LATTICE DYNAMICAL SYSTEMS ASSOCIATED TO A FRACTIONAL LAPLACIAN

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Abstract. We derive optimal well-posedness results and explicit representations of solutions in terms of special functions for the linearized version of the equation

$$(*)\begin{cases}\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + f(n-ct, u(n,t)), & t>0, \quad 0<\alpha,\beta<1, \quad n\in\mathbb{Z},\\ u(n,0)=\varphi(n), & n\in\mathbb{Z},\end{cases}$$

for some constant $c\geq 0$, where $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative in time of order β and $(-\Delta_d)^\alpha$ denotes the discrete fractional Laplacian of order $\alpha\in(0,1]$. We also prove a comparison principle. A special case of this equation is the discrete Fisher-KPP equation with and without delay. We show that if $0\leq\varphi(n)\leq\gamma$ for every $n\in\mathbb{Z}$, and the function $f(x,\cdot)$ is concave on $[0,\gamma]$, $f(x,s)$ is non-negative for every $x\in\mathbb{R}$, $s\in[0,\gamma]$, and satisfies

$$f(x,0)=0 \text{ and } f(x,\gamma)\leq 0, \quad \forall\, x\in\mathbb{R},$$

for some $\gamma>0$, then the system $(*)$ has a nonnegative unique solution u satisfying $0\leq u(n,t)\leq\gamma$ for every $n\in\mathbb{Z}$ and $t\geq 0$. Our results includes cubic nonlinearities and incorporates new results for the discrete Newell-Whitehead-Segel equation. We use Lévy stable processes as well as Mittag-Leffler, Wright and modified Bessel functions to describe the solutions of the linear lattice model, providing a useful framework for further study. For the nonlinear model, we use a generalization of the upper-lower solution method for reaction-diffusion equations in order to prove existence and uniqueness of solutions.

1. Introduction

The nonlinear partial differential equation of diffusion type:

$$u_t(x,t) = D\Delta u(x,t) + ru(x,t)(1-u(x,t)), \qquad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

where $D>0$ is the diffusion coefficient and $r\in\mathbb{R}$, is known as the Fisher-KPP equation and was originally studied by R. Fisher in the paper [20] in connection with population dynamics. Let $0<\alpha<1$ and consider the fractional Laplacian $(-\Delta)^\alpha$. Then the above equation is the limiting case of the fractional diffusion equation:

$$u_t(x,t) = -D(-\Delta)^\alpha u(x,t) + ru(x,t) - ru^2(x,t), \qquad t>0,\ x\in\mathbb{R}. \tag{1.2}$$

The latter equation has been investigated by Stan and Vázquez [40]. They analyzed the propagation properties of nonnegative and bounded solutions of this problem in the spirit of the Fisher-KPP theory. These authors do in fact study a more general model, namely the porous media equation

$$u_t(x,t) = -(-\Delta)^\alpha u^m(x,t) + f(u(x,t)), \qquad t>0,\ x\in\mathbb{R}, \tag{1.3}$$

where m is a positive parameter. The Fisher equation (1.2) corresponds to the case $m=1$ in (1.3).

Equations of the above type have met with increasing interest due to their relevance in such areas as spatial ecology, physics, chemistry, biology and the emerging area of global warming (see e.g. [2, 25]). Additional areas of application of fractional diffusion models appear in the recent book [36] by Meerschaert

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and Sikorskii where the general theory is presented from a probabilistic standpoint. The following discrete version of Fisher's equation

$$u_t(n,t) = \Delta_d u(n,t) + ru(n,t) - u^2(n,t), \quad t>0,\ n\in\mathbb{Z},$$

where $(\Delta_d)v(n,t) = v(n+1,t) - 2v(n,t) + v(n-1,t)$ denotes the one step central difference (in the space variable), was first studied by Zinner, Harris and Hudson [45] in the context of traveling waves. The version with a retarded term, i.e.

$$u_t(n,t) = \Delta_d u(n,t) + r(n-ct)u(n,t) - u^2(n,t), \quad t>0,\ n\in\mathbb{Z},$$

was recently studied by Hu and Li [25] in terms of long term behavior of solutions.

On the other hand, from numerical analysis, it is well known that semi-discretizations is a useful way of looking at the general behavior of evolutionary partial differential equations. For instance, in the numerical optimization of low eigenvalues of the Laplacian [3] or controllability [4]. One step semi-discrete one-dimensional equations naturally appear in some fields of Physics [5, 6] and in fracture mechanics and biology [11], [35, Section 5]. For instance, the model

$$u_t(n,t) = \Delta_d u(n,t) - A\sin u(n,t) + F, \quad n\in\mathbb{Z}, \quad t>0,$$

where A and F are positive parameters, has been investigated in fracture mechanics and is related to the Frenkel-Kontorova model [19] for dislocations, while in neurobiology it is a qualitative model for propagation of an impulse along a myelinated nerve axon [7]. Cardiac cells provide another suitable biological context [27] of application. Another interesting example is provided by the discrete Nagumo equation

$$u_t(n,t) = \rho\,\Delta_d u(n,t) + u(n,t)(u(n,t)-a)(1-u(n,t)), \quad n\in\mathbb{Z},\ t>0, \tag{1.4}$$

where $\rho>0$ and $0<a<\frac{1}{2}$. The discrete Nagumo equation has become a challenging object of study because it exhibits differences with the continuous model that has led to new insights regarding e.g. the formation of spiral waves [26]. Studies about the existence of traveling wave solutions for sufficiently large ρ was proved by Zinner [44]. However, in the fractional case, the discrete Nagumo equation has only been analyzed very recently from a numerical point of view [31, 10].

In view of this context, an obvious question that arises naturally is the following: Can one establish optimal well posedness of the general model

$$u_t(n,t) = -(-\Delta_d)^\alpha u(n,t) + f(n-ct, u(n,t)), \qquad t>0,\ n\in\mathbb{Z}, \tag{1.5}$$

where $0<\alpha\le 1$? Here $(-\Delta_d)^\alpha$ represents the discrete fractional Laplacian of order α defined by

$$(-\Delta_d)^\alpha f(n) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{t\Delta_d}f(n) - f(n))\frac{dt}{t^{1+\alpha}}, \tag{1.6}$$

for each $0<\alpha<1$, where

$$e^{t\Delta_d}f(n) := \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m), \quad f\in l^2(\mathbb{Z}) \tag{1.7}$$

represents the heat semigroup in the discrete setting. Here I_n denotes the modified Bessel function. We observe that the discrete fractional Laplacian was introduced in [15, Section 3] in connection with harmonic analysis. The link with the above representation is proved in [16, Theorem 2]. A comparison between the fractional discrete Laplacian and the *discretized* continuous fractional Laplacian is carried out in [17]. The discrete fractional Laplacian is a nonlocal operator on $\mathbb{Z}$.

Our aim in this paper is to provide a positive answer to the question raised above. Our results concerning equation (1.5) will be obtained viewing it as a particular case of a more general study on conditions for a nonlinear term f in order to have optimal well-posedness of solutions for semi-discretizations of two-parameters equation

$$\mathbb{D}_t^\beta u(x,t) = -(-\Delta)^\alpha u(x,t) + f(x,u(x,t)), \quad 0<\alpha\le 1, \qquad t>0,\ x\in\mathbb{R}, \tag{1.8}$$

where $0 < \beta \leq 1$ and $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative in time of order $\beta > 0$.

We note that the formulation (1.8) includes counterparts of well-known fractional partial differential equations. For example, the choice $\beta = 0$ corresponds to the fractional Keller-Segel model, which was studied for the first time by Escudero in the reference [18], and then in the paper [30] to include memory effects in time (that is, $\beta > 0$). See also the article [8] where the authors studied the blow up of solutions. The discrete model (1.8) for $\beta \in \{1, 2\}$ was recently studied in the paper [32].

We provide new insights on the analytical structure of the solutions, highlighting the role of two special distributions: The Lévy probability density function and the Wright function.

Our first contribution in this paper is the description of explicit solutions for the linear nonhomogeneous equation

$$\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), \quad 0 < \alpha \leq 1, \; 0 < \beta \leq 1, \quad t \geq 0, \; n \in \mathbb{Z}, \tag{1.9}$$

We then consider the associated nonlinear problem

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + f(n-ct, u(n,t)), & 0 < \alpha \leq 1, \; 0 < \beta \leq 1, \quad t \geq 0, \quad n \in \mathbb{Z}, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.10}$$

for some constant $c \geq 0$. We show that if $0 \leq \varphi(n) \leq \gamma$ for every $n \in \mathbb{Z}$, and the function $f(x,\cdot)$ is concave on $[0,\gamma]$, $f(x,s)$ is non-negative for every $x \in \mathbb{R}$, $s \in [0,\gamma]$, and satisfies

$$f(x,0) = 0 \text{ and } f(x,\gamma) \leq 0, \quad \forall\, x \in \mathbb{R},$$

for some $\gamma > 0$, then the system (1.10) has a nonnegative unique solution u satisfying $0 \leq u(n,t) \leq \gamma$ for every $n \in \mathbb{Z}$ and $t \geq 0$.

The paper is organized as follows. In Section 2, we present some preliminary material on the special functions that will be used for the representation of solutions. These include the densities of the Lévy stable processes $(f_{t,\alpha}(\cdot))$ (with $0 < \alpha < 1$, $t > 0$), the Mittag-Leffler functions $(E_{\alpha,\beta})$ which are fundamental objects in fractional calculus and the Wright functions $(\Phi_\gamma(\cdot))$ (also known as M- Wright functions). We also need some identities involving Bessel and modified Bessel functions $I_n(\cdot), n \in \mathbb{Z}$. The special functions described in Section 2 are used in Section 3 to derive explicit representations for the solutions of the nonhomogeneous problem (1.9). In particular, for $0 < \alpha \leq 1$ and $\beta = 1$, we show that (1.9) admits the solution

$$\begin{aligned} u(n,t) &= e^{-(-\Delta_d)^\alpha t} u(n,0) + \int_0^t e^{-(-\Delta_d)^\alpha (t-s)} g(n,s)ds \\ &= \sum_{m\in\mathbb{Z}} \left[\int_0^\infty I_{n-m}(2\lambda) e^{-2\lambda} f_{t,\alpha}(\lambda) d\lambda \right] \varphi(m) \\ &\quad + \sum_{m\in\mathbb{Z}} \int_0^\infty I_{n-m}(2\lambda) e^{-2\lambda} \left(\int_0^t f_{t-s,\alpha}(\lambda) g(m,s) ds \right) d\lambda \end{aligned}$$

where $T_t^\alpha := e^{-(-\Delta_d)^\alpha t}$ is a uniformly continuous Markovovian one-parameter semigroup in the space $\ell^2(\mathbb{Z};\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. This complements a recent result in [16] where it was proved that this semigroup is only strongly continuous, and generalizes [25, Corollary 2.1] (see also [25, Remark 2.6]) where the representation in terms of the modified Bessel functions was proved in case $\alpha = 1$. We note that positivity is important in this context in view of the application areas. The optimal well-posedness of the nonlinear problem is studied in Section 4, where we prove our main result on optimal well-posedness of the nonlinear model (1.10). This generalizes a recent result due to Hu and Li [25, Theorem 3.1]. We finish this work with the analysis of optimal wellposedness for the fractional discrete Fisher and Newell-Whitehead-Segel equations. This includes equations with delay $r(x)$. More precisely, we prove that if $r : \mathbb{R} \to \mathbb{R}$ is continuous, non-decreasing, bounded and piecewise continuously differentiable satisfying $0 < r(\infty) < \infty$ and $0 \leq \varphi(n) \leq r(\infty)$ then the system

$$\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + r(n-ct)u(n,t) - u^2(n,t), \quad 0 < \alpha \leq 1, \; 0 < \beta \leq 1, \quad t \geq 0, \; n \in \mathbb{Z},$$

with initial condition $u(n,0)=\varphi(n)$ has a unique continuous solution $0 \le u(n,t) \le r(\infty)$. This result is new even in case $0<\alpha<1$ and $\beta=1$. A similar results holds for the fractional discrete Newell- Whitehead-Segel equation

$$\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + u(n,t)[r(n-ct)-u^2(n,t)], \quad 0<\alpha\le 1,\ 0<\beta\le 1, \quad t\ge 0,\ n\in\mathbb{Z}.$$

It is remarkable that this result is new even in the integer case $\beta=\alpha=1$. Moreover, the comparison principle holds for both systems (see Corollaries 4.3 and 4.4).

2. Preliminaries

In this section we introduce some notations and give some known results as they are needed throughout the article.

Let $(-\Delta)^\alpha$ be the fractional Laplacian of order $0<\alpha\le 1$ (see e.g. [9] and references therein). We recall from [15, Section 3] (see also [16]) that the discrete counterpart of the fractional Laplacian is defined for $0<\alpha<1$ as follows:

$$(-\Delta_d)^\alpha f(n) := \sum_{k\in\mathbb{Z}} K^\alpha(n-k)f(k), \quad n\in\mathbb{Z}, \quad f\in\ell^2(\mathbb{Z}), \tag{2.1}$$

where

$$K^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(4\sin^2(\theta/2)\right)^\alpha e^{-in\theta}\,d\theta = \frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}, \quad n\in\mathbb{Z}. \tag{2.2}$$

We refer to [16, Remark 1] for more details. We note that the kernel K^α defined as the expression in the second equality in (2.2) has also appeared very recently [31, Formula (9)] in connection with a successful numerical method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion model. See also Tarasov [41, 42] for other interesting approach. Also, it is remarkable that this definition coincides with the fractional centered difference defined by Ortigueira [38] for approximating the Riesz fractional derivative in the case $0<\alpha\le 1$. This method was proved to be $O(h^2)$ accurate in [12, Section 2].

We observe that in the border case $\alpha=1$ we have that

$$K^1(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2(\theta/2)e^{-in\theta}\,d\theta = \frac{2}{\pi}\int_{-\pi}^{\pi}\sin^2(\theta/2)\cos(n\theta)\,d\theta = \begin{cases} -1 & \text{if } n=-1\\ 2 & \text{if } n=0\\ -1 & \text{if } n=1\\ 0 & \text{otherwise,}\end{cases}$$

so that

$$\Delta_d f(n) = f(n+1)-2f(n)+f(n-1) \quad n\in\mathbb{Z},$$

which is consistent with the definition given by several authors. We also mention that the infinite sum in (2.1) shows that $(-\Delta_d)^\alpha$ is a nonlocal operator for $0<\alpha<1$. It is interesting to note that in the case $\alpha=1/2$ the representation is particularly simple as the following example shows.

Example 2.1. *The following identity holds*

$$-(-\Delta)^{1/2}f(n) = \frac{4}{\pi}\sum_{k\in\mathbb{Z}}\frac{f(n-k)}{(2k-1)(2k+1)}, \quad n\in\mathbb{Z}. \tag{2.3}$$

Indeed, it is enough to use (2.2) *and the Euler's reflection formula* $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$. *This example also shows that the definition of the fractional Laplacian with the discretization of the generator for the Poisson semigroup in case* $\alpha=1/2$ *agrees with the formula* (2.3).

Remark 2.2. *We observe that the discrete fractional Laplacian operator* $(-\Delta_d)^\alpha$ *is a bounded linear operator on* $\ell^p(\mathbb{Z},X)$ *for any Banach space* X *and* $1\le p\le\infty$. *Indeed, by* [16, p.121] *we have* $K^\alpha(n)\sim\frac{C}{n^{2\alpha+1}}$ *(actually this is readily verified using Raabe's test) and hence the result follows from the estimate* $\|(-\Delta_d)^\alpha f\|_p = \|K^\alpha * f\|_p \le \|K^\alpha\|_1\|f\|_p$.

We also recall that for a given sequence $f = (f(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, the discrete Fourier transform is defined by

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{in\theta} = \sum_{n\in\mathbb{Z}} f(n)z^n \equiv \hat{f}(z), \quad \theta \in \mathbb{T}, \quad z = e^{i\theta},$$

where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ is the one-dimensional torus, that we identify with the interval $[-\pi, \pi]$.

The inverse discrete Fourier transform is obtained for a given function φ by the formula

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Therefore

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\mathbb{Z}}(f)(\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} z^{n-1}\hat{f}(z)dz, \quad n \in \mathbb{Z}.$$

It is easy to verify that

$$\mathcal{F}_{\mathbb{Z}}(f * g) = \mathcal{F}_{\mathbb{Z}}(f)\mathcal{F}_{\mathbb{Z}}(g),$$

where $*$ denote the usual convolution in $\mathbb{Z}$.

In what follows, we denote

$$J(z) := z + \frac{1}{z} - 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.4}$$

Remark 2.3. *The following properties hold:*

(i) $-J(z) \geq 0$ *for all* $z \in \mathbb{C},\ |z| = 1$.
(ii) $\hat{K}^\alpha(z) = (-J(z))^\alpha$ *for all* $z \in \mathbb{C},\ |z| = 1$.

We will need the following function, which is a probability density function related to stable Lévy processes, defined for $0 < \alpha < 1$ by

$$f_{t,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^\alpha}dz, & \sigma > 0, \quad t > 0, \quad \lambda \geq 0, \\ 0 & \lambda < 0, \end{cases} \tag{2.5}$$

where the branch of z^α is taken so that $Re(z^\alpha) > 0$ for $Re(z) > 0$. This branch is single-valued in the z-plane cut along the negative real axis. These functions were introduced by K. Yosida [43] to define fractional powers of closed linear operators. They correspond to the density functions associated with the stable Lévy processes in the rotational invariant case. They are also related with the Hurst exponent for fractional Brownian motion.

Remark 2.4. *The following properties hold:*

(i) $\displaystyle\int_0^\infty e^{-\lambda a} f_{t,\alpha}(\lambda)d\lambda = e^{-ta^\alpha}, \quad t > 0, \quad a > 0, \quad 0 < \alpha < 1.$
(ii) $f_{t,\alpha}(\lambda) \geq 0, \quad \lambda > 0,\ t > 0, \quad 0 < \alpha < 1.$
(iii) $\displaystyle\int_0^\infty f_{t,\alpha}(\lambda)d\lambda = 1, \quad t > 0, \quad 0 < \alpha < 1.$
(iv) $\displaystyle f_{t+s,\alpha}(\lambda) = \int_0^\lambda f_{t,\alpha}(\lambda-\mu)f_{s,\alpha}(\mu)d\mu, \quad \lambda > 0, \quad t, s > 0, \quad 0 < \alpha < 1.$
(v) $\displaystyle\int_0^\infty e^{\lambda z} f_{\lambda,\alpha}(t)d\lambda = t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha), \quad z \in \mathbb{C}, \quad t > 0, \quad 0 < \alpha < 1.$

For a proof of $(i) - (iv)$, see [43, p.260-262]. *Concerning (iv) we have to observe in* [43, Proposition 1, p.260] *that by definition $f_{t,\alpha}(\lambda) = 0$ for $\lambda < 0$. For the interesting property (v) we refer to the recent paper* [1, Theorem 3.2 (iii)].

In the above remark and throughout the paper, for $0 < \alpha \leq 1$ and $\gamma > 0$, $E_{\alpha,\gamma}$ denotes the well-known Mittag-Leffler function defined for every $z \in \mathbb{C}$ by

$$E_{\alpha,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \gamma)}.$$

In the literature, the notation $E_{\alpha,1}(z) = E_\alpha(z)$, $z \in \mathbb{C}$ is frequently used.

We will also make use of the Wright function (also known as the M-Wright function) with parameter γ which is defined by

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi i} \int_\Gamma e^{\gamma - 1} e^{\mu - z\mu^\gamma} d\mu, \quad 0 < \gamma < 1,$$

where Γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. It is of interest because of its relationship with the Mittag-Leffler function.

Remark 2.5. *The following properties hold:*

(i) $E_{\gamma,1}(z) = \displaystyle\int_0^\infty \Phi_\gamma(t) e^{zt} dt, \quad z \in \mathbb{C},\ 0 < \gamma < 1.$

(ii) $\Phi_\gamma(t) \geq 0, \quad t \geq 0.$

(iii) $\displaystyle\int_0^\infty \Phi_\gamma(t) dt = 1.$

The Wright function and the Mittag-Leffler function are related through the Laplace transform (see Remark 2.5 (i), [21] and [22]).

More information on the Mittag-Leffler and Wright functions can be found in the references [21, 33, 34] and the recent book [22]. Some interesting relations concerning these functions and their applications to the abstract Cauchy problem appear in [1].

We recall some properties of the Bessel functions: the generating formula (see [23, Formula 8.511])

$$\sum_{n\in\mathbb{Z}} J_n(x) z^n = e^{\frac{x}{2}(z - \frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0, \tag{2.6}$$

for the Bessel function $J_n(x)$. Analogously, we have the identity

$$\sum_{n\in\mathbb{Z}} I_n(x) z^n = e^{\frac{x}{2}(z + \frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0, \tag{2.7}$$

for the modified Bessel function $I_n(x)$. This function verifies, among others, the following properties

$$I_n(x) \geq 0, \quad n \in \mathbb{Z},\ x \geq 0, \tag{2.8}$$

and

$$\sum_{n\in\mathbb{Z}} I_n(2x) e^{2x} = 1. \tag{2.9}$$

3. The linear case: Explicit representation of solutions

The knowledge of explicit solutions for linear equations is a critical step to understand the physical mechanism of the phenomena described by nonlinear evolution equations. In this section, we give an explicit representation of solutions for the following time/space fractional diffusion equation

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), & 0 < \alpha, \beta \leq 1, \quad t > 0, \quad n \in \mathbb{Z}, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{3.1}$$

Here $(-\Delta_d)^\alpha$ is the discrete fractional Laplace operator defined in (2.1) and $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative given by

$$\mathbb{D}_t^\beta v(t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} v'(s)\, ds.$$

We adopt the following notion of solutions.

Definition 3.1. *Let $\varphi \in \ell^2(\mathbb{Z})$ and $g \in L^1_{loc}([0,\infty), \ell^2(\mathbb{Z}))$ be given. A function $u \in C([0,\infty); \ell^2(\mathbb{Z}))$ is said to be a solution of the system* (3.1) *if for every $n \in \mathbb{Z}$ and $t \geq 0$, we have*

$$\begin{aligned} u(n,t) &= \varphi(n) - (-\Delta_d)^\alpha \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(n,s)\, ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(n,s)\, ds \\ &= \varphi(n) - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} (-\Delta_d)^\alpha u(n,s)\, ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(n,s)\, ds. \end{aligned}$$

Remark 3.2. *We notice that since $(-\Delta_d)^\alpha$ is a bounded operator on $\ell^2(\mathbb{Z})$ (by Remark 2.2), we have that if a solution u (in the sense of Definition 3.1) exists, then $u \in C^\infty([0,\infty); \ell^2(\mathbb{Z}))$ and hence, is a classical solution if $g \in C^\infty([0,\infty); \ell^2(\mathbb{Z}))$.*

We observe that this remains valid if we replace the space $\ell^2(\mathbb{Z})$ with $\ell^2(\mathbb{Z}, \mathcal{H})$ for any Hilbert space $\mathcal{H}$.

For $0 < \alpha, \beta < 1$, we introduce the function

$$G_{\alpha,\beta}(n,t) := \int_0^\infty I_n(2\lambda t^{\beta/\alpha}) e^{-2\lambda t^{\beta/\alpha}} \left(\int_0^\infty \Phi_\beta(s) f_{s,\alpha}(\lambda) ds \right) d\lambda, \quad n \in \mathbb{Z},\ t \geq 0. \tag{3.2}$$

Our main result in this section is the following theorem.

Theorem 3.3. *Let $\varphi \in \ell^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ such that $g(\cdot, t) \in \ell^\infty(\mathbb{Z})$ and $\sup_{s \in [0,\, t]} \|g(\cdot, s)\|_\infty < \infty$ for each $t \geq 0$. Then the function*

$$\begin{aligned} u(n,t) &= \sum_{m \in \mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) \\ &\quad + \sum_{m \in \mathbb{Z}} \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s) \left(\int_0^\infty I_{n-m}(2\gamma) e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\, d\gamma \right) d\lambda g(m,s)\, ds, \end{aligned} \tag{3.3}$$

solves the initial value problem (3.1).

Proof. Let us denote $u(n,t) = u_h(n,t) + u_{nh}(n,t)$ with

$$u_h(n,t) := \sum_{m \in \mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m),$$

and

$$u_{nh}(n,t) := \sum_{m \in \mathbb{Z}} \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s) \left(\int_0^\infty I_{n-m}(2\gamma) e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\, d\gamma \right) d\lambda g(m,s)\, ds.$$

We first check that the series converges for $\varphi \in \ell^\infty(\mathbb{Z})$. Indeed, we have the estimate

$$\begin{aligned} |u_h(n,t)| &= \left| \sum_{m \in \mathbb{Z}} \left[\int_0^\infty I_m(2\lambda t^{\beta/\alpha}) e^{-2\lambda t^{\beta/\alpha}} \left(\int_0^\infty \Phi_\beta(s) f_{s,\alpha}(\lambda) ds \right) d\lambda \right] \varphi(n-m) \right| \\ &\leq \sum_{m \in \mathbb{Z}} \left| \int_0^\infty I_m(2\lambda t^{\beta/\alpha}) e^{-2\lambda t^{\beta/\alpha}} \left(\int_0^\infty \Phi_\beta(s) f_{s,\alpha}(\lambda) ds \right) d\lambda \right| \|\varphi\|_\infty \\ &= \int_0^\infty \left(\int_0^\infty \Phi_\beta(s) f_{s,\alpha}(\lambda) ds \right) d\lambda \|\varphi\|_\infty = \int_0^\infty \left(\int_0^\infty \Phi_\beta(s) f_{s,\alpha}(\lambda) d\lambda \right) ds \|\varphi\|_\infty \\ &= \int_0^\infty \Phi_\beta(s) \left(\int_0^\infty f_{s,\alpha}(\lambda) d\lambda \right) ds \|\varphi\|_\infty = \|\varphi\|_\infty, \end{aligned}$$

where we have used (2.8), (2.9), Fubini's theorem, Remark 2.4 (ii)-(iii) and Remark 2.5 (ii)-(iii). For the second term, we have the following estimate:

$$
\begin{aligned}
|u_{nh}(n,t)| &= \left|\sum_{m\in\mathbb{Z}} \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty I_{n-m}(2\gamma)e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\,d\gamma\right) d\lambda g(m,s)\,ds\right| \\
&\le \sum_{m\in\mathbb{Z}} \left|\int_0^t \int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty I_{n-m}(2\gamma)e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\,d\gamma\right) d\lambda\right| \|g(\cdot,s)\|_\infty ds \\
&= \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty \sum_{m\in\mathbb{Z}} I_{n-m}(2\gamma)e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\,d\gamma\right) d\lambda \|g(\cdot,s)\|_\infty ds \\
&= \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty f_{\lambda,\alpha}(\gamma)\,d\gamma\right) d\lambda \|g(\cdot,s)\|_\infty ds \\
&= \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s) d\lambda \|g(\cdot,s)\|_\infty ds \\
&= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(0)\|g(\cdot,s)\|_\infty\,ds \\
&\le \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(0)\,ds \sup_{s\in[0,\,t]} \|g(\cdot,s)\|_\infty \\
&\le \frac{t^\beta}{\Gamma(\beta+1)} \sup_{s\in[0,\,t]} \|g(\cdot,s)\|_\infty,
\end{aligned}
$$

where we have used Remark 2.4 (iii)-(v) and (2.8). This proves the claim.

Next, taking the discrete Fourier transform of (3.1) and using Remark 2.3 (ii) we get that

$$
\begin{cases}
\mathbb{D}_t^\beta \hat{u}(z,t) = -(-J(z))^\alpha \hat{u}(z,t) + \hat{g}(z,t), \\
\hat{u}(z,0) = \hat{\varphi}(z).
\end{cases}
\tag{3.4}
$$

The unique solution of (3.4) is given by means of the Mittag-Leffler functions as follows

$$
\hat{u}(z,t) = E_{\beta,1}(-(-J(z))^\alpha t^\beta)\hat{\varphi}(z) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(-J(z))^\alpha (t-s)^\beta)\hat{g}(z,s)\,ds.
$$

By Remark 2.5 (i) and Remark 2.4 (v) we obtain that

$$
\hat{u}(z,t) = \int_0^\infty e^{-(-J(z))^\alpha t^\beta s}\Phi_\beta(s) ds \hat{\varphi}(z) + \int_0^t \int_0^\infty e^{-\lambda(-J(z))^\alpha} f_{\lambda,\beta}(t-s)\,d\lambda \hat{g}(z,s)\,ds.
$$

Note that $-J(z) > 0$ for $|z| = 1$ and hence we can insert the Lévy function (see Remark 2.4 (i)) in the above integrals to get

$$
\begin{aligned}
\hat{u}(z,t) =& \int_0^\infty \int_0^\infty e^{\lambda J(z) t^{\beta/\alpha}} f_{s,\alpha}(\lambda)\Phi_\beta(s)\,d\lambda\,ds\,\hat{\varphi}(z) \\
&+ \int_0^t \int_0^\infty \left(\int_0^\infty e^{\gamma J(z)} f_{\lambda,\alpha}(\gamma)\,d\gamma\right) f_{\lambda,\beta}(t-s)\,d\lambda \hat{g}(z,s)\,ds \\
=& \int_0^\infty e^{\lambda(z+\frac{1}{z})t^{\beta/\alpha}} e^{-2\lambda t^{\beta/\alpha}} \int_0^\infty f_{s,\alpha}(\lambda)\Phi_\beta(s)\,ds\,d\lambda\,\hat{\varphi}(z) \\
&+ \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty e^{\gamma(z+\frac{1}{z})} e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\,d\gamma\right) d\lambda \hat{g}(z,s)\,ds,
\end{aligned}
$$

where in the last equality we have applied Fubini's theorem. Inserting the generating formula (2.7) for the modified Bessel function, we get that

$$\hat{u}(z,t) = \sum_{m\in\mathbb{Z}} \left[\int_0^\infty I_m(2\lambda t^{\beta/\alpha}) e^{-2\lambda t^{\beta/\alpha}} \left(\int_0^\infty f_{s,\alpha}(\lambda)\Phi_\beta(s)\, ds \right) d\lambda \right] z^m \hat{\varphi}(z)$$
$$+ \sum_{m\in\mathbb{Z}} \int_0^t \int_0^\infty f_{\lambda,\beta}(t-s) \left(\int_0^\infty I_m(2\gamma) e^{-2\gamma} f_{\lambda,\alpha}(\gamma)\, d\gamma \right) d\lambda z^m \hat{g}(z,s)\, ds.$$

Inverting the Fourier transform we get the expression (3.3) and the proof is finished. □

We now discuss several special cases as corollaries of Theorem 3.3.

3.1. **The heat equation.** This corresponds to the case $(\alpha,\beta)=(1,1)$. We first consider the discretization in space of the one-dimensional heat equation with initial condition

$$\begin{cases} v_t(n,t) = \Delta_d v(n,t), \ t\geq 0, & n\in\mathbb{Z}, \\ v(n,0) = \varphi(n), & n\in\mathbb{Z}, \end{cases} \tag{3.5}$$

where we recall that $\Delta_d v(n,t) = v(n+1,t) - 2v(n,t) + v(n-1,t)$ is the central difference. This equation and its solution has been recently deeply studied in connection with the discrete fractional Laplacian by Ciaurri et al. in [15] where they have obtained the solution (also called fundamental solution, or Green function)

$$v(n,t) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t)\varphi(m), \quad n\in\mathbb{Z}, \quad t\geq 0,$$

where $\varphi\in\ell^\infty(\mathbb{Z})$ and I_n denotes the modified Bessel function. In addition, one analysis for the following operator, called the heat semigroup of the discrete Laplacian

$$W_t f(n) := \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m) \tag{3.6}$$

and its properties, is performed. We also use the notation $W_t f(n) = e^{t\Delta_d} f(n)$, $f\in\ell^2(\mathbb{Z})$. In particular, in [15, Proposition 1] it was proved that W_t is a Markov semigroup. We refer to [16] for the connection of this semigroup with continuous fractional processes.

We derive the following complementary result.

Corollary 3.4. *Let $g:\mathbb{Z}\times\mathbb{R}_+\to\mathbb{C}$ be given and assume that $g(\cdot,t)\in\ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,\,t]}\|g(\cdot,s)\|_\infty<\infty$ for each $t\geq 0$. Then the non homogeneous heat equation*

$$\begin{cases} v_t(n,t) = \Delta_d v(n,t) + g(n,t), & t\geq 0, \quad n\in\mathbb{Z}, \\ v(n,0) = \varphi(n), & n\in\mathbb{Z}, \end{cases} \tag{3.7}$$

admits a unique solution in the space $\ell^2(\mathbb{Z})$ given explicitly by

$$v(n,t) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t)\varphi(m) + \sum_{m\in\mathbb{Z}} \left(\int_0^t e^{-2(t-s)} I_{n-m}(2(t-s)) g(m,s) ds \right). \tag{3.8}$$

Proof. Note that the operator Δ_d is bounded in the space $\ell^2(\mathbb{Z})$. Hence, the operator Δ_d generates the uniformly continuous semigroup $(e^{\Delta_d t})_{t\geq 0}$ on $\ell^2(\mathbb{Z})$, and consequently the general solution of the first order equation (3.7) in the variable t can be explicitly written by Duhamel's formula

$$v(n,t) = e^{\Delta_d t} v(n,0) + \int_0^t e^{\Delta_d(t-s)} g(n,s) ds.$$

Since the solution for the homogeneous problem (3.5) is given by the semigroup W_t, and it has the same generator Δ_d as the semigroup $(e^{\Delta_d t})_{t\geq 0}$, it follows from the uniqueness of solutions that the equality $e^{\Delta_d t} = W_t$ holds for every $t\geq 0$. The conclusion follows. □

Remark 3.5. *We observe the following situations.*

(a) *As observed in the proof of the above corollary, the semigroup W_t has a bounded generator. In consequence, the semigroup is not only strongly continuous, as was directly proved in* [15]*, but even more, namely, it is uniformly continuous.*

(b) *The family W_t also defines a uniformly continuous semigroup on $\ell^p(\mathbb{Z})$ for every $1 \leq p \leq \infty$ with the same generator. In view of this, the above results remain valid if we replace $\ell^2(\mathbb{Z})$ with $\ell^p(\mathbb{Z})$ $(1 \leq p \leq \infty)$.*

Remark 3.6. *We notice that prior the functions $f_{t,\gamma}$ and Φ_γ are defined for $0 < \gamma < 1$, but by* [33, 34] *the definition can be extended to $\gamma = 1$ so that $f_{t,1} = \delta_t$, $\Phi_1(t) = \delta_t$, that it, the δ-Dirac measure concentrated at t. Hence, taking the limit of the identity* (3.3) *as $\alpha, \beta \uparrow 1$ we get that*

$$\begin{aligned}
&\lim_{\beta\uparrow 1}\lim_{\alpha\uparrow 1}\sum_{m\in\mathbb{Z}}\int_0^\infty I_{n-m}(2\lambda t^{\beta/\alpha})e^{-2\lambda t^{\beta/\alpha}}\left(\int_0^\infty \Phi_\beta(s)f_{s,\alpha}(\lambda)ds\right)\varphi(m)d\lambda\\
&+\lim_{\beta\uparrow 1}\lim_{\alpha\uparrow 1}\sum_{m\in\mathbb{Z}}\int_0^t\int_0^\infty f_{\lambda,\beta}(t-s)\left(\int_0^\infty I_{n-m}(2\gamma)e^{-2\gamma}f_{\lambda,\alpha}(\gamma)\,d\gamma\right)d\lambda g(m,s)\,ds\\
=&\sum_{m\in\mathbb{Z}}e^{-2t}I_{n-m}(2t)\varphi(m)+\sum_{m\in\mathbb{Z}}\left(\int_0^t e^{-2(t-s)}I_{n-m}(2(t-s))g(m,s)ds\right),
\end{aligned}$$

which is exactly the identity (3.8).

3.2. **The case $0 < \alpha < 1$ and $\beta = 1$.** In this subsection, we analyze the system

$$\begin{cases} u_t(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), \quad 0<\alpha<1, \quad t>0, & n\in\mathbb{Z},\\ u(n,0) = \varphi(n), & n\in\mathbb{Z}.\end{cases} \tag{3.9}$$

Before we give the explicit representation of solutions of the system (3.9), we construct the semigroup generated by the operator $-(-\Delta)^\alpha$.

For each $g \in \ell^2(\mathbb{Z})$ we define the operator

$$T_t^\alpha g(n) := \sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]g(m), \quad n\in\mathbb{Z},\ t>0. \tag{3.10}$$

Proposition 3.7. *For all $0 < \alpha < 1$ and $g \in \ell^2(\mathbb{Z})$ we have that $T_t^\alpha g \in \ell^2(\mathbb{Z})$ and*

$$\|T_t^\alpha g\|_2 \leq \|g\|_2.$$

Proof. Let $g \in \ell^2(\mathbb{Z})$ be given. By Minkowski's integral inequality, we have

$$\begin{aligned}
\|T_t^\alpha g\|_2 &= \left(\sum_{n\in\mathbb{Z}}|T_t^\alpha g(n)|^2\right)^{\frac12} = \left(\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_m(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]g(n-m)\right|^2\right)^{\frac12}\\
&\leq \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}\left|\left[\int_0^\infty I_m(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]g(n-m)\right|^2\right)^{\frac12}\\
&= \sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_m(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]\left(\sum_{n\in\mathbb{Z}}\left|g(n-m)\right|^2\right)^{\frac12}\\
&= \sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_m(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]\|g\|_2.
\end{aligned}$$

By the positivity of the modified Bessel and Lévy functions, (2.7), and Remark 2.4 (iii) we obtain that

$$\|T_t^\alpha g\|_2 \le \int_0^\infty \sum_{m\in\mathbb{Z}} I_m(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda \|g\|_2 = \int_0^\infty f_{t,\alpha}(\lambda)d\lambda\|g\|_2 = \|g\|_2,$$

proving the assertions. □

We are able to prove the following result that extends [15, Proposition 1] to the range $0<\alpha<1$.

Theorem 3.8. *For each $0<\alpha<1$ the one parameter family $\{T_t^\alpha\}_{t\ge 0}$ defines a uniformly continuous Markov semigroup on $\ell^2(\mathbb{Z})$.*

Proof. We first prove the semigroup property. In fact, by definition we have

$$\begin{aligned} T_t^\alpha(T_s^\alpha g)(n) &= \sum_{m\in\mathbb{Z}}\left(\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right)(T_s^\alpha g)(m)\\ &= \sum_{m\in\mathbb{Z}}\left(\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right)\sum_{j\in\mathbb{Z}}\left(\int_0^\infty I_{m-j}(2\mu)e^{-2\mu}f_{s,\alpha}(\mu)d\mu\right)g(j)\\ &= \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_0^\infty\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)I_{m-j}(2\mu)e^{-2\mu}f_{s,\alpha}(\mu)d\lambda d\mu\, g(j). \end{aligned}$$

Applying Neumann's identity

$$I_n(t_1+t_2) = \sum_{m\in\mathbb{Z}} I_{n-m}(t_1)I_m(t_2), \quad n\in\mathbb{Z},$$

we obtain, after a change of variable

$$\begin{aligned} T_t^\alpha(T_s^\alpha g)(n) &= \sum_{j\in\mathbb{Z}}\int_0^\infty\int_0^\infty I_{n-j}(2(\lambda+\mu))e^{-2(\lambda+\mu)}f_{t,\alpha}(\lambda)f_{s,\alpha}(\mu)d\lambda d\mu\, g(j)\\ &= \sum_{j\in\mathbb{Z}}\int_0^\infty\left[\int_0^\infty I_{n-j}(2(\lambda+\mu))e^{-2(\lambda+\mu)}f_{t,\alpha}(\mu)d\mu\right]f_{s,\alpha}(\lambda)d\lambda\, g(j)\\ &= \sum_{j\in\mathbb{Z}}\int_0^\infty\left[\int_\lambda^\infty I_{n-j}(2\varphi)e^{-2\varphi}f_{t,\alpha}(\varphi-\lambda)d\varphi\right]f_{s,\alpha}(\lambda)d\lambda\, g(j). \end{aligned}$$

Using Fubini's theorem and Remark 2.4 (iv), we get that

$$\begin{aligned} T_t^\alpha(T_s^\alpha g)(n) &= \sum_{j\in\mathbb{Z}}\int_0^\infty\left[\int_0^\varphi I_{n-j}(2\varphi)e^{-2\varphi}f_{t,\alpha}(\varphi-\lambda)f_{s,\alpha}(\lambda)d\lambda\right]d\varphi\, g(j)\\ &= \sum_{j\in\mathbb{Z}}\int_0^\infty I_{n-j}(2\varphi)e^{-2\varphi}\left[\int_0^\varphi f_{t,\alpha}(\varphi-\lambda)f_{s,\alpha}(\lambda)d\lambda\right]d\varphi\, g(j)\\ &= \sum_{j\in\mathbb{Z}}\int_0^\infty I_{n-j}(2\varphi)e^{-2\varphi}f_{t+s,\alpha}(\varphi)d\varphi\, g(j), \end{aligned}$$

proving the semigroup property $T_t^\alpha(T_s^\alpha g)(n) = T_{t+s}g(n)$. Moreover, for $\mathbb{1}(n)\equiv 1$ we obtain by (2.7) and Remark 2.4 (iii) that

$$T_t^\alpha\mathbb{1}(n) = \mathbb{1}(n), \quad n\in\mathbb{Z}.$$

On the other hand, since $I_n(x)\ge 0$ for all $x\ge 0, n\in\mathbb{Z}$ (see e.g. [15, formula (28)]) it follows from Remark 2.4 (ii) that the semigroup is positive. It remains to show that

$$T_0^\alpha f(n) = f(n), \quad n\in\mathbb{Z}. \tag{3.11}$$

Indeed, we follow an idea due to Yosida [43, p.263-364] in order to handle $f_{t,\alpha}(\lambda)$. We have by [43, p.263, Formula (17)] with $\theta_\alpha := \frac{\pi}{1+\alpha}$ that

$$f_{t,\alpha}(\lambda) = \frac{1}{\pi}\int_0^\infty e^{(\lambda r - tr^\alpha)\cos\theta_\alpha}\sin[(\lambda r - tr^\alpha)\sin\theta_\alpha + \theta_\alpha]dr.$$

Inserting this formula in the definition of T_t^α we obtain

$$T_t^\alpha f(n) = \frac{1}{\pi}\sum_{m\in\mathbb{Z}}\left[\int_0^\infty\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}e^{(\lambda r - tr^\alpha)\cos\theta_\alpha}\sin[(\lambda r - tr^\alpha)\sin\theta_\alpha + \theta_\alpha]drd\lambda\right]f(m).$$

We made the change of variables

$$s = t^{1/\alpha}r \quad \text{and} \quad p = \lambda t^{-1/\alpha},$$

to obtain

$$\begin{aligned} T_t^\alpha f(n) =& \frac{1}{\pi}\sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_{n-m}(pt^{1/\alpha})e^{-2pt^{1/\alpha}}\right. \\ &\left.\int_0^\infty e^{(ps-s^\alpha)\cos\theta_\alpha}\sin[(ps-s^\alpha)\sin\theta_\alpha + \theta_\alpha]dsdp\right]f(m). \end{aligned}$$

Now, we observe that the second integral on the right is exactly $\pi f_{1,\alpha}(p)$. Therefore,

$$T_t^\alpha f(n) = \sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_{n-m}(pt^{1/\alpha})e^{-2pt^{1/\alpha}}f_{1,\alpha}(p)dp\right]f(m).$$

Taking into account that $I_0(0) = 1$ and $I_k(0) = 0$ for $k \neq 0$ we obtain (3.11). Finally, we prove that T_t^α is uniformly continuous. Indeed, the semigroup T_t^α satisfies $T_t^\alpha g = -(-\Delta_d)^\alpha T_t^\alpha g$ for all $g \in \ell^2(\mathbb{Z})$ and consequently the generator of T_t^α is the operator $A = -(-\Delta_d)^\alpha$ which is bounded on $\ell^2(\mathbb{Z})$ (see Remark 2.2). The proof is finished. □

As one immediate consequence of Proposition 3.7 and Theorem 3.8 we obtain the following result.

Corollary 3.9. *Assume that* $g(\cdot,t) \in \ell^\infty(\mathbb{Z})$ *and* $\sup_{s\in[0,\,t]}\|g(\cdot,s)\|_\infty < \infty$ *for each* $t \geq 0$. *Then for every* $0 < \alpha < 1$ *and* $\varphi \in \ell^\infty(\mathbb{Z})$ *the function*

$$\begin{aligned} u(n,t) =& \sum_{m\in\mathbb{Z}}\left[\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}f_{t,\alpha}(\lambda)d\lambda\right]\varphi(m) \\ &+ \sum_{m\in\mathbb{Z}}\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda}\left(\int_0^t f_{t-s,\alpha}(\lambda)g(m,s)ds\right)d\lambda, \quad n\in\mathbb{Z},\ t\geq 0, \end{aligned} \tag{3.12}$$

solves the initial value problem (3.9).

Proof. By Remark 2.2 the operator $-(-\Delta_d)^\alpha$ is bounded on $\ell^2(\mathbb{Z})$. Consequently, we have that the semigroup $(e^{-(-\Delta_d)^\alpha t})_{t\geq 0}$ exists and the solution of the inhomogeneous problem can be given by Duhamel's formula

$$u(n,t) = e^{-(-\Delta_d)^\alpha t}u(n,0) + \int_0^t e^{-(-\Delta_d)^\alpha (t-s)}g(n,s)ds.$$

Since the generator of the semigroup $(T_t^\alpha)_{t\geq 0}$ is the same operator $-(-\Delta_d)^\alpha$ we conclude that $T_t^\alpha = e^{-(-\Delta_d)^\alpha t}$ for every $t \geq 0$ and the result follows. □

Remark 3.10. *Using the argument in Remark 3.6 we have that for every $g \in \ell^2(\mathbb{Z})$, $n \in \mathbb{Z}$ and $t > 0$,*

$$\begin{aligned}\lim_{\alpha\uparrow 1} T_t^\alpha g(n) &= \lim_{\alpha\uparrow 1} \sum_{m\in\mathbb{Z}} \left[\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda\right] g(m)\\ &= \sum_{m\in\mathbb{Z}} \left[\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda} \lim_{\alpha\uparrow 1} f_{t,\alpha}(\lambda)d\lambda\right] g(m)\\ &= \sum_{m\in\mathbb{Z}} I_{n-m}(2t)e^{-2t} g(m) = W_t g(n).\end{aligned}$$

3.3. **The case $0 < \beta < 1$ and $\alpha = 1$.** In this section, we study the solutions for the system

$$\begin{cases}\mathbb{D}_t^\beta u(n,t) = \Delta_d u(n,t) + g(n,t), \quad 0 < \alpha < 1, \ \ t > 0, & n \in \mathbb{Z},\\ u(n,0) = \varphi(n), & n \in \mathbb{Z}.\end{cases} \tag{3.13}$$

We have the following explicit representation of solutions to the system (3.13).

Theorem 3.11. *Assume that $g(\cdot, t) \in \ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,\,t]} \|g(\cdot, s)\|_\infty < \infty$ for each $t \geq 0$. Then for every $\varphi \in \ell^\infty(\mathbb{Z})$ the function*

$$\begin{aligned}u(n,t) = &\sum_{m\in\mathbb{Z}} \left[\int_0^\infty I_{n-m}(2st^\beta)e^{-2st^\beta}\Phi_\beta(s)ds\right]\varphi(m)\\ &+ \sum_{m\in\mathbb{Z}} \int_0^t \left(\int_0^\infty I_{n-m}(2\lambda)e^{-2\lambda} f_{\lambda,\beta}(t-s)\, d\lambda\right) g(m,s)\, ds\end{aligned} \tag{3.14}$$

solves the initial value problem (3.13).

Proof. We take the Fourier transform in the variable n of the system (3.13) and we obtain the following equation in the variable t

$$\begin{cases}\mathbb{D}_t^\beta \hat{u}(z,t) = J(z)\hat{u}(z,t) + \hat{g}(z,t), \ \ t > 0,\\ \hat{u}(z,0) = \hat{\varphi}(z),\end{cases} \tag{3.15}$$

where we recall that $J(z) = z + \frac{1}{z} - 2$. It is well-known that the system (3.15) has the unique solution

$$\hat{u}(z,t) = E_\beta(J(z)t^\beta)\hat{\varphi}(z) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(J(z)(t-s)^\beta)\hat{g}(z,s)\, ds,$$

where $E_{\beta,1}$ and $E_{\beta,\beta}$ denote the Mittag Leffler functions. Using Remark 2.5 (i) we obtain that

$$\begin{aligned}\hat{u}(z,t) &= \int_0^\infty e^{J(z)t^\beta s}\Phi_\beta(s)ds\hat{\varphi}(z) + \int_0^t \left(\int_0^\infty e^{\lambda J(z)} f_{\lambda,\beta}(t-s)\, d\lambda\right)\hat{g}(z,s)\, ds\\ &= \int_0^\infty e^{(z+\frac{1}{z})t^\beta s}e^{-2t^\beta s}\Phi_\beta(s)ds\hat{\varphi}(z) + \int_0^t \left(\int_0^\infty e^{\lambda(z+\frac{1}{z})}e^{-2\lambda} f_{\lambda,\beta}(t-s)\, d\lambda\right)\hat{g}(z,s)\, ds.\end{aligned}$$

Inserting the generating formula (2.7) for the modified Bessel function in the last two integrals, we get

$$\begin{aligned}\hat{u}(z,t) = &\sum_{m\in\mathbb{Z}} \left(\int_0^\infty \Phi_\beta(s)I_m(2t^\beta s)e^{-2t^\beta s}ds\right) z^m\hat{\varphi}(z)\\ &+ \sum_{m\in\mathbb{Z}} \int_0^t \left(\int_0^\infty I_m(2\lambda)e^{-2\lambda} f_{\lambda,\beta}(t-s)\, d\lambda\right) z^m\hat{g}(z,s)\, ds\end{aligned}$$

and hence the result follows by inversion of the Fourier transform. $\square$

4. Well Posedness of the nonlinear problem

In this section we establish the well posedness of the nonlinear model given by

$$\begin{cases}\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + f(n-ct, u(n,t)), & t>0, \quad 0<\alpha,\beta<1, \quad n\in\mathbb{Z},\\ u(n,0)=\varphi(n), & n\in\mathbb{Z},\end{cases} \tag{4.1}$$

for some constant $c \ge 0$. For each $\gamma>0$ we define the set

$$\mathcal{L}_\gamma := \{u \in \ell^\infty(\mathbb{Z}) \,:\, 0\le u(n)\le \gamma \text{ for all } n\in\mathbb{Z}\}.$$

The following result is one of the main tools in order to establish the well-posedness of the model (4.1).

Lemma 4.1. *Let $\gamma>0$. Suppose that $f:\ \mathbb{R}\times[0,\infty)\to\mathbb{R}$ is measurable, the restriction of $f(x,\cdot)$ to $[0,\gamma]$ belongs to $C^1[0,\gamma]$, is concave on $[0,\gamma]$ and satisfies $f(x,0)=0$ for every $x\in\mathbb{R}$. Let $\rho>0$ be such that*

$$\rho + \partial_s f(x,\gamma) \ge 0, \ \ \forall\, x\in\mathbb{R}. \tag{4.2}$$

Then the function $F:\ \mathbb{R}\times[0,\gamma]\to\mathbb{R}$ defined by $F(x,s):=\rho s + f(x,s)$ is non-decreasing (in the second variable) and non-negative for every $x\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$ be fixed and $0\le s_1\le s_2\le\gamma$. Since $f(x,\cdot)|_{[0,\gamma]}$ is differentiable, it follows from the Mean Value Theorem that there exists $c>0$ with $s_1\le c\le s_2$ such that $\partial_s f(x,c)(s_2-s_1) = f(x,s_2)-f(x,s_1)$. Moreover, since $f(x,\cdot)|_{[0,\gamma]}$ is concave we have that $\partial_s f(x,\cdot)$ is non-increasing on $[0,\gamma]$. Therefore $\partial_s f(x,c)\ge \partial_s f(x,\gamma)$ and the condition (4.2) implies that

$$\begin{aligned}0 &\le (\rho+\partial_s f(x,\gamma))(s_2-s_1) \le (\rho+\partial_s f(x,c))(s_2-s_1)\\ &= \rho(s_2-s_1)+\partial_s f(x,c)(s_2-s_1) = \rho(s_2-s_1)+f(x,s_2)-f(x,s_1)\\ &= F(x,s_2)-F(x,s_1).\end{aligned}$$

We have shown that $F(x,\cdot)$ is non-decreasing on $[0,\gamma]$. Since $F(x,0)=0+f(x,0)=0$, we also have that $F(x,s)\ge 0$ for all $s\in[0,\gamma]$ and $x\in\mathbb{R}$, and this proves the lemma. ☐

We observe that using the techniques of resolvent families (see e.g. [28, 29]) we have the following situation. Let $\rho>0$ be fixed and $(e^{-((-\Delta_d)^\alpha-\rho)t})_{t\ge 0}$ be the semigroup generated by the operator $-(-\Delta_d)^\alpha-\rho$ on $\ell^2(\mathbb{Z})$. For every $\varphi\in\ell^2(\mathbb{Z})$ we define the operators

$$\begin{aligned}E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi(n) &:= t^{-\beta}\int_0^\infty \Phi_\beta(t^{-\beta}\tau)e^{-((-\Delta_d)^\alpha+\rho)\tau}\varphi(n)\,d\tau\\ &= t^{-\beta}\int_0^\infty \Phi_\beta(t^{-\beta}\tau)e^{-(-\Delta_d)^\alpha\tau}e^{-\rho\tau}\varphi(n)\,d\tau,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}t^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi(n) &:= \beta t^{-\beta-1}\int_0^\infty \tau\Phi_\beta(t^{-\beta}\tau)e^{-((-\Delta_d)^\alpha+\rho)\tau}\varphi(n)\,d\tau\\ &= \beta t^{-\beta-1}\int_0^\infty \tau\Phi_\beta(t^{-\beta}\tau)e^{-(-\Delta_d)^\alpha\tau}e^{-\rho\tau}\varphi(n)\,d\tau.\end{aligned} \tag{4.4}$$

Then a solution of the nonlinear system (4.1) is a fixed point of the equation

$$\begin{aligned}u(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi(n)\\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,u(n,s))ds\end{aligned} \tag{4.5}$$

where we recall that $F(x,\xi):=\rho\xi+f(x,\xi)$. In the notation of [28, 29], $E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)t^\beta) = S_\alpha^0(t)$ and $t^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi = P_\alpha^0(t)$, where in each case the generator is $A=-(-\Delta_d)^\alpha-\rho I$.

The following theorem is the first main result of this section.

Theorem 4.2. *(Well-posedness) Suppose that $f : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is measurable, the restriction of $f(x,\cdot)$ to $[0,\gamma]$ belongs to $C^1[0,\gamma]$, is concave on $[0,\gamma]$ for every $x \in \mathbb{R}$ and satisfies*

$$f(x,0) = 0 \text{ and } f(x,\gamma) \le 0, \tag{4.6}$$

for all $x \in \mathbb{R}$ and for some $\gamma > 0$, and $\partial_s f(x,0)$ is a nondecreasing function of x. Let $\varphi \in \mathcal{L}_\gamma$. Then there exists a unique solution $u \in C(\mathbb{R}_+, \mathcal{L}_\gamma)$ to the problem (4.5), and hence a unique solution to the nonlinear system (4.1).

Proof. Define

$$\begin{aligned}\mathcal{K}_\beta(u)(n,t) :=& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, u(n,s)) ds,\end{aligned}$$

where $F(x,\xi) = \rho\xi + f(x,\xi)$ and $\rho > 0$ is such that (4.2) holds.

We first consider the sequence $\underline{u}^k(n,t)$ given by $\underline{u}^0(n,t) = 0$ and $\underline{u}^{k+1} = \mathcal{K}_\beta(\underline{u}^k)$ for $k \in \mathbb{N} \cup \{0\}$. Since $F(x,0) = f(x,0) = 0$ for every $x \in \mathbb{R}$ we have that

$$\begin{aligned}\underline{u}^1(n,t) =& \mathcal{K}_\beta(\underline{u}^0)(n,t)\\ =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, 0)\, ds\\ =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n) \ge 0,\end{aligned}$$

where we have used the fact that $E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)$ is a positive operator. Moreover, since $F(x,s)$ is non negative for every $x \in \mathbb{R}$ and $0 \le s \le \gamma$, we have that

$$\begin{aligned}\underline{u}^1(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ \le& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \underline{u}^1(n,s))\, ds = \underline{u}^2(n,t).\end{aligned}$$

Proceeding by induction we easily get that

$$0 = \underline{u}^0(n,t) \le \underline{u}^1(n,t) \le \underline{u}^2(n,t) \le \cdots \le \underline{u}^k(n,t) \le \cdots$$

On the other hand, define $\overline{u}^0(n,t) = \gamma$ and $\overline{u}^{k+1} = \mathcal{K}(\overline{u}^k)$ for $k \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}\overline{u}^1(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \overline{u}^0(n,s))\, ds\\ =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n)\\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \gamma)\, ds.\end{aligned} \tag{4.7}$$

Using (4.3) and the fact that the semigroup $(e^{-(-\Delta_d)^\alpha t})_{t\ge 0}$ generated by the operator $-(-\Delta_d)^\alpha$ is Markovian, we get that

$$\begin{aligned}E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n) &= t^{-\beta} \int_0^\infty \Phi_\beta(t^{-\beta}\tau) e^{-(-\Delta_d)^\alpha \tau} e^{-\rho\tau} \varphi(n)\, d\tau\\ &\le t^{-\beta} \int_0^\infty \Phi_\beta(t^{-\beta}\tau) e^{-\rho\tau} \varphi(n)\, d\tau = E_{\beta,1}(-\rho t^\beta)\varphi(n).\end{aligned} \tag{4.8}$$

Similarly, it follows from (4.4) that for every non-negative ψ, we have that

$$\begin{aligned} t^{\beta-1}E_{\beta,\beta}(-(-\Delta_d)^\alpha+\rho)t^\beta)\psi(n) &= \beta t^{-\beta-1}\int_0^\infty \tau\Phi_\beta(t^{-\beta}\tau)e^{-(-\Delta_d)^\alpha\tau}e^{-\rho\tau}\psi(n)\,d\tau \\ &\leq \beta t^{-\beta-1}\int_0^\infty \tau\Phi_\beta(t^{-\beta}\tau)e^{-\rho\tau}\psi(n)\,d\tau \\ &= t^{\beta-1}E_{\beta,\beta}(-\rho t^\beta)\psi(n). \end{aligned} \tag{4.9}$$

Using (4.8), (4.9) and the fact that $0 \leq F(x,\gamma) \leq \rho\gamma$ (this follows from the fact that by hypothesis $f(x,\gamma) \leq 0$) we get from (4.7) that

$$\begin{aligned} \overline{u}^1(n,t) &\leq E_{\beta,1}(-\rho t^\beta)\varphi(n) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-\rho(t-s)^\beta)F(n-cs,\gamma)\,ds \\ &= E_{\beta,1}(-\rho t^\beta)\varphi(n) + \rho\gamma\int_0^t s^{\beta-1}E_{\beta,\beta}(-\rho s^\beta)\,ds. \end{aligned} \tag{4.10}$$

Since $\varphi(n) \leq \gamma$, $E_{\beta,1}(0) = 1$ and

$$\frac{d}{dt}E_{\beta,1}(-\rho t^\beta) = -\rho t^{\beta-1}E_{\beta,\beta}(-\rho t^\beta),$$

it follows from (4.10) by integrating that

$$\overline{u}^1(n,t) \leq E_{\beta,1}(-\rho t^\beta)\gamma - E_{\beta,1}(-\rho t^\beta)\gamma + \gamma = \gamma. \tag{4.11}$$

Moreover, by Lemma 4.1, the inequality (4.11) implies that $F(n-cs,\overline{u}^1) \leq F(n-cs,\gamma)$. Hence,

$$\begin{aligned} \overline{u}^1(n,t) =&E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi(n) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,\gamma)\,ds \\ \geq&E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)t^\beta)\varphi(n) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,\overline{u}^1(n,s))\,ds. \end{aligned}$$

We have shown that $\overline{u}^1(n,t) \geq \overline{u}^2(n,t)$. Proceeding by induction we also get that

$$\overline{u}^1(n,t) \geq \overline{u}^2(n,t) \geq \cdots \overline{u}^k(n,t) \geq \cdots.$$

We have proved the following monotonicity of the sequences $\underline{u}^k$ and $\overline{u}^k$:

$$0 \leq \underline{u}^1(n,t) \leq \underline{u}^2(n,t) \leq \ldots \leq \overline{u}^2(n,t) \leq \overline{u}^1(n,t) \leq \gamma.$$

It is clear that $\underline{u}^k(n,t) \to \underline{u}(n,t)$ and $\overline{u}^k(n,t) \to \overline{u}(n,t)$ as $k \to \infty$. Moreover $\underline{u}$, $\overline{u} \in C(\mathbb{R}_+,\mathcal{L}_\gamma)$ and are solutions of (4.5).

It remains to prove uniqueness. In fact, an application of the Mean Value Theorem gives

$$\begin{aligned} 0 &\leq \overline{u}(n,t) - \underline{u}(n,t) \\ &= \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)[F(n-cs,\overline{u}(n,s)) - F(n-cs,\underline{u}(n,s))]ds \\ &= \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)\partial_s F(n-cs,d)(\overline{u}(n,s)-\underline{u}(n,s))ds, \end{aligned}$$

where $0 \leq \underline{u} \leq d \leq \overline{u} \leq \gamma$. Since $\partial_s F(n-cs,d) = \rho + \partial_s f(n-cs,d)$ and by Lemma 4.1 we have that $\partial_s f(n-cs,\cdot)$ is non-increasing, we obtain that $\partial_s f(n-cs,\gamma) \leq \partial_s f(n-cs,d) \leq \partial_s f(n-cs,0)$ and hence

$\partial_s F(n-cs,d) \leq \partial_s F(n-cs,0)$. Observe that $\partial_s F(\cdot,s)$ is nondecreasing (in the first variable). This together with the above inequality gives

$$\begin{aligned} 0 &\leq \overline{u}(n,t) - \underline{u}(n,t) \\ &\leq \partial_s F(n,0) \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta)(\overline{u}(n,s) - \underline{u}(n,s))ds. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \sup_n [\overline{u}(n,t) - \underline{u}(n,t)] \\ &\leq \partial_s F(n,0) \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) \sup_n [\overline{u}(n,s) - \underline{u}(n,s)]ds \\ &\leq \partial_s F(n,0) \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\rho(t-s)^\beta) \sup_n [\overline{u}(n,s) - \underline{u}(n,s)]ds, \end{aligned}$$

where we have used (4.9). Therefore, using the Gronwall inequality for integral equations (see e.g. [14]) we get that $\underline{u} \equiv \overline{u}$ and the proof is finished. $\square$

We have the following result as a corollary of the preceding theorem.

Corollary 4.3. *(Comparison Principle) Assume the hypothesis in Theorem 4.2. Let* u *and* v *be two solutions of* (4.5) *with initial data* $\varphi, \psi \in \mathcal{L}_\gamma$ *respectively. If* $\varphi(n) \leq \psi(n)$ *for all* $n \in \mathbb{Z}$*, then* $u(n,t) \leq v(n,t)$ *for all* $t \geq 0$ *and* $n \in \mathbb{Z}$*.*

Proof. Consider the sequences $\underline{u}^k(n,t)$ and $\underline{v}^k(n,t)$ given by $\underline{u}^0(n,t) = 0$, $\underline{v}^0(n,t) = 0$ and $\underline{u}^{k+1} = \mathcal{K}_\beta(\underline{u}^k)$, $\underline{v}^{k+1} = \mathcal{K}_\beta(\underline{v}^k)$, $k \in \mathbb{N} \cup \{0\}$, with initial conditions φ and ψ respectively. By the uniqueness of solutions of the system (4.1) proved in Theorem 4.2, we have that $\underline{u}^k(n,t) \to u(n,t)$ and $\underline{v}^k(n,t) \to v(n,t)$, for every $n \in \mathbb{Z}$, $t > 0$ and $k \in \mathbb{N}$. Since $F(x, \underline{u}^0) = 0$ for every $x \in \mathbb{R}$, we have that

$$\begin{aligned} \underline{u}^1(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n) \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \underline{u}^0(n,s))ds \\ =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\varphi(n), \end{aligned}$$

and

$$\begin{aligned} \underline{v}^1(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\psi(n) \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \underline{v}^0(n,s))ds \\ =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)t^\beta)\psi(n). \end{aligned}$$

Since $E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)$ and $t^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)t^\beta)$ are positive operators, the hypothesis on the initial data implies that

$$\underline{u}^1(n,t) \leq \underline{v}^1(n,t) \tag{4.12}$$

Now, we have

$$\begin{aligned} \underline{u}^2(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha + \rho)\varphi(n) \\ &+ \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-((-\Delta_d)^\alpha + \rho)(t-s)^\beta) F(n-cs, \underline{u}^1(n,s))ds \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\underline{v}^2(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)\psi(n) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,\underline{v}^1(n,s))ds. \end{aligned} \tag{4.14}$$

Since $F(x,\cdot)$ is non-decreasing for every $x\in\mathbb{R}$ (by Lemma 4.1), it follows from (4.12) that $F(n-cs,\underline{u}^1)\le F(n-cs,\underline{v}^1)$ for every $x\in\mathbb{R}$. We then conclude from equations (4.13) and (4.14) that $\underline{u}^2(n,t)\le\underline{v}^2(n,t)$. Proceeding by induction, one obtains that

$$\underline{u}^k(n,t)\le\underline{v}^k(n,t) \text{ for every } k\in\mathbb{N}. \tag{4.15}$$

Taking the limit of (4.15) as $k\to\infty$, we get that $u(n,t)\le v(n,t)$ for all $t>0$ and $n\in\mathbb{Z}$ and the proof of the corollary is finished. □

Next, we give the following comparison result, which extends [24, Corollary 3.1].

Corollary 4.4. *Assume the hypothesis of Theorem 4.2. Let $u,v\in C(\mathbb{R}_+,\mathcal{L}_\gamma)$ be such that $u(n,t)\ge\mathcal{K}_\beta(u)(n,t)$ and $v(n,t)\le\mathcal{K}_\beta(v)(n,t)$ for all $t>0$ and $n\in\mathbb{Z}$. Suppose that $u(n,0)\ge v(n,0)$ for all $n\in\mathbb{Z}$. Then $u(n,t)\ge v(n,t)$ for all $t\ge 0$ and $n\in\mathbb{Z}$.*

Proof. By Lemma 4.1 we have that $F(x,\cdot)$ is non-decreasing for every $x\in\mathbb{R}$ and therefore

$$\begin{aligned}\mathcal{K}^2(u)(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)u(n,0) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,\mathcal{K}_\beta(u)(n,s))ds \\ \le& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)u(n,0) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,u(n,s))ds \\ =&\mathcal{K}_\beta(u)(n,t),\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}^2(v)(n,t) =& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)v(n,0) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,\mathcal{K}_\beta(v)(n,s))ds \\ \ge& E_{\beta,1}(-((-\Delta_d)^\alpha+\rho)v(n,0) \\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-((-\Delta_d)^\alpha+\rho)(t-s)^\beta)F(n-cs,v(n,s))ds \\ =&\mathcal{K}_\beta(v)(n,t).\end{aligned}$$

Proceeding by induction, one obtains that

$$u(n,t)\ge\mathcal{K}_\beta(u)(n,t)\ge\mathcal{K}_\beta^2(u)(n,t)\ge...\ge\lim_{k\to\infty}\mathcal{K}_\beta^k(u)(n,t)=:\overline{u}(n,t)$$

and

$$v(n,t)\le\mathcal{K}_\beta(v)(n,t)\le\mathcal{K}_\beta^2(v)(n,t)\le...\le\lim_{k\to\infty}\mathcal{K}_\beta^k(v)(n,t)=:\overline{v}(n,t).$$

Observe that both $\overline{u}$ and $\overline{v}$ are solutions of the nonlinear equation (4.5) with initial data $u(n,0)$ and $v(n,0)$, respectively. Then, by the comparison principle (Corollary 4.3) we obtain that

$$v(n,t)\le\overline{v}(n,t)\le\overline{u}(n,t)\le u(n,t),$$

for all $t\ge 0$ and $n\in\mathbb{Z}$ and the proof is finished. □

We conclude the paper with the following set of examples.

Example 4.5 (Generalized Fisher-KPP Equation)**.** *Let $r:\ \mathbb{R}\to\mathbb{R}$ be continuous, non-decreasing, bounded and picewise continuously differentiable satisfying $0<r(\infty)<\infty$. Define*

$$f:\ \mathbb{R}\times\mathbb{R}_+\to\mathbb{R},\quad f(x,s)=s(r(x)-s). \tag{4.16}$$

Let $\gamma:=r(\infty)$. It is clear that f is measurable, the restriction of $f(x,\cdot)$ to $[0,r(\infty)]$ belongs to $C^1[0,r(\infty)]$ and that $f(x,0)=0$ for every $x\in\mathbb{R}$. Calculating we have that $\partial_s f(x,s)=r(x)-2s$ and $\partial_s^2 f(x,0)(x,s)=-2<0$. Hence, $f(x,\cdot)$ is concave on $[0,r(\infty)]$. Letting $\rho:=3r(\infty)$ we also have that (4.2) *is satisfied. We have shown that f satisfies all the hypothesis in Lemma 4.1. It is clear that $\partial_s f(x,0)$ is a nondecreasing function of x. Finally $f(x,r(\infty))=r(\infty)(r(x)-r(\infty))\le 0$ and hence the condition* (4.6) *is also satisfied. The system* (4.1) *with f given by* (4.16) *is the discrete generalized Fisher-KPP equation. This includes the discrete generalized Fisher-KPP equation with delay by taking $f(n-ct,s)=s(r(n-st)-s)$, $n\in\mathbb{Z}$, $t\ge 0$ and for some constant $c\ge 0$. The classical Fisher-KPP equation corresponds to the case where $r(x)=r(\infty)$ is constant.*

Next, we consider a generalized Newell-Whitehead-Segel type equation.

Example 4.6 (Generalized Newell-Whitehead-Segel Equation)**.** *Let r be as in Example* (4.5)*, $p\ge 1$ and define*

$$f:\ \mathbb{R}\times\mathbb{R}_+\to\mathbb{R},\quad f(x,s)=s(r(x)-s^p). \tag{4.17}$$

Let $\gamma:=r(\infty)^{1/p}$. Then f is measurable, the restriction of $f(x,\cdot)$ to $[0,\gamma]$ belongs to $C^1[0,\gamma]$ and $f(x,0)=0$ for every $x\in\mathbb{R}$. Moreover, $\partial_s f(x,s)=r(x)-(p+1)s^p$ and $\partial_s f(x,s)=-p(p+1)s^{p-1}\le 0$. Hence, $f(x,\cdot)$ is concave on $[0,\gamma]$. The inequality (4.2) *is satisfied with the choice of $\rho:=(p+2)\gamma^p=(p+2)r(\infty)^{1/p}$. Hence, f satisfies all the hypothesis in Lemma 4.1. In addition we have that*

$$f(x,\gamma)=\gamma(r(x)-\gamma^p)=r(\infty)^{1/p}(r(x)-r(\infty))\le 0.$$

Hence, (4.6) *is also satisfied. It is also clear that $\partial_s f(x,0)$ is a nondecreasing function of x. The case $r\equiv 1$ and $p=2$ corresponds to the Newell-Whitehead-Segel equation which describes the so called Rayleigh-Benard convection. The Newell-Whitehead-Segel equation is a well-known universal equation to govern evolution of nearly one-dimensional nonlinear patterns produced by a finite-wavelength instability in isotropic two-dimensional media* [37, 39]*.*

Example 4.7. *Let $a>0$ be a fixed constant and define*

$$f:\ \mathbb{R}_+\to\mathbb{R},\ f(s)=s(a-s)(a+s).$$

Then f is measurable, the restriction of f to $[0,a]$ belongs to $C^1[0,a]$ and $f(0)=0$. Moreover, $f'(s)=a^2-3s^2$ and $f''(s)=-6s\le 0$. Hence, f is concave on $[0,a]$. Equation (4.2) *is satisfied with the choice of $\rho:=3a^2$. Hence, f satisfies all the hypotheses in Lemma 4.1. In addition we have that $f(a)=0$ and* (4.6) *is also satisfied.*

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