, the term $[A_n - A]S(t-\tau)\Psi(\tau)$ may be treated like the difference Φ_n defined by (17). So, we obtain

$$\begin{aligned}\mathbb{E}\left(N_{n,2}^2(t)\right) &= \mathbb{E}\left(\int_0^t \left[\sum_{j=1}^{+\infty} |[A_n - A]S(t-\tau)\Psi(\tau)\, f_j|_H^2\right] d\tau\right)\\ &\leq \mathbb{E}\left(\sum_{j=1}^{+\infty}\int_0^T |[A_n - A]S(t-\tau)\Psi(\tau)\, f_j|_H^2\, d\tau\right) < +\infty,\end{aligned}$$

for any $t \in [0,T]$.

By the convergence (26), for any fixed j,

$$\int_0^T |[A_n - A]S(t-\tau)\Psi(\tau)\, f_j|_H^2 d\tau$$

tends to zero for $n \to +\infty$.

Summing up our considerations, we have

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}(N_{n,2}^2(t)) = 0\ .$$

So, we can deduce that

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}|A_n W_n^{\Psi}(t) - AW^{\Psi}(t)|_H^2 = 0,$$

and then (16) holds. ■

Proof of Theorem 21 In order to prove Theorem 21, we have to show only the condition (10). Let us note that the convolution $W^{\Psi}(t)$ has integrable trajectories. Because the closed unbounded linear operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [17, Chapter 5], we obtain that $AW^{\Psi}(\cdot) \in L^1([0,T];H)$, P-a.s. Hence, properties of convolution provide integrability of the function $a(T-\tau)AW^{\Psi}(\tau)$ with respect to τ, what finishes the proof. ■

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