Solutions to stochastic fractional oscillation equations

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Abstract

We formulate a fractional stochastic oscillation equation as a generalization of Bagley's fractional differential equation. We do this in analogous way as in the case of Basset's equation which gives rise to fractional stochastic relaxation equations. We analyze solutions under some conditions of spatial regularity of the operators considered.

Keywords: Fractional differential equations, stochastic equations, regularized resolvents.
2010 *Mathematics Subject Classification.* 60H15, 34A08, 47D06.

1. Introduction

In this paper we consider the stochastic fractional differential equation

$$u(t) + \int_0^t (t-\tau)[AD_\tau^\alpha u(\tau) + u(\tau)]d\tau = W(t),\ t \geq 0,\ 1 < \alpha \leq 2, \tag{1.1}$$

where A is the generator of a bounded analytic semigroup defined on a Hilbert space H, D_τ^α denotes the Caputo fractional derivative of order α and $W(t)$ is an H-valued Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, P)$.

Equation (1.1) is a stochastic counterpart of the deterministic fractional equation

$$u''(t) + AD_t^\alpha u(t) + u(t) = f(t),\ t \geq 0,\ 1 < \alpha \leq 2, \tag{1.2}$$

where f is a vector-valued function. Eq. (1.2) models an oscillation process with fractional damping term. In the scalar case, that is $A = a,\ a \in \mathbb{R}$, it was formerly treated by Caputo [4], who provided a preliminary analysis by Laplace transform methods. The special case $\alpha = 3/2$ has been discussed by Bagley [1]. Beyer and Kempfle [3] discussed equation (1.2) for $t \in \mathbb{R}$ to investigate the uniqueness and causality of the solution, i.e. the present state of the process is determined by its history and the present force f, but does not depend on the future. A careful analysis of the solutions to (1.2) in the scalar case has been realized by Gorenflo and Mainardi [6].

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[1] The authors are supported by Laboratorio de Analisis Estocástico, Proyecto Anillo ACT-13.

In the present paper we apply an operator theoretical method based on vector-valued Laplace transform to solve the stochastic fractional differential equation (1.1) and get some insight into their fundamental solutions. The obtained results are complementary with those in our previous paper [8], where we studied stochastic relaxation equations.

We deal with the Caputo fractional derivative, defined as

$$D^\alpha u(t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u''(\tau)}{(t-\tau)^{\alpha-1}} d\tau, \qquad 1 < \alpha < 2\,, \tag{1.3}$$

where $u \in C^2(\mathbb{R}_+, H)$ and $D^2 u(t) := u''(t)$. The following rule holds for the Laplace transform (see [2, p.12])

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \lambda^{\alpha-1} u(0) - \lambda^{\alpha-2} u'(0), \quad Re\, \lambda > 0, \quad 1 < \alpha \le 2.$$

The generalized Mittag-Leffler function (see [5, Vol. 3]), is defined as follows

$$E_{\alpha,\beta}(z) \;\; := \;\; \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta}\, e^\mu}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counter-clockwise. This is an entire function which is a generalization of the exponential function: $E_{1,1}(z) = e^z$ and the cosine function: $E_{2,1}(z^2) = \cosh(z)$; $E_{2,1}(-z^2) = \cos(z)$. We shall use the below property of the Mittag-Leffler function

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha) dt \;\; = \;\; \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \tag{1.4}$$

where $Re\ \lambda > \omega^{1/\alpha}, \omega > 0$. We recall (see, e.g. [13]) that a Laplace transformable function $a(t)$, $t \ge 0$, is said to be of *positive type* if $Re\ \hat{a}(\lambda) \ge 0$ for all $Re\ \lambda > 0$, and 1-*regular* if there is a constant $c > 0$ such that

$$|\lambda \hat{a}'(\lambda)| \le c |\hat{a}(\lambda)| \quad \text{for all} \quad Re\ \lambda > 0. \tag{1.5}$$

Above $\hat{a}(\lambda)$ denotes the Laplace transform of the function $a(t)$.

2. α-regularized families

Let B be a complex Banach space with the norm $|\cdot|_B$ and let $\mathcal{B}(B)$ denote the space of linear bounded operators on B. In the whole paper we denote by $||\cdot||$ the operator norm. First, we shall study existence and uniqueness of solution to the deterministic analog of equation (1.1) which consists in equation (1.2) two times integrated, that is

$$u(t) + \int_0^t (t-\tau)[AD^\alpha u(\tau) + u(\tau)] d\tau = \int_0^t (t-\tau) f(\tau) d\tau, \quad 1 < \alpha \le 2. \tag{2.1}$$

Because $u(0) = 0$ and $u'(0) = 0$ in case $1 < \alpha \le 2$, by definition of the Caputo derivative we have

$$\int_0^t (t-\tau) D_t^\alpha u(\tau) d\tau = \int_0^t g_{2-\alpha}(t-\tau) u(\tau) d\tau, \quad t > 0. \tag{2.2}$$

Then the problem (2.1), equivalently, reads

$$u(t) + \int_0^t g_{2-\alpha}(t-\tau) A u(\tau) d\tau + \int_0^t (t-\tau) u(\tau) d\tau = \int_0^t (t-\tau) f(\tau) d\tau, \quad t > 0, \tag{2.3}$$

where $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta \geq 0$. Hence, by a solution of (2.1) we can understand a function $u \in C(\mathbb{R}_+; D(A))$ that satisfies (2.3).

Definition 2.1. *Let A be a closed linear operator with domain $D(A)$ defined on Banach space B. We call A the generator of an α-regularized family if there exists $\omega \geq 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(B)$ such that $\{\frac{1+\lambda^2}{\lambda^\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x \quad := \quad \lambda^{-\alpha}(\frac{1+\lambda^2}{\lambda^\alpha} - A)^{-1}x = \int_0^\infty e^{-\lambda t} R(t)xdt, \tag{2.4}$$

where $Re\lambda > \omega,\ x \in B$. $R(t), t \geq 0$, is called the α-regularized family generated by A.

By taking Laplace transform to (2.1) we see that A generates an α-regularized family if and only if (2.1) has a unique solution. Hence the equation (2.1) is well-posed.

Let $k \in C(\mathbb{R}_+)$ be an scalar kernel and let $a \in L^1_{loc}(\mathbb{R}_+)$. Assume that A is a closed linear operator. Following [10], a strongly continuous operator family $R(t)$ is called an (a, k)-*regularized* resolvent if and only if the following holds:

(i) $R(t)Ax = AR(t)x$ for all $x \in D(A), t \geq 0$ and $R(0) = k(0)I$.

(ii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)xds, x \in D(A), t \geq 0$.

The notion of (a, k)-regularized resolvent unify several notions of strongly continuous operators families. For a recent review as well as extensions of the theory, we refer to the monograph [9]. Observe that if $R(t)$ is Laplace transformable, as well as $k(t)$ and $a(t)$, (e.g. when $R(t)$ is exponentially bounded), then we have

$$\hat{R}(\lambda) = \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} \tag{2.5}$$

for all λ with $Re(\lambda)$ sufficiently large. Let us note that because of the uniqueness of the Laplace transform, α-regularized family corresponds to an (a, k)-regularized resolvent, with $k(t) = \sin(t)$ and $a_\alpha(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$. In fact, we have $\hat{a}_\alpha(\lambda) = \frac{\lambda^\alpha}{\lambda^2+1}$ and $\hat{k}(\lambda) = \frac{1}{1+\lambda^2}$ so, we easily see that (2.5) coincides with (2.4).

We can derive relations between α-regularized resolvent and its generator. The following result is a direct consequence of [10, Proposition 3.1 and Lemma 2.2].

Proposition 2.2. *Let $R(t), t \geq 0$, be an α-regularized family on B with the generator A. Then the following conditions hold:*

1. *$R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A),\ t \geq 0$.*
2. *For $x \in D(A)$ and $t \geq 0$,*

$$R(t)x = \sin(t)x + \int_0^t a_\alpha(t-s)AR(s)xds\ . \tag{2.6}$$

3. *For $x \in B$ and $t \geq 0$, $\int_0^t a_\alpha(t-s)AR(s)xds \in D(A)$ and*

$$R(t)x = \sin(t)x + A\int_0^t a_\alpha(t-s)R(s)xds\ . \tag{2.7}$$

The following lemma will be used in the proof of Theorem 3.1.

Lemma 2.3. *The Mittag-Leffler function $a_\alpha(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2), t > 0$, is 1-regular and of positive type for all $1 < \alpha \leq 2$.*

3. Convergence of α-regularized family

The below result gives sufficient condition on A to provide well-posedness of (2.1).

Theorem 3.1. *Suppose $1 < \alpha \leq 2$ and let A be the generator of a bounded analytic semigroup on B. Then:*

1. *A is the generator of an α-regularized resolvent family $(R(t))_{t\geq 0}$ on B.*
2. *There exist bounded operators A_n and α-regularized resolvent families $R_n(t)$ generated by A_n satisfying $||R_n(t)|| \leq Me^t$ for all $t > 0,\ n \in \mathbb{N}$, such that*

$$R_n(t)x \to R(t)x \quad as \quad n \to +\infty \tag{3.1}$$

for all $x \in B,\ t \geq 0$. Additionally, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof. Because $a_\alpha(t),\ t > 0$, is of positive type by Lemma 2.3, we obtain by [13, Corollary 3.1] that $\frac{1}{\hat{a}_\alpha(\lambda)} \in \rho(A)$ for all $Re\ \lambda > 0$. Moreover, there is a constant $M \geq 1$ such that $H(\lambda) := (I - \hat{a}_\alpha(\lambda)A)^{-1}/\lambda$ satisfies $||H(\lambda)|| \leq \frac{M}{|\lambda|}$ for all $Re\ \lambda > 0$. From the above, and since $a_\alpha(t)$ is 1-regular by Lemma 2.3, we get by [13, Theorem 3.1] that A generates a resolvent family $(S(t))_{t\geq 0}$ such that $\hat{S}(\lambda) = H(\lambda),\ Re\ \lambda > 0$. Additionally, there is a constant $C \geq 1$ such that $||S(t)|| \leq C,\ t > 0$. Let $x \in B$ and define

$$R(t)x := \int_0^t \cos(t-\tau)S(\tau)x d\tau, \quad t > 0.$$

We claim that $R(t)$ is an α-regularized family generated by A. In fact, clearly $R(t)$ is Laplace transformable, and we have

$$\hat{R}(\lambda) = \frac{1}{1+\lambda^2}(I - \hat{a}_\alpha(\lambda)A)^{-1} = \frac{1}{1+\lambda^2}\frac{1}{\hat{a}_\alpha(\lambda)}(\frac{1}{\hat{a}_\alpha(\lambda)} - A)^{-1} = \lambda^{-\alpha}(\frac{1+\lambda^2}{\lambda^\alpha} - A)^{-1}$$

for all $\lambda \in \mathbb{C}$ with $Re(\lambda)$ sufficiently large. Hence, the claim follows from Definition 2.1.

Because A generates a bounded analytic semigroup, the resolvent set $\rho(A)$ of A contains the ray $(0, \infty)$ and $||R(\lambda, A)^k|| \leq \frac{M}{|\lambda|^k}$ for $\lambda > 0, \quad k \in \mathbb{N}$. Define $A_n := nAR(n, A) = n^2R(n, A) - nI,\ n > 0,$ the *Yosida approximation* of A. Then, for $n > 0, \quad ||e^{A_nt}|| \leq M$. Because each A_n is bounded and hence generates bounded analytic semigroup e^{A_nt}, it follows that there exist resolvent families $(S_n(t))_{t\geq 0}$ (or $(a, 1)$-regularized resolvent families) such that $||S_n(t)|| \leq M$, see [13, Corollary 3.1 and Theorem 3.1]. So, as before, for each $n \in \mathbb{N}$ and $x \in B$ the formula

$$R_n(t)x = \int_0^t \cos(t-\tau)S_n(\tau)x d\tau, \quad t > 0$$

defines α-regularized resolvent families generated by A_n such that

$$||R_n(t)|| \leq M(1+t), \quad t > 0. \tag{3.2}$$

Now, we recall the fact that $R(\mu, A_n)x \to R(\mu, A)x$ as $n \to \infty$ for all μ sufficiently large (see e.g. [12, Lemma 7.3]). In consequence $\hat{k}(\lambda)(I - \hat{a}_\alpha(\lambda)A_n)^{-1}x \to \hat{k}(\lambda)(I - \hat{a}_\alpha(\lambda)A)^{-1}x$ as $n \to \infty$ for all λ sufficiently large. Then the uniform stability condition (3.2) and [11, Theorem 2.2] implies that $R_n(t)x \to R(t)x$ as $n \to +\infty$ for all $x \in B$, uniformly for t on every compact subset of $\mathbb{R}_+$.

Now, we are able to formulate the following result.

Theorem 3.2. *Assume $1 < \alpha \leq 2$. Let A be the generator of a bounded analytic semigroup on B and let $f : \mathbb{R}_+ \to B$ be locally integrable on $\mathbb{R}_+$. Then there exists an α-regularized family $R(t)$ such that*

$$u(t) := \int_0^t R(t-\tau)f(\tau)d\tau \tag{3.3}$$

is a solution of the equation (2.1).

Proof. From the assumptions $f(t) \in D(A)$ and A is closed, so $u(t) \in D(A)$. Let us note the following identities

$$(t * k)(t) = t - k(t), \quad t \geq 0, \quad \text{and} \tag{3.4}$$

$$a_\alpha(t) + (t * a_\alpha)(t) = g_{2-\alpha}(t), \quad t > 0. \tag{3.5}$$

Moreover, by Theorem 3.1 there exists an α-regularized family $R(t)$. So, by the equation (2.7) and by using (3.3) and (3.5), we have

$$\begin{aligned} u(t) \;&-\; A\int_0^t g_{2-\alpha}(t-\tau)u(\tau)d\tau = (R*f)(t) - A(g_{2-\alpha} * R * f)(t) \\ &= [(k + Aa_\alpha * R)*f](t) - A(g_{2-\alpha}*R*f)(t) = (k*f)(t) + [A(a_\alpha - g_{2-\alpha})*R*f](t) \\ &= (k*f)(t) - A[(t*a_\alpha)*R*f](t)\ . \end{aligned}$$

Using the resolvent equation (2.7) and then (3.3) and (3.4), we get

$$\begin{aligned} &\int_0^t (t-\tau)f(\tau)d\tau - \int_0^t (t-\tau)u(\tau)d\tau = (t*f)(t) - (t*R*f)(t) \\ = \;& (t*f)(t) - [t*(k + Aa_\alpha * R)*f](t) = (t*f)(t) - (t*k*f)(t) - A(t*a_\alpha*R*f)(t) \\ = \;& (t*f)(t) - [(t-k)*f](t) - A(t*a_\alpha*R*f)(t) = (k*f)(t) - A(t*a_\alpha*R*f)(t), \end{aligned}$$

so the proof is done.

4. Stochastic fractional oscillation equations

Here H denotes a Hilbert space with a norm $|\cdot|_H$. By (2.2), the equation (1.2) reads

$$X(t) + \int_0^t g_{2-\alpha}(t-\tau)A\,X(\tau)\,d\tau + \int_0^t (t-\tau)X(\tau)\,d\tau = W(t),\ 1 < \alpha \leq 2,\ t > 0. \tag{4.1}$$

The following types of solutions to the equation (4.1) can be introduced.

Definition 4.1. *An H-valued predictable process $X(t)$, $t > 0$, is said to be a strong solution to (4.1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in (0,T]$; for any $t \in (0,T]$, $\int_0^t |g_{2-\alpha}(t-\tau)\,A\,X(\tau) - (t-\tau)X(\tau)|_H d\tau < +\infty$, $P-a.s.$, $1 < \alpha \leq 2$, and for any $t \in (0,T]$ the equation (4.1) holds $P-a.s.$*

Let A^* denote the adjoint of A with domain $D(A^*)$ equipped with the graph norm $|\cdot|_{A^*}$.

Definition 4.2. *An H-valued predictable process $X(t)$, $t > 0$, is said to be a weak solution to (4.1), if $\int_0^t |g_{2-\alpha}(t-\tau)\, X(\tau)|_H d\tau < +\infty$ and $\int_0^t |X(\tau)|_H\, d\tau < +\infty$, P-a.s. and if for all $\xi \in D(A^*)$ and all $t \in (0,T]$, the following equation holds P.a.s.*

$$\langle X(t), \xi\rangle_H = \left\langle \int_0^t g_{2-\alpha}(t-\tau)\, X(\tau) d\tau, A^*\xi \right\rangle_H - \left\langle \int_0^t (t-\tau) X(\tau) d\tau, \xi \right\rangle_H + \langle W(t), \xi\rangle_H .$$

If we follow the proof of Theorem 3.2 in the stochastic case we arrive at the below definition.

Definition 4.3. *Assume that A is the generator of an α-regularized family $R(t)$, $t \geq 0$, on H. Let W takes values in $D(A)$ with probability 1 for almost all $t \in [0,T]$. An H-valued predictable process of the form $X(t) = \int_0^t R(t-\tau)\, dW(\tau)$, $t \in (0,T]$, is said to be a mild solution to (4.1).*

Proposition 4.4. *Let assumptions of Definition 4.3 hold. Then the convolution*

$$W^R(t) := \int_0^t R(t-\tau)\, dW(\tau), \quad t \in (0,T]\,, \tag{4.2}$$

is a weak solution to the equation (4.1).

Proof. In the proof we follow the scheme used in [7] and [8], so we omit it.

Corollary 4.5. *Assume that the operator A is bounded. Then the following formula holds*

$$W^R(t) = \int_0^t g_{2-\alpha}(t-\tau) A W^R(\tau)\, d\tau - \int_0^t (t-\tau) W^R(\tau)\, d\tau + W(t), \; 1 < \alpha \leq 2. \tag{4.3}$$

Hence, the convolution $W^R(t)$ given by (4.3) is the strong solution to the equation (4.1). By $L(D(A))$ denote the set of all linear bounded operators acting from $D(A)$ into $D(A)$.

Theorem 4.6. *Assume that A is the generator of an α-regularized family $R(t)$, $t \geq 0$, on H and*

$$\int_0^T ||AR(t)||^2_{L(D(A))} dt < +\infty.$$

Let W takes values in $D(A)$, P-a.s., for almost all $t \in [0,T]$. Then the mild solution (4.2) is a strong solution to (4.1).

Proof. Because the idea of the proof is similar to that used in [8], we omit the proof.

Our last result, shows that if we regularize by sine-convolution the stochastic term, the resulting equation still have strong solution.

Theorem 4.7. *Assume that X is a strong solution to (1.2). Then $Y(t) = \displaystyle\int_0^t \sin(t-\tau) X(\tau) d\tau$ is a strong solution to the equation*

$$Y(t) + A\int_0^t (g_{2-\alpha}(t-\tau) + (t-\tau)) Y(\tau)\, d\tau = \int_0^t \sin(t-\tau) dW(\tau)\,, \quad 1 < \alpha \leq 2, \quad t > 0.$$

Proof. By assumption, X satisfies

$$X(t) + \int_0^t [A\, D^\alpha X(\tau) + (t-\tau)X(\tau)]\, d\tau = W(t), \quad 1<\alpha\leq 2, \quad t>0, \quad P-a.s.$$

Convolving the above equation with $k(t)=\sin(t)$, we obtain

$$(k*X)(t) + A(t*k*D^\alpha X)(t) + (t*k*X)(t) = (k*W)(t), \quad 1<\alpha\leq 2, \quad t>0, \quad P-a.s.$$

We note that the following identity holds

$$(k*D^\alpha X)(t) = D^\alpha k * X + \frac{k'(0)}{\Gamma(2-\alpha)} t^{1-\alpha} * X, \quad t>0, \tag{4.4}$$

where

$$(D^\alpha k)(t) = t^{1-\alpha} E_{2,2-\alpha}(-t) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} = a_\alpha(t) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \ t>0. \tag{4.5}$$

Since $k'(0)=1$, we obtain from (4.4) and (4.5): $(k*D^\alpha X)(t) = (a_\alpha * X)(t), \quad t\geq 0$. We note next that $t*a_\alpha = c*k$, where $c(t)=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$. Hence

$$(t*k*D^\alpha X)(t) = (c*X)(t), \quad t\geq 0. \tag{4.6}$$

Replacing $Y=k*X$ and putting the formula (4.6) into the equation (4.4), we obtain (4.4).

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