SPECTRAL CRITERIA FOR SOLVABILITY OF BOUNDARY VALUE PROBLEMS AND POSITIVITY OF SOLUTIONS OF TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

VALENTIN KEYANTUO, CARLOS LIZAMA, AND MAHAMADI WARMA

Abstract. We investigate mild solutions of the fractional order nonhomogeneous Cauchy problem

$$D_t^\alpha u(t) = Au(t) + f(t),\ t > 0$$

where $0 < \alpha < 1$. When A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, we obtain an explicit representation of mild solutions of the above problem in terms of the semigroup. We then prove that this problem under the boundary condition $u(0) = u(1)$ admits a unique mild solution for each $f \in C([0,1];X)$ if and only if the operator $I - S_\alpha(1)$ is invertible. Here, we use the representation $S_\alpha(t)x = \int_0^\infty \Phi_\alpha(s)T(st^\alpha)xds,\ t > 0$ in which Φ_α is a Wright type function. For the first order case, that is, $\alpha = 1$, the corresponding result was proved by J. Prüss [25] in 1984. In case X is a Banach lattice and the semigroup $(T(t))_{t\geq 0}$ is positive, we obtain existence of solutions of the semilinear problem

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad t > 0, \quad 0 < \alpha < 1.$$

1. Introduction

Let X be a complex Banach space and A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in X. We consider the following linear differential equation

$$D_t^\alpha u(t) = Au(t) + f(t), \quad t > 0, \quad 0 < \alpha < 1, \tag{1.1}$$

where D_t^α is the Caputo fractional derivative.

In the integer case $\alpha = 1$, it is well known that there exists a strong connection between the spectrum of $(T(t))_{t\geq 0}$ and solutions of the inhomogeneous differential equation (1.1) satisfying the condition $u(0) = u(1)$, where f is a forcing term. A complete characterization of the class of generators A such that for any given $f \in C([0,1];X)$, Equation (1.1) with the condition $u(0) = u(1)$ has a unique solution was obtained by Prüss [25] in 1984, extending earlier results by Haraux (see [13]).

Denoting the resolvent set of an operator L by $\rho(L)$, the result of Prüss reads as follows: $1 \in \rho(T(1))$ if and only if for any $f \in C([0,1];X)$, the equation $u' = Au + f$ admits exactly one mild solution satisfying $u(0) = u(1)$.

After Prüss theorem, many interesting consequences and related results have appeared. For example, the corresponding connection with the spectrum of strongly

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continuous sine functions [26], cosine functions [6] and connections with maximal L^p-regularity are discussed in [3, 16, 17].

More recently, Nieto [21] studied periodic boundary valued solutions of the equation (1.1) considering the scalar case. Nieto's considers the Riemann-Liouville fractional derivative, and the meaning he gives to a "periodic" boundary condition is the following:

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = \lim_{t\to 1^-} t^{1-\alpha}u(t), \quad 0<\alpha<1. \tag{1.2}$$

Further results along these lines are given in [5]. One disadvantage of this condition is that continuity of $u(t)$ for $t \geq 0$ forces the condition $u(1) = 0$. It thus appears that Riemann-Liouville is not the most appropriate choice when one considers periodic boundary valued problems. In contrast, the Caputo derivative needs higher regularity conditions of $u(t)$ than the Riemann-Liouville derivative.

Our objective in this paper is twofold: First, we reformulate Nieto's results for the vector-valued case of equation (1.1) considering Caputo's fractional derivative and the natural periodic boundary condition

$$u(0) = u(1). \tag{1.3}$$

Even more, we are successful in extend to the range $0 < \alpha < 1$ the above mentioned *characterization* given by Prüss, in terms of the following strongly continuous resolvent family associated to (1.1):

$$S_\alpha(t)x = \int_0^\infty \Phi_\alpha(s)T(st^\alpha)x ds, \quad t\geq 0, \quad 0<\alpha<1, \quad x\in X, \tag{1.4}$$

where Φ_α is a Wright type function defined by: $\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)}$ for every $z \in \mathbb{C}$ (see Section 2 below for more properties of the function Φ_α). We observe that $\Phi_\alpha(t)$ is a probability density function on $[0,\infty)$, whose Laplace transform is the Mittag-Leffler function in the whole complex plane.

A remarkable consequence of our extension result given in Theorem 4.3 is the following: If A generates a uniformly stable semigroup, then for each $f \in C([0,1];X)$ equation (1.1) admits exactly one mild solution fulfilling the boundary conditions $u(0) = u(1)$. In order to do this, we study mild solutions of (1.1) and show that any mild solution has the representation

$$u(t) = S_\alpha(t)u(0) + \int_0^t P_\alpha(t-s)f(s)ds, \ t>0,$$

where $(S_\alpha(t))$ is given by (1.4) and $(P_\alpha(t))$ is a second operator family associated with $(T(t))_{t\geq 0}$ (see Section 2 below).

Secondly, we study positivity of mild solutions and obtain a simple spectral condition that ensures positivity thereof in the periodic boundary value case. More precisely, let $\alpha\in(0,1)$ and A be the generator of a positive C_0-semigroup $(T(t))_{t\geq 0}$. Suppose $(I-S_\alpha(1))^{-1}x \geq 0$ for all $x\in X_+$ and assume u is a mild solution of (1.1) and that

$$D_t^\alpha u(t) - Au(t) \geq 0, \quad u(0)=u(1). \tag{1.5}$$

Then $u(t)\geq 0$ for all $t\in[0,1]$.

Finally, we study in Banach lattices existence of mild solutions for the semilinear problem:

$$D_t^\alpha u(t) = Au(t) + f(t,u(t)), \ t>0, \ \ 0<\alpha<1,$$

under the hypothesis that A generates a positive C_0-semigroup. This is an extension of recent results given by Zhang [29] in the integer case $\alpha = 1$ (cf. Theorem 5.4 below).

Typical operators to which the results apply are elliptic operators in divergence form: Namely, let Ω be an open subset of $\mathbb{R}^N$. We consider on $L^p(\Omega)$ the operator formally given by

$$Au = \sum_{i,j=1}^{N} D_i(a_{i,j}(x)D_j u) = \operatorname{div}(a(x)\nabla u),$$

in which $(a_{i,j})_{1\le i,j\le N}$ are bounded real valued functions. Under various boundary conditions (including Dirichlet, Neumann, Robin and Wentzell), the results apply (see Section 4).

While in the present paper, we concentrate on periodic boundary conditions, we mention the recent papers [14, 30, 27] dealing with fractional differential equations. The first two deal with nonlocal Cauchy problems while the third considers the fractional evolution problem governed by an almost sectional operator and proceeds to construct the corresponding evolution operators by mean of a certain functional calculus.

The paper is organized as follows. In Section 2, we present some preliminaries on the resolvent families needed in the sequel. In Section 3, assuming that A generates a C_0-semigroup, we represent the resolvent families of Section 2 using the subordination principle. In Section 4 we study mild solutions in general and in the periodic boundary valued case in particular. Positivity of mild solutions as well as the semilinear equation are considered in the final Section 5.

2. Preliminaries

The algebra of bounded linear operators on a Banach space X will be denoted by $\mathcal{B}(X)$, the resolvent set of a linear operator A by $\rho(A)$ and the spectral radius of a bounded operator S will be denoted by $r_\sigma(S)$. Let $\tau > 0$ be a real number. The space of continuous functions $f : [0, \tau] \to X$ is denoted by $C([0, \tau]; X)$ and its norm by $||f|| := \sup\{||f(t)|| : t \in [0, \tau]\}$. We denote $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0$, where Γ is the usual gamma function. It will be convenient to write $g_0 := \delta_0$, the Dirac measure concentrated at 0. Note the semigroup property: $g_{\alpha+\beta} = g_\alpha * g_\beta$ for all $\alpha, \beta > 0$.

The Riemann-Liouville fractional integral of order α, $0 < \alpha < 1$, of a function $u : [0, 1] \to X$ is given by:

$$I^\alpha u(t) := (g_\alpha * u)(t) := \int_0^t g_\alpha(t-s)u(s)ds, \tag{2.1}$$

for example when u is locally integrable on (0, 1).

The Caputo fractional derivative of order $0 < \alpha < 1$ of a function u is defined by

$$D_t^\alpha u(t) := I^{1-\alpha}u'(t) = \int_0^t g_{1-\alpha}(t-s)u'(s)ds \tag{2.2}$$

where u' is the distributional derivative of $u(\cdot)$, under appropriate assumptions. The definition can be extended in a natural way to $\alpha > 0$. Then, when $\alpha = n$ is a natural number, we get $D_t^n := \frac{d^n}{dt^n}$.

The Laplace transform of a locally integrable function $f : [0, \infty) \longrightarrow X$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt = \lim_{R\to\infty} \int_0^R e^{-\lambda t} f(t)dt,$$

provided the integral converges for some $\lambda \in \mathbb{C}$. If for example f is exponentially bounded, that is, $\|f(t)\| \le Me^{\omega t}$, $t \ge 0$ for some $M > 0$ and $\omega \in \mathbb{R}$ then the integral converges absolutely for $\mathrm{Re}(\lambda) > \omega$ and defines an analytic function there.

Regarding the fractional derivative, we have the following important property: For $f \in C([0, \infty); X)$ such that $g_{1-\alpha} * f \in W^{1,1}((0, \infty); X)$,

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0). \tag{2.3}$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

The Mittag-Leffler function (see e.g. [11, 24]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counterclockwise. It is an entire function which provides a generalization of several usual functions, for example:

- Exponential function: $E_{1,1}(z) = e^z$;
- Cosine functions: $E_{2,1}(z^2) = \cosh(z)$ and $E_{2,1}(-z^2) = \cos(z)$;
- Sine functions: $zE_{2,2}(z^2) = \sinh(z)$ and $zE_{2,2}(-z^2) = \sin(z)$.

Let $k \in \mathbb{N} \cup \{0\}$. The Laplace transform of the k^{th}-order derivative of the Mittag-Leffler function is given by ([24]):

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{k!\lambda^{\alpha-\beta}}{(\lambda^\alpha \mp \omega)^{k+1}}, \quad \mathrm{Re}(\lambda) > |\omega|^{1/\alpha}. \tag{2.4}$$

Using this formula, we obtain for $0 < \alpha < 1$:

$$D_t^\alpha E_{\alpha,1}(zt^\alpha) = zE_{\alpha,1}(zt^\alpha), \quad t > 0,\ z \in \mathbb{C}, \tag{2.5}$$

and the identity

$$\frac{d}{dt} E_{\alpha,1}(zt^\alpha) = zt^{\alpha-1} E_{\alpha,\alpha}(zt^\alpha). \tag{2.6}$$

To see this, it is sufficient to write

$$\mathcal{L}\left(t^{\alpha-1} E_{\alpha,\alpha}(zt^\alpha)\right)(\lambda) = \frac{1}{\lambda^\alpha - z} = \frac{1}{z}\left[\lambda \frac{\lambda^{\alpha-1}}{\lambda^\alpha - z} - 1\right],$$

and invert the Laplace transform.

The following two definitions are taken from [1] and [4], respectively.

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an (α, α)-resolvent family if there exists $\omega \ge 0$ and a strongly continuous function $P_\alpha : [0, \infty) \to \mathcal{B}(X)$ (resp. $P_\alpha : (0, \infty) \to \mathcal{B}(X)$ in case $0 < \alpha < 1$) such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} P_\alpha(t)x dt, \quad Re(\lambda) > \omega, \quad x \in X.$$

In this case, $P_\alpha(t)$ is called the (α,α)-resolvent family generated by A.

Definition 2.2. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We call A the generator of an $(\alpha,1)$-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re(\lambda) > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the $(\alpha,1)$-resolvent family generated by A.

In the above definitions, the integrals involved are taken in the sense of Riemann, more precisely as improper Riemann integrals.

By the uniqueness theorem for the Laplace transform, a $(1,1)$-resolvent family is the same as a C_0-semigroup, a $(2,2)$-resolvent family corresponds to the concept of sine family, while a $(2,1)$-resolvent family corresponds to a cosine family. See e.g. [2] and the references therein for an overview on these concepts. A systematic study in the fractional case is carried out in [4].

Some properties of $(P_\alpha(t))$ and $(S_\alpha(t))$ are included in the following lemma. Their proof uses techniques from the general theory of (a,k)-regularized resolvent families [18] (see also [1, 4]). It will be of crucial use in the investigation of mild solutions in Section 4.

Lemma 2.3. *The following properties hold:*

(i) $S_\alpha(0) = I$.
(ii) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.*
(iii) *For all $x \in D(A)$: $S_\alpha(t)x = x + \int_0^t g_\alpha(t-s)AS_\alpha(s)xds$, $t \geq 0$.*
(iv) *For all $x \in X$: $(g_\alpha * S_\alpha)(t)x \in D(A)$ and*

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds,\ t \geq 0.$$

(v) *$P_\alpha(t)D(A) \subset D(A)$ and $AP_\alpha(t)x = P_\alpha(t)Ax$ for all $x \in D(A)$, $t > 0$.*
(vi) *For all $x \in D(A)$: $P_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)AP_\alpha(s)xds$, $t > 0$.*
(vii) *For all $x \in X$: $(g_\alpha * P_\alpha)(t)x \in D(A)$ and*

$$P_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)P_\alpha(s)xds,\ t > 0.$$

(viii) *For $0 < \alpha < 1$, $\lim_{t\to 0} t^{1-\alpha}P_\alpha(t) = \frac{1}{\Gamma(\alpha)}I$, and $P_1(0) = I$ and for $\alpha > 1$, $P_\alpha(0) = 0$.*

Proof. Let ω be as in Definition 2.1 or Definition 2.2. Let $\mu, \lambda > \omega$ and $x \in D(A)$. Then $x = (I - \mu^{-\alpha}A)^{-1}y$ for some $y \in X$. Since $(I - \lambda^{-\alpha}A)^{-1}$ and $(I - \mu^{-\alpha}A)^{-1}$ are bounded and commute, and since the operator A is closed, we obtain from the definition of $P_\alpha(t)$,

$$\begin{aligned}\hat{P}_\alpha(\lambda)x = \int_0^\infty e^{-\lambda t} P_\alpha(t)x dt =& \hat{P}_\alpha(\lambda)(I-\mu^{-\alpha}A)^{-1}y \\ =& \lambda^{-\alpha}(I-\lambda^{-\alpha}A)^{-1}(I-\mu^{-\alpha}A)^{-1}y \\ =& (I-\mu^{-\alpha}A)^{-1}\lambda^{-\alpha}(I-\lambda^{-\alpha}A)^{-1}y \\ =& (I-\mu^{-\alpha}A)^{-1}\hat{P}_\alpha(\lambda)y \\ =& \int_0^\infty e^{-\lambda t}(I-\mu^{-\alpha}A)^{-1}P_\alpha(t)y dt\end{aligned}$$

and, analogously, from the definition of $S_\alpha(t)$:

$$\begin{aligned}\hat{S}_\alpha(\lambda)x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt =& \hat{S}_\alpha(\lambda)(I-\mu^{-\alpha}A)^{-1}y \\ =& \lambda^{-1}(I-\lambda^{-\alpha}A)^{-1}(I-\mu^{-\alpha}A)^{-1}y \\ =& (I-\mu^{-\alpha}A)^{-1}\lambda^{-1}(I-\lambda^{-\alpha}A)^{-1}y \\ =& (I-\mu^{-\alpha}A)^{-1}\hat{S}_\alpha(\lambda)y \\ =& \int_0^\infty e^{-\lambda t}(I-\mu^{-\alpha}A)^{-1}S_\alpha(t)y dt.\end{aligned}$$

Hence, by uniqueness of the Laplace transform:

$$P_\alpha(t)x = (I-\mu^{-\alpha}A)^{-1}P_\alpha(t)(I-\mu^{-\alpha}A)x$$

and

$$S_\alpha(t)x = (I-\mu^{-\alpha}A)^{-1}S_\alpha(t)(I-\mu^{-\alpha}A)x$$

for all $t > 0$. From this two equalities and the continuity of S_α on $[0,\infty)$, we immediately get (ii) and (v).

On the other hand, from the convolution theorem we obtain, for $x \in D(A)$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t} g_\alpha(t)x dt =& \lambda^{-\alpha}x = \hat{P}_\alpha(\lambda)(I-\lambda^{-\alpha}A)x \\ =& \hat{P}_\alpha(\lambda)x - \lambda^{-\alpha}\hat{P}_\alpha(\lambda)Ax \\ =& \int_0^\infty e^{-\lambda t}\left[P_\alpha(t)x - \int_0^t g_\alpha(t-s)P_\alpha(s)Ax ds\right] dt\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-\lambda t} x dt =& \lambda^{-1}x = \hat{S}_\alpha(\lambda)(I-\lambda^{-\alpha}A)x \\ =& \hat{S}_\alpha(\lambda)x - \lambda^{-\alpha}\hat{S}_\alpha(\lambda)Ax \\ =& \int_0^\infty e^{-\lambda t}\left[S_\alpha(t)x - \int_0^t g_\alpha(t-s)S_\alpha(s)Ax ds\right] dt.\end{aligned}$$

The uniqueness theorem for the Laplace transform yield (iii) and (vi).

We now prove (iv) and (vii). Let $x \in X$ and define $y = (\lambda - A)^{-1}x \in D(A)$, where $\lambda \in \rho(A)$ is fixed. Let $z = (g_\alpha * S_\alpha)(t)x$, $t \geq 0$. We have to show that

$z \in D(A)$ and $Az = S_\alpha(t)x - x$. Indeed, using (ii) and (iii) we obtain that

$$\begin{aligned} z &= (\lambda - A)(g_\alpha * S_\alpha)(t)y = \lambda(g_\alpha * S_\alpha)(t)y - A(g_\alpha * S_\alpha)(t)y \\ &= \lambda(g_\alpha * S_\alpha)(t)y - (S_\alpha(t)y - y) \in D(A) \end{aligned}$$

and

$$\begin{aligned} Az =& \lambda A(g_\alpha * S_\alpha)(t)y - AS_\alpha(t)y + Ay \\ =& \lambda(g_\alpha * AS_\alpha)(t)y - S_\alpha(t)Ay + (\lambda y - x) \\ =& \lambda(g_\alpha * AS_\alpha)(t)y - S_\alpha(t)(\lambda y - x) + \lambda y - x \\ =& \lambda[(g_\alpha * AS_\alpha)(t)y - S_\alpha(t)y + y] + S_\alpha(t)x - x \\ =& S_\alpha(t)x - x, \end{aligned}$$

proving the claim. Analogously, we prove that $(g_\alpha * P_\alpha)(t)x \in D(A)$ and

$$P_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)P_\alpha(s)xds.$$

From the continuity of S_α on $[0,\infty)$ and from (iv) we obtain (i). That $P_1(0) = I$ and $P_\alpha(0) = 0$ for $\alpha > 1$ follow from (vi) by using the fact that $g_1(0) = 1$, $g_\alpha(0) = 0$ for $\alpha > 1$, and that the operator A is closed. We notice that (iv) implies that the domain $D(A)$ of the operator A is necessarily dense in X. Now, if $x \in D(A)$, the first assertion in (viii), that is, $\lim_{t\to 0} t^{1-\alpha}P_\alpha(t)x = \frac{1}{\Gamma(\alpha)}x$, for $0 < \alpha < 1$, follows from (vi), and we obtain that $\lim_{t\to 0} t^{1-\alpha}P_\alpha(t) = \frac{1}{\Gamma(\alpha)}I$, for $0 < \alpha < 1$, by using this and the fact that $D(A)$ is dense in X. The proof of the lemma is finished. □

Note that it follows from (vii) and $(viii)$ that $(P_\alpha(t))$ exhibits a singular behavior at the origin if $0 < \alpha < 1$. However, $t \mapsto \|P_\alpha(t)\|$ is in $L^1_{loc}[0,\infty)$ since by (viii) we have that if t is near zero, then

$$\|P_\alpha(t)\| \le \frac{1}{\Gamma(\alpha)t^{1-\alpha}}. \tag{2.7}$$

Recall the definition of the Wright type function [12, p.10, Formula (28)]:

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{2\pi i}\int_\gamma \mu^{\alpha-1}e^{\mu - z\mu^\alpha}d\mu, \quad 0 < \alpha < 1,$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. By [4, p.14], $\Phi_\alpha(t)$ is a probability density function, that is,

$$\Phi_\alpha(t) \ge 0, \quad t > 0; \qquad \int_0^\infty \Phi_\alpha(t)dt = 1.$$

We also have $\Phi_\alpha(0) = \frac{1}{\Gamma(1-\alpha)}$, and as $t \to +\infty$, Φ_α has the following asymptotic expansion

$$\Phi_\alpha(t) = Y^{\alpha-1/2}e^{-Y}\left(\sum_{m=0}^{M-1} A_m Y^{-m} + O(Y^{-M})\right), \tag{2.8}$$

for any $M \in \mathbb{N}$, with $Y = (1-\alpha)(\alpha^\alpha t)^{1/(1-\alpha)}$, where A_m are real numbers.

The following identity holds: For every $\alpha \in (0,1)$ and $s > 0$,

$$e^{-\lambda^\alpha s} = \mathcal{L}\left(\alpha \frac{s}{t^{\alpha+1}} \Phi_\alpha(st^{-\alpha})\right)(\lambda). \tag{2.9}$$

See [12, Formulas (40) and (41)]. We note that the above Laplace transform was formerly first given by Pollard and Mikusinski (see [12] and references therein). For more details on the Wright type functions, we refer to the monographs [4, 12, 19, 28] and the references therein.

Let X be a Banach lattice with positive cone X_+. We recall that a semigroup $(T(t))_{t\geq 0}$ on X is positive if for any $x \in X_+$ and $t \geq 0$, $T(t)x \geq 0$. Similarly, an operator $(A, D(A))$ is resolvent positive if there is $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $(\lambda - A)^{-1}x \geq 0$ for all $\lambda > \omega$ and any $x \in X_+$.

It is a well known fact that a strongly continuous semigroup is positive if and only if its generator is resolvent positive. We finally shall need the following result due to Zhang [29]:

Theorem 2.4. *Let X be a Banach lattice, $A : X \to X$ be a nonlinear operator. Suppose that there exists a positive linear bounded operator $B : X \to X$ with $r_\sigma(B) < 1$ and*

$$-B(x-y) \leq A(x) - A(y) \leq B(x-y),$$

for all $x, y \in X, x \geq y$. Then the equation $x = A(x)$ has a unique solution in X.

3. Subordination

Let A be a linear closed densely defined operator in a complex Banach space X. If A generates a C_0-semigroup $(T(t))_{t\geq 0}$ then A generates an $(\alpha, 1)$-resolvent family $(S_\alpha(t))$ for all $0 < \alpha < 1$ and they are related by the formula [4]:

$$S_\alpha(t)x = \int_0^\infty t^{-\alpha}\Phi_\alpha(st^{-\alpha})T(s)x ds, \quad t \geq 0, \quad x \in X. \tag{3.1}$$

A change of variables, shows that the above is equivalent to

$$S_\alpha(t)x = \int_0^\infty \Phi_\alpha(\tau)T(\tau t^\alpha)x d\tau, \quad t \geq 0, \quad x \in X. \tag{3.2}$$

In particular, it follows from the above representation formulas that $(S_\alpha(t))$ is analytic and $S_\alpha(0) = I$.

Concerning (α, α)-resolvent families, we prove the following important theorem, which is the main result of this section.

Theorem 3.1. *Let $0 < \alpha < 1$. If A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then A generates an (α, α)-resolvent family $(P_\alpha(t))$ given for every $x \in X$ and $t > 0$ by*

$$P_\alpha(t)x = \alpha \int_0^\infty \frac{s}{t^{\alpha+1}} \Phi_\alpha(st^{-\alpha})T(s)x ds = \alpha \int_0^\infty \frac{\tau}{t^{1-\alpha}} \Phi_\alpha(\tau)T(\tau t^\alpha)x d\tau. \tag{3.3}$$

Moreover, for all $x \in D(A)$, $P_\alpha(t)x \in D(A)$ and

$$S'_\alpha(t)x = AP_\alpha(t)x, \quad t > 0, \, x \in X; \tag{3.4}$$

$$(g_{1-\alpha} * P_\alpha)(t)x = x + \int_0^t AP_\alpha(s)x ds = S_\alpha(t)x, \, t \geq 0. \tag{3.5}$$

Proof. Since A generates a C_0-semigroup $(T(t))_{t\geq 0}$, there exists $\omega > 0$ such that $\{\mu : \mathrm{Re}(\mu) > \omega\} \subset \rho(A)$ and

$$(\mu - A)^{-1}x = \int_0^\infty e^{-\mu t}T(t)xdt, \quad x \in X, \, \mathrm{Re}(\mu) > \omega.$$

In particular, $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega^{1/\alpha}\} \subset \rho(A)$. It is clear that $(P_\alpha(t))$ is strongly continuous [and in fact analytic] for $t > 0$.

We next show that $\hat{P}_\alpha(\lambda) = (\lambda^\alpha - A)^{-1}$ for $\mathrm{Re}(\lambda)$ large enough. In fact, by (2.9) and Fubini's theorem, we obtain for every $x \in X$,

$$\begin{aligned}\hat{P}_\alpha(\lambda)x &= \int_0^\infty e^{-\lambda t}\alpha \int_0^\infty \frac{s}{t^{\alpha+1}}\Phi_\alpha(st^{-\alpha})T(s)xdsdt \\ &= \int_0^\infty \left(\int_0^\infty \alpha e^{-\lambda t}\frac{s}{t^{\alpha+1}}\Phi_\alpha(st^{-\alpha})dt\right)T(s)xds \\ &= \int_0^\infty e^{-\lambda^\alpha s}T(s)xds = (\lambda^\alpha - A)^{-1}x,\end{aligned}$$

for all $\mathrm{Re}(\lambda)$ sufficiently large, proving the claim. We conclude that $P_\alpha(t)$ is an (α, α) resolvent family with generator A.

On the other hand, from (3.2) and the fact that A is closed, we obtain for all $x \in D(A)$ that $P_\alpha(t)x \in D(A)$ and the identity

$$\begin{aligned}S'_\alpha(t)x =& \int_0^\infty \alpha\tau t^{\alpha-1}\Phi_\alpha(\tau)T'(\tau t^\alpha)xd\tau = \int_0^\infty \alpha\tau t^{\alpha-1}\Phi_\alpha(\tau)AT(\tau t^\alpha)xd\tau \\ =& AP_\alpha(t)x, \, t > 0,\end{aligned}$$

proving (3.4). Integrating the above identity, we obtain

$$x + \int_0^t AP(s)xds = S_\alpha(t)x, \quad t \geq 0.$$

Finally, from (vi) in Lemma 2.3, we get

$$(g_{1-\alpha} * P_\alpha)(t)x = (g_{1-\alpha} * g_\alpha)(t)x + (g_{1-\alpha} * g_\alpha * AP_\alpha)(t)x = x + (1 * AP_\alpha)(t)x,$$

proving the theorem. □

Remark 3.2. We observe that the paper [27] uses a different approach for the evolution operators S_α and P_α. More precisely, the authors consider an almost sectorial operator A in a Banach space and give a direct construction using the Mittag-Leffer functions.

4. Boundary conditions

In this section, we give a spectral characterization of existence of mild solutions of Equation (1.1) with the boundary condition $u(0) = u(1)$. The approach is based on the representation of solutions using the solution families $(S_\alpha(t))$ and $(P_\alpha(t))$ of the previous section. Assume that A generates a C_0 semigroup $(T(t))_{t\geq 0}$. Let $S_\alpha(t)$ and $P_\alpha(t)$ be given by (3.1) and (3.3) respectively. The linear fractional equation

$$D_t^\alpha u(t) = Au(t) + f(t), \quad t > 0 \tag{4.1}$$

with initial condition

$$u(0) = u_0, \tag{4.2}$$

has the unique (classical) solution u given by

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(s)ds, \tag{4.3}$$

whenever $f \in W^{1,1}(\mathbb{R}_+; X)$ and $u_0 \in D(A)$. Indeed, note that by (3.4),

$$u'(t) = S'_\alpha(t)u_0 + P_\alpha(t)f(0) + (P_\alpha * f')(t), \quad t > 0. \tag{4.4}$$

Hence, using (3.4), (3.5) and Lemma 2.3 we obtain

$$\begin{aligned} D_t^\alpha u(t) =& (g_{1-\alpha} * u')(t) \\ =& (g_{1-\alpha} * AP_\alpha)(t)u_0 + (g_{1-\alpha} * P_\alpha)(t)f(0) + (g_{1-\alpha} * P_\alpha * f')(t) \\ =& AS_\alpha(t)u_0 + S_\alpha(t)f(0) + (S_\alpha * f')(t) \\ =& AS_\alpha(t)u_0 + f(t) + (S'_\alpha * f)(t) \\ =& AS_\alpha(t)u_0 + A(P_\alpha * f)(t) + f(t) \\ =& Au(t) + f(t). \end{aligned}$$

Note that for $\alpha = 1$ the representation (4.3) is nothing else but the well known variation of constant formula for the abstract Cauchy problem of first order, $S_1(t) \equiv P_1(t)$ and corresponds exactly to $T(t)$, the C_0-semigroup generated by A.

Definition 4.1. *Let A with domain $D(A)$ be a closed linear operator on a Banach space X. Let $f \in L^1_{\text{loc}}((0,\infty); X)$ and $0 < \alpha < 1$. Let I^α be the operator defined in (2.1). A function $u \in C([0,\infty); X)$ is called a mild solution of the equation*

$$D_t^\alpha u(t) = Au(t) + f(t), \;\; t > 0, \;\; u(0) = x, \tag{4.5}$$

if $I^\alpha u(t) \in D(A)$, $t \geq 0$ and

$$u(t) = x + A\int_0^t g_\alpha(t-s)u(s)\,ds + \int_0^t g_\alpha(t-s)f(s)\,ds.$$

Equivalently,

$$u(t) = x + AI^\alpha u(t) + I^\alpha f(t), \;\; t \geq 0.$$

We have the following representation of mild solutions.

Lemma 4.2. *Suppose that the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ and let $f \in L^1_{loc}((0,\infty); X)$ such that the mapping $t \mapsto \int_0^t f(s)ds$ is exponentially bounded. Let $u \in C([0,\infty); X)$, $0 < \alpha < 1$ and $u(0) := x \in X$. Then the following assertions are equivalent.*

(i) *$I^\alpha u(t) \in D(A)$, $t \geq 0$ and $u(t) = x + AI^\alpha u(t) + I^\alpha f(t)$, $t \geq 0$, that is, u is a mild solution of (4.5).*

(ii) *$u(t) = S_\alpha(t)x + \displaystyle\int_0^t P_\alpha(t-s)f(s)\,ds$ for all $t > 0$.*

Note that the mapping $t \mapsto \int_0^t f(s)\,ds$ is exponentially bounded if for example, the function $f \in \cup_{p\geq 1} L^p((0,\infty); X)$ or f itself is exponentially bounded.

Proof of Lemma 4.2. (i) $\Rightarrow$ (ii): Assume that assertion (i) holds. Then $u(t) - u(0) = AI^\alpha u(t) + I^\alpha f(t)$. Taking the Laplace transform of this equality, we get that, $\hat{u}(\lambda) - 1/\lambda u(0) = A\lambda^{-\alpha}\hat{u}(\lambda) + \lambda^{-\alpha}\hat{f}(\lambda)$, that is, $\hat{u}(\lambda) - A\lambda^{-\alpha}\hat{u}(\lambda) = 1/\lambda u(0) +$

$\lambda^{-\alpha}\hat{f}(\lambda)$. Therefore, $(I - A\lambda^{-\alpha})\hat{u}(\lambda) = 1/\lambda u(0) + \lambda^{-\alpha}\hat{f}(\lambda)$, and $\lambda^{-\alpha}(\lambda^\alpha - A)\hat{u}(\lambda) = 1/\lambda u(0) + \lambda^{-\alpha}\hat{f}(\lambda)$. Hence,

$$\begin{aligned}\hat{u}(\lambda) &= \frac{1}{\lambda}\lambda^\alpha(\lambda^\alpha - A)^{-1}u(0) + \lambda^\alpha\lambda^{-\alpha}(\lambda^\alpha - A)^{-1}\hat{f}(\lambda)\\ &= \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}u(0) + (\lambda^\alpha - A)^{-1}\hat{f}(\lambda).\end{aligned}$$

Taking the inverse Laplace transform of this equality, we get the assertion (ii).

(ii) ⇒ (i): As a consequence of (*iv*) and (*vii*) in Lemma 2.3 we have

$$I^\alpha u(t) = (g_\alpha * u)(t) = (g_\alpha * S_\alpha)(t)x + (g_\alpha * P_\alpha * f)(t) \in D(A)$$

and

$$\begin{aligned}A(g_\alpha * u)(t) =& A(g_\alpha * S_\alpha)(t)x + A(g_\alpha * P_\alpha * f)(t)\\ =& S_\alpha(t)x - x + [A(g_\alpha * P_\alpha) * f](t)\\ =& S_\alpha(t)x - x + (P_\alpha - g_\alpha) * f(t)\\ =& S_\alpha(t)x - x + (P_\alpha * f)(t) - (g_\alpha * f)(t)\\ =& u(t) - x - (g_\alpha * f)(t),\end{aligned}$$

proving the lemma. □

Uniqueness of the classical solution follows from the lemma upon observing that any classical solution is necessarily a mild solution.

The following problem was considered by J. Prüss [25] when $\alpha = 1$ and A generates a strongly continuous semigroup. If one starts with $f \in C([0,1];X)$ and solves the problem $u'(t) = Au(t) + f(t)$ with the boundary condition $u(0) = u(1)$, then the resulting solution can be extended to a periodic continuous function on $\mathbb{R}$. We observe that Haraux [13] had considered similar problems earlier.

For the fractional differential equation (4.1), we obtain a mild solution on $[0,\infty)$. In the next result (Theorem 4.3), we obtain a necessary and sufficient condition that the mild solution will satisfy the boundary condition $u(0) = u(1)$.

We remark that the concept of periodic boundary valued solutions for fractional differential equations have been introduced in the literature by Belmekki, Nieto and Rodríguez-López in the paper [5] as described in the introduction. In this line of research, we note that the paper [15] by Kaslik and Sivasundaram considers existence and nonexistence of periodic solutions of fractional differential equations for various definitions of the fractional derivative.

We consider the following problem

$$D_t^\alpha u(t) = Au(t) + f(t), \quad t \in (0,1), \;\; 0 < \alpha < 1, \;\; u(0) = u(1). \tag{4.6}$$

Theorem 4.3. *Let X be a Banach space and assume that A generates a C_0 semigroup $(T(t))_{t\geq 0}$. Let $(S_\alpha(t))$ be the subordinated $(\alpha,1)$-resolvent family. Then $1 \in \rho(S_\alpha(1))$ if and only if for any $f \in C([0,1];X)$, equation* (4.6) *admits precisely one mild solution.*

Proof. Suppose $1 \in \rho(S_\alpha(1))$. Note that if the solution u of the differential equation in (4.6) satisfies the condition $u(0) = u(1)$, then Lemma 4.2(ii) implies

$$(I - S_\alpha(1))u(0) = \int_0^1 P_\alpha(1-s)f(s)ds. \tag{4.7}$$

Hence,

$$u(0) = (I - S_\alpha(1))^{-1} \int_0^1 P_\alpha(1-s)f(s)ds. \tag{4.8}$$

We notice that the existence of solutions follows, if one define $u(0)$ by (4.8).

Conversely, define $K_\alpha : C([0,1];X) \to C([0,1];X)$ by means of $(K_\alpha f)(t) = u(t)$, where $u(t)$ denotes the unique mild solution of (4.6). It is clear that K_α is linear and everywhere defined. Moreover, it is not difficult to show, using the Closed Graph Theorem, that K_α is bounded. Now, for $x \in X$, consider

$$f_\alpha(t)x := (\alpha-1)g_{2-\alpha}(t) - \alpha(g_{1-\alpha} * S_\alpha)(t)x$$

and define $Q_\alpha x := (K_\alpha f_\alpha)(0)x$. Clearly, $Q_\alpha : X \to X$ is linear and bounded. We claim that

$$\int_0^t P_\alpha(t-s)f_\alpha(s)xds = tS_\alpha(t)x, \quad t > 0. \tag{4.9}$$

Indeed, the Laplace transform of f_α is given by

$$\hat{f}_\alpha(\lambda) = (\alpha-1)\lambda^{\alpha-2} - \alpha\lambda^{\alpha-1}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}. \tag{4.10}$$

Let $I_f(t)$ be the left hand side of (4.9). Then, taking the Laplace transform and using (4.10) we get that

$$\begin{aligned} \hat{I}_f(\lambda) =& (\lambda^\alpha - A)^{-1}\hat{f}_\alpha(\lambda) \\ =& (\alpha-1)\lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} - \alpha\lambda^{\alpha-1}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-2} \\ =& \frac{d}{d\lambda}\left(\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\right) = \frac{d}{d\lambda}\left(S_\alpha(\lambda)\right). \end{aligned}$$

From the uniqueness of the Laplace transform we obtain (4.9) and the claim is proved. Now using (4.7) and (4.9) we get that

$$\begin{aligned} (I - S_\alpha(1))(Q_\alpha x + x) =& \int_0^1 P_\alpha(1-s)f_\alpha(s)xds + x - S_\alpha(1)x \\ =& S_\alpha(1)x + x - S_\alpha(1)x = x. \end{aligned} \tag{4.11}$$

This shows that $I - S_\alpha(1)$ is surjective. Now, let $x_0 \in X$ be such that $(I - S_\alpha(1))x_0 = 0$. Then using (4.9), we get that

$$x_0 = S_\alpha(1)x_0 = \int_0^1 P_\alpha(1-s)f_\alpha(s)x_0 ds. \tag{4.12}$$

Using (4.12), (4.7) (which follows from Lemma 4.2(ii)) and (4.9) again, we get that

$$0 = (I - S_\alpha(1))x_0 = \int_0^1 P_\alpha(1-s)f_\alpha(s)x_0 ds = S_\alpha(1)x_0.$$

We have shown that $x_0 = 0$ proving that $I - S_\alpha(1)$ is injective. Hence, $I - S_\alpha(1)$ is invertible and the proof is finished. □

Remark 4.4. An alternative proof of the injectivity of $(I - S_\alpha(1))$ in the preceding proof runs as follows: Let $x_0 \in X$ be such that $(I - S_\alpha(1))x_0 = 0$ and set $u(t) := S_\alpha(t)x_0$. Then u is a mild solution of (4.6) with the forcing term $f = 0$. Since the function $u = 0$ is also of mild solution of (4.6) (with the forcing term $f = 0$), the uniqueness of the solution yields $x_0 = u(0) = 0$, proving that $I - S_\alpha(1)$ is injective.

We remark that the condition $1 \in \rho(S_\alpha(1))$ is trivially satisfied if $\|S_\alpha(1)\| < 1$.

Corollary 4.5. *Suppose that the operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\lim_{t\to\infty}\|T(t)\| = 0$. Then, for $f \in C([0,1];X)$, equation (4.6) admits exactly one mild solution.*

Proof. First, observe that $S_\alpha(1)x = \int_0^\infty \Phi_\alpha(s)T(s)xds$, $x \in X$ and recall that $\Phi_\alpha(s) \geq 0$, $s \geq 0$, and $\int_0^\infty \Phi_\alpha(s)ds = 1$. Since $\Phi_\alpha(z)$ is a nonzero analytic function, it follows that for each $\tau > 0$, we have $\int_\tau^\infty \Phi_\alpha(s)ds > 0$. We first assume that $(T(t))_{t\geq 0}$ is contractive, that is $\|T(t)\| \leq 1$, $t \geq 0$. Then for $x \in X$,

$$\begin{aligned}\|S_\alpha(1)x\| = \left\|\int_0^\infty \Phi_\alpha(s)T(s)xds\right\| &\leq \int_0^\infty \Phi_\alpha(s)\|T(s)x\|ds\\ &\leq \int_0^\tau \Phi_\alpha(s)ds\|x\| + \int_\tau^\infty \Phi_\alpha(s)\|T(s)x\|ds.\end{aligned}$$

Let $0 < \varepsilon < 1$. We can choose $\tau > 0$ such that $\|T(s)\| < \varepsilon$ for all $s \geq \tau$. It follows that

$$\begin{aligned}\|S_\alpha(1)x\| &\leq \int_0^\tau \Phi_\alpha(s)ds\|x\| + \int_\tau^\infty \Phi_\alpha(s)\|T(s)x\|ds\\ &\leq \left(\int_0^\tau \Phi_\alpha(s)ds + \varepsilon\int_\tau^\infty \Phi_\alpha(s)ds\right)\|x\|\\ &\leq (1 + (\varepsilon - 1)\int_\tau^\infty \Phi_\alpha(s)ds)\|x\|.\end{aligned}$$

Therefore, $\|S_\alpha(1)\| < 1$ and hence, $I - S_\alpha(1)$ is invertible. If $(T(t))_{t\geq 0}$ is not contractive, we renorm the space X with $|||x||| = \sup_{t\geq 0}\|T(t)x\|$. This norm is equivalent to the original one and $|||T(t)||| \leq 1,\ t \geq 0$. The proof is complete. $\square$

Note that by [9, Prop. V.1.2 &V.1.7], the assumption on $(T(t))_{t\geq 0}$ is equivalent to the fact that $(T(t))_{t\geq 0}$ is exponentially stable.

In the following examples, the semigroups are exponentially stable.

Example 4.6. (1): Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Define the operator A_D on $L^2(\Omega)$ by

$$D(A_D) := \{u \in H_0^1(\Omega),\ \Delta u \in L^2(\Omega)\},\ \ A_D u := \Delta u.$$

Then A_D is a realization of the Laplace operator on $L^2(\Omega)$ with Dirichlet boundary conditions and it generates a C_0-semigroup on $L^2(\Omega)$ which is exponentially stable. Moreover the semigroup interpolates on all $L^p(\Omega)$ and each semigroup on $L^p(\Omega)$ $(1 \leq p < \infty)$ is also exponentially stable (for a complete description we refer for example to [7]).

(2) Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ and let $\gamma \in L^\infty(\partial\Omega)$ satisfy $\gamma(x) \geq \gamma_0 > 0$ for some constant γ_0. Define the bilinear form a_γ on $L^2(\Omega)$ by

$$a_\gamma(u,v) = \int_\Omega \nabla u \nabla v\ dx + \int_{\partial\Omega} \gamma uv\ d\sigma, \quad u,v \in H^1(\Omega).$$

Then the operator A_γ on $L^2(\Omega)$ associated with the form a_γ in the sense that,

$$\begin{cases} D(A_\gamma) &= \{u \in H^1(\Omega),\ \exists\, f \in L^2(\Omega),\ a_\gamma(u,v) = (f,v)_{L^2(\Omega)}\ \forall\, v \in H^1(\Omega)\}\\ A_\gamma u &= -f,\end{cases}$$

is a realization of the Laplace operator with Robin boundary conditions. As in part (1), this operator generates an exponentially stable C_0-semigroup in $L^2(\Omega)$ which interpolates on $L^p(\Omega)$ and each semigroup is exponentially stable in $L^p(\Omega)$ $(1 \le p < \infty)$.

(3): Let Ω and γ be as in part (2). Let A be an elliptic operator in divergence form of the type

$$Au = \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j u) = \operatorname{div}(a(x)\nabla u),$$

where a_{ij} $(i,j = 1,\dots N)$ are real valued bounded measurable functions such that $a_{ij}(x) = a_{ji}(x)$ and there exists a constant $\alpha_0 > 0$ such that $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \alpha_0|\xi|^2$ holds for all $\xi \in \mathbb{R}^N$ and almost every x in Ω. Let $\Delta_\Gamma u = \operatorname{div}_\Gamma(\nabla_\Gamma u)$ be the Laplace-Beltrami operator on the boundary, where $\nabla_\Gamma u$ denotes the tangential gradient at the boundary $\partial\Omega$. Define the bilinear symmetric form $\mathcal{A}$ with domain $\mathcal{H}^1(\Omega) := \{U = (u, u|_{\partial\Omega}) :\ u \in H^1(\Omega),\ u|_{\partial\Omega} \in H^1(\partial\Omega)\}$ on the product space $L^2(\Omega) \times L^2(\partial\Omega)$ by

$$\mathcal{A}(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)D_i u D_j v\, dx + \int_{\partial\Omega} \nabla_\Gamma u \nabla_\Gamma v\, d\sigma + \int_{\partial\Omega} \gamma(x)uv\, d\sigma.$$

It is straightforward to show that $\mathcal{A}$ is closed and the operator on $L^2(\Omega) \times L^2(\partial\Omega)$ associated with it, generates a contraction C_0-semigroup with generator A_2 given by

$$\begin{cases} D(A_2) & = \{U \in \mathcal{H}^1(\Omega),\ Au \in L^2(\Omega),\ \ \Delta_\Gamma u - \partial_\nu^a u - \gamma u \ \text{ exists in } \ L^2(\partial\Omega)\} \\ A_2 u & = (Au, \Delta_\Gamma u - \partial_\nu^a u - \gamma u), \end{cases}$$

while $\vec{\nu}$ is the unit outer normal and $\partial_\nu^a u := (a(x)\nabla u) \cdot \vec{\nu}$ is the conormal derivative of u with respect to the matrix $a(x) = (a_{ij}(x))_{1 \le i,j \le N}$. Moreover, the semigroup interpolates on all $L^p(\Omega) \times L^p(\partial\Omega)$, and each semigroup is contractive and exponentially stable for every $p \in [1,\infty)$.

5. Positivity of solutions and the semilinear equation

Throughout this section, X will be a real Banach lattice.

It was shown by Nieto [21, 22] that if $\alpha \in (0,1), \lambda \in \mathbb{R}, E_{\alpha,\alpha}(\lambda) < \frac{1}{\Gamma(\alpha)}$ and u is such that

$$\mathbb{D}_t^\alpha u(t) - \lambda u(t) \ge 0, \tag{5.1}$$

where $\mathbb{D}_t^\alpha$ is the Riemann-Liouville fractional derivative, then $u(t) \ge 0$ for $t \in (0,1]$. Motivated by Nieto's result, we show in this section that if we consider the Caputo fractional derivative in equation (5.1) then the same type of result holds assuming $E_{\alpha,1}(\lambda) < 1$ instead of $E_{\alpha,\alpha}(\lambda) < \frac{1}{\Gamma(\alpha)}$.

We begin with the following maximum principle based only in initial conditions.

Proposition 5.1. *Let $\alpha \in (0,1)$ and A be the generator of a positive C_0-semigroup $(T(t))_{t\ge0}$. Assume u is a mild solution of (4.5) such that*

$$D_t^\alpha u(t) - Au(t) \ge 0, \quad u(0) \ge 0. \tag{5.2}$$

Then $u(t) \geq 0$ *for all* $t \in [0,1]$.

Proof. Let $x \in X_+$. By the subordination formulae (3.1) and (3.3) and the fact that Φ_α is a probability density on $[0,\infty)$, we obtain $S_\alpha(t)x \geq 0$ and $P_\alpha(t)x \geq 0$ respectively. Using the representation of the solution given in Lemma 4.2 (ii), we see that $u(t) \geq 0$ for all $t \in [0,1]$. □

Proposition 5.2. *Let* $\alpha \in (0,1)$ *and* A *be the generator of a positive* C_0*-semigroup* $(T(t))_{t\geq 0}$. *Suppose* $(I - S_\alpha(1))^{-1}x \geq 0$ *for all* $x \in X_+$. *Assume* u *is a mild solution of* (4.5) *and that*

$$D_t^\alpha u(t) - Au(t) \geq 0, \quad u(0) = u(1). \tag{5.3}$$

Then $u(t) \geq 0$ *for all* $t \in [0,1]$.

Proof. Under the given hypothesis, we have $u(0) \geq 0$ by equation (4.8). Hence the result follows from Proposition 5.1. □

We mention the condition in (5.2) or in (5.3) is equivalent to $f(t) \geq 0$, since by hypothesis, u is a mild solution of (4.5).

Note that if the spectral radius of $S_\alpha(1)$, that is, $r_\sigma(S_\alpha(1))$, is less than 1, then the inverse $(I - S_\alpha(1))^{-1}$ is given by the Neumann series $\sum_{n=0}^{\infty}(S_\alpha(1))^n$, from which it follows that $\sum_{n=0}^{\infty}(S_\alpha(1))^n \geq 0$, since $S_\alpha(1) \geq 0$.

The following corollary shows that the result obtained by Nieto (mentioned at the beginning of this section) using the Riemann-Liouville fractional derivative, also holds for the Caputo fractional derivative.

Corollary 5.3. *Let* $\alpha \in (0,1)$ *and* $\lambda \in \mathbb{R}$. *Suppose* $E_{\alpha,1}(\lambda) < 1$. *Assume that* u *is a mild solution of* (4.6) *with* $A = \lambda I$ *and satisfies*

$$D_t^\alpha u(t) - \lambda u(t) \geq 0, \ \ \forall\, t \in (0,1), \quad u(0) = u(1). \tag{5.4}$$

Then $u(t) \geq 0$ *for all* $t \in [0,1]$.

Proof. By Definition 2.2 and formula (2.4) we have $S_\alpha(t) = E_{\alpha,1}(\lambda t^\alpha)$. Moreover, the semigroup generated by A is given $T(t) = e^{\lambda t}I$ which is positive. The condition $E_{\alpha,1}(\lambda) < 1$ implies that $(I - S_\alpha(1))^{-1}x = (1 - E_{\alpha,1}(\lambda))^{-1}x \geq 0$ for all $x \in X_+$. Hence, the result follows from Proposition 5.2. □

Now, we consider on a Banach lattice X, the semilinear problem:

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad 0 \leq t \leq \tau, \quad 0 < \alpha < 1, \quad u(0) = x \in X, \tag{5.5}$$

where A is the generator of a positive semigroup $(T(t))_{t\geq 0}$, $f : [0,\tau] \times X \to X$ is a locally integrable given function, and $\tau > 0$ is a fixed real number.

If $u \in C([0,\tau]; X)$ satisfies the integral equation

$$u(t) = S_\alpha(t)u(0) + \int_0^t P_\alpha(t-s) f(s, u(s))ds, \quad t > 0, \tag{5.6}$$

then u is called the mild solution of the semilinear problem (5.5) on $[0,\tau]$. The following is the main result in this section.

Theorem 5.4. *Let* A *generate a positive* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on a Banach lattice* X. *Let* $\tau > 0$ *be a fixed real number. Suppose that there exist constants* $M \geq 0, N \geq 0$ *such that for all* $s \in [0,\tau]$ *and* $x_1, x_2 \in X$ *with* $x_1 \geq x_2$*:*

$$-M(x_1 - x_2) \leq f(s, x_1) - f(s, x_2) \leq N(x_1 - x_2). \tag{5.7}$$

Then equation (5.5) *has a unique mild solution in* $C([0,\tau];X)$.

Proof. Define the operator $Q_\alpha : C([0,\tau];X) \to C([0,\tau];X)$ by

$$(Q_\alpha u)(t) = S_\alpha(t)u(0) + \int_0^t P_\alpha(t-s)f(s,u(s))ds, \quad t \in [0,\tau].$$

Then $u \in C([0,\tau];X)$ is the mild solution of (5.5) if and only if $u = Q_\alpha u$. Thus, existence of mild solutions is achieved by proving that Q_α has a fixed point.

For any $u, v \in C([0,\tau];X)$, $u \le v$ and $t \in [0,\tau]$ we have

$$(Q_\alpha v)(t) - (Q_\alpha u)(t) \ge -\int_0^t P_\alpha(t-s)M(v(s)-u(s))ds,$$

and

$$(Q_\alpha v)(t) - (Q_\alpha u)(t) \le \int_0^t P_\alpha(t-s)N(v(s)-u(s))ds.$$

Let $C := \max\{M, N\}$, then

$$-B_\alpha(v(t)-u(t)) \le (Q_\alpha v)(t) - (Q_\alpha u)(t) \le B_\alpha(v(t)-u(t)), \quad t \in [0,\tau],$$

where

$$(B_\alpha x)(t) := \int_0^t P_\alpha(t-s)x(s)ds.$$

Note that since the semigroup $(T(t))_{t\ge0}$ is positive, then $(P_\alpha(t))$ is positive, and hence, B_α is a positive operator. We shall show that $r_\sigma(B_\alpha) = 0$. Indeed, using (2.7), we have that for any $t \in [0,\tau]$,

$$\begin{aligned} ||(B_\alpha x)(t)|| \le& \int_0^t ||P_\alpha(t-s)||||x(s)||ds \\ \le& \frac{1}{\Gamma(\alpha)} \sup_{s\in[0,\tau]} \|x(s)\| \int_0^t (t-s)^{\alpha-1}\, ds = \frac{t^\alpha}{\Gamma(\alpha)\alpha}\|x\|. \end{aligned}$$

Let $\mathbb{B}$ denote the beta function defined by

$$\mathbb{B}(\gamma,\rho) = \int_0^1 s^{\gamma-1}(1-s)^{\rho-1}\, ds.$$

It is well-known that $\mathbb{B}(\gamma,\rho) = \dfrac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}$. Therefore,

$$\begin{aligned} ||(B_\alpha^2 x)(t)|| \le& \int_0^t ||P_\alpha(t-s)||||(B_\alpha x)(s)||ds \\ \le& \frac{1}{\Gamma(\alpha)\alpha}\|x|| \int_0^t t-s)^{\alpha-1}s^\alpha ds = \frac{t^{2\alpha}}{\Gamma(\alpha)\alpha}||x|| \int_0^1 (1-s)^{\alpha-1}s^\alpha\, ds \\ =& \frac{t^{2\alpha}}{\Gamma(\alpha)\alpha}||x||\mathbb{B}(\alpha+1,\alpha) = \frac{t^{2\alpha}}{\Gamma(\alpha)\alpha}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}||x|| \\ =& t^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha+1)}||x||. \end{aligned}$$

By induction, it is easy to prove that for any $n \in \mathbb{N}$ we have

$$||(B_\alpha^n x)(t)|| \le \frac{(t^\alpha)^n\,(\Gamma(\alpha))^{n-1}}{\Gamma(n\alpha+1)}||x||, \quad t \in [0,\tau].$$

Hence, $||B_\alpha^n x|| \leq \frac{(\tau^\alpha)^n (\Gamma(\alpha))^{n-1}}{\Gamma(n\alpha+1)}||x||$, and consequently $||B_\alpha^n|| \leq \frac{(\tau^\alpha)^n (\Gamma(\alpha))^{n-1}}{\Gamma(n\alpha+1)}$. Therefore,

$$r_\sigma(B_\alpha) := \lim_{n\to\infty} ||B_\alpha^n||^{1/n} \leq \lim_{n\to\infty} \frac{\tau^\alpha (\Gamma(\alpha))^{1-1/n}}{(\Gamma(n\alpha+1))^{1/n}} = 0.$$

Hence, by Theorem 2.4 the result follows. □

Remark 5.5. The above result remains valid if instead of $f(t,u(t))$ we consider the more general nonlinearity of [29], namely

$$f(t,u(t), \int_0^t k_1(t,s)g_1(s,u(s))ds, \int_0^\tau k_2(t,s)g_2(s,u(s))ds)$$

where k_1 and k_2 are nonnegative and satisfy $k_1 \in C(D)$, $k_2 \in C([0,\tau]\times[0,\tau])$, with $D := \{(t,s) \in [0,\tau]\times[0,\tau] : \ s \leq t\}$, the functions $g_j(s,\cdot)$ $(s \in [0,\tau],\ j = 1,2)$ are nondecreasing, and there are constants $M_j, N_j, L \geq 0$ such that for every $s \in [0,\tau]$ and $x_j, y_j, z_j \in X$ $(j = 1,2)$, $x_1 \geq x_2, y_1 \geq y_2, z_1 \geq z_2$, we have

$$-M_1(x_2-x_1) - M_2(y_2-y_1) \leq f(s,x_2,y_2,z_2) - f(s,x_1,y_1,z_1),$$
$$f(s,x_2,y_2,z_2) - f(s,x_1,y_1,z_1) \leq N_1(x_2-x_1) + N_2(y_2-y_1),$$
$$g_1(s,x_2) - g_1(s,x_1) \leq L(x_2-x_1).$$

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University of Puerto Rico, Department of Mathematics, Faculty of Natural Sciences, Rio Piedras Campus, P.O. Box 70377, San Juan PR 00936-8377
E-mail address: valentin.keyantuo1@upr.edu

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile.
E-mail address: carlos.lizama@usach.cl

University of Puerto Rico, Department of Mathematics, Faculty of Natural Sciences, Rio Piedras Campus, P.O. Box 70377, San Juan PR 00936-8377
E-mail address: mjwarma@gmail.com, mahamadi.warma1@upr.edu