Solutions to stochastic fractional relaxation equations ⋆

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Abstract: Stochastic fractional relaxation equations are studied. Sufficient conditions for existence of mild, weak and strong solutions to the equations are provided. A new family of strongly continuous operators, called α-regularized family, corresponding to the relaxation equations, has been introduced.

Keywords: Stochastic fractional relaxation equation, stochastic convolution, α-regularized family.

1. INTRODUCTION

The aim of the paper is to obtain the existence of solutions to the stochastic version of the generalized Basset equation. The Basset equation arises in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, see e.g. Mainardi (1997). It corresponds to a fractional relaxation equation and can be interpreted as an integral equation, which is our viewpoint in this paper.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis and let H be a complex Hilbert space with a norm $|\cdot|_H$. We study the problem in the following abstract form in H

$$X(t) - \int_0^t [AD^\alpha X(\tau) - X(\tau)]d\tau = W(t), \qquad (1)$$
$$0 < \alpha < 1, \qquad t > 0,$$

where A is a closed linear operator with a dense domain $D(A) \subset H$ equipped with the graph norm $|\cdot|_A$ and W is an H-valued Wiener process.

The fractional derivative of order α is assumed to be the Caputo derivative. The fractional differential equation (1) with $\alpha = 1/2$, $H = \mathbb{C}$ and A a real number, corresponds to the original Basset problem, which in its deterministic form can be written as $u'(t) - AD^{1/2}u(t) + u(t) = f(t)$, $t > 0$, with initial condition $u(0) = 0$ and f an appropriate H-valued function.

Sufficient conditions for existence of strong solutions to stochastic Volterra equations were recently obtained in Karczewska&Lizama (2007). This was done using a method which involves the use of a resolvent family associated to the deterministic version of the stochastic Volterra equation. It is possible to handle the equation (1) by using a similar method. Now, the main tool will be the notion of α-regularized family introduced below and an argument of approximation. So, we can follow the methods used in Karczewska&Lizama (2007) to obtain existence of solutions to the stochastic equation (1).

Our plan for the paper is the following. In section 2 we recall some facts from fractional calculus and in section 3 we formulate deterministic results which will play the key role for the paper. Section 4 is devoted to solutions to (1), while in section 5 we give sufficient conditions to have existence of α-regularized families.

2. MITTAG-LEFFLER FUNCTION

In the paper we deal with the Caputo fractional derivative, defined as

$$D^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(\tau)}{(t-\tau)^\alpha} d\tau, \qquad 0 < \alpha < 1\ , \quad (2)$$

where $u \in C^1(\mathbb{R}_+, H)$ and Γ is the gamma function. One of the main advantages of the Caputo fractional derivative is that initial conditions are expressed in terms of initial values of integer order derivatives. So, the Caputo fractional derivative appears more suitable to be treated by the Laplace transform technique in that it requires the knowledge of the (bounded) initial values of the function and of its integer derivatives. The following rule holds for the Laplace transform (see (Bazhlekova, 2001, p.12) and references therein)

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \lambda^{\alpha-1} u(0), \quad Re\,\lambda > 0, \quad 0 < \alpha < 1.$$

The generalized Mittag-Leffler function (see (Erdélyi et al., 1955, Vol. 3, Chapter 18)), is defined as follows

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta}\, e^\mu}{\mu^\alpha - z} d\mu,$$
$$\alpha, \beta > 0, \quad z \in \mathbb{C},$$

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where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counter-clockwise. This is an entire function which is a generalization of the exponential function: $E_{1,1}(z) = e^z$ and the cosine function: $E_{2,1}(z^2) = \cosh(z)$; $E_{2,1}(-z^2) = \cos(z)$, and plays an important role in the theory of fractional differential equations.

One of the most important properties of the Mittag-Leffler functions is associated with their Laplace transform

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \qquad (3)$$

$$\text{where} \quad Re\,\lambda > \omega^{1/\alpha}, \quad \omega > 0.$$

We recall (see e.g. Prüss (1993)) that a Laplace transformable function $a(t)$, $t \ge 0$, is said to be of *positive type* if $Re\,\hat{a}(\lambda) \ge 0$ for all $Re\,\lambda > 0$, and 1-*regular* if there is a constant $c > 0$ such that

$$|\lambda \hat{a}'(\lambda)| \le c|\hat{a}(\lambda)| \quad \text{for all} \quad Re\,\lambda > 0. \qquad (4)$$

Above $\hat{a}(\lambda)$ denotes the Laplace transform of the function $a(t)$. Both types of functions are very important in the theory of viscoelasticity, see (Prüss, 1993, p.38 and p.69) and references therein. In the whole paper we denote by $||\cdot||$ the operator norm.

3. α-REGULARIZED FAMILY

Let B be a complex Banach space with the norm $|\cdot|_B$ and let $\mathcal{B}(B)$ denote the space of linear bounded operators on B. In this section we study existence and uniqueness of solution to the equation

$$u(t) = \int_0^t [AD^\alpha u(\tau) - u(\tau)]d\tau + \int_0^t f(\tau)d\tau, \qquad (5)$$

$$t > 0, \quad 0 < \alpha < 1.$$

Because $u(0) = 0$, by definition of the Caputo derivative we have

$$\int_0^t D_t^\alpha u(\tau)d\tau = \int_0^t g_{1-\alpha}(t-\tau)u(\tau)d\tau. \qquad (6)$$

Then the problem (5) is equivalent to

$$u(t) - \int_0^t g_{1-\alpha}(t-\tau)Au(\tau)d\tau + \int_0^t u(\tau)d\tau = \int_0^t f(\tau)d\tau, \qquad (7)$$

$$t > 0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta \ge 0.$$

In consequence, by a solution of (5) (equivalently (7)) we can understand a function $u \in C(\mathbb{R}_+; D(A))$ that satisfies (7).

Note that letting $v := u + (1 * u)$ we have $u = v - (e^{-t} * v)$ and then the equation (7) is equivalent to the Volterra equation

$$v(t) = A(a_\alpha * v)(t) + g(t), \quad v(0) = 0, \quad g = (1 * f) \qquad (8)$$

where

$$a_\alpha(t) = t^{-\alpha} - (t^{-\alpha} * e^{-t}) = t^{-\alpha} E_{1,1-\alpha}(-t).$$

In order to give an operator theoretical approach to the equation (5) we introduce the following definition.

Definition 1. Let A be a closed linear operator with domain $D(A)$ defined on Banach space B. We call A *the generator of an α-regularized family* if there exists $\omega \ge 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(B)$ such that $\{\frac{1+\lambda}{\lambda^\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \lambda^{-\alpha}(\frac{1+\lambda}{\lambda^\alpha} - A)^{-1}x = \int_0^\infty e^{-\lambda t}R(t)xdt, \qquad (9)$$

$$Re\lambda > \omega, \quad x \in B.$$

In this case, $R(t), t \ge 0$, is called *the α-regularized family generated by A.*

By taking Laplace transform to (5) we see that A generates an α-regularized family if and only if (5) has a unique solution. This means that the equation (5) is well-posed. Also note that because of the uniqueness of the Laplace transform, α-regularized family corresponds to an (a, k)-regularized family Lizama (2000), with $k(t) = e^{-t}$ and $a_\alpha(t) = t^{-\alpha}E_{1,1-\alpha}(-t)$. In fact, we have $\hat{a}_\alpha(\lambda) = \frac{\lambda^\alpha}{\lambda+1}$ and $\hat{k}(\lambda) = \frac{1}{1+\lambda}$.

Analogously like in the case of C_0-semigroups, we can derive relations between α-regularized family and its generator. The following result is a direct consequence of (Lizama, 2000, Proposition 3.1 and Lemma 2.2).

Proposition 2. Let $R(t), t \ge 0$, be an α-regularized family on B with the generator A. Then the following conditions hold:

(1) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$, $t \ge 0$.

(2) Let $x \in D(A)$ and $t \ge 0$. Then

$$R(t)x = e^{-t}x + \int_0^t a_\alpha(t-s)AR(s)xds. \qquad (10)$$

(3) Let $x \in B$ and $t \ge 0$. Then

$$\int_0^t a_\alpha(t-s)AR(s)xds \in D(A)$$

and

$$R(t)x = e^{-t}x + A\int_0^t a_\alpha(t-s)R(s)xds. \qquad (11)$$

The following lemma will be used in the proof of Theorem 4.

Lemma 3. The Mittag-Leffler function

$$a_\alpha(t) = t^{-\alpha}E_{1,1-\alpha}(-t), \quad t > 0, \quad 0 < \alpha < 1,$$

is of positive type and 1-regular.

Proof. By (3), we obtain $\hat{a}_\alpha(\lambda) = \frac{\lambda^\alpha}{\lambda+1}$. Let $\lambda = re^{i\theta}$ with $-\pi/2 \le \theta \le \pi/2$ and $r > 0$. Then $Re\,\hat{a}_\alpha(\lambda) = \frac{r^\alpha}{|\lambda+1|^2}[r\cos((\alpha-1)\theta) + \cos(\alpha\theta)]$ and hence

$a_\alpha(t)$ is of positive type, because $0<\alpha<1$. On the other hand,

$$\lambda\frac{\hat{a}'_\alpha(\lambda)}{\hat{a}_\alpha(\lambda)}=(\alpha-1)\frac{\lambda}{\lambda+1}+\frac{\alpha}{\lambda+1},$$

which shows that $a_\alpha(t)$ is 1-regular.

The below result gives an important sufficient condition on the operator A to provide well-posedness of the equation (5). Moreover, we determine a sequence of approximate operators which is crucial in the treatment of the stochastic version of the equation (5).

Theorem 4. Let A be the generator of a bounded analytic semigroup on B. Then

(1) A is the generator of an α-regularized resolvent family $(R(t))_{t\geq 0}$ on B.
(2) There exist bounded operators A_n and α-regularized resolvent families $R_n(t)$ generated by A_n satisfying $||R_n(t)||\leq Me^t$ for all $t>0,\ n\in\mathbb{N}$, such that
$$R_n(t)x\to R(t)x \quad \text{as} \quad n\to+\infty \tag{12}$$
for all $x\in B,\ t\geq 0$. Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof. Because $a_\alpha(t),\ t>0$, is of positive type by Lemma 3, we obtain by (Prüss, 1993, Corollary 3.1, p.69) that $\frac{1}{\hat{a}_\alpha(\lambda)}\in\rho(A)$ for all $Re\ \lambda>0$. Moreover, there is a constant $M\geq 1$ such that $H(\lambda):=(I-\hat{a}_\alpha(\lambda)A)^{-1}/\lambda$ satisfies

$$||H(\lambda)||\leq\frac{M}{|\lambda|} \quad \text{for all} \quad Re\ \lambda>0.$$

From the above, and since $a_\alpha(t)$ is 1-regular by Lemma 3, we get by (Prüss, 1993, Theorem 3.1 p.73) that A generates a resolvent family $(S(t))_{t\geq 0}$ such that

$$\hat{S}(\lambda)=H(\lambda),\quad Re\ \lambda>0.$$

Additionally, there is a constant $C\geq 1$ such that the estimate $||S(t)||\leq C,\ t>0$ is valid. Let $x\in B$ and define

$$R(t)x:=S(t)x-\int_0^t e^{-(t-\tau)}S(\tau)x d\tau,\quad t>0.$$

We can see that $R(t)$ is an α-regularized family generated by A (see Lizama (2000)). On the other hand, since A generates a bounded analytic semigroup, the resolvent set $\rho(A)$ of A contains the ray $(0,\infty)$ and

$$||R(\lambda,A)^k||\leq\frac{M}{|\lambda|^k}\qquad \text{for } \lambda>0,\qquad k\in\mathbb{N}.$$

Define

$$A_n:=nAR(n,A)=n^2R(n,A)-nI,\qquad n>0,$$

the *Yosida approximation* of A.

$$\text{Then }\ ||e^{tA_n}||=e^{-nt}||e^{n^2R(n,A)t}||\leq$$
$$\leq e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}t^k}{k!}||R(n,A)^k||\leq$$
$$\leq M\,e^{(-n+\frac{n^2}{n})t}=M.$$

Hence, for $n>0$,

$$||e^{A_nt}||\leq M. \tag{13}$$

Because each A_n is bounded and hence generates bounded analytic semigroup e^{A_nt} verifying (13), it follows that there exist resolvent families $(S_n(t))_{t\geq 0}$ (or $(a,1)$-regularized resolvent families) such that $||S_n(t)||\leq M$, see (Prüss, 1993, Corollary 3.1 p.69 and Theorem 3.1 p.73). Then, for each $n\in\mathbb{N}$ and $x\in B$ the formula

$$R_n(t)x=S_n(t)x-\int_0^t e^{-(t-\tau)}S_n(\tau)xd\tau,\quad t>0$$

defines α-regularized resolvent families generated by A_n such that

$$||R_n(t)||\leq Me^t\ . \tag{14}$$

Now, we recall the fact that $R(\mu,A_n)x\to R(\mu,A)x$ as $n\to\infty$ for all μ sufficiently large (see e.g. (Pazy, 1983, Lemma 7.3)). In consquence

$$\hat{k}(\lambda)(I-\hat{a}(\lambda)A_n)^{-1}x\to\hat{k}(\lambda)(I-\hat{a}(\lambda)A)^{-1}x$$

as $n\to\infty$ for all λ sufficiently large. Then the uniform stability condition (14) and (Lizama, 2001, Theorem 2.2) implies that

$$R_n(t)x\to R(t)x\quad \text{as}\quad n\to+\infty$$

for all $x\in B$, uniformly for t on every compact subset of $\mathbb{R}_+$.

Theorem 5. Let A be the generator of a bounded analytic semigroup on B and let $f:\mathbb{R}_+\to B$ be locally integrable on $\mathbb{R}_+$. Then there exists an α-regularized family $R(t)$ such that

$$u(t):=\int_0^t R(t-\tau)f(\tau)d\tau \tag{15}$$

is a solution of the equation (5).

Proof. Since $f(t)\in D(A)$ and A is closed then $u(t)\in D(A)$. Because

$$(1*D^\alpha u)(t)=\int_0^t g_{1-\alpha}(t-\tau)u(\tau)d\tau, \tag{16}$$

we observe that (5) is equivalent to

$$u(t)-A\int_0^t g_{1-\alpha}(t-\tau)u(\tau)d\tau+\int_0^t u(\tau)d\tau=\int_0^t f(\tau)d\tau,\ t>0.$$

To show that $u(t)$ satisfies (7) we note the following identities

$$(1*k)(t)=1-k(t),\quad t\geq 0, \tag{17}$$

and

$$a_\alpha(t)+(1*a_\alpha)(t)=g_{1-\alpha}(t),\quad t>0. \tag{18}$$

By Theorem 4 there exists an α-regularized family $R(t)$. Then by the equation (11) and by using (15) and (18), we obtain

$$\begin{aligned}u(t)-A\int_0^t g_{1-\alpha}(t-\tau)u(\tau)d\tau &=\\ &=(R*f)(t)-A(g_{1-\alpha}*R*f)(t)\\ &=[(k+Aa_\alpha*R)*f](t)-A(g_{1-\alpha}*R*f)(t)\\ &=(k*f)(t)+[A(a_\alpha-g_{1-\alpha})*R*f](t)\\ &=(k*f)(t)-A[(1*a_\alpha)*R*f](t)\ .\end{aligned}$$

On the other hand, again using the resolvent equation (11) and then (15) and (17), we get

$$\begin{aligned}\int_0^t f(\tau)d\tau - \int_0^t u(\tau)d\tau &= (1*f)(t) - (1*R*f)(t)\\ &= (1*f)(t) - [1*(k + Aa_\alpha*R)*f](t)\\ &= (1*f)(t) - (1*k*f)(t) - A(1*a_\alpha*R*f)(t)\\ &= (1*f)(t) - [(1-k)*f](t) - A(1*a_\alpha*R*f)(t)\\ &= (k*f)(t) - A(1*a_\alpha*R*f)(t),\end{aligned}$$

so the proof is finished.

4. SOLUTIONS TO STOCHASTIC FRACTIONAL RELAXATION EQUATIONS

Let H denote a Hilbert space with a norm $|\cdot|_H$. By (16) the equation (1) can be rewritten as

$$X(t) - \int_0^t g_{1-\alpha}(t-\tau)A\,X(\tau)\,d\tau + \int_0^t X(\tau)\,d\tau = W(t), \quad 0<\alpha<1, \quad t>0. \tag{19}$$

The following types of solutions to (19) can be considered.

Definition 6. An H-valued predictable process $X(t)$, $t>0$, is said to be a *strong solution* to (19), if X has a version such that $P(X(t)\in D(A)) = 1$ for almost all $t\in(0,T]$; for any $t\in(0,T]$,

$$\int_0^t |g_{1-\alpha}(t-\tau)\,A\,X(\tau) - X(\tau)|_H d\tau < +\infty,\ P-a.s. \tag{20}$$

and for any $t\in(0,T]$ the equation (19) holds $P-a.s.$

Let A^* denote the adjoint of the operator A with its domain $D(A^*)$ equipped with the graph norm $|\cdot|_{A^*}$.

Definition 7. An H-valued predictable process $X(t)$, $t>0$, is said to be a *weak solution* to (19), if $\int_0^t |g_{1-\alpha}(t-\tau)\,X(\tau)|_H d\tau < +\infty$ and $\int_0^t |X(\tau)|_H\,d\tau < +\infty$, P-a.s. and if for all $\xi\in D(A^*)$ and all $t\in(0,T]$, the following equation holds

$$\langle X(t),\xi\rangle_H = \left\langle \int_0^t g_{1-\alpha}(t-\tau)\,X(\tau)d\tau, A^*\xi\right\rangle_H - \left\langle\int_0^t X(\tau)d\tau,\xi\right\rangle_H + \langle W(t),\xi\rangle_H, \quad P-a.s. \tag{21}$$

Let us note that a strong solution is also a weak solution to (19), but not vice versa. This is worth to emphasize that strong solutions exist very rarely, even in semigroup case.

If we follow the proof of Theorem 5 in the stochastic case we arrive at the below definition of solution to (19).

Definition 8. Assume that A is the generator of an α-regularized family $R(t)$, $t\geq 0$, on H. Let W takes values in $D(A)$ with probability 1 for almost all $t\in[0,T]$. An H-valued predictable process of the form

$$X(t) = \int_0^t R(t-\tau)\,dW(\tau), \quad t\in(0,T], \tag{22}$$

is said to be a *mild solution* to (19).

The following proposition shows that the mild solution is a weak solution to (19).

Proposition 9. Let the assumptions of Definition 8 be satisfied. Then the stochastic convolution

$$W^R(t) := \int_0^t R(t-\tau)\,dW(\tau), \qquad t\in(0,T]\ , \tag{23}$$

fulfills the equation (21).

Proof. Below we adapt the scheme used in Karczewska (2007). We denote

$$Z(t) := \left\langle \int_0^t g_{1-\alpha}(t-\tau)\,W^R(\tau)d\tau\ ,\ A^*\xi\right\rangle_H .$$

Let us note that the convolution $W^R(t)$, $t\in(0,T]$ has integrable trajectories on the finite interval $(0,T]$.

Then, from (23), Dirichlet's formula and stochastic Fubini's theorem we have

$$\begin{aligned}Z(t) &= \left\langle \int_0^t g_{1-\alpha}(t-\tau)\left[\int_0^\tau R(\tau-\sigma)\,dW(\sigma)\right]d\tau\ ,\ A^*\xi\right\rangle_H\\ &= \left\langle \int_0^t \left[\int_\sigma^t g_{1-\alpha}(t-\tau)R(\tau-\sigma)\,d\tau\right]dW(\sigma)\ ,\ A^*\xi\right\rangle_H\\ &= \left\langle A\int_0^t \left[\int_0^{t-\sigma} g_{1-\alpha}(t-\sigma-z)R(z)\,dz\right]dW(\sigma)\ ,\ \xi\right\rangle_H .\end{aligned}$$

Using (18), the definition of convolution and the equation (11) we obtain

$$\begin{aligned}Z(t) &= \langle \int_0^t A[\int_0^{t-\sigma}(a_\alpha(t-\sigma-z)\\ &\quad +(1*a_\alpha)(t-\sigma-z))R(z)\,dz]dW(\sigma)\ ,\ \xi\rangle_H\\ &= \langle\int_0^t A[(a_\alpha*R)(t-\sigma)+((1*a_\alpha)*R)(t-\sigma)]dW(\sigma),\xi\rangle_H\\ &= \left\langle\int_0^t [R(t-\sigma)-k(t-\sigma)]dW(\sigma)\ ,\ \xi\right\rangle_H\\ &\quad + \left\langle\int_0^t A[(1*a_\alpha*R)(t-\sigma)]dW(\sigma),\xi\right\rangle_H\end{aligned}$$

$$= \left\langle W^R(t), \xi \right\rangle_H - \left\langle \int_0^t k(t-\sigma)\, dW(\sigma), \xi \right\rangle_H$$
$$+ \left\langle \int_0^t A[(1*a_\alpha * R)(t-\sigma)] dW(\sigma), \xi \right\rangle_H .$$

Therefore, analogously like in the proof of Theorem 5 we have

$$Z(t) = \left\langle W^R(t), \xi \right\rangle_H + \left\langle \int_0^t W^R(\tau)\, d\tau, \xi \right\rangle_H$$
$$- \left\langle W(t), \xi \right\rangle_H , \quad P-a.s.$$

So, the mild solution fulfills (21).

Summing up our considerations we have that the convolution (23) is a weak solution to the equation (19).

Corollary 10. Assume that the operator A is bounded. Then the following formula holds

$$W^R(t) = \int_0^t g_{1-\alpha}(t-\tau) A W^R(\tau)\, d\tau \tag{24}$$
$$- \int_0^t W^R(\tau)\, d\tau + W(t), \quad 0<\alpha<1, \quad t>0.$$

Proof. In fact, if A is bounded, then $D(A)=H$ and the equation (21) holds for any $\xi \in H$. Hence, the conditions of definition of strong solution are satisfied.

Let $L(D(A))$ denote the set of all linear bounded operators acting from $D(A)$ into $D(A)$ equipped with the operator norm denoted by $||\cdot||_{L(D(A))}$.

We can formulate the following result on strong solution to (19).

Theorem 11. Assume that A is the generator of an α-regularized family $R(t)$, $t \geq 0$, on H and $\int_0^T ||AR(t)||^2_{L(D(A))} dt < +\infty$. Let W takes values in $D(A)$, P-a.s., for almost all $t \in [0,T]$. Then the mild solution (23) is a strong solution to (19).

Proof. From Proposition 9 and Corollary 10, if A were bounded operator then the convolution $W^R(t)$ would be strong solution to (19). Because the formula (24) holds for any bounded operator, then (24) holds for the Yosida approximation A_n of the operator A, too. Let us recall (see proof of Theorem 4) that $A_n := nAR(n,A) = A\,J_n$ with $J_n := nR(n,A),\ n>0$.

By $R_n(t),\ t>0$, we denote the α-regularized families generated by A_n. We define

$$W^{R_n}(t) := \int_0^t R_n(t-\tau)\, dW(\tau), \quad t>0. \tag{25}$$

So, by Corollary 10 we have

$$W^{R_n}(t) = \int_0^t g_{1-\alpha}(t-\tau) A_n W^{R_n}(\tau)\, d\tau \tag{26}$$
$$- \int_0^t W^{R_n}(\tau)\, d\tau + W(t), \quad P-a.s. ,$$

for $0<\alpha<1,\ t>0$.

The idea of the proof is the following. We prove convergences $W^{R_n}(t)$ to $W^R(t)$ and $A_n W^{R_n}(t)$ to $AW^R(t)$ in mean-square sense, as $n \to +\infty$. Such convergences imply convergences with probability one at least for some subsequence $\{n_k\},\ k\in\mathbb{N}$, of the sequence $\{n\},\ n\in\mathbb{N}$. In consequence, we obtain that W_R fulfills (19).

Next, properties of convolution provide integrability of the function $g_{1-\alpha}(t-\tau)\, AW^R(\tau)$, $0<\alpha<1$, with respect to τ, for any $t\in(0,T]$. This finishes the proof.

Theorem 12. Assume that X is a strong solution to (1). Then $Y(t) = \int_0^t e^{-(t-\tau)} X(\tau) d\tau$ is a strong solution to the equation

$$Y(t) = A \int_0^t (g_{1-\alpha}(t-\tau)+1) Y(\tau)\, d\tau \tag{27}$$
$$+ \int_0^t e^{-(t-\tau)} dW(\tau) , \quad 0<\alpha<1, \quad t>0.$$

Proof. By assumption, X satisfies

$$X(t) = \int_0^t [A\, D^\alpha X(\tau) - X(\tau)]\, d\tau + W(t), \tag{28}$$
$$0<\alpha<1, \quad t>0, \quad P-a.s.$$

Convolving the above equation with $k(t)=e^{-t}$, we obtain

$$(k*X)(t) - A(1*k*D^\alpha X)(t) + (1*k*X)(t) = \tag{29}$$
$$= (k*W)(t), \quad 0<\alpha<1, \quad t>0, \ \ P-a.s.$$

We note that the following identity holds

$$(k*D^\alpha X)(t) = D^\alpha k * X + k(0) t^{-\alpha} * X, \quad t>0, \tag{30}$$

where

$$(D^\alpha k)(t) = t^{-\alpha} E_{1,1-\alpha}(-t) - t^{-\alpha} = a_\alpha(t) - t^{-\alpha}, \ t>0. \tag{31}$$

Since $k(0)=1$, we obtain from (30) and (31)

$$(k*D^\alpha X)(t) = (a_\alpha * X)(t), \quad t\geq 0. \tag{32}$$

We note next that $1*a_\alpha = c*k$, where $c(t)=t^{-\alpha}$.

Hence

$$(1*k*D^\alpha X)(t) = (c*X)(t), \quad t\geq 0. \tag{33}$$

Replacing $Y=k*X$ and putting the formula (33) into the equation (29), we obtain (27).

5. CONDITIONS FOR WELL-POSEDNESS

In this section B denotes, as previously, a Banach space with the norm $|\cdot|_B$.

In order to have a concrete description of α-regularized families in some cases, we observe that given α-regularized family $R(t)$ with the generator A and $\mu \in \sigma_p(A)$ fulfills

$$R(t)x = r(t,\mu)x, \tag{34}$$

where $x \neq 0$ is an eigenvector corresponding to the eigenvalue μ, and $r(t,\mu)$ is the solution to the scalar equation

$$r(t,\mu) = k(t) + \mu \int_0^t a_\alpha(t-s) r(s,\mu) ds, \quad t \geq 0 \tag{35}$$

(recall that $k(t) = e^{-t}$ and $a_\alpha(t) = t^{-\alpha} E_{1,1-\alpha}(-t)$).

In fact, we have

$$A a_\alpha * r(t,\mu)x = a_\alpha * r(t,\mu)Ax = \mu a_\alpha * r(t,\mu)$$

$$= r(t,\mu)x - k(t)x,$$

and the assertion follows from the uniqueness of the Laplace transform.

Let us note that the relation (34) with the equation (35) are analogous to the well-known relations for Volterra equations of scalar type (see, e.g., (Prüss, 1993, p. 36)). Following the same idea of the proof as in (Prüss, 1993, Proposition 1.5) we obtain the below result.

Proposition 13. Suppose $R(t)$ is an α-regularized resolvent family with the generator A. Then

$$|r(t,\mu)| \leq ||R(t)|| \text{ for all } \mu \in \sigma(A) \text{ and } t \geq 0.$$

Proof. Let $\mu \in \sigma(A) \setminus \sigma_r(A)$. Then there exist sequences $(x_n) \in D(A)$ and $(y_n) \subset X$ such that $|x_n|_B = 1$ and $(\mu - A)x_n = y_n$.

Define $u_n(t) := R(t)x_n - r(t,\mu)x_n$. Then

$$\begin{aligned} & u_n(t) - \mu a_\alpha * u_n(t) \qquad (36) \\ &= R(t)x_n - r(t,\mu)x_n - \mu a_\alpha * [R(t)x_n - r(t,\mu)x_n] \\ &= R(t)x_n - \mu a_\alpha * R(t)x_n + \mu a_\alpha * r(t,\mu)x_n - r(t,\mu)x_n \\ &= R(t)x_n - \mu a_\alpha * R(t)x_n - k(t)x_n \\ &= A a_\alpha * R(t)x_n - \mu a_\alpha * R(t)x_n \\ &= a_\alpha * R(t)(Ax_n - \mu x_n) = -a_\alpha * R(t)y_n. \end{aligned}$$

Then $u_n(t) = l(\cdot,\mu) * R(t)y_n$ where $l(t,\mu)$ is the solution to the scalar equation

$$l(t,\mu) + a_\alpha(t) = \mu \int_0^t a_\alpha(t-s) l(s,\mu) ds.$$

It follows that $|u_n(t)|_B \to 0$ as $n \to \infty$ uniformly on compact intervals. Therefore

$$|r(t,\mu)| \leq |u_n(t)|_B + |R(t)x_n|_B \leq |u_n(t)|_B + ||R(t)||,$$

implies $|r(t,\mu)| \leq ||R(t)||$ for all $\mu \in \sigma(A) \setminus \sigma_r(A)$ and $t \geq 0$.

Now, let $\mu \in \sigma_r(A)$. Then there exists $x^* \in D(A^*)$ with $|x^*|_{A^*} = 1$ such that $A^* x^* = \mu x^*$. Hence

$$< R(t)x, x^* > = < k(t)x, x^* > + \mu a_\alpha * < R(t)x, x^* > .$$

Therefore $< R(t)x, x^* > = r(t,\mu) < x, x^* >$ by uniqueness of the Laplace transform. We conclude that

$$\begin{aligned} |r(t,\mu)| &= \sup_{|x|_B \leq 1} |r(t,\mu) < x, x^* > | \\ &= \sup_{|x|_B \leq 1} | < R(t)x, x^* > | \leq ||R(t)||. \end{aligned}$$

As a consequence of the above Proposition 13 and Theorem 4 we obtain a more precise estimate for $r(t,\mu)$.

Corollary 14. Assume that A is the generator of a bounded analytic semigroup. Then there exists $M > 0$ such that

$$|r(t,\mu)| \leq M e^t \quad \text{for all} \quad \mu \in \sigma(A) \quad \text{and} \quad t \geq 0.$$

If A is a normal operator defined on a Hilbert space H, the functional calculus can be used to explicitly define an α-regularized family $R(t)$ (compare (Prüss, 1993, Theorem 1.1)).

Theorem 15. Let A be a normal operator in a Hilbert space H. Assume that there is a locally bounded lower semicontinuous function ψ such that

$$|r(t,\mu)| \leq \psi(t) \text{ for all } \quad \mu \in \sigma(A) \quad \text{and} \quad t \geq 0.$$

Then A is the generator of an α-regularized family $R(t)$ given by

$$R(t) = \int_{\sigma(A)} r(t,\mu) dE(\mu),$$

where E denotes the spectral measure of A.

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