ON STOCHASTIC FRACTIONAL VOLterra EQUATIONS IN HILBERT SPACE

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Abstract. In this paper, stochastic Volterra equations, particularly fractional, in Hilbert space are studied. Sufficient conditions for existence of strong solutions are provided.

1. Introduction. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a stochastic basis and \(H\) a separable Hilbert space. In this paper we consider the stochastic Volterra equations in \(H\) of the form

\[ X(t) = X(0) + \int_0^t a(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad t \geq 0. \]

(1)

In (1), \(X(0)\) is an \(H\)-valued \(\mathcal{F}_0\)-measurable random variable and \(a \in L^1_{\text{loc}}(\mathbb{R}^+)\) is a scalar kernel. The operator \(A\) is closed linear unbounded in \(H\) with a dense domain \(D(A)\) equipped with the graph norm \(| \cdot |_{D(A)}\), i.e. \(|h|_{D(A)} := (|h|^2_H + |Ah|^2_H)^{1/2}\), where \(\cdot ; H\) denotes the norm in \(H\). \(W\) is a cylindrical Wiener process (see e.g. [3] or [7] for the definition, properties and the stochastic integral with respect to that process) on another separable Hilbert space \(U\), with the covariance operator \(Q \in L(U)\). \(Q\) is a linear symmetric positive operator with \(\text{Tr} Q = +\infty\) and \(\Psi\) is an appropriate process defined below.

Equations (1) contain important special cases, e.g. heat, wave and integro-differential equations. Moreover, (1) are motivated by a wide class of model problems and correspond to abstract stochastic versions of several deterministic problems, mentioned, e.g. in [13] (see also the references therein).

In order to provide a sense for the integral \(\int_0^t \Psi(\tau)dW(\tau)\), the process \(\Psi(t)\), \(t \geq 0\), has to be an operator-valued process (see, e.g. [7]). We define the subspace \(U_0 := Q^{1/2}(U)\) of the space \(U\) endowed with the inner product \(\langle u, v \rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U\). By \(L^2_0 := L_2(U_0, H)\) we denote the set of all Hilbert-Schmidt operators acting from \(U_0\) into \(H\); the set \(L^2_0\) equipped with the norm \(||C||_{L_2(U_0, H)} := \left(\sum_{k=1}^{+\infty} |Cu_k|^2_H\right)^{1/2}\) is a separable Hilbert space.

2000 Mathematics Subject Classification. Primary: 60H15, 60H20; Secondary: 60H05, 45D05.

Key words and phrases. Stochastic fractional Volterra equation, \(\alpha\)-times resolvent, strong solution, stochastic convolution, convergence of resolvent families.

Both authors are partially supported by Laboratorio de Análisis Estocástico, PBCT-ACT13.
We say that the function $u(t) = \int_0^t a(t-\tau) A u(\tau) d\tau + f(t)$, $t \geq 0$, (2) where $f$ is an $H$-valued function.

Definition 1. A family $(S(t))_{t \geq 0}$ of bounded linear operators in $H$ is called resolvent for (2) if the following conditions are satisfied:

1. $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
2. $S(t)$ commutes with the operator $A$:

   $$S(t)(D(A)) \subset D(A)$$

   and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
3. the following resolvent equation holds

   $$S(t)x = x + \int_0^t a(t-\tau) A S(\tau)x d\tau$$

   for all $x \in D(A)$, $t \geq 0$.

We will assume in the sequel that the resolvent family $(S(t))_{t \geq 0}$ to (2) exists.

Let us emphasize that the family $(S(t))_{t \geq 0}$ does not create in general any semigroup and that $S(t)$, $t \geq 0$, are generated by the pair $(A, a(t))$, that is, the operator $A$ and the kernel function $a(t)$, $t \geq 0$.

A consequence of the strong continuity of $S(t)$ is that $\sup_{t \leq T} ||S(t)|| < +\infty$ for any $T \geq 0$.

Definition 2. We say that the function $a \in L^1(0, T)$ is completely positive on $[0, T]$, if for any $\mu \geq 0$, the solutions of the equations

   $$s(t) + \mu(a \ast s)(t) = 1$$

   and $r(t) + \mu(a \ast r)(t) = a(t)$

   satisfy $s(t) \geq 0$ and $r(t) \geq 0$ on $[0, T]$.

The class of completely positive kernels, introduced in [2], arise naturally in applications, see [13]. For instance, the functions $a(t) \equiv 1$, $a(t) = t$, $a(t) = e^{-t}$, $t \geq 0$, are completely positive.

Definition 3. Suppose $S(t)$, $t \geq 0$, is a resolvent. $S(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

   $$||S(t)|| \leq M e^{\omega t}, \text{ for all } t \geq 0;$$
\((M, \omega)\) is called a type of \(S(t)\).

Let us note that contrary to \(C_0\)-semigroups, not every resolvent family needs to be exponentially bounded; for counterexamples we refer to [4].

In the paper, the key role is played by the following, not yet published, result providing a convergence of resolvents.

**Theorem 1.** Let \(A\) be the generator of a \(C_0\)-semigroup in \(H\) and suppose the kernel function \(a\) is completely positive. Then \((A, a)\) admits an exponentially bounded resolvent \(S(t)\). Moreover, there exist bounded operators \(A_n\) such that \((A_n, a)\) admit resolvent families \(S_n(t)\) satisfying \(||S_n(t)|| \leq Me^{\omega t} (M \geq 1, \omega_0 \geq 0)\) for all \(t \geq 0, n \in \mathbb{N}\), and

\[
S_n(t)x \to S(t)x \quad \text{as} \quad n \to +\infty \quad (5)
\]

for all \(x \in H, t \geq 0\).

Additionally, the convergence is uniform in \(t\) on every compact subset of \(\mathbb{R}_+\).

**Proof.** The first assertion follows directly from [12, Theorem 5] (see also [13, Theorem 4.2]). Since \(A\) generates a \(C_0\)-semigroup \(T(t)\), \(t \geq 0\), the resolvent set \(\rho(A)\) of \(A\) contains the ray \([w, \infty)\) and

\[
||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \quad \text{for} \quad \lambda > w, \quad k \in \mathbb{N},
\]

where \(R(\lambda, A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)\).

Define

\[
A_n := nAR(n, A) = n^2R(n, A) - nI, \quad n > w \quad (6)
\]

the Yosida approximation of \(A\), where \(R(n, A) = (nI - A)^{-1}\). For details, see e.g. [11].

Then

\[
||e^{tA_n}|| = e^{-nt}||e^{n^2R(n, A)t}|| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!} ||R(n, A)^k||
\]

\[
\leq Me^{(n + \frac{n^2}{w})t} = Me^{\frac{2nt}{w}}.
\]

Hence, for \(n > 2w\) we obtain

\[
||e^{A_n t}|| \leq Me^{2wt}.
\]

(7)

Taking into account the above estimate and the complete positivity of the kernel function \(a\), we can follow the same steps as in [12, Theorem 5] to obtain that there exist constants \(M_1 > 0\) and \(w_1 \in \mathbb{R}\) (independent of \(n\), due to (7)) such that

\[
||[H_n(\lambda)]^{(k)}|| \leq \frac{M_1}{(\lambda - w_1)^{k+1}} \quad \text{for} \quad \lambda > w_1,
\]

where \(H_n(\lambda) := (\lambda - \lambda \hat{a}(\lambda)A_n)^{-1}\). Here and in the sequel the hat indicates the Laplace transform. Hence, the generation theorem for resolvent families implies that for each \(n > 2\omega\), the pair \((A_n, a)\) admits resolvent family \(S_n(t)\) such that

\[
||S_n(t)|| \leq M_1 e^{w_1 t}.
\]

(8)

In particular, the Laplace transform \(\hat{S}_n(\lambda)\) exists and satisfies

\[
\hat{S}_n(\lambda) = H_n(\lambda) = \int_0^\infty e^{-\lambda t}S_n(t)dt, \quad \lambda > w_1.
\]
Now recall from semigroup theory that for all \( \mu \) sufficiently large we have
\[
R(\mu, A_n) = \int_0^\infty e^{-\mu t} e^{A_n t} \, dt
\]
as well as,
\[
R(\mu, A) = \int_0^\infty e^{-\mu t} T(t) \, dt .
\]
Since \( \hat{a}(\lambda) \to 0 \) as \( \lambda \to \infty \), we deduce that for all \( \lambda \) sufficiently large, we have
\[
H_n(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R\left( \frac{1}{\hat{a}(\lambda)}, A_n \right) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda)) t} e^{A_n t} \, dt ,
\]
and
\[
H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)} R\left( \frac{1}{\hat{a}(\lambda)}, A \right) = \frac{1}{\lambda \hat{a}(\lambda)} \int_0^\infty e^{(-1/\hat{a}(\lambda)) t} T(t) \, dt .
\]
Hence, from the identity
\[
H_n(\lambda) - H(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)} [R\left( \frac{1}{\hat{a}(\lambda)}, A_n \right) - R\left( \frac{1}{\hat{a}(\lambda)}, A \right)]
\]
and the fact that \( R(\mu, A_n) \to R(\mu, A) \) as \( n \to \infty \) for all \( \mu \) sufficiently large (see, e.g. [11, Lemma 7.3]), we obtain that
\[
H_n(\lambda) \to H(\lambda) \quad \text{as} \quad n \to \infty . \tag{9}
\]
Finally, due to (8) and (9) we can use the Trotter-Kato theorem for resolvent families of operators (cf. [9, Theorem 2.1]) and the conclusion follows. \( \square \)

**Remark 1.**
(a) The convergence (5) is an extension of the result due to Clément and Nohel [2].

(b) The above theorem gives a partial answer to the following open problem for a resolvent family \( S(t) \) generated by a pair \((A, a)\): do there exist bounded linear operators \( A_n \) generating resolvent families \( S_n(t) \) such that \( S_n(t)x \to S(t)x \)? In particular case \( a(t) \equiv 1, A_n \) are provided by the Hille-Yosida approximation of \( A \) and additionally \( S_n(t) = e^{tA_n} \).

2. **Probabilistic results.** In the sequel we shall use the following **Probability Assumptions**, abbr. (PA):

1. \( X(0) \) is an \( H \)-valued, \( \mathcal{F}_0 \)-measurable random variable;
2. \( \Psi \in \mathcal{N}^2(0, T; L^2_0) \) and the interval \([0, T]\) is fixed.

The following types of definitions of solutions to (1) are possible, see [8].

**Definition 4.** Assume that (PA) hold. An \( H \)-valued predictable process \( X(t) \), \( t \in [0, T] \), is said to be a **strong solution** to (1), if \( X \) has a version such that \( P(X(t) \in D(A)) = 1 \) for almost all \( t \in [0, T] \); for any \( t \in [0, T] \)
\[
\int_0^t |a(t-\tau)AX(\tau)|_H \, d\tau < +\infty, \quad P\text{-a.s.} \tag{10}
\]
and for any \( t \in [0, T] \) the equation (1) holds \( P\text{-a.s.} \).

Let \( A^* \) be the adjoint of \( A \) with a dense domain \( D(A^*) \subset H \) and the graph norm \(| \cdot |_{D(A^*)} \) defined as follows: \(|h|_{D(A^*)} := (|h|^2_H + |A^* h|^2_H)^{1/2} \) for \( h \in D(A^*) \).
Definition 5. Let (PA) hold. An $H$-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a weak solution to (1), if $P(\int_0^T |a(t - \tau)X(\tau)|_H d\tau < +\infty) = 1$ and if for all $\xi \in D(A^*)$ and all $t \in [0, T]$ the following equation holds

$$\langle X(t), \xi \rangle_H = \langle X(0), \xi \rangle_H + \int_0^t a(t - \tau)X(\tau) d\tau, A^*\xi \rangle_H + \left( \int_0^t \Psi(\tau)dW(\tau), \xi \right)_H, \quad P\text{-a.s.}$$

Definition 6. Assume that $X(0)$ is $\mathcal{F}_0$-measurable random variable. An $H$-valued predictable process $X(t)$, $t \in [0, T]$, is said to be a mild solution to the stochastic Volterra equation (1), if $\mathbb{E}(\int_0^T |S(t - \tau)\Psi(\tau)|_{L^2}^2 d\tau) < +\infty$ for $t \leq T$ and, for arbitrary $t \in [0, T]$,

$$X(t) = S(t)X(0) + \int_0^t S(t - \tau)\Psi(\tau) dW(\tau), \quad P\text{-a.s.} \tag{11}$$

The integral appearing in (11) will be called stochastic convolution and denoted by

$$W^\Psi(t) := \int_0^t S(t - \tau)\Psi(\tau) dW(\tau), \quad t \geq 0, \tag{12}$$

where $\Psi \in \mathcal{N}^2(0, T; L^2_H)$. We will show in the sequel that the convolution $W^\Psi$ is a weak solution to (1) and next we will provide sufficient conditions under which $W^\Psi$ is a strong solution to (1), as well.

Let us recall (from [3] and [8]) some properties of the convolution $W^\Psi(t)$, $t \geq 0$.

Proposition 1. (see, e.g. [3, Proposition 4.15])
Assume that $A$ is a closed linear unbounded operator with the dense domain $D(A) \subset H$ and $\Phi(t)$, $t \in [0, T]$ is an $L_2(U_0, H)$-predictable process. If $\Phi(t)U(t) \subset D(A)$, $P$-a.s. for all $t \in [0, T]$ and

$$P \left( \int_0^T |\Phi(s)|_{L^2_H}^2 ds < \infty \right) = 1, \quad P \left( \int_0^T |A\Phi(s)|_{L^2_H}^2 ds < \infty \right) = 1,$$

then

$$P \left( \int_0^T \Phi(s) dW(s) \in D(A) \right) = 1$$

and

$$A \int_0^T \Phi(s) dW(s) = \int_0^T A\Phi(s) dW(s), \quad P\text{-a.s.}$$

For the proofs of Propositions 2, 3 and 4 we refer to [8].

Proposition 2. Assume that (2) admits resolvent operators $S(t)$, $t \geq 0$. Then, for arbitrary process $\Psi \in \mathcal{N}^2(0, T; L^2_H)$, the process $W^\Psi(t)$, $t \geq 0$, given by (12) has a predictable version.

Proposition 3. Assume that $\Psi \in \mathcal{N}^2(0, T; L^2_H)$. Then the process $W^\Psi(t)$, $t \geq 0$, defined by (12) has square integrable trajectories.
Proposition 4. If $\Psi \in \mathcal{N}^2(0, T; L^2_0)$, then the stochastic convolution $W^\Psi$ fulfills the equation

$$\langle W^\Psi(t), \xi \rangle_H = \int_0^t \langle a(t - \tau) W^\Psi(\tau), A^* \xi \rangle_H + \int_0^t \langle \xi, \Psi(\tau) dW(\tau) \rangle_H, \quad P - a.s.$$ for any $t \in [0, T]$ and $\xi \in D(A^*)$.

Proposition 4 shows that the convolution $W^\Psi$ is a weak solution to (1) (see [8]) and enables us to formulate the following results.

Proposition 5. Let $A$ be the generator of $C_0$-semigroup in $H$ and suppose that the function $a$ is completely positive. If $\Psi$ and $A\Psi$ belong to $\mathcal{N}^2(0, T; L^2_0)$ and in addition $\Psi(t)(U_0) \subset D(A)$, $P$-a.s., then the following equality holds

$$W^\Psi(t) = \int_0^t a(t - \tau) A W^\Psi(\tau) d\tau + \int_0^t \Psi(\tau) dW(\tau), \quad P - a.s. \quad (13)$$

Proof. Because formula (13) holds for any bounded operator, then it holds for the Yosida approximation $A_n$ of the operator $A$, too, that is

$$W^\Psi_n(t) = \int_0^t a(t - \tau) A_n W^\Psi_n(\tau) d\tau + \int_0^t \Psi(\tau) dW(\tau),$$

where

$$W^\Psi_n(t) := \int_0^t S_n(t - \tau) \Psi(\tau) dW(\tau)$$

and

$$A_n W^\Psi_n(t) = A_n \int_0^t S_n(t - \tau) \Psi(\tau) dW(\tau).$$

Recall that by assumption $\Psi \in \mathcal{N}^2(0, T; L^2_0)$. Because the operators $S_n(t)$ are deterministic and bounded for any $t \in [0, T]$, $n \in \mathbb{N}$, then the operators $S_n(t - \cdot) \Psi(\cdot)$ belong to $\mathcal{N}^2(0, T; L^2_0)$, too. In consequence, the difference

$$\Phi_n(t - \cdot) := S_n(t - \cdot) \Psi(\cdot) - S(t - \cdot) \Psi(\cdot) \quad (14)$$

belongs to $\mathcal{N}^2(0, T; L^2_0)$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. This means that

$$\mathbb{E} \left( \int_0^t |\Phi_n(t - \tau)|_{L^2_0}^2 d\tau \right) < +\infty \quad (15)$$

for any $t \in [0, T]$.

Let us recall (see [7]) that the cylindrical Wiener process $W(t)$, $t \geq 0$, can be written in the form

$$W(t) = \sum_{j=1}^{+\infty} f_j \beta_j(t), \quad (16)$$

where $\{f_j\}$ is an orthonormal basis of $U_0$ and $\beta_j(t)$ are independent real Wiener processes. From (16) we have

$$\int_0^t \Phi_n(t - \tau) dW(\tau) = \sum_{j=1}^{+\infty} \int_0^t \Phi_n(t - \tau) f_j d\beta_j(\tau). \quad (17)$$

Then, from (15)

$$\mathbb{E} \left[ \int_0^t \left( \sum_{j=1}^{+\infty} |\Phi_n(t - \tau) f_j|_H^2 \right) d\tau \right] < +\infty \quad (18)$$
for any $t \in [0, T]$. Next, from (17), properties of stochastic integral and (18) we obtain for any $t \in [0, T]$, that
\[
\mathbb{E} \left| \int_0^t \Phi_n(t, \tau) dW(\tau) \right|^2_H = \mathbb{E} \left| \sum_{j=1}^{+\infty} \int_0^t \Phi_n(t, \tau) f_j d\beta_j(\tau) \right|^2_H 
\]
\[
\mathbb{E} \left[ \sum_{j=1}^{+\infty} \int_0^t |\Phi_n(t, \tau) f_j|^2_H d\tau \right] \leq \mathbb{E} \left[ \sum_{j=1}^{+\infty} \int_0^T |\Phi_n(T, \tau) f_j|^2_H d\tau \right] < +\infty.
\]

By Theorem 1, the convergence (5) of resolvent families is uniform in $t$ on every compact subset of $\mathbb{R}_+$, particularly on the interval $[0, T]$. Then, for any fixed $j$, \(\int_0^T [S_n(T-\tau) - S(T-\tau)] \Psi(\tau) f_j^2_H d\tau \longrightarrow 0\), as $n \to \infty$. (19)

So, using (18) and (19) we can write
\[
\sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t \Phi_n(t, \tau) dW(\tau) \right|^2_H \leq \sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t [S_n(t, \tau) - S(T-\tau)] \Psi(\tau) dW(\tau) \right|^2_H 
\]
\[
\leq \mathbb{E} \left[ \sum_{j=1}^{+\infty} \int_0^T [S_n(T-\tau) - S(T-\tau)] \Psi(\tau) f_j^2_H d\tau \right] \to 0
\]
as $n \to +\infty$.

Hence, by the Lebesgue dominated convergence theorem
\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E} |W_n^\Psi(t) - W^\Psi(t)|^2_H = 0. \tag{20}
\]

By assumption, $\Psi(t)(U_0) \subset D(A)$, $P$-a.s. Because $S(t)(D(A)) \subset D(A)$, $S(t-\tau)\Psi(\tau)(U_0) \subset D(A)$, $P$-a.s., for any $\tau \in [0, t]$, $t \geq 0$. Hence, by Proposition 1, $P(W^\Psi(t) \in D(A)) = 1$.

For any $n \in \mathbb{N}$, $t \geq 0$, we can estimate
\[
|A_n W_n^\Psi(t) - AW^\Psi(t)|^2_H < 3[N_{n,1}(t) + N_{n,2}(t)], \tag{21}
\]
where
\[
N_{n,1}(t) := |A_n W_n^\Psi(t) - A_n W^\Psi(t)|_H,
\]
\[
N_{n,2}(t) := |A_n W^\Psi(t) - AW^\Psi(t)|_H.
\]

Using the convergence of resolvents (5) and the Yoshida approximation properties, we can follow the same steps as above for proving
\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}(N_{n,1}(t)) \to 0
\]
and
\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}(N_{n,2}(t)) \to 0.
\]

Therefore, we can deduce that
\[
\lim_{n \to +\infty} \sup_{t \in [0, T]} \mathbb{E}|A_n W_n^\Psi(t) - AW^\Psi(t)|^2_H = 0,
\]
and then (13) holds. \(\square\)
Theorem 2. Suppose that assumptions of Proposition 5 hold. Then the equation (1) has a strong solution. Precisely, the convolution $W^\Psi$ given by (12) is the strong solution to (1).

Proof. In order to prove Theorem 2, we have to show only the condition (10). Let us note that the convolution $W^\Psi$ has integrable trajectories. Because the closed unbounded linear operator $A$ becomes bounded on $(D(A), \|\cdot\|_{D(A)})$, see [14, Chapter 5], we obtain that $AW^\Psi(\cdot) \in L^1([0,T]; H)$, P-a.s. Hence, properties of convolution provide integrability of the function $a(T-\tau)AW^\Psi(\tau)$ with respect to $\tau$, what finishes the proof. \qed

3. Fractional Volterra equations. As we have already written, (2) contains some class of equations. For instance when $a(t) = \frac{\alpha^{-1}}{\Gamma(1+\alpha)}$, $\alpha > 0$, we obtain integro-differential equations studied by many authors, see e.g. [1] and references therein. These facts lead us to the fractional stochastic Volterra equations of the form

$$X(t) = X(0) + \int_0^t g_\alpha(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)dW(\tau), \quad t \geq 0,$$

where $g_\alpha(t) = \frac{\alpha^{-1}}{\Gamma(1+\alpha)}$, $\alpha > 0$. Let us emphasize that for $\alpha \in (0,1]$, $g_\alpha$ are completely positive, but for $\alpha > 1$, $g_\alpha$ are not completely positive.

Now, the pairs $(A, g_\alpha(t))$ generate $\alpha$-times resolvents $S_\alpha(t)$, $t \geq 0$ which are analogous to resolvents defined in section 1; for more details, see [1].

Remark 2. Observe that the $\alpha$-times resolvent family corresponds to a $C_0$-semigroup in case $\alpha = 1$ and a cosine family in case $\alpha = 2$. (Let us recall, e.g. from [5], that a family $C(t)$, $t \geq 0$, of linear bounded operators on $H$ is called cosine family if for every $t, s \geq 0$, $t > s$: $C(t+s) + C(t-s) = 2C(t)C(s)$.) In consequence, when $1 < \alpha < 2$ such resolvent families interpolate $C_0$-semigroups and cosine functions. In particular, for $A = \Delta$, the integro-differential equations corresponding to such resolvent families interpolate the heat equation and the wave equation, see, e.g. [6].

We consider two cases:

(A1): $A$ is the generator of $C_0$-semigroup and $0 < \alpha < 1$;

(A2): $A$ is the generator of a strongly continuous cosine family and $\alpha \in (0,2)$.

In this part of the paper, the results concerning a weak convergence of $\alpha$-times resolvents play the key role. Using the very recent result due to Li and Zheng [10], we can formulate the approximation theorems for fractional Volterra equations.

Theorem 3. Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ in $H$ such that $\|T(t)\| \leq Me^{\omega t}$, $t \geq 0$. Then, for each $0 < \alpha < 1$ there exist bounded operators $A_n$ and $\alpha$-times resolvent families $S_{\alpha,n}(t)$ for $A_n$ satisfying $\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha} t}$, for all $t \geq 0$, $n \in \mathbb{N}$, and

$$S_{\alpha,n}(t)x \to S_{\alpha}(t)x \quad \text{as} \quad n \to +\infty$$

for all $x \in H$, $t \geq 0$. Moreover, the convergence is uniform in $t$ on every compact subset of $\mathbb{R}_+$.

Outline of the proof: The first assertion follows from [1, Theorem 3.1], that is, for each $0 < \alpha < 1$ there is an $\alpha$-times resolvent family $(S_{\alpha}(t))_{t \geq 0}$ for $A$ given by

$$S_{\alpha}(t)x = \int_0^\infty \varphi_{t,\alpha}(s)T(s)xds, \quad t > 0,$$
where \( \varphi_{t,\gamma}(s) := t^{-\gamma} \Phi_\gamma(st^{-\gamma}) \) and \( \Phi_\gamma(z) \) is the Wright function defined as
\[
\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1.
\]
Define
\[
A_n := nAR(n, A) = n^2 R(n, A) - nI, \quad n > w,
\]
the Yosida approximation of \( A \).

Since each \( A_n \) is bounded, it follows that for each \( 0 < \alpha < 1 \) there exists an \( \alpha \)-times resolvent family \( (S_{\alpha,n}(t))_{t \geq 0} \) for \( A_n \) given as
\[
S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s) e^{sA_n} ds, \quad t > 0.
\]

We recall that the Laplace transform of the Wright function corresponds to \( E_\gamma(-z) \) where \( E_\gamma \) denotes the Mittag-Leffler function. In particular, \( \Phi_\gamma(z) \) is a probability density function. It follows that for \( t \geq 0 \):
\[
\|S_{\alpha,n}(t)\| \leq \int_0^\infty \varphi_{t,\alpha}(s)\|e^{sA_n}\|ds
\]
\[
\leq M \int_0^\infty \varphi_{t,\alpha}(s)e^{s\omega s} ds = M \int_0^\infty \Phi_\alpha(\tau)e^{2\omega t^\alpha \tau} d\tau = ME_\alpha(2\omega t^\alpha).
\]

The continuity in \( t \geq 0 \) of the Mittag-Leffler function and its asymptotic behavior, imply that for \( \omega \geq 0 \) there exists a constant \( C > 0 \) such that
\[
E_\alpha(\omega t^\alpha) \leq Ce^{\omega^{1/\alpha} t}, \quad t \geq 0, \quad \alpha \in (0, 2).
\]
This gives
\[
\|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha} t}, \quad t \geq 0.
\]

Now we recall the fact that \( R(\lambda, A_n)x \to R(\lambda, A)x \) as \( n \to \infty \) for all \( \lambda \) sufficiently large (e.g. [11, Lemma 7.3]), so we can conclude from [10, Theorem 4.2] that
\[
S_{\alpha,n}(t)x \to S_{\alpha}(t)x \quad \text{as} \quad n \to +\infty
\]
for all \( x \in H \), uniformly for \( t \) on every compact subset of \( \mathbb{R}_+ \). \( \square \)

An analogous convergence for \( \alpha \)-times resolvents can be proved in another case, too.

**Theorem 4.** Let \( A \) be the generator of a \( C_0 \)-cosine family \( (T(t))_{t \geq 0} \) in \( H \). Then, for each \( 0 < \alpha < 2 \) there exist bounded operators \( A_n \) and \( \alpha \)-times resolvent families \( S_{\alpha,n}(t) \) for \( A_n \) satisfying \( \|S_{\alpha,n}(t)\| \leq MCe^{(2\omega)^{1/\alpha} t} \), for all \( t \geq 0 \), \( n \in \mathbb{N} \), and \( S_{\alpha,n}(t)x \to S_{\alpha}(t)x \) as \( n \to +\infty \) for all \( x \in H \), \( t \geq 0 \). Moreover, the convergence is uniform in \( t \) on every compact subset of \( \mathbb{R}_+ \).

Now, we are able to formulate the result analogous to that in section 2.

**Theorem 5.** Assume that (A1) or (A2) holds. If \( \Psi \) and \( A\Psi \) belong to \( N^2(0, T; L_0^\infty) \) and in addition \( \Psi(t)(U_0) \subset D(A) \), \( P \)-a.s., then the equation (1) has a strong solution. Precisely, the convolution
\[
W_{\Psi}^\tau(t) := \int_0^t S_{\alpha}(t - \tau) \Psi(\tau) dW(\tau)
\]
is the strong solution to (1).
Outline of the proof: First, analogously like in section 2, we show that the convolution $W_\alpha^\Psi(t)$ fulfills the following equation

$$W_\alpha^\Psi(t) = \int_0^t g_\alpha(t-\tau)A W_\alpha^\Psi(\tau)\,d\tau + \int_0^t \Psi(\tau)dW(\tau). \quad (24)$$

Next, we have to show the condition

$$\int_0^T \left| g_\alpha(T-\tau)A W_\alpha^\Psi(\tau) \right|_H \,d\tau < +\infty, \quad P-a.s., \quad \alpha > 0, \quad (25)$$

that is, the condition (10) adapted for the fractional Volterra equation (22).

The convolution $W_\alpha^\Psi(t)$ has integrable trajectories, that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator $A$ becomes bounded on $(D(A),\|\cdot\|_{D(A)})$, see [14, Chapter 5]. Hence, $AW_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Therefore, the function $g_\alpha(T-\tau)A W_\alpha^\Psi(\tau)$ is integrable with respect to $\tau$, what completes the proof. □

Acknowledgement The authors thank the referee for a careful reading of the manuscript and useful remarks.

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