ASYMPTOTIC BEHAVIOR OF FRACTIONAL ORDER SEMILINEAR EVOLUTION EQUATIONS

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ABSTRACT. Fractional calculus is a subject of great interest in many areas of mathematics, physics and sciences, including stochastic processes, mechanics, chemistry and biology. We will call an operator A on a Banach space X ω-sectorial ($\omega \in \mathbb{R}$) of angle θ if there exists $\theta \in [0,\pi/2)$ such that $S_\theta := \{\lambda \in \mathbb{C}\setminus\{0\}, |\arg(\lambda)| < \theta+\pi/2\} \subset \rho(A)$ (the resolvent set of A) and $\sup\{|\lambda-\omega|\|(\lambda-A)^{-1}\|, \lambda \in \omega+S_\theta\} < \infty$. Let A be ω-sectorial of angle $\beta\pi/2$ with $\omega < 0$ and f an X-valued function. Using the theory of regularized families, and Banach's fixed point theorem, we prove existence and uniqueness of mild solutions for the semilinear fractional order differential equation

$$D_t^{\alpha+1}u(t)+\mu D_t^\beta u(t) = Au(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}u'(0)+\mu\frac{t^{-\beta}}{\Gamma(1-\beta)}u(0)+f(t,u(t)),\ t>0,$$

where $0<\alpha\le\beta\le 1,\ \mu>0$, with the property that the solution decomposes, uniquely, into a periodic term (resp. almost periodic, almost automorphic, compact almost automorphic) and a second term that decays to zero. The general result on the asymptotic behavior is obtained by first establishing a sharp estimate on the solution family associated to the linear equation.

1. INTRODUCTION

Our concern in this paper is the existence, uniqueness and asymptotic behavior of bounded solutions for fractional order differential equations of the form

$$D_t^\alpha(u'(t)-u'(0))+\mu D_t^\beta(u(t)-u(0))-Au(t) = f(t,u(t)), \quad t>0,\mu\ge 0 \tag{1.1}$$

where $0<\alpha\le\beta\le 1$, the (unbounded) closed linear operator $A : D(A)\subset X\to X$ is the generator of a strongly continuous family $S_{\alpha,\beta}(t)$ of bounded and linear operators defined on a complex Banach space X, f is a vector-valued function, and D_t^α denotes the Riemann-Liouville fractional derivative of order α. Notice that equation (1.1) can be rewritten as

$$D_t^{\alpha+1}u(t)+\mu D_t^\beta u(t) = Au(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}u'(0)+\mu\frac{t^{-\beta}}{\Gamma(1-\beta)}u(0)+f(t,u(t)), \quad t>0,$$

or, equivalently,

$$\mathbb{D}_t^{\alpha+1}u(t)+\mu\mathbb{D}_t^\beta u(t)-Au(t) = f(t,u(t)), \quad t>0,$$

in terms of the Caputo fractional derivative, with prescribed initial conditions $u(0)$ and $u'(0)$.

Fractional order differential equations represent a subject of increasing interest in different contexts and areas of research, see e.g. [1, 5, 18, 19, 21, 33, 35], the survey paper [15]

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and the references therein. Our motivation to study equation (1.1) comes from recent investigations where a related class appears in connection with partial differential equations and Cauchy-time processes, a type of iterated stochastic processes (see [4]). Note that when $A = 2\Delta - \varepsilon^2\Delta^2$ (where Δ is the Laplace operator on $\mathbb{R}^N$), $\alpha = \beta = 1$ and $\mu = 1/\varepsilon^2$ the above equation was recently considered by Nane [31, Theorem 2.2]. In particular, in case $\mu = 0, \alpha = \beta = 1,\ A = -\Delta^2$ and $u'(0) = -\frac{2}{\pi}\Delta u_0$ with u_0 belonging to the domain $D(\Delta)$ of the operator Δ the equation

$$u''(t) = -\Delta^2 u(t) - \frac{2}{\pi t}\Delta u_0 + f(t), \quad t > 0, \tag{1.2}$$

has been studied in [31, Theorem 2.1] in connection with partial differential equations and iterated processes. A precise interplay between integral and fractional order differential equations was investigated in [20]. Very recently, the article [26] studied existence and uniqueness of solutions for the abstract Equation (1.1) in the special case $\alpha = \beta$ and the article [37] studied the nonlinear two-term time fractional diffusion wave equation (1.1) with $0 < \alpha < \beta - 1$, $0 < \beta < 1$ and $A = \frac{d^2}{dx^2}$.

Observe that one cannot apply semigroup theory directly to solve problem (1.1) in terms of a variation of constant formula. However, our methods based on the theory of (a,k)-regularized families [24] allow us to construct a solution. In fact, we will show that it is possible to give an abstract operator approach to equation (1.1) by defining first an ad-hoc solution family of strongly continuous operators $S_{\alpha,\beta}(t)$ for (1.1) in case $f \equiv 0$. It turns out, that it is a particular case of an (a,k)-regularized family, that we will identify (Lemma 2.5 below). Then, we will be able to show that the solution of equation (1.1) can be written in terms of a kind of variation of constants formula (cf. Theorem 3.1 below). Our method can be viewed as an extension of the ideas in the reference [5] for the abstract fractional order Cauchy problem.

We outline the plan of the paper as follows. In section 2, we introduce some preliminaries on fractional order derivatives, the Mittag-Leffler function and the concept of (α,β)-regularized families, which give us the necessary framework to apply an operator theoretical approach in the analysis of solutions for the abstract fractional order differential equation (1.1). In section 3 we consider the linear case, that is $f(t,u(t)) = f(t)$ and show existence and uniqueness of solutions of our problem. Section 4 contains a sharp estimate of mild solutions and their asymptotic behavior. The estimate is obtained for the linear equation and implies integrability of the corresponding solution family. Our estimate is a significant extension of a previous one obtained by Cuesta [10] who studied the equation with $\mu = 0$. This estimate is crucial for the derivation of the results on the asymptotic behavior of the semi-linear problem. The existence, uniqueness and the asymptotic behavior of mild solutions of the semi-linear problem is investigated in Section 5. Existence is proved by means of the contraction mapping theorem. Finally, we conclude the paper in Section 6 by giving concrete examples of operators where the situation in the previous sections can be applied.

2. PRELIMINARIES

Let $\alpha > 0, m = \lceil \alpha \rceil$ (the least integer larger than or equal to m) and $u : [0,\infty) \to X$, where X is a complex Banach space. We denote by $\mathbb{R}_+$ the closed interval $[0,\infty)$. The Riemann-Liouville fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s)u(s)ds, \quad t > 0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0,$$

and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin. When $\alpha = n$ is an integer, we define $D_t^n := \frac{d^n}{dt^n}, n = 1, 2, ...$ Note that $D_t^\alpha g_\beta = g_{\beta-\alpha}$ for $\beta \geq \alpha$. In particular, $D_t^\alpha 1 = g_{1-\alpha}$ for $0 < \alpha \leq 1$ and $D_t^\alpha g_\alpha = \delta_0$. We note that we write $D_t^\alpha 1$ for the fractional derivative of the Heaviside function.

The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\mathscr{L}(f)(\lambda) := \hat{f}(\lambda) := \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

when the limit exists.

In particular if f is such that $\int_0^t f(s)ds$ is is exponentially bounded, i.e., there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\| \int_0^t f(s)ds\| \leq Me^{\omega t}$ for all $t \geq 0$, then $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for $\mathrm{Re}(\lambda) > \omega$, and the integral is absolutely convergent. This remains true if we make the stronger assumption that f is exponentially bounded (see [2, Chapter I]). We have

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0)\lambda^{m-1-k}. \tag{2.1}$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$. The Mittag-Leffler function (see e.g. [13, 14] and [15]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{H_a} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where H_a is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counter-clockwise. The function $E_{\alpha,\beta}$ is an entire function which provides a generalization of several usual functions. For a recent review, we refer to [17].
An interesting property related with the Laplace transform of the Mittag-Leffler function is the following (cf. [15, (A.27) p.267]):

$$\mathscr{L}(t^{\beta-1}E_{\alpha,\beta}(-\rho^\alpha t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha}, \quad \mathrm{Re}(\lambda) > |\rho|^{1/\alpha}; \quad \alpha > 0, \beta > 0, \rho \in \mathbb{R}. \tag{2.2}$$

We also recall from [34, Proposition 0.1] the following result, which will be useful in what follows. We denote by $\mathbb{C}_+$ the open right-half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Proposition 2.1. *Let Y be a Banach space. Suppose $h : \mathbb{C}_+ \to Y$ is holomorphic and satisfies*

$$||\lambda h(\lambda)|| + ||\lambda^2 h'(\lambda)|| \leq M$$

for all $Re(\lambda) > 0$ and some $M > 0$. Then

$$||h^{(n)}(\lambda)|| \leq \frac{Mn!}{\lambda^{n+1}}$$

for all $\lambda > 0$ and $n = 0, 1, 2...$

We denote by

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is continuous}, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

the Banach space of X-valued bounded and continuous functions on $\mathbb{R}$, with natural norm.

Now we turn our attention to the family of function spaces built on X and which will play a key role in our study.

Let $P_T(X) := \{f \in BC(X) : f(t+T) = f(t)\ \forall t \in \mathbb{R}\}$ be the space of all vector-valued periodic functions, with fixed period $T > 0$. We denote by $AP(X)$ the space of almost periodic functions (in the sense of Bohr) which consists of all functions $f \in BC(X)$ such that for each $\varepsilon > 0$ there exists a $T > 0$ such that every subinterval of $\mathbb{R}$ of length T contains at least one point τ such that $||f(t+\tau) - f(t)||_\infty \le \varepsilon$. This definition is equivalent to the so-called Bochner criterion (cf. [32, Theorem 3.1.8]), namely, $f \in AP(X)$ if and only if for every sequence of reals (s'_n) there exists a subsequence (s_n) such that $(f(\cdot + s_n))$ is uniformly convergent on $\mathbb{R}$.

The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that a continuous bounded function f belongs to $AA_c(X)$ if and only if for any sequence (s'_n) of real numbers, there exists a subsequence $(s_n) \subset (s'_n)$ such that $\lim_{n\to\infty} f(t+s_n) =: \overline{f}(t)$ and $\lim_{n\to\infty} \overline{f}(t-s_n) = f(t)$ uniformly over compact subsets of $\mathbb{R}$.

The space of almost automorphic functions is defined as follows

$$\begin{aligned} AA(X) := \{f \in BC(X) :\ &\text{for all } (s'_n), \text{ there exists } (s_n) \subset (s'_n) \text{ such that} \\ &\lim_{n\to\infty} f(t+s_n) =: \overline{f}(t) \text{ and } \lim_{n\to\infty} \overline{f}(t-s_n) = f(t),\ \forall t \in \mathbb{R}\}, \end{aligned}$$

and is endowed with the norm $||\cdot||_\infty$. Almost automorphic functions were introduced by Bochner in connection to some aspects of differential geometry [6, 7, 8, 9]. For more details about this topic we refer to the book [32] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. We note that more general classes of function spaces have been introduced and recently applied to semi-linear differential equations (see [23] and references therein).

We have that $P_T(X), AP(X), AA_c(X)$ and $AA(X)$ are Banach spaces with the norm $||\cdot||_\infty$ and the following inclusions hold:

$$P_T(X) \subset AP(X) \subset AA_c(X) \subset AA(X) \subset BC(X).$$

Now we consider the space

$$C_0(X) := \{f : \mathbb{R}_+ \to X :\ f \text{ is continuous and } \lim_{t\to\infty} ||f(t)|| = 0\},$$

where again $\mathbb{R}_+ = [0,\infty)$, and define the space of asymptotically periodic functions as

$$AP_T(X) := P_T(X) \oplus C_0(X).$$

Analogously, we define the space of asymptotically almost periodic functions

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural inclusions

$$AP_T(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

Note that all these inclusions are proper.

Definition 2.2. ([34]) *A strongly measurable family of operators* $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ *is called uniformly integrable (or strongly integrable) if* $\int_0^\infty ||T(t)||dt < \infty$.

Throughout the following, for any uniformly integrable family of such operators $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$, we will use the notation $\|T\| := \int_0^\infty ||T(t)||dt < \infty$.

Note that exponentially stable C_0-semigroups are examples of uniformly integrable families of operators.

Throughout, we will use $\mathscr{M}(X)$ to denote any of the spaces $AP_T(X), AAP(X), AAA_c(X)$ and $AAA(X)$ defined above.

We recall from [30] the following result.

Theorem 2.3. *Let* $\{S(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ *be a uniformly integrable and strongly continuous family. Assume that* f *belongs to the space* $\mathscr{M}(X)$, *and set*

$$w(t) := \int_{-\infty}^{t} S(t-s)f(s)\,ds. \tag{2.3}$$

Then w *belongs to the same space as* f.

In order to give an operator theoretical approach to equation (1.1) we introduce the following definition.

Definition 2.4. *Let* $\mu \geq 0$ *and* $0 \leq \alpha, \beta \leq 1$ *be given. Let* A *be a closed linear operator with domain* $D(A)$ *defined on a Banach space* X. *We call* A *the generator of an* $(\alpha,\beta)_\mu$-*regularized family if there exist* $\omega \geq 0$ *and a strongly continuous function* $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathscr{L}(X)$ *such that* $\{\lambda^{\alpha+1} + \mu\lambda^{\beta} : Re(\lambda) > \omega\} \subset \rho(A)$ *and*

$$H(\lambda)x := \lambda^{\alpha}(\lambda^{\alpha+1} + \mu\lambda^{\beta} - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x dt, \quad Re(\lambda) > \omega, \quad x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\mu = 0$ and $\alpha = 0$, this corresponds to the case of a C_0-semigroup whereas the case $\mu = 0$, $\alpha = 1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to the monograph [2].

Lemma 2.5. *Let* $0 \leq \alpha, \beta \leq 1$, $\mu \geq 0$. *There exist Laplace transformable functions* $a_{\alpha,\beta} \in C^1(\mathbb{R}_+)$ *and* $k_{\alpha,\beta} \in C^1(\mathbb{R}_+)$ *such that* $\hat{a}_{\alpha,\beta}(\lambda) = \frac{1}{\lambda^{\alpha+1}+\mu\lambda^{\beta}}$ *and* $\hat{k}_{\alpha,\beta}(\lambda) = \frac{\lambda^{\alpha}}{\lambda^{\alpha+1}+\mu\lambda^{\beta}}$.

Proof. Suppose first that $\alpha > 0$. Then we can choose

$$k_{\alpha,\beta}(t) := E_{\alpha+1-\beta,1}(-\mu^{1/\alpha+1-\beta}t^{\alpha+1-\beta})$$

because $\alpha + 1 > \beta$. We also define $a_{\alpha,\beta}(t) := \int_0^t g_\alpha(s)k_{\alpha,\beta}(t-s)ds$. Suppose now that $\alpha = 0$ and $\beta < 1$. Then we define

$$k_{0,\beta}(t) = a_{0,\beta}(t) := t^{-\beta}E_{1-\beta,1-\beta}(-(\mu)^{1/1-\beta}t^{1-\beta}).$$

Finally, in case $\alpha = 0$ and $\beta = 1$ we take $k_{0,1}(t) = a_{0,1}(t) := 1 + \mu$. □

As a consequence of the previous lemma, we note that for $\alpha, \beta \in [0,1]$ fixed, we have

$$H(\lambda) = \frac{\hat{k}_{\alpha,\beta}(\lambda)}{\hat{a}_{\alpha,\beta}(\lambda)} \left(\frac{1}{\hat{a}_{\alpha,\beta}(\lambda)} - A\right)^{-1},$$

for Re(λ) large enough. It follows that an $(\alpha,\beta)_\mu$-regularized family corresponds to an (a,k)-regularized family with $a(t) \equiv a_{\alpha,\beta}(t)$ and $k(t) \equiv k_{\alpha,\beta}(t)$. The concept of (a,k)-regularized family was introduced in [24] and studied in a series of papers in the past years (see [25, 27, 28, 29, 36]). For more information of this subject we refer to the cited references and to the recent paper by Kostic [22] which have studied and extended the notion of (a,k)-regularized families, discussing its basic structural properties and covering subjects like regularity, perturbations, duality, spectral properties and subordination principles. Moreover, several illustrative examples are provided.

Recall that $a_{\alpha,\beta} = g_\alpha * k_{\alpha,\beta}$. Notice that $k_{\alpha,\beta}(0) = 1$ and $k_{\alpha,\beta}$ is differentiable with

$$\frac{dk_{\alpha,\beta}(t)}{dt} = -\mu t^{\alpha-\beta} E_{\alpha+1-\beta,\alpha+1-\beta}(-\mu^{1/(\alpha+1-\beta)} t^{\alpha+1-\beta}), \quad t > 0.$$

Then the following result is a direct consequence of [24, Proposition 3.1 and Lemma 2.2].

Proposition 2.6. *Let $0 < \alpha,\beta \le 1$ and $\mu \ge 0$. Let $S_{\alpha,\beta}(t)$ be an $(\alpha,\beta)_\mu$-regularized family on X with generator A. Then the following assertions hold true:*

(a) *$S_{\alpha,\beta}(t)$ is strongly continuous and $S_{\alpha,\beta}(0) = I$.*

(b) *For all $x \in D(A)$ and $t \ge 0$ we have $S_{\alpha,\beta}(t)x \in D(A)$ and $AS_{\alpha,\beta}(t)x = S_{\alpha,\beta}(t)Ax$.*

(c) *Let $x \in X$ and $t \ge 0$. Then $\int_0^t a_{\alpha,\beta}(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and*

$$S_{\alpha,\beta}(t)x = k_{\alpha,\beta}(t)x + A\int_0^t (g_\alpha * k_{\alpha,\beta})(t-s)S_{\alpha,\beta}(s)xds. \tag{2.4}$$

(d) *For all $x \in D(A)$ we have $S_{\alpha,\beta}(\cdot)x \in C^1(\mathbb{R}_+;X)$.*

The following characterization of generators of $(\alpha,\beta)_\mu$-regularized families, analogous to the Hille-Yosida Theorem for C_0- semigroups, follows directly from [24, Theorem 3.4].

Theorem 2.7. *Let A be a closed linear densely defined operator in a Banach space X. Let $0 < \alpha,\beta \le 1$ and $\mu \ge 0$. Then the following assertions are equivalent.*

(i) *The operator A is the generator of an $(\alpha,\beta)_\mu$-regularized family $(S_{\alpha,\beta}(t))_{t\ge 0}$ satisfying $||S_{\alpha,\beta}(t)|| \le Me^{\omega t}$ for all $t \ge 0$ and for some constants $M > 0$ and $\omega \in \mathbb{R}$.*

(ii) *There exist constants $\omega \in \mathbb{R}$ and $M > 0$ such that*
   (P1) *$\lambda^{\alpha+1} + \mu\lambda^\beta \in \rho(A)$ for all $\lambda > \omega$ and*
   (P2) *$H(\lambda) := \lambda^\alpha(\lambda^{\alpha+1} + \mu\lambda^\beta - A)^{-1}$ satisfies the estimates*

$$||H^{(n)}(\lambda)|| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega, \quad n = 0,1,2...$$

We also note the following result on perturbation of generators of $(\alpha,\beta)_\mu$-regularized families.

Theorem 2.8. *Let A be the generator of an $(\alpha,\beta)_\mu$-regularized family and $B \in \mathscr{L}(X)$. Then $A+B$ is also a generator of an $(\alpha,\beta)_\mu$-regularized family.*

Proof. According to [29, Corollary 3.2], we only need to show that there exists a function b such that $b * k_{\alpha,\beta} = a_{\alpha,\beta}$. Choosing $b(t) = g_\alpha(t)$ we obtain the result. □

Note that this is in contrast with the result for integrated semigroups [2] where in general, bounded perturbations do not preserve the generation property.

Finally, let us recall that a closed and densely defined operator A is said to be ω-sectorial of angle θ if there exist $\theta \in [0,\pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\} \setminus \{\omega\}, \tag{2.5}$$

and

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta. \tag{2.6}$$

These are generators of holomorphic semigroups. In case $\omega = 0$ we merely say that A is sectorial of angle θ. We should mention that in the general theory of sectorial operators, it is only required that (2.6) hold in a sector of angle γ for some $\gamma > 0$. Our restriction corresponds to the class of operators used in the present paper.

3. Existence and Uniqueness: The Linear case

In this section we study the existence and uniqueness of solutions for the linear fractional evolution equation

$$D_t^\alpha(u'(t) - u'(0)) + \mu D_t^\beta(u(t) - u(0)) - Au(t) = f(t), \quad t \geq 0, \tag{3.1}$$

where $0 < \alpha, \beta \leq 1, \mu \geq 0$ and A is a closed linear operator which generates an exponentially bounded $(\alpha, \beta)_\mu$-regularized family. Notice that in terms of the Caputo fractional derivative, which we denote by $\mathbb{D}_t^\alpha$, equation (3.1) is equivalent to

$$\mathbb{D}_t^{\alpha+1}u(t) + \mu \mathbb{D}_t^\beta u(t) - Au(t) = f(t), \quad 0 < \alpha, \beta \leq 1, \quad \mu \geq 0. \tag{3.2}$$

where then one has to specify $u(0)$ and $u'(0)$ (see e.g. [5, (1.20)]). We say that $u \in C^1(\mathbb{R}_+; X)$ is a *strong solution* of equation (3.1) if $u(t) \in D(A)$ for all $t \geq 0$ and u satisfies (3.1). The following theorem was proved in [26] in the special case $\alpha = \beta$.

Theorem 3.1. *Let $0 < \alpha \leq \beta \leq 1$, $\mu \geq 0$ and A be the generator of an $(\alpha, \beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ satisfying $\|S_{\alpha,\beta}(t)\| \leq Me^{\omega t}$, $t \geq 0$. For each exponentially bounded function $f : \mathbb{R}_+ \to D(A)$, the unique exponentially bounded solution of Equation* (3.1) *is given by*

$$\begin{aligned} u(t) =& S_{\alpha,\beta}(t)u(0) + (1 * S_{\alpha,\beta})(t)u'(0) + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta})(t)u(0) \\ &+ (g_\alpha * S_{\alpha,\beta} * f)(t), \end{aligned} \tag{3.3}$$

provided $u(0), u'(0) \in D(A)$.

Proof. Without loss of generality, we may assume that $\|f(t)\| \leq Me^{\omega t}$, $t \geq 0$. Taking the Laplace transform of (3.2) (equivalently (3.1)) we obtain

$$(\lambda^{\alpha+1} + \mu\lambda^\beta - A)\hat{u}(\lambda) = \lambda^\alpha u(0) + \lambda^{\alpha-1}u'(0) + \mu\lambda^{\beta-1}u(0) + \hat{f}(\lambda), \quad \mathrm{Re}(\lambda) > \omega.$$

Hence, by Definition 2.4

$$\begin{aligned} \hat{u}(\lambda) =& \hat{S}_{\alpha,\beta}(\lambda)u(0) + (\widehat{1 * S_{\alpha,\beta}})(\lambda)u'(0) \\ &+ \mu(\widehat{g_{1+\alpha-\beta} * S_{\alpha,\beta}})(\lambda)u(0) + (\widehat{g_\alpha * S_{\alpha,\beta}} * f)(\lambda), \quad \mathrm{Re}(\lambda) > \omega. \end{aligned}$$

Inversion of the Laplace transform shows that $u(t)$ has the form (3.3). To check that it is a strong solution of Equation (3.1), we have to prove that $u \in C^1(\mathbb{R}_+; X)$ and $u(t) \in D(A)$ for all $t \geq 0$. Indeed, by Proposition 2.6 (a) and (d) we obtain the identity

$$\begin{aligned} u'(t) =& S'_{\alpha,\beta}(t)u(0) + S_{\alpha,\beta}(t)u'(0) + \mu g_{1+\alpha-\beta}(t)u(0) \\ &+ \mu(g_{1+\alpha-\beta} * S'_{\alpha,\beta})(t)u(0) + (g_\alpha * f)(t) + (g_\alpha * S'_{\alpha,\beta} * f)(t), \end{aligned}$$

because $u(0) \in D(A)$ and $f(t) \in D(A)$ for $t \geq 0$. On the other hand, by Proposition 2.6 (b), the fact that A is closed and the assumptions $u(0), u'(0) \in D(A)$ and $f(t) \in D(A)$ we obtain $u(t) \in D(A)$. For uniqueness, let v be an exponentially bounded solution of equation (3.1).

Then it is Laplace transformable and we obtain $\hat{v}(\lambda) = \hat{u}(\lambda)$ for all λ sufficiently large. Uniqueness of the Laplace transform then gives $u \equiv v$. □

A sufficient condition to obtain generators of an $(\alpha,\beta)_\mu$-regularized family are given in the following general result.

Theorem 3.2. *Let $0 < \alpha \le \beta \le 1$, $\mu > 0$ and A be a sectorial operator of angle $\beta\pi/2$. Then A generates a bounded $(\alpha,\beta)_\mu$-regularized family.*

Proof. Define $g(\lambda) = \lambda^{\alpha+1} + \mu\lambda^\beta$ and let $\lambda = re^{i\theta}$ with $|\theta| < \pi/2$ and $r > 0$. We observe that

$$\arg(g(re^{i\theta})) = \operatorname{Im}\log(g(re^{i\theta})) = \operatorname{Im}\int_0^\theta \frac{d}{dt}\log(g(re^{it}))dt = \operatorname{Im}\int_0^\theta \frac{g'(re^{it})ire^{it}}{g(re^{it})}dt$$

with

$$\lambda\frac{g'(\lambda)}{g(\lambda)} = \alpha + \frac{(\beta-\alpha)\mu}{\lambda^{\alpha+1-\beta}+\mu} + \frac{\lambda^{\alpha+1-\beta}}{\lambda^{\alpha+1-\beta}+\mu}.$$

Next, observe that for $\mu > 0$ and all $z \in \mathbb{C}$ with $\operatorname{Re}(z) \ge 0$ we have $\frac{\mu}{|z+\mu|} \le 1$ and $\frac{|z|}{|z+\mu|} \le 1$. Moreover, since $0 < \alpha \le \beta \le 1$ we have $0 < 1+\alpha-\beta \le 1$ and hence $\operatorname{Re}(\lambda^{1+\alpha-\beta}) \ge 0$. It follows that

$$\left|\operatorname{Im}\int_0^\theta \frac{\mu}{r^{\alpha+1-\beta}e^{i(\alpha+1-\beta)t}+\mu}dt\right| \le \int_0^\theta \left|\frac{\mu}{r^{\alpha+1-\beta}e^{i(\alpha+1-\beta)t}+\mu}\right|dt \le \theta,$$

and

$$\left|\operatorname{Im}\int_0^\theta \frac{r^{\alpha+1-\beta}e^{i(\alpha+1-\beta)t}}{r^{\alpha+1-\beta}e^{i(\alpha+1-\beta)t}+\mu}dt\right| \le \theta.$$

Therefore

$$|\arg(g(re^{i\theta}))| \le \alpha|\theta| + (\beta-\alpha)|\theta| + |\theta| < \beta\frac{\pi}{2} + \frac{\pi}{2}. \tag{3.4}$$

We conclude that $g(\lambda) \in S_{\beta\frac{\pi}{2}}$ for all $\operatorname{Re}(\lambda) > 0$. From the above, we have that $H(\lambda) = \dfrac{\lambda^{\alpha-\beta}}{\lambda^{\alpha+1-\beta}+\mu}g(\lambda)(g(\lambda)-A)^{-1}$ is well defined and satisfies the estimate

$$||\lambda H(\lambda)|| = ||\frac{\lambda^{\alpha+1-\beta}}{\lambda^{\alpha+1-\beta}+\mu}g(\lambda)(g(\lambda)-A)^{-1}|| \le \frac{|\lambda|^{\alpha+1-\beta}}{|\lambda^{\alpha+1-\beta}+\mu|}M \le M_1 \text{ for all } \operatorname{Re}(\lambda) > 0.$$

In particular, note that $||\lambda^{\beta-\alpha}H(\lambda)|| \le \frac{M}{|\lambda^{\alpha+1-\beta}+\mu|}$. For $H'(\lambda)$ one obtains that

$$\lambda^2 H'(\lambda) = \alpha\lambda H(\lambda) - (\alpha+1)\lambda^2 H(\lambda)^2 - \beta\mu\lambda^{\beta-\alpha+1}H(\lambda)H(\lambda),$$

and hence we conclude that there exists a constant $C > 0$ such that for all $\operatorname{Re}(\lambda) > 0$,

$$\begin{aligned}||\lambda^2 H'(\lambda)|| &\le \alpha||\lambda H(\lambda)|| + (\alpha+1)||\lambda^2 H(\lambda)^2|| + \beta M\frac{\mu}{|\lambda^{\alpha+1-\beta}+\mu|}||\lambda H(\lambda)|| \\ &\le \alpha M_1 + (\alpha+1)M_1^2 + \beta M_1 C.\end{aligned}$$

By Proposition 2.1 we have that (P2) in Theorem 2.7 is satisfied with $\omega = 0$, proving the claim. □

Recall that a closed and densely defined operator A is said to be ω-sectorial of angle θ if $\omega I + A$ is sectorial of angle θ. We note that the concept of operator ω-sectorial of angle θ was used by Cuesta [10] to establish estimates of solution operators that imply their integrability in case $\omega < 0$. Starting with the work [1], this result has been used by some authors to establish necessary conditions on the operator A for the existence of bounded mild solutions to some evolution equations (cf [11]).

As a consequence of Proposition 3.2 and Theorem 2.8 we obtain the following result.

Corollary 3.3. *Let $0<\alpha\le\beta\le 1$, $\mu>0$ and A be an ω-sectorial operator of angle $\beta\pi/2$. Then A generates an exponentially bounded $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$.*

Proof. The claim follows from the decomposition $A=(\omega I+A)-\omega I$. ☐

Using the definition of $S_{\alpha,\beta}(t)$ by means of the Laplace transform, the following result is readily established.

Corollary 3.4. *Let $0<\alpha\le\beta\le 1$, $\mu>0$ and A be an ω- sectorial operator of angle $\beta\pi/2$. For each exponentially bounded function $f:\mathbb{R}_+\to D(A)$, the unique exponentially bounded solution of Equation* (3.1) *is given by*

$$\begin{aligned} u(t) =& S_{\alpha,\beta}(t)u(0)+(1*S_{\alpha,\beta})(t)u'(0)+\mu(g_{1+\alpha-\beta}*S_{\alpha,\beta})(t)u(0)\\ &+(g_\alpha*S_{\alpha,\beta}*f)(t),\quad 0<\alpha,\beta\le 1,\ \mu\ge 0 \end{aligned} \tag{3.5}$$

provided $u(0),u'(0)\in D(A)$, and where $S_{\alpha,\beta}(t)$ is the $(\alpha,\beta)_\mu$-regularized family generated by A.

Combining the previous results, we can give the following corollary on existence of solutions of Equation (3.1).

Corollary 3.5. *Let $0<\alpha\le\beta\le 1$, $\mu>0$ and A be an ω-sectorial operator of angle $\beta\pi/2$. Then for each exponentially bounded function $f:\mathbb{R}_+\to D(A)$, there exists a unique exponentially bounded solution of Equation* (3.1) *such that $u(0),u'(0)\in D(A)$.*

In the particular case of $\omega=0$ we have the following result.

Corollary 3.6. *Let $0<\alpha\le\beta\le 1$, $\mu>0$ and A be a sectorial operator of angle $\beta\pi/2$. Then there exists a unique polynomially bounded solution of equation (3.1) for all polynomially bounded function $f:\mathbb{R}_+\to D(A)$, such that $u(0),u'(0)\in D(A)$.*

4. Asymptotic behavior of mild solutions

In this section we study bounded mild solutions for the linear fractional differential equation

$$\mathbb{D}_t^{\alpha+1}u(t)+\mu\mathbb{D}_t^{\beta}u(t)-Au(t)=h(t),\quad t\ge 0,\quad 0<\alpha\le\beta\le 1,\ \mu\ge 0, \tag{4.1}$$

with initial conditions $u(0)=x$, $u'(0)=y$ (specified in X) and A is a ω-sectorial operator of angle $\beta\pi/2$.

Recall that a function $u\in C^1(\mathbb{R}_+;X)$ is called a strong solution of (4.1) on $\mathbb{R}_+$ if $u(t)\in D(A)$ and (4.1) holds on $\mathbb{R}_+$. If merely $u(t)\in X$ instead of the domain of A, we say that u is a *mild solution* of the linear equation (4.1). Since A is ω-sectorial of angle $\beta\pi/2$, by Theorem 3.1 and Theorem 3.2, a mild solution for (4.1) always exists and is given by:

$$u(t)=S_{\alpha,\beta}(t)x+(g_1*S_{\alpha,\beta})(t)y+\mu(g_{1+\alpha-\beta}*S_{\alpha,\beta}(t))x+(S_{\alpha,\beta}*f)(t), \tag{4.2}$$

where $0<\alpha\le\beta\le 1$, $\mu>0$, and $x,y\in X$; $f:=h*g_\alpha$ and $S_{\alpha,\beta}(t)$ is the $(\alpha,\beta)_\mu$-regularized family generated by A (cf. Corollaries 3.3 and 3.4).

In order to study the asymptotic behavior of mild solutions, we need to prove the integrability of the $(\alpha,\beta)_\mu$-regularized family generated by A. The following theorem establishes a sharp estimate on the solution family corresponding to the linear equation. It extends in a significant way a fundamental result of Cuesta [10, Theorem 1] and yields the integrability result.

Theorem 4.1. *Let $0<\alpha\le\beta\le 1$, $\mu>0$ and $\omega<0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$, then A generates an $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ satisfying the estimate*

$$||S_{\alpha,\beta}(t)||\le\frac{C}{1+|\omega|(t^{\alpha+1}+\mu t^{\beta})},\quad t\ge 0,\tag{4.3}$$

for some constant $C>0$ depending only on α,β.

Proof. Define $g(\lambda):=\lambda^{\alpha+1}+\mu\lambda^{\beta}$ and let $\lambda=re^{i\theta}$ with $|\theta|<\pi/2$ and $r>0$. It follows from (3.4) in the proof of Theorem 3.2 that $g(\lambda)\in S_{\beta\pi/2}$ for all $\lambda\in\mathbb{C}$, with $\mathrm{Re}(\lambda)>0$. Since A is ω-sectorial of angle $\beta\pi/2$ it follows that $\lambda^{\alpha+1}+\mu\lambda^{\beta}\in\rho(A)$ for $\mathrm{Re}(\lambda)>0$ and

$$||(\lambda^{\alpha+1}+\mu\lambda^{\beta}-A)^{-1}||\le\frac{M}{|\lambda^{\alpha+1}+\mu\lambda^{\beta}-\omega|},\tag{4.4}$$

for all $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)>0$.

Define

$$S_{\alpha,\beta}(t)=\frac{1}{2\pi i}\int_{\gamma}e^{\lambda t}\lambda^{\alpha}(\lambda^{\alpha+1}+\mu\lambda^{\beta}-A)^{-1}d\lambda,\tag{4.5}$$

where γ is a positively oriented path, lying inside the sector $\omega+S_\theta$, $0\le\theta<\beta\pi/2$, whose support Γ is the set of all $\lambda\in\mathbb{C}$ such that $\lambda^{\alpha+1}+\mu\lambda^{\beta}$ lies on the boundary of $B_{1/(t^{\alpha+1}+\mu t^{\beta})}$, for $t>0$, where

$$B_\delta:=\{\delta+S_\theta\}\cup S_\phi,$$

δ is a positive constant and $0<\phi<\theta$.

Notice that with this definition, the resolvent $(\lambda^{\alpha+1}+\mu\lambda^{\beta}-A)^{-1}$ is well defined and therefore the representation (4.5) of $S_{\alpha,\beta}(t)$ is meaningful.

By Definition 2.4, taking the inverse Laplace transform, we have that (4.5) gives the $(\alpha,\beta)_\mu$-regularized family generated by A.

Let us split γ into two paths γ_1 and γ_2 whose supports Γ_1 and Γ_2 are the sets formed by the complex λ such that $\lambda^{\alpha+1}+\mu\lambda^{\beta}$ lies on the intersection of Γ and the boundaries of $\frac{1}{t^{\alpha+1}+\mu t^{\beta}}+S_\theta$ and S_ϕ respectively, i.e.

$$\Gamma_1=\Gamma\cap\overline{\{1/(t^{\alpha+1}+\mu t^{\beta})+S_\theta\}}\quad\text{and}\quad\Gamma_2=\Gamma\cap\overline{S_\phi}.$$

Therefore $\Gamma=\Gamma_1\cup\Gamma_2$ and $S_{\alpha,\beta}(t)=I_1(t)+I_2(t)$, for $t\ge 0$, where

$$I_j(t)=\frac{1}{2\pi i}\int_{\gamma_j}e^{\lambda t}\lambda^{\alpha}(\lambda^{\alpha+1}+\mu\lambda^{\beta}-A)^{-1}d\lambda,\quad j=1,2.$$

By (4.4) we have that the integral I_j, $(j=1,2)$ can be estimated by

$$\begin{aligned}||I_j(t)|| &\le \frac{1}{2\pi}\int_{\gamma_j}|e^{t\lambda}|\cdot|\lambda|^{\alpha}||(\lambda^{\alpha+1}+\mu\lambda^{\beta}-A)^{-1}|||d\lambda|\\ &\le \frac{M}{2\pi}\int_{\gamma_j}|e^{t\lambda}|\frac{|\lambda|^{\alpha}}{|\lambda^{\alpha+1}+\mu\lambda^{\beta}-\omega|}||d\lambda|.\end{aligned}$$

Furthermore, since

$$\frac{1}{|\lambda^{\alpha+1}+\mu\lambda^{\beta}-\omega|}\le\frac{t^{\alpha+1}}{(1+|\omega|(t^{\alpha+1}+\mu t^{\beta}))},\quad\lambda\in\Gamma_1,$$

we have

$$||I_1(t)|| \leq \frac{Mt^{\alpha+1}}{2\pi(1+|\omega|(t^{\alpha+1}+\mu t^{\beta}))}\int_{\gamma_1}|e^{t\lambda}|\cdot|\lambda|^{\alpha}|d\lambda|.$$

Define $f(\varphi) := \arg(g(re^{i\varphi}))$ for $\lambda = re^{i\varphi}$, where $|\varphi| < \beta\pi/2$. Then $f(\varphi) = \arctan(h(\varphi))$ where

$$h(\varphi) = \frac{r^{\alpha+1}\sin(\alpha\varphi+\varphi)+\mu r^{\beta}\sin(\beta\varphi)}{r^{\alpha+1}\cos(\alpha\varphi+\varphi)+\mu r^{\beta}\cos(\beta\varphi)}.$$

A simple calculation shows that

$$h'(\varphi) = \frac{(\alpha+1)r^{2\alpha+2}+\mu^2\beta r^{2\beta}+\mu(\alpha+\beta+1)r^{\alpha+\beta+1}\cos((\alpha+1-\beta)\varphi)}{[r^{\alpha+1}\cos(\alpha\varphi+\varphi)+\mu r^{\beta}\cos(\beta\varphi)]^2}.$$

Since $0 < \alpha \leq \beta \leq 1$, we deduce $0 < |\varphi|(\alpha+1-\beta) \leq |\varphi| < \beta\pi/2 \leq \pi/2$ and hence $\cos((\alpha+1-\beta)\varphi) > 0$. We conclude that f is injective and therefore letting the function $\gamma_1(s) = -se^{if^{-1}(\theta)}, s \in \mathbb{R}$ we get

$$\begin{aligned}
||I_1(t)|| &\leq \frac{Mt^{\alpha+1}}{\pi(1+|\omega|(t^{\alpha+1}+\mu t^{\beta}))}\int_0^{\infty} e^{-t\cos(f^{-1}(\theta))s}\cdot s^{\alpha}ds \\
&\leq \frac{M}{\pi\cos(f^{-1}(\theta))^{\alpha+1}(1+|\omega|(t^{\alpha+1}+\mu t^{\beta}))}
\end{aligned}$$

for $t \geq 0$.

We now proceed to estimate I_2. To this end note that for $\omega < 0$ we have

$$|\lambda^{\alpha+1}+\mu\lambda^{\beta}-\omega| \geq |\omega|, \quad \lambda \in \Gamma_2.$$

Thus, for $\gamma_2(s) = -se^{if^{-1}(\phi)}, s \in \mathbb{R}$ we obtain

$$\begin{aligned}
||I_2(t)|| &\leq \frac{M}{2\pi}\int_{\gamma_2}|e^{t\lambda}|\frac{|\lambda|^{\alpha}}{|\lambda^{\alpha+1}+\mu\lambda^{\beta}-w|}||d\lambda| \\
&\leq \frac{M}{2\pi|\omega|}\int_{\gamma_2}|e^{t\lambda}||\lambda|^{\alpha}||d\lambda| \\
&\leq \frac{M}{\pi|\omega|}\int_0^{\infty} e^{-t\cos(f^{-1}(\phi))s}\cdot s^{\alpha}ds \\
&\leq \frac{M\Gamma(\alpha+1)}{\pi|\omega|\cos(f^{-1}(\phi))^{\alpha+1}t^{\alpha+1}},
\end{aligned}$$

and hence the conclusion follows. □

Corollary 4.2. *Let $0 < \alpha < \beta \leq 1$. Under the hypotheses of the above theorem, we have that $(g_{\alpha} * S_{\alpha,\beta})(t) \to 0$ as $t \to \infty$.*

Proof. Notice that by the estimate (4.3) we have

$$\begin{aligned}
||(g_{\alpha} * S_{\alpha,\beta})(t)|| &\leq \int_0^t g_{\alpha}(t-\tau)||S_{\alpha,\beta}(\tau)||d\tau \\
&\leq \Gamma(1-\beta)\int_0^t g_{\alpha}(t-\tau)g_{1-\beta}(\tau)\tau^{\beta}||S_{\alpha,\beta}(\tau)||d\tau \\
&\leq \frac{\Gamma(1-\beta)}{|\omega|\mu}\int_0^t g_{\alpha}(t-\tau)g_{1-\beta}(\tau)d\tau = \frac{\Gamma(1-\beta)}{|\omega|\mu}g_{\alpha-\beta+1}(t),
\end{aligned}$$

for all $t > 0$. Hence there exists a constant $C > 0$ such that $||(g_\alpha * S_{\alpha,\beta})(t)|| \le Ct^{\alpha-\beta}$. Since $\alpha - \beta < 0$, we obtain the claim. $\square$

We recall that $\mathcal{M}(X)$ is any one of the spaces $AP_T(X), AAP(X), AAA_c(X), AAA(X)$ defined in Section 2. Using the above theorem, we can prove the following theorem which is one of the main results in this section.

Theorem 4.3. *Let $0 < \alpha \le \beta \le 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$. Then for each $f \in \mathcal{M}(X)$ there exists a unique mild solution u of equation (4.1) such that $u \in \mathcal{M}(X)$.*

Proof. Let $f \in \mathcal{M}(X)$ be given. By Theorem 4.1, A generates a uniformly integrable $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ on the Banach space X, and the unique mild solution for (4.1) is given by (4.2), that is:

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + (S_{\alpha,\beta} * f)(t),$$

where $0 < \alpha \le \beta \le 1$, $\mu > 0$, and $x, y \in X$. Let $\Lambda \in \{P_T(X), AP(X), AA_c(X), AA(X)\}$. We claim that $S_{\alpha,\beta} * f \in \mathcal{M}(X)$. In fact, for $f = g + h$ where $g \in \Lambda$ and $h \in C_0(X)$, we have that

$$(S_{\alpha,\beta} * f)(t) = \int_{-\infty}^{t} S_{\alpha,\beta}(t-s)g(s)ds + \int_0^t S_{\alpha,\beta}(t-s)h(s)ds - \int_{-\infty}^{0} S_{\alpha,\beta}(t-s)g(s)ds.$$

By Theorem 2.3 and Corollary 4.2 we conclude that the first term on the right hand side of the above equality belongs to Λ. Let us show that

$$H(t) := \int_0^t S_{\alpha,\beta}(t-s)h(s)ds - \int_{-\infty}^{0} S_{\alpha,\beta}(t-s)g(s)ds$$

belongs to $C_0(X)$. Since $h \in C_0(X)$, for each $\varepsilon > 0$ there exists a constant $T > 0$ such that $||h(s)|| \le \varepsilon$ for all $s \ge T$. Then for all $t \ge 2T$, we deduce

$$\begin{aligned}
||H(t)|| \le & \int_0^{t/2} ||S_{\alpha,\beta}(t-s)||\,||h(s)||ds + \int_{t/2}^{t} ||S_{\alpha,\beta}(t-s)||\,||h(s)||ds \\
& + \int_{-\infty}^{0} ||S_{\alpha,\beta}(t-s)||\,||g(s)||ds \\
\le & ||h||_\infty \int_0^{t/2} ||S_{\alpha,\beta}(t-s)||ds + \varepsilon \int_{t/2}^{t} ||S_{\alpha,\beta}(t-s)||ds \\
& + ||g||_\infty \int_{-\infty}^{0} ||S_{\alpha,\beta}(t-s)||ds \\
\le & (||h||_\infty + ||g||_\infty) \int_{t/2}^{\infty} ||S_{\alpha,\beta}(\tau)||d\tau + \varepsilon ||S_{\alpha,\beta}||.
\end{aligned}$$

Therefore $\lim_{t\to\infty} H(t) = 0$, that is, $H \in C_0(X)$. This proves the claim. Notice that, by the inequality (4.3), we also have $S_{\alpha,\beta} \in \mathcal{M}(X)$. We now prove that $g_1 * S_{\alpha,\beta} \in \mathcal{M}(X)$. In fact, by (4.3) we have $\sup_{t>\tau} ||tS_{\alpha,\beta}(t)|| < \infty$, for each $\tau > 0$. Moreover, by (4.4), we also have $||\hat{S}_{\alpha,\beta}(\lambda)|| \to 0$ as $\lambda \to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [2, Theorem 4.2.9]) we conclude that $||(g_1 * S_{\alpha,\beta})(t)|| \to 0$ as $t \to \infty$. It remains only to show that $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in \mathcal{M}(X)$ for $\alpha < \beta$. To see this, we estimate $||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)||$ as follows. Let $0<\varepsilon<\beta-\alpha$ be given, then

$$||g_{1+\alpha-\beta}*S_{\alpha,\beta}(t)|| = ||\Gamma(\beta-\alpha-\varepsilon)\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\varepsilon}(\tau)\tau^{\alpha-\beta+\varepsilon+1}S_{\alpha,\beta}(\tau)d\tau||$$
$$\leq \Gamma(\beta-\alpha-\varepsilon)\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\varepsilon}(\tau)\tau^{\alpha-\beta+\varepsilon+1}||S_{\alpha,\beta}(\tau)||d\tau$$

where, thanks to (4.3), we have that

$$\Gamma(\beta-\alpha-\varepsilon)\tau^{\alpha-\beta+\varepsilon+1}||S_{\alpha,\beta}(\tau)|| \quad \leq \quad \frac{M\tau^{\alpha-\beta+\varepsilon-1}}{1+|\omega|\tau^{\alpha+1}} = \frac{M\tau^{-\beta+\varepsilon}}{\frac{1}{\tau^{\alpha+1}}+|\omega|}, \quad \tau>0.$$

Since $\varepsilon<\beta$, there exists a constant $C>0$ such that $\tau^{\alpha-\beta+\varepsilon+1}||S_{\alpha,\beta}(\tau)||\leq C$. Therefore,

$$||g_{1+\alpha-\beta}*S_{\alpha,\beta}(t)|| \quad \leq \quad C\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\varepsilon}(\tau)d\tau = Cg_{1-\varepsilon}(t) = Ct^{-\varepsilon},$$

which shows that $||g_{1+\alpha-\beta}*S_{\alpha,\beta}(t)||\to 0$ as $t\to\infty$. □

5. Asymptotic behavior of solutions for the semilinear problem

Define the Nemytskii superposition operator

$$\mathscr{N}(\varphi)(\cdot) := f(\cdot,\varphi(\cdot))$$

for $\varphi\in\mathscr{M}(X)$.

We define the set $\mathscr{M}(\mathbb{R}_+\times X;X)$ to consist of all functions $f:\mathbb{R}_+\times X\to X$ such that $f(\cdot,x)\in\mathscr{M}(X)$ uniformly for each $x\in K$, where K is any bounded subset of X. From now on, we also denote

$$C_0(\mathbb{R}_+\times X,X) = \{h\in BC(\mathbb{R}_+\times X,X) : \lim_{t\to\infty}||f(t,x)||=0 \text{ uniformly for } x \text{ in any bounded subset of } X\}.$$

Theorem 5.1 ([30]). *Let $f\in\mathscr{M}(\mathbb{R}_+\times X,X)$ be given and assume that there exists a constant $L_f>0$ such that*

$$||f(t,x)-f(t,y)||\leq L_f||x-y||, \tag{5.1}$$

for all $t\in\mathbb{R}_+,x,y\in X$. Let $\varphi\in\mathscr{M}(X)$. Then $\mathscr{N}(\varphi)$ belongs to the same space as φ.

In what follows we study existence and uniqueness of solutions in $\mathscr{M}(X)$ for the semilinear fractional order differential equation

$$\mathbb{D}_t^{\alpha+1}u(t)+\mu\mathbb{D}_t^{\beta}u(t)-Au(t)=\mathbb{D}_t^{\alpha}f(t,u(t)), \quad t\geq 0, \quad 0<\alpha\leq\beta\leq 1, \quad \mu>0, \tag{5.2}$$

with $u(0)=x, u'(0)=y$ (specified in X) and where A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega<0$.

In view of the linear case, the following definition of mild solution is natural. Note that in the borderline case $\mu=0$ and $\alpha=1$ it corresponds to the notion of mild solution for the semi-linear problem

$$u''(t)=Au(t)+f(t,u(t))$$

under the hypothesis that A is the generator of a cosine family $C(t)$. In fact, in this case $S_{1,0}(t)\equiv C(t)$ and the associate sine family is equal to $(g_1*S_{1,0})(t)$.

Definition 5.2. *Suppose* $0 < \alpha \leq \beta \leq 1$, $\mu > 0$. *A function* $u : \mathbb{R}_+ \to X$ *is said to be a mild solution to Equation (5.2) if it satisfies*

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \tag{5.3}$$

for each $t \in \mathbb{R}_+$ *and some* $(x,y) \in X \times X$.

We next give a result on existence of mild solutions for the semi-linear problem.

Theorem 5.3. *Let* $0 < \alpha \leq \beta \leq 1$ *and* $\mu > 0$. *Assume that* A *is an* ω*-sectorial operator of angle* $\beta\pi/2$ *and* $\omega < 0$. *Let* $f : \mathbb{R}_+ \times X \to X$ *be a function on* $\mathscr{M}(\mathbb{R}_+ \times X; X)$ *and assume that there exists a bounded integrable function* $L_f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$||f(t,x) - f(t,y)|| \leq L_f(t)||x-y||, \tag{5.4}$$

for all $x,y \in X$ *and* $t \geq 0$. *Then Equation (5.2) has a unique mild solution* $u \in \mathscr{M}(X)$.

Proof. Let $S_{\alpha,\beta}(t)$ be the $(\alpha,\beta)_\mu$ regularized family generated by A (cf. Theorem 4.1). We define the operator $K_{\alpha,\beta}$ on the space $\mathscr{M}(X)$ by

$$(K_{\alpha,\beta}u)(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds. \tag{5.5}$$

From the proof of Theorem 4.3, we know that $(K_{\alpha,\beta}u)(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x \in \mathscr{M}(X)$. Moreover from Theorem 5.1 we conclude that the function $s \mapsto f(s,u(s))$ is in $\mathscr{M}(X)$. Then, by hypothesis and in the same way as in the proof of Theorem 4.3, we arrive at the conclusion that $\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds$ is also in $\mathscr{M}(X)$ and thus $K_{\alpha,\beta}$ is well defined.

Let u,v be in $\mathscr{M}(X)$. By induction, we find the following estimate:

$$\begin{aligned} ||(K^n_{\alpha,\beta}u)(t) - (K^n_{\alpha,\beta}v)(t)|| &\leq \frac{||S_{\alpha,\beta}||_1^n}{(n-1)!}||u-v||_\infty \int_0^t L_f(s)\left(\int_0^s L_f(\tau)d\tau\right)^{n-1} ds \\ &= \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty \left(\int_0^t L_f(\tau)d\tau\right)^n \\ &\leq \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty ||L_f||_1^n \end{aligned}$$

from which we deduce that $K_{\alpha,\beta}$ has a unique fixed point in $\mathscr{M}(X)$ and this completes the proof. □

6. APPLICATIONS

In this section, we give concrete examples of operators where the situation of the previous sections can be applied. For more details on these examples we refer to [3, 37] and the references therein.

We start with examples of ω-sectorial operators of angle $\theta \leq \pi/2$ where ω is not necessary negative.

Example 6.1. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $\mathscr{A}$ be the elliptic operator formally given by

$$\mathscr{A}u = -\sum_{i,j=1}^{N} D_j a_{ij} D_i u + \sum_{i=1}^{N} b_i D_i u - \sum_{i=1}^{N} D_i(c_i u) + c_0 u$$

with real-valued coefficients. We suppose that $a_{ij} \in L^\infty(\Omega)$ $(i,j = 1,\dots,N)$, $b_i, c_i \in W^{1,\infty}(\Omega)$ $(i = 1,\dots,N)$ and $c_0 \in L^\infty(\Omega)$ are real valued functions such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$ where $\gamma > 0$ is a fixed constant. Let $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$ with Ω having the extension property of Sobolev functions. Define the form $a : V \times V \to \mathbb{R}$ by

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} D_i u D_j v\, dx + \sum_{i=1}^{N} \int_\Omega b_i D_i uv\, dx + \sum_{i=1}^{N} \int_\Omega c_i u D_i v\, dx + \int_\Omega c_o uv\, dx$$

It is easy to see that the form a is continuous and coercive and there exists $w \in \mathbb{R}$ such that

$$a(u,u) + w\|u\|^2_{L^2(\Omega)} \geq 2^{-1}\gamma\|u\|^2_V.$$

Let A be the operator on $L^2(\Omega)$ associated with the form a. It is well known (see [3]) that A generates a holomorphic semigroup on $L^2(\Omega)$ of angle $\theta_a \leq \pi/2$. Moreover, the semigroup interpolates on all $L^p(\Omega)$ $(1 \leq p < \infty)$ and each semigroup is holomorphic with the same angle θ_a on $L^p(\Omega)$. Let A_p denotes the generator of the semigroup on $L^p(\Omega)$. Since the semigroup on $L^p(\Omega)$ is holomorphic of angle θ_a, we have that the operator A_p is an ω-sectorial operator (with $\omega \in \mathbb{R}$) of angle θ_a. More precisely,

$$\theta_a = \pi/2 - \inf\{\theta > 0 : \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\overline{\xi_j} \in \Sigma_\theta \text{ for all } \xi \in \mathbb{C}^N, \text{ for a.e. } x \in \Omega\}.$$

In particular, θ_a can be taken to be $\beta\pi/2$ with $\beta < 1$ if the coefficients a_{ij} are not symmetric. In contrast, note that $\theta_a = \pi/2$ if the coefficients a_{ij} are symmetric, that is, $a_{ij}(x) = a_{ji}(x)$ for all $i,j = 1,\dots,N$ and for a.e. $x \in \Omega$.

Next, we give examples where ω is negative.

Example 6.2.
(a) Assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set. Let the form $a_D : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be given by

$$a_D(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx.$$

Then the operator A_D on $L^2(\Omega)$ associated with the form a_D is a realization of the Laplace operator with Dirichlet boundary conditions. This is a particular case of the preceding example. Let $A_{D,p}$ be the corresponding operator on $L^p(\Omega)$ as in Example 6.1. Then $A_{D,p}$ is an ω-sectional operator of angle $\pi/2$ on $L^p(\Omega)$ $(1 \leq p < \infty)$ with $\omega < 0$.

(b) Assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set with a Lipschitz continuous boundary $\partial\Omega$ and let $\gamma \in L^\infty(\partial\Omega)$ satisfy $\gamma(x) \geq \gamma_0 > 0$ for some constant γ_0. Define the bilinear form a_γ on $L^2(\Omega)$ by

$$a_\gamma(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} \gamma uv\, d\sigma, \quad u,v \in H^1(\Omega).$$

Then the operator A_γ on $L^2(\Omega)$ associated with the form a_γ is a realization of the Laplace operator with Robin boundary conditions. This operator also satisfies the generation property in Example 6.1. Let $A_{\gamma,p}$ be the corresponding operator on $L^p(\Omega)$ as in Example 6.1. Then $A_{\gamma,p}$ is also an ω-sectional operator of angle $\pi/2$ on $L^p(\Omega)$ $(1 \leq p < \infty)$ with $\omega < 0$.

Example 6.3. Let us consider the nonlinear two-term time fractional diffusion wave equation with time operator in the Caputo sense and a nonlinear force term $F \in L^\infty_{loc}([0,T) \times \mathbb{R})$, $T > 0, x \in \mathbb{R}$,

$$\mu_1 \mathbb{D}_t^{\alpha_1} u(x,t) + \mu_2 \mathbb{D}_t^{\beta_1} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = F(t,u(t)), \quad t > 0, \quad 0 < \alpha_1, \beta_1 \leq 2, \tag{6.1}$$

subject to the conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x); \quad f,g \in L^p(\mathbb{R}), \quad 1 \leq p \leq \infty.$$

Existence of super viscosity and sub viscosity solutions on the Lebesgue spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, for equation (6.1) was recently studied by Stojanovic and Gorenflo in [37]. Their method is based on the successive application of the Laplace and Fourier transform in order to arrive to a linear Abel Volterra-type equation. Then Gronwall's inequality yields the super viscosity solution assuming a Lipschitz condition for the force term. Taking $X = L^2[0,\pi]$, $\mu_1 = 1, \mu_2 = \mu, \alpha_1 = \alpha+1, \beta_1 = \beta$ with $0 < \alpha \leq \beta \leq 1$ and $F(t,x) = \tau x + \mathbb{D}_t^\alpha b(t)\sin(x)$ where $\tau < 0$ is fixed, we consider the equation

$$\mathbb{D}_t^{\alpha+1} u(x,t) + \mu \mathbb{D}_t^{\beta} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + \tau u(x,t) + \mathbb{D}_t^\alpha [b(t)\sin(u(t))], \quad t > 0, \tag{6.2}$$

(where $0 < \alpha \leq \beta \leq 1$) which is of the form (5.2) with $Au = \frac{\partial^2}{\partial x^2} u + \tau u$. Setting the Dirichlet boundary conditions $u(0,t) = u(2\pi,t) = 0$ we consider A with domain $D(A) := \{u \in L^2[0,2\pi] : u'' \in L^2[0,2\pi]; u(0) = u(2\pi) = 0\}$ and $f(t,x) = b(t)\sin(x)$. By Example 6.1, we see that A is ω-sectorial with $\omega = \tau < 0$ and angle $\pi/2$ (and hence of angle $\beta\pi/2$ for all $\beta \leq 1$). Moreover, we have

$$\|f(t,u) - f(t,v)\|_2^2 = \int_0^\pi |b(t)|^2 |\sin(u(s)) - \sin(v(s))|^2 ds \leq |b(t)|^2 \|u-v\|_2^2.$$

Therefore, if $b(t)$ is bounded, integrable and belongs to $\mathscr{M}(X)$, then Theorem 5.3 applies. If we consider Equation (6.2) using our concept of mild solutions we arrive at the following result.

Proposition 6.4. *Suppose that $b \in L^1(\mathbb{R}_+)$ and $b(t) \to 0$ as $t \to \infty$. Then equation (6.2) with initial and zero boundary conditions has a unique mild solution $u(t,x)$ which decomposes as a sum of a first part which is almost automorphic (possibly zero) and a second part that goes to* 0 *as $t \to \infty$.*

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