EXISTENCE, REGULARITY AND REPRESENTATION OF SOLUTIONS OF TIME FRACTIONAL DIFFUSION EQUATIONS

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Abstract. Using regularized resolvent families, we investigate the solvability of the fractional order inhomogeneous Cauchy problem

$$\mathbb{D}_t^\alpha u(t) = Au(t) + f(t),\ t > 0,\ \ 0 < \alpha \le 1,$$

where $\mathbb{D}_t^\alpha$ is the Caputo fractional derivative of order α, A a closed linear operator on some Banach space X, $f : [0,\infty) \to X$ is a given function. We define an operator family associated with this problem and study its regularity properties. When A is the generator of a β-times integrated semigroup $(T_\beta(t))$ on a Banach space X, explicit representations of mild and classical solutions of the above problem in terms of the integrated semigroup are derived. The results are applied to the fractional diffusion equation with non-homogeneous, Dirichlet, Neumann and Robin boundary conditions and to the time fractional order Schrödinger equation $\mathbb{D}_t^\alpha u(t,x) = e^{i\theta}\Delta_p u(t,x) + f(t,x),\ t > 0,\ x \in \mathbb{R}^N$ where $\pi/2 \le \theta < (1-\alpha/2)\pi$ and Δ_p is a realization of the Laplace operator on $L^p(\mathbb{R}^N)$, $1 \le p < \infty$.

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1. Introduction

Many phenomena in physical sciences and engineering, and increasingly in other areas, are modeled by evolutionary partial differential or integral equations. Due to the importance of these models for the design and prediction of behavior of concrete systems, a wide range of mathematical methods have been developed over three centuries to study the properties and the qualitative behavior of the corresponding differential equations.

The following model is typical of the above statement:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(t,x),\ t>0,\ x\in\Omega, \\ \dfrac{\partial u(t,z)}{\partial \nu} + \gamma(z)u(t,z) = g(t,z),\ t>0,\ \ z\in\partial\Omega, \\ u(0,x) = u_0(x),\ \ x\in\Omega. \end{cases} \tag{1.1}$$

This corresponds to the inhomogeneous heat equation with non-homogeneous Robin, Neumann or Dirichlet boundary conditions. Other equations of interest are the transport equation, the Navier Stokes equations and the Schrödinger equation. In many cases, one has to study nonlinear models associated to the above systems.

Increasingly, it has been realized that properties of many phenomena occurring in real life problems are not adequately described by evolution equations of integer order (typically 1 or 2) in time. Such is the case for phenomena with memory effects, anomalous diffusion, polymer science, rheology, material science, fractals and control theory. The references [10, 15, 16, 17, 28, 30, 32, 35] cover several of these phenomena and demonstrate the importance of the fractional model. We remark that time fractional evolution equations are a special case of more general classes of integral and integro-differential equations. They are treated in a thorough way in the monograph [37] by J. Prüss, including several applications to models in physics, most notably viscoelasticity.

We shall be concerned with the following linear differential equation of fractional order:

$$\mathbb{D}_t^\alpha u(t) = Au(t) + f(t), \quad t>0, \quad 0<\alpha\le 1, \tag{1.2}$$

in which $\mathbb{D}_t^\alpha$ is the Caputo fractional derivative (see (2.2) below). Here X is a complex Banach space, A is a closed linear operator in X and $f:\ [0,\infty)\to X$ is a given function. The use of the Caputo fractional derivative for the evolution problem has the advantage that the initial condition is formulated in terms of the value of the solution u at 0. This has physically significant interpretations in concrete problems. In the case of the Riemann-Liouville fractional derivative, one needs to know the solution in a right neighborhood of 0.

Our aim is to construct a basic theory for the solutions of this equation along with applications to some partial differential equations modeling phenomena from science and engineering. To study the existence, uniqueness and regularity of the solutions of Problem (1.2), in general, one needs an operator family associated with the problem. For example, the theory of strongly continuous semigroups has been developed to deal with the case $\alpha = 1$. In case A does not generate a semigroup (if $\alpha = 1$), the concept of exponentially bounded β-times integrated semigroups has been used in the treatment of Problem (1.2). In [4, Section 2.1], an operator family called S_α has been introduced to deal with the fractional case, that is, $0 < \alpha \leq 1$. More precisely, the family $(S_\alpha(t))$ associated with the closed linear operator A on a Banach space X has been defined to be a strongly continuous family $S_\alpha : [0, \infty) \to \mathcal{L}(X)$ such that, $\| \int_0^t S_\alpha(s)x \, ds\| \leq Me^{\omega t}$ for some constants $M, \omega \geq 0$, $\{\lambda^\alpha : \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$, and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X.$$

It turns out that $(S_1(t))$ is a strongly continuous semigroup. Unfortunately, this theory does not include the case of exponentially bounded β-times integrated semigroups. Consequently, the results obtained in [4, Chapters 2 and 3] cannot be applied to deal with the following problem in $L^p(\Omega)$ (if $g \neq 0$) which is the fractional order version of Problem (1.1):

$$\begin{cases} \mathbb{D}_t^\alpha u(t,x) = Au(t,x) + f(t,x), \ \ t > 0, \ x \in \Omega, \ \ 0 < \alpha < 1, \\ \dfrac{\partial u(t,z)}{\partial \nu_A} + \gamma(z)u(t,z) = g(t,z), \ t > 0, \ \ z \in \partial\Omega, \\ u(0,x) = u_0(x), \ \ x \in \Omega. \end{cases} \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^N$ is an open set with boundary $\partial\Omega$, A is a uniformly elliptic operator with bounded measurable coefficients formally given by

$$Au = \sum_{j=1}^N D_j \Big(\sum_{i=1}^N a_{i,j} D_i u + b_j u\Big) - \Big(\sum_{i=1}^N c_i D_i u + du\Big) \tag{1.4}$$

and

$$\frac{\partial u}{\partial \nu_A} = \sum_{j=1}^N \Big(\sum_{i=1}^N a_{ij} D_i u + b_j u\Big) \cdot \nu_j,$$

where ν denotes the unit outer normal vector of Ω at $\partial\Omega$ and γ is a nonnegative measurable function in $L^\infty(\partial\Omega)$.

In this paper, we introduce an appropriate operator family in a general Banach space associated with Problem (1.2) that will cover all the above mentioned cases. This family will be called $(\alpha, 1)^\beta$-resolvent family $(S_\alpha^\beta(t))$ (see Definition 4.2 below) where $0 < \alpha \le 1$ and $\beta \ge 0$ is a real parameter associated with the operator A. The case $\beta = 0$ and $\alpha = 1$ corresponds to the heat equation with A generating a semigroup. The family S_α^0 $(0 < \alpha \le 1)$ corresponds to the family S_α introduced in the reference [4] and mentioned above (see also [20, 21] for related results). The family S_α^β $(\beta > 0)$ and $\alpha = 1$ corresponds to the theory of exponentially bounded β-times integrated semigroups and is well-understood (see the monograph [2, Section 3.2] and its bibliography). We use this framework to treat the homogeneous ($f = 0$ in (1.2)) as well as the inhomogeneous problems (under appropriate conditions on f in (1.2)). Some related work appears in the reference [11] where however the operator family S_α^β and its analytical and operator theoretic properties are not considered. We shall in fact consider the case where the operator A is a L^p-realization of a more general uniformly elliptic operator in divergence form (as the one in (1.4)) with various boundary conditions (Dirichlet, Neumann or Robin). We obtain a representation of mild and classical solutions in terms of the operator family S_α^β. Our results apply to the situation where the closed linear operator A satisfies the following condition: There exist $\omega \ge 0$ and $\gamma \ge -1$ such that

$$\|(\lambda - A)^{-1}\| \le M|\lambda|^\gamma, \quad \mathrm{Re}(\lambda) > \omega. \tag{1.5}$$

In particular, this includes the case of almost sectorial operators (in which $-1 < \gamma < 0$) studied in [40] using a certain functional calculus. In fact, several operators of interest (as the one involved in the non-homogeneous boundary conditions in Problem (1.3) or the Schrödinger operator $i\Delta_p$ on $L^p(\mathbb{R}^N)$, $p \ne 2$) which do not generate strongly continuous semigroups are generators of integrated semigroups. Operators of this type have been used in [26, 27] in the study of age structured population models in the L^p-context. More examples will be presented in Section 8 below.

The paper is organized as follows. In Section 2, we present some preliminaries on fractional derivatives, the Wright type functions and the Mittag-Leffler functions. In Section 3 we use the Laplace transform to motivate the introduction of the operator family which will be used in the sequel. Section 4 is devoted to the definition and several properties of the resolvent family S_α^β. In the short Section 5 we characterize the resolvent family S_α^β through the regularized fractional Cauchy problem. The homogeneous (fractional) abstract Cauchy problem is solved in Section 6. The conditions on the initial data that ensure solvability of the problem agree with the classical case $\alpha = 1$. We take up the inhomogeneous (fractional) abstract Cauchy problem in Section 7. We are able to deal satisfactory with this problem under natural conditions on the initial data and the inhomogeneity. In the case of generators of integrated semigroups relating to the case $\alpha = 1$, the inhomogeneous problem is studied in [2, Section 3.2]; our results agree with this. In fact, we are able to deal with the full range $0 < \alpha \leq 1$. In the final Section 8 we present various examples of problems where the results of the previous sections apply.

2. Preliminaries

The algebra of bounded linear operators on a Banach space X will be denoted by $\mathcal{L}(X)$, the resolvent set of a linear operator A by $\rho(A)$. We denote by g_α the function $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0,\ \alpha > 0$, where Γ is the usual gamma function. It will be convenient to write $g_0 := \delta_0$, the Dirac measure concentrated at 0. Note the semigroup property:

$$g_{\alpha+\beta} = g_\alpha * g_\beta,\ \ \alpha,\ \beta \geq 0. \tag{2.1}$$

The Riemann-Liouville fractional integral of order $\alpha > 0$, of a locally integrable function $u :\ [0, \infty) \to X$ is given by:

$$I_t^\alpha u(t) := (g_\alpha * u)(t) := \int_0^t g_\alpha(t-s)u(s)ds.$$

The Caputo fractional derivative of order $\alpha > 0$ of a function u is defined by

$$\mathbb{D}_t^\alpha u(t) := I_t^{m-\alpha} u^{(m)}(t) = \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds, \tag{2.2}$$

where $m := \lceil \alpha \rceil$ is the smallest integer greatest than or equal to α, $u^{(m)}$ is the m^{th}-order distributional derivative of $u(\cdot)$, for example if we assume that $u(\cdot)$ has locally integrable distributional derivatives up to order m. Then, when $\alpha = n$ is a natural number, we obtain $\mathbb{D}_t^n := \dfrac{d^n}{dt^n}$. In relation to the Riemann-Liouville fractional derivative of order α, namely D_t^α, we have:

$$\mathbb{D}_t^\alpha f(t) = D_t^\alpha \Big(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t) \Big),\ t > 0, \tag{2.3}$$

where $m := \lceil \alpha \rceil$ has been defined above, and for a locally integrable function $u : [0, \infty) \to X$,

$$D_t^{\alpha} u(t) := \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s) u(s)\, ds, \quad t > 0.$$

The Laplace transform of a locally integrable function $f : [0, \infty) \to X$ is defined by

$$\mathcal{L}(f)(\lambda) := \widehat{f}(\lambda) := \int_0^{\infty} e^{-\lambda t} f(t) dt = \lim_{R \to \infty} \int_0^R e^{-\lambda t} f(t) dt,$$

provided the integral converges for some $\lambda \in \mathbb{C}$. If for example f is exponentially bounded, that is, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|f(t)\| \leq M e^{\omega t}$, $t \geq 0$, then the integral converges absolutely for $\mathrm{Re}(\lambda) > \omega$ and defines an analytic function there. The most general existence theorem for the Laplace transform in the vector-valued setting is given by [2, Theorem 1.4.3].

Regarding the fractional derivative, we have for $\alpha > 0$ and $m := \lceil \alpha \rceil$, the following important properties:

$$\widehat{\mathbb{D}_t^{\alpha} f}(\lambda) = \lambda^{\alpha} \widehat{f}(\lambda) - \sum_{k=0}^{m-1} \lambda^{\alpha-k-1} f^{(k)}(0), \tag{2.4}$$

and

$$\widehat{D_t^{\alpha} f}(\lambda) = \lambda^{\alpha} \widehat{f}(\lambda) - \sum_{k=0}^{m-1} \lambda^{m-k-1} \left(\frac{d^k}{dt^k} I_t^{m-\alpha} f\right)(0). \tag{2.5}$$

The power function λ^{α} is uniquely defined as $\lambda^{\alpha} = |\lambda|^{\alpha} e^{i \arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

Next, we recall some useful properties of convolutions that will be frequently used throughout the paper. For every $f \in C([0, \infty); X)$, $k \in \mathbb{N}$ and $\alpha \geq 0$, we have that for every $t \geq 0$,

$$\frac{d^k}{dt^k} \left[(g_{k+\alpha} * f)(t) \right] = (g_{\alpha} * f)(t). \tag{2.6}$$

If $f \in C^1([0, \infty); X)$, then for every $\alpha > 0$, we have that for every $t \geq 0$,

$$\frac{d}{dt} \left[(g_{\alpha} * f)(t) \right] = g_{\alpha}(t) f(0) + (g_{\alpha} * f')(t). \tag{2.7}$$

Let $k \in \mathbb{N}$. If $u \in C^{k-1}([0,\infty);X)$ and $v \in C^k([0,\infty);X)$, then for every $t \geq 0$,

$$\begin{aligned}\frac{d^k}{dt^k}\left[(u*v)(t)\right] &= \sum_{j=0}^{k-1} u^{(k-1-j)}(t)v^{(j)}(0) + (u*v^{(k)})(t) \\ &= \sum_{j=0}^{k-1} \frac{d^{k-1}}{dt^{k-1}}\Big[(g_j*u)(t)v^{(j)}(0)\Big] + (u*v^{(k)})(t). \qquad (2.8)\end{aligned}$$

The Mittag-Leffler function (see e.g. [16, 17, 35, 38]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \qquad (2.9)$$

By [35, Formula (1.135)] if $0 < \alpha < 2$, $N \in \mathbb{N}\setminus\{1\}$ and μ is a real number such that

$$\frac{\alpha\pi}{2} < \mu < \min\{\pi, \alpha\pi\},$$

then $E_{\alpha,\beta}$ has the following asymptotic expansion:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_{j=1}^{N} \frac{1}{\Gamma(\beta-\alpha j)} \frac{1}{z^j} + O\Big[\frac{1}{z^{N+1}}\Big] \qquad (2.10)$$

as $|z| \to \infty$, $|\arg(z)| \leq \mu$ and

$$E_{\alpha,\beta}(z) = -\sum_{j=1}^{N} \frac{1}{\Gamma(\beta-\alpha j)} \frac{1}{z^j} + O\Big[\frac{1}{z^{N+1}}\Big] \qquad (2.11)$$

as $|z| \to \infty$ and $\mu \leq |\arg(z)| \leq \pi$. The following Laplace transform formula related to the Mittag-Leffler function (see e.g. [35, Formula (180), page 21]) will be useful:

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{k!\lambda^{\alpha-\beta}}{(\lambda^\alpha \mp \omega)^{k+1}}, \quad \mathrm{Re}(\lambda) > |\omega|^{1/\alpha}. \qquad (2.12)$$

Using this formula, we obtain for $0 < \alpha < 1$:

$$\mathbb{D}_t^\alpha E_{\alpha,1}(zt^\alpha) = zE_{\alpha,1}(zt^\alpha), \quad t > 0, z \in \mathbb{C}, \qquad (2.13)$$

and the identity

$$\frac{d}{dt} E_{\alpha,1}(zt^\alpha) = zt^{\alpha-1} E_{\alpha,\alpha}(zt^\alpha).$$

To see the latter, it is sufficient to write

$$\mathcal{L}\Big(t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha)\Big)(\lambda) = \frac{1}{\lambda^\alpha - z} = \frac{1}{z}\Big[\lambda\frac{\lambda^{\alpha-1}}{\lambda^\alpha - z} - 1\Big],$$

and invert the Laplace transform. It follows from (2.13) that for every $z \in \mathbb{C}$, the function $u(t) := E_{\alpha,1}(zt^\alpha)$ is the solution of the scalar valued problem

$$\mathbb{D}_t^\alpha u(t) = zu(t), \quad t > 0,$$

satisfying $u(0) = 1$. Letting $v(t) := t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha)x$, $t > 0$, $x \in X$ and using the Laplace transform, it is easy to see that

$$v(t) = g_\alpha(t)x + z(g_\alpha * v)(t). \tag{2.14}$$

It follows from (2.10) and (2.11) (see also [4, Formula (2.9)]), that if $\omega \geq 0$ is a real number, then there exist some constants $C_1, C_2 \geq 0$ such that

$$E_{\alpha,1}(\omega t^\alpha) \leq C_1 e^{t\omega^{\frac{1}{\alpha}}}, \text{ and } E_{\alpha,\alpha}(\omega t^\alpha) \leq C_2 e^{t\omega^{\frac{1}{\alpha}}}, \ \ t \geq 0, \ \alpha \in (0,2), \tag{2.15}$$

and the estimates in (2.15) are sharp. Recall the definition of the Wright type function [17, Formula (28)] (see also [35, 38, 42]):

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)}, \tag{2.16}$$

$0 < \alpha < 1$. This has sometimes also been called the Mainardi function. By [4, p.14] or [17], $\Phi_\alpha(t)$ is a probability density function, that is,

$$\Phi_\alpha(t) \geq 0, \quad t > 0 \ \text{ and } \quad \int_0^\infty \Phi_\alpha(t)dt = 1,$$

and its Laplace transform is the Mittag-Leffler function in the whole complex plane. We also have $\Phi_\alpha(0) = \frac{1}{\Gamma(1-\alpha)}$, and as $t \to +\infty$, Φ_α has the following asymptotic expansion

$$\Phi_\alpha(t) = Y^{\alpha-1/2}e^{-Y}\Big(\sum_{m=0}^{M-1} A_m Y^{-m} + O(Y^{-M})\Big), \ \ 0 < \alpha < 1, \tag{2.17}$$

for any $M \in \mathbb{N}$, with $Y = (1-\alpha)(\alpha^\alpha t)^{1/(1-\alpha)}$, where A_m are real numbers (see e.g. [17, Theorems 2.1.1, 2.1.3 and 2.1.4] and [42])

Concerning the Laplace transform of the Wright type functions, the following identities hold:

$$e^{-\lambda^\alpha s} = \mathcal{L}\left(\alpha \frac{s}{t^{\alpha+1}} \Phi_\alpha(st^{-\alpha})\right)(\lambda), \quad 0 < \alpha < 1, \tag{2.18}$$

and

$$\lambda^{\alpha-1}e^{-\lambda^\alpha s} = \mathcal{L}\Big(\frac{1}{t^\alpha}\Phi_\alpha(st^{-\alpha})\Big)(\lambda), \quad 0 < \alpha < 1. \tag{2.19}$$

See [17, Formulas (40) and (42)] and [4, Formula (3.10)]. We notice that (2.18) was formerly first given by Pollard and Mikusinski (see [36, 17] and references therein).

The following formula on the moments of the Wright function will be useful:

$$\int_0^\infty x^p \Phi_\alpha(x)dx = \frac{\Gamma(p+1)}{\Gamma(\alpha p+1)}, \quad p > 0, \quad 0 < \alpha < 1. \tag{2.20}$$

The preceding formula (2.20) is derived from the representation (2.16) and can be found in [17, formula after (38)]. Note that in this reference the notation $M(x,\alpha) := \Phi_\alpha(x)$ is used. For more details on the Wright type functions, we refer to the papers [4, 17, 28, 42] and the references therein. We note that the Wright functions have been used by Bochner to construct fractional powers of semigroup generators (see e.g. [43, Chapter IX]).

3. Motivations

In this section we discuss heuristically the solvability of the fractional order Cauchy problem (1.2). We proceed through the use of the Laplace transform and derive some representation formulas that will serve as motivation for the theoretical framework of the subsequent sections.

We assume that $0 < \alpha < 1$. In view of (2.3), we may rewrite (1.2) as:

$$u(t) = A(g_\alpha * u)(t) + (g_\alpha * f)(t) + u(0),\ t > 0. \tag{3.1}$$

Suppose that u is exponentially bounded, e.g. $\|u(t)\| \leq Me^{\omega t}$ ($\omega \in \mathbb{R}$ and $M \geq 0$) or more generally $\|(g_1 * u)(t)\| \leq Me^{\omega t}$, and satisfies (1.2). If $(g_1 * f)(t)$ is also exponentially bounded, then we can take the Laplace transform on both sides of (3.1), and this yields for $\mathrm{Re}(\lambda) > \omega$,

$$(\lambda^\alpha - A)\widehat{u}(\lambda) = \widehat{f}(\lambda) + \lambda^{\alpha-1}u(0). \tag{3.2}$$

The above relation (3.2) can be rewritten as:

$$\widehat{u}(\lambda) = (\lambda^\alpha - A)^{-1}\widehat{f}(\lambda) + \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}u(0), \tag{3.3}$$

provided that $\{\lambda^\alpha :\ \mathrm{Re}(\lambda) > \omega\} \subseteq \rho(A)$.

If we now assume that the operator A generates an exponentially bounded β-times integrated semigroup $(T_\beta(t))$ on X for some $\beta \geq 0$, then there exist some constants $\omega, M \geq 0$ such that $\|T_\beta(t)x\| \leq Me^{\omega t}\|x\|$, $x \in X$, $t > 0$, $\{\lambda \in \mathbb{C} :\ \mathrm{Re}(\lambda) > \omega\} \subset \rho(A)$ and

$$(\lambda - A)^{-1}x = \lambda^\beta \int_0^\infty e^{-\lambda t} T_\beta(t)xdt, \quad \mathrm{Re}(\lambda) > \omega, \quad x \in X. \tag{3.4}$$

Then $\{\lambda^\alpha : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ and we get from (3.4) that,

$$(\lambda^\alpha - A)^{-1}x = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda^\alpha t}T_\beta(t)xdt, \ \ \operatorname{Re}(\lambda) > \omega, \ \ x \in X. \tag{3.5}$$

Substituting (3.5) into (3.3), we get that for $\operatorname{Re}(\lambda) > \omega$,

$$\widehat{u}(\lambda) = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda^\alpha t}T_\beta(t)\widehat{f}(\lambda)\,dt + \lambda^{\alpha\beta}\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha t}T_\beta(t)u(0)\,dt. \tag{3.6}$$

Using (2.18) we get from (3.6) and Fubini's theorem that for $\operatorname{Re}(\lambda) > \omega$,

$$\begin{aligned}\widehat{u}(\lambda) =& \lambda^{\alpha\beta}\int_0^\infty \alpha\int_0^\infty e^{-\lambda s}\frac{t}{s^{\alpha+1}}\Phi_\alpha(ts^{-\alpha})T_\beta(t)\widehat{f}(\lambda)dtds\\ &+ \lambda^{\alpha-1}\lambda^{\alpha\beta}\int_0^\infty \alpha\int_0^\infty e^{-\lambda s}\frac{t}{s^{\alpha+1}}\Phi_\alpha(ts^{-\alpha})T_\beta(t)u(0)dtds\\ =&\lambda^{\alpha\beta}\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{s^{\alpha+1}}\Phi_\alpha(ts^{-\alpha})T_\beta(t)\widehat{f}(\lambda)dsdt\\ &+ \lambda^{\alpha\beta}\widehat{g_{1-\alpha}}(\lambda)\int_0^\infty e^{-\lambda s}\int_0^\infty \frac{\alpha t}{s^{\alpha+1}}\Phi_\alpha(ts^{-\alpha})T_\beta(t)u(0)dtds.\end{aligned} \tag{3.7}$$

Setting

$$T_\alpha^\beta(t)x := \int_0^\infty \frac{\alpha s}{t^{\alpha+1}}\Phi_\alpha(st^{-\alpha})T_\beta(s)xds \ \text{ and } \ \widetilde{T}_\alpha^\beta(t)x := (g_{1-\alpha} * T_\alpha^\beta)(t)x$$

$t > 0, \ x \in X$, we get from (3.7) that, for $\operatorname{Re}(\lambda) > \omega$,

$$\widehat{u}(\lambda) = \lambda^{\alpha\beta}\widehat{T_\alpha^\beta}(\lambda)u(0) + \lambda^{\alpha\beta}\widehat{\widetilde{T}_\alpha^\beta * f}(\lambda).$$

The above identity shows that we can find a solution $u(t)$ of (1.2) whenever we can prove the existence of an operator family whose Laplace transform coincides with $\lambda^{\alpha\beta}\widehat{T}_\alpha^\beta(\lambda)$.

4. Resolvent families and their properties

The following two definitions are motivated by the discussion in the previous Section 3. They are an extension of the one considered in [1] and [4], respectively.

Definition 4.1. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $0 < \alpha \leq 1, \beta \geq 0$. We say that A is the generator of an $(\alpha,\alpha)^\beta$-resolvent family if there exists a strongly continuous function $P_\alpha^\beta : [0,\infty) \to \mathcal{L}(X)$ (resp. $P_\alpha^\beta : (0,\infty) \to \mathcal{L}(X)$ in case $0 <$*

$\alpha(\beta+1)<1$) such that, $\|(g_1 * P_\alpha^\beta)(t)\| \le Me^{\omega t}$, $t>0$, for some constants $M,\omega \ge 0$, $\{\lambda^\alpha : Re(\lambda)>\omega\} \subset \rho(A)$, and

$$(\lambda^\alpha - A)^{-1}x = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}P_\alpha^\beta(t)x dt, \quad Re(\lambda)>\omega, \quad x\in X.$$

In this case, P_α^β is called the $(\alpha,\alpha)^\beta$-resolvent family generated by A.

Definition 4.2. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $0<\alpha\le 1, \beta \ge 0$. We call A the generator of an $(\alpha,1)^\beta$-resolvent family if there exists a strongly continuous function $S_\alpha^\beta : [0,\infty) \to \mathcal{L}(X)$ such that, $\|(g_1 * S_\alpha^\beta)(t)\| \le Me^{\omega t}$, $t \ge 0$, for some constants $M,\omega\ge 0$, $\{\lambda^\alpha : Re(\lambda)>\omega\}\subset \rho(A)$, and*

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}S_\alpha^\beta(t)x dt, \quad Re(\lambda)>\omega, \quad x\in X.$$

In this case, S_α^β is called the $(\alpha,1)^\beta$-resolvent family generated by A.

We will say that P_α^β (resp. S_α^β) is exponentially bounded if there exist some constants $M,\omega\ge 0$ such that $\|P_\alpha^\beta(t)\| \le Me^{\omega t}, \quad \forall\ t>0$, (resp. $\|S_\alpha^\beta(t)\|\le Me^{\omega t}, \quad \forall\ t\ge 0$).

It follows from the uniqueness theorem for the Laplace transform that an operator A can generate at most one $(\alpha,1)^\beta$ (resp. $(\alpha,\alpha)^\beta$)-resolvent family for given parameters $0<\alpha\le 1$ and $\beta\ge 0$.

We shall write $(\alpha,1)$ and (α,α) for $(\alpha,1)^0$ and $(\alpha,\alpha)^0$, respectively.

Before we give some properties of the resolvent families defined above, we need the following preliminary result.

Lemma 4.3. *Let $f:\ [0,\infty)\to X$ be such that there exist some constants $M\ge 0$ and $\omega\ge 0$ such that $\|(g_1 * f)(t)\|\le Me^{\omega t}$, $t>0$. Then for every $\alpha\ge 1$, there exist some constants $M_1\ge 0$ and $\omega_1\ge 0$ such that $\|(g_\alpha * f)(t)\| \le M_1e^{\omega_1 t}$, $t>0$.*

Proof. Assume that f satisfies the hypothesis of the lemma and let $\alpha\ge 1$. We just have to consider the case $\alpha>1$. Then for every $t>0$,

$$\begin{aligned}\|(g_\alpha * f)(t)\| = \|(g_{\alpha-1} * g_1 * f)(t)\| &\le \int_0^t g_{\alpha-1}(s)Me^{\omega(t-s)}\, ds \\ &= Me^{\omega t}\int_0^t \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}e^{-\omega s}\, ds \le Me^{\omega t}\frac{t^{\alpha-1}}{\Gamma(\alpha)} \le M_1e^{\omega_1 t},\end{aligned}$$

for some constants $M_1,\omega_1\ge 0$ and the proof is finished. □

Remark 4.4. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X and let $0 < \alpha \leq 1$, $\beta \geq 0$.

(a) Using Lemma 4.3, we have that if A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β, then it generates an $(\alpha, 1)^\beta$-resolvent family S_α^β given by

$$S_\alpha^\beta(t)x = (g_{1-\alpha} * P_\alpha^\beta)(t)x, \quad t \geq 0, \ \ x \in X. \tag{4.1}$$

(b) By the uniqueness theorem for the Laplace transform, a $(1,1)$-resolvent family is the same as a C_0-semigroup, a $(1,1)^\beta$-resolvent family is the same as an exponentially bounded $\beta-$times integrated semigroup. We refer to the monograph [2] (especially Chapter 3 and Chapter 6) and the corresponding references for a study of the concept of integrated semigroups. Integrated semigroups have been applied systematically to the study of age-dependent population models in the papers [26, 27] by Magal and Ruan, and [39] by Thieme. A detailed study of the fractional Cauchy problem is carried out in [4, 5] for the case $\beta = 0$ (see also [40] for the situation with almost sectorial operators).

Some properties of $(P_\alpha^\beta(t))$ and $(S_\alpha^\beta(t))$ are included in the following lemmas. Their proof uses techniques from the general theory of (a,k)-regularized resolvent families [24] (see also [1, 4, 25]). It will be of crucial use in the investigation of solutions of fractional order Cauchy problems in Sections 5, 6, and 7. The proof of the analogous results in the case of strongly continuous semigroups may be found in [2, Chapter 3]. The case $\beta = 0$ is included in [21]. For the sake of completeness we include the full proof.

Lemma 4.5. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $0 < \alpha \leq 1, \beta \geq 0$ and assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β. Then the following properties hold:*

(a) $S_\alpha^\beta(t)D(A) \subset D(A)$ *and* $AS_\alpha^\beta(t)x = S_\alpha^\beta(t)Ax$ *for all* $x \in D(A)$, $t \geq 0$.

(b) *For all* $x \in D(A)$, $S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + \int_0^t g_\alpha(t-s)AS_\alpha^\beta(s)xds$, $t \geq 0$.

(c) *For all* $x \in X$, $(g_\alpha * S_\alpha^\beta)(t)x \in D(A)$ *and*

$$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + A\int_0^t g_\alpha(t-s)S_\alpha^\beta(s)xds, \ t \geq 0.$$

(d) $S_\alpha^\beta(0) = g_{\alpha\beta+1}(0)$. *Thus,* $S_\alpha^\beta(0) = I$ *if* $\beta = 0$ *and* $S_\alpha^\beta(0) = 0$ *if* $\beta > 0$.

Proof. Let ω be as in Definition 4.2. Let $\lambda, \mu > \omega$ and $x \in D(A)$. Then $x = (I - \mu^{-\alpha}A)^{-1}y$ for some $y \in X$. Since $(I - \mu^{-\alpha}A)^{-1}$ and $(I - \lambda^{-\alpha}A)^{-1}$ are bounded and commute, and given that the operator A is closed, we obtain from the definition of S_α^β that,

$$\begin{aligned}\widehat{S}_\alpha^\beta(\lambda)x &= \int_0^\infty e^{-\lambda t} S_\alpha^\beta(t)x\,dt = \widehat{S}_\alpha^\beta(\lambda)(I-\mu^{-\alpha}A)^{-1}y\\ &= (I-\mu^{-\alpha}A)^{-1}\lambda^{-\alpha\beta}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}y = (I-\mu^{-\alpha}A)^{-1}\widehat{S}_\alpha^\beta(\lambda)y\\ &= \int_0^\infty e^{-\lambda t}(I-\mu^{-\alpha}A)^{-1}S_\alpha^\beta(t)y\,dt.\end{aligned}$$

By the uniqueness theorem for the Laplace transform and by continuity, we get that

$$S_\alpha^\beta(t)x = (I-\mu^{-\alpha}A)^{-1}S_\alpha^\beta(t)y = (I-\mu^{-\alpha}A)^{-1}S_\alpha^\beta(t)(I-\mu^{-\alpha}A)x, \quad (4.2)$$

$\forall\ t \geq 0$. It follows from (4.2) that $S_\alpha^\beta(t)x \in D(A)$. Hence, $S_\alpha^\beta(t)D(A) \subset D(A)$ for every $t \geq 0$. It follows also from (4.2) that $AS_\alpha^\beta(t)x = S_\alpha^\beta(t)Ax$ for all $x \in D(A)$ and $t \geq 0$ and we have shown the assertion (a).

Let $x \in D(A)$. Using the convolution theorem, we get that

$$\begin{aligned}\int_0^\infty e^{-\lambda t} g_{\alpha\beta+1}(t)x\,dt &= \lambda^{-\alpha\beta-1}x = \lambda^{-\alpha\beta}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}(I-\lambda^{-\alpha}A)x\\ &= \widehat{S}_\alpha^\beta(\lambda)(I-\lambda^{-\alpha}A)x = \widehat{S}_\alpha^\beta(\lambda)x - \lambda^{-\alpha}\widehat{S}_\alpha^\beta(\lambda)Ax\\ &= \int_0^\infty e^{-\lambda t}\Big[S_\alpha^\beta(t)x - \int_0^t g_\alpha(t-s)S_\alpha^\beta(s)Ax\,ds\Big].\end{aligned}$$

By the uniqueness theorem for the Laplace transform we obtain part (b).

Next, let $\lambda \in \rho(A)$ be fixed, $x \in X$ and set $y := (\lambda - A)^{-1}x \in D(A)$. Let $z := (g_\alpha * S_\alpha^\beta)(t)x$, $t \geq 0$. We have to show that $z \in D(A)$ and $Az = S_\alpha^\beta(t)x - g_{\alpha\beta+1}(t)x$. Using part (b) we obtain that

$$\begin{aligned}z &= (\lambda - A)(g_\alpha * S_\alpha^\beta)(t)y = \lambda(g_\alpha * S_\alpha^\beta)(t)y - A(g_\alpha * S_\alpha^\beta)(t)y\\ &= \lambda(g_\alpha * S_\alpha^\beta)(t)y - (S_\alpha^\beta(t)y - g_{\alpha\beta+1}(t)y) \in D(A).\end{aligned}$$

Therefore,

$$\begin{aligned}Az &= \lambda A(g_\alpha * S_\alpha^\beta)(t)y - AS_\alpha^\beta(t)y + g_{\alpha\beta+1}(t)Ay\\ &= \lambda\left[(g_\alpha * AS_\alpha^\beta)(t)y - S_\alpha^\beta(t)y + g_{\alpha\beta+1}(t)y\right] + S_\alpha^\beta(t)x - g_{\alpha\beta+1}(t)x\\ &= S_\alpha^\beta(t)x - g_{\alpha\beta+1}(t)x,\end{aligned}$$

and we have shown part (c).

Finally, it follows from the strong continuity of $S_\alpha^\beta(t)$ on $[0,\infty)$ and from the assertion (c) that $S_\alpha^\beta(0)x = g_{\alpha\beta+1}(0)x$ for every $x \in X$. This implies all the properties in (d) and the proof is finished. $\square$

The corresponding result for the family P_α^β is given in the following lemma. Its proof runs similar to the proof of Lemma 4.5.

Lemma 4.6. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $0 < \alpha \le 1, \beta \ge 0$ and assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family P_α^β. Then the following properties hold.*

(a) *$P_\alpha^\beta(t)D(A) \subset D(A)$ and $AP_\alpha^\beta(t)x = P_\alpha^\beta(t)Ax$ for all $x \in D(A)$, $t > 0$.*

(b) *For all $x \in D(A)$, $P_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + \int_0^t g_\alpha(t-s)AP_\alpha^\beta(s)xds$, $t > 0$.*

(c) *For all $x \in X$, $(g_\alpha * P_\alpha^\beta)(t)x \in D(A)$ and $P_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + A\int_0^t g_\alpha(t-s)P_\alpha^\beta(s)xds$, $t > 0$.*

(d) *If $\beta > 0$, then for every $x \in \overline{D(A)}$, we have*

$$\frac{1}{\Gamma(\alpha(\beta+1))}\lim_{t\to 0} t^{1-\alpha(\beta+1)}P_\alpha^\beta(t)x = x$$

if $\alpha(\beta+1) < 1$, $P_\alpha^\beta(0)x = x$ if $\alpha(\beta+1) = 1$ and $P_\alpha^\beta(0)x = 0$ if $\alpha(\beta+1) > 1$.

If $\alpha(\beta+1) \ge 1$, then all the above equalities hold for all $t \ge 0$.

Proof. Let just give a short justification of the assertion (d). If $x \in D(A)$, then all the properties in (d) follow from (b). We obtain the assertion (d) for every $x \in \overline{D(A)}$ by density of $D(A)$ in $\overline{D(A)}$. $\square$

We notice that it follows from Lemma 4.6 (d) that $P_\alpha^\beta(t)$ exhibits a singular behavior at the origin if $\alpha(\beta+1) < 1$. In any case, $t \mapsto \|P_\alpha^\beta(t)x\|$ is in $L^1_{loc}[0,\infty)$ for every $x \in X$.

Remark 4.7. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $0 < \alpha \le 1$ and $\beta \ge 0$.

(a) If A generates an $(\alpha,1)^0 = (\alpha,1)$-resolvent family S_α, then it follows from Lemma 4.5 (c) that $D(A)$ is necessary dense in X. In that case, if A also generates an $(\alpha,\alpha)^0 = (\alpha,\alpha)$-resolvent family P_α, then all the properties in Lemma 4.6 (d) hold for every $x \in X$.

(b) We notice that if A generates an $(\alpha,1)^\beta$-resolvent family S_α^β and $D(A)$ is dense in X then this does not necessary imply that $\beta = 0$ (see Section 8).

(c) The examples presented below in Corollary 4.14 and in Section 8 show that in general ($\beta > 0$) the domain of A is not necessary dense in X.

(d) An operator family $\mathcal{S}(t)$, $t > 0$, is called non-degenerate if for $x \in X$ the property $\mathcal{S}(t)x = 0$ for all $t \in (0, \tau]$ (where $\tau \in (0, \infty]$) implies that $x = 0$. The $(\alpha, 1)^\beta$ and $(\alpha, \alpha)^\beta$-resolvent families S_α^β and P_α^β are non-degenerate. This is a direct consequence of Lemma 4.5(c) and Lemma 4.6(c).

The following result shows some regularity properties of the family S_α^β in case A generates a family P_α^β.

Lemma 4.8. *Let A be a closed linear operator on a Banach space X and let $0 < \alpha \leq 1, \beta \geq 0$. Assume that A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β and an $(\alpha, 1)^\beta$-resolvent family S_α^β. Then for every $x \in D(A)$ the mapping $t \mapsto S_\alpha^\beta(t)x$ is differentiable on $(0, \infty)$ and*

$$(S_\alpha^\beta)'(t)x = g_{\alpha\beta}(t)x + P_\alpha^\beta(t)Ax, \quad t > 0. \tag{4.3}$$

Proof. Let $x \in D(A)$. Then it is clear that the right-hand side of (4.3) belongs to $C((0, \infty), \mathcal{L}(X))$. Applying the Laplace transform to both sides of (4.3) and using the fact that $S_\alpha^\beta(0) = 0$, we have that for $\mathrm{Re}(\lambda) > \omega$ (where ω is the real number from the definition of S_α^β and P_α^β),

$$\widehat{(S_\alpha^\beta)'}(\lambda)(x) = \lambda \widehat{S_\alpha^\beta}(\lambda)(x) = \lambda\lambda^{-\alpha\beta}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{-\alpha\beta}\lambda^\alpha(\lambda^\alpha - A)^{-1}x.$$

On the other hand we have that for $\mathrm{Re}(\lambda) > \omega$,

$$\begin{aligned}\widehat{g_{\alpha\beta}}(\lambda)x + \widehat{P_\alpha^\beta}(\lambda)Ax &= \lambda^{-\alpha\beta}x + \lambda^{-\alpha\beta}(\lambda^\alpha - A)^{-1}Ax \\ &= \lambda^{-\alpha\beta}x - \lambda^{-\alpha\beta}x + \lambda^{-\alpha\beta}\lambda^\alpha(\lambda^\alpha - A)^{-1}x = \lambda^{-\alpha\beta}\lambda^\alpha(\lambda^\alpha - A)^{-1}x.\end{aligned}$$

By the uniqueness theorem for the Laplace transform and continuity of the right-hand side of (4.3), we conclude that the identity (4.3) holds. □

The following result presents the extrapolation property of the families S_α^β and P_α^β in terms of the parameter β.

Proposition 4.9. *Let A be a closed linear operator on a Banach space X and let $0 < \alpha \leq 1, \beta \geq 0$. Then the following assertions hold.*

(a) *If A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β, then it generates an $(\alpha, \alpha)^{\beta'}$-resolvent family $P_\alpha^{\beta'}$ for every $\beta' \geq \beta$ and*

$$P_\alpha^{\beta'}(t)x = (g_{\alpha(\beta'-\beta)} * P_\alpha^\beta)(t)x, \quad \forall\, t > 0,\ x \in X. \tag{4.4}$$

(b) *If A generates an $(\alpha,1)^\beta$-resolvent family S_α^β, then it generates an $(\alpha,1)^{\beta'}$-resolvent family $S_\alpha^{\beta'}$ for every $\beta'\ge\beta$ and*

$$S_\alpha^{\beta'}(t)x = (g_{\alpha(\beta'-\beta)} * S_\alpha^\beta)(t)x, \ \ \forall\, t\ge 0,\ x\in X. \tag{4.5}$$

Proof. Let A be a closed linear operator on X, $0<\alpha\le 1$ and $\beta\ge 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family P_α^β. Then, by definition, there exists $\omega\ge 0$ such that $\{\lambda^\alpha : \mathrm{Re}(\lambda)>\omega\}\subset\rho(A)$ and

$$(\lambda^\alpha - A)^{-1}x = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}P_\alpha^\beta(t)x dt, \quad \mathrm{Re}(\lambda)>\omega, \quad x\in X. \tag{4.6}$$

Let $\beta'\ge\beta$ and let $P_\alpha^{\beta'}$ be given by the right hand side in (4.4). Then using Lemma 4.6(c) we have that for every $x\in X$ and $t>0$,

$$\begin{aligned} P_\alpha^{\beta'}(t)x :&= (g_{\alpha(\beta'-\beta)} * P_\alpha^\beta)(t)x \\ &= g_{\alpha(\beta'+1)}(t)x + A\left(g_{\alpha(\beta'-\beta+1)} * P_\alpha^\beta\right)(t)x. \end{aligned}$$

Hence, $P_\alpha^{\beta'}$ is strongly continuous from $[0,\infty)$ into $\mathcal{L}(X)$ if $\alpha(\beta'+1)\ge 1$ and from $(0,\infty)$ into $\mathcal{L}(X)$ if $0<\alpha(\beta'+1)<1$. By (4.4), we have that for every $x\in X$ and $t>0$,

$$(g_1 * P_\alpha^{\beta'})(t)x = (g_{\alpha(\beta'-\beta)+1} * P_\alpha^\beta)(t)x,$$

and since by hypothesis $\|(g_1 * P_\alpha^\beta)(t)x\| \le Me^{\omega t}\|x\|$ for some constants $M,\omega\ge 0$, it follows from Lemma 4.3 that there exist some constants $M',\omega'\ge 0$ such that $\|(g_1 * P_\alpha^{\beta'})(t)x\|\le M'e^{\omega' t}\|x\|$. Next, using (4.6), we have that for $\mathrm{Re}(\lambda)>\omega$, $x\in X$ and $\beta'\ge\beta$,

$$\begin{aligned} (\lambda^\alpha - A)^{-1}x &= \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}P_\alpha^\beta(t)x dt = \lambda^{\alpha\beta'}\lambda^{\alpha(\beta-\beta')}\int_0^\infty e^{-\lambda t}P_\alpha^\beta(t)x dt \\ &= \lambda^{\alpha\beta'}\int_0^\infty e^{-\lambda t}(g_{\alpha(\beta-\beta')} * P_\alpha^\beta)(t)x dt. \end{aligned}$$

Hence, A generates an $(\alpha,\alpha)^{\beta'}$-resolvent family $P_\alpha^{\beta'}$ given by (4.4) and we have shown (a).

(b) The proof of this part follows the lines of the proof of part (a) where now we use Lemma 4.5. □

The following example shows that a generation of an $(\alpha,1)^\beta$ or $(\alpha,\alpha)^\beta$-resolvent family does not imply a generation of an $(\alpha',1)^\beta$ or $(\alpha',\alpha')^\beta$-resolvent family for $0<\alpha<\alpha'\le 1$. That is, in general, an extrapolation property in terms of the parameter α does not always hold.

Example 4.10. Let $1 \le p < \infty$ and Δ_p a realization of the Laplacian in $L^p(\mathbb{R}^N)$. It is well-known that Δ_p generates an analytic C_0-semigroup of contractions. Hence, for every $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\|(\lambda - \Delta_p)^{-1}\| \le \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\pi-\varepsilon}, \tag{4.7}$$

where for $0 < \gamma < \pi$, $\Sigma_\gamma := \{z \in \mathbb{C} : 0 < |\arg(z)| < \gamma\}$. Let $0 < \alpha < 1$, $\theta \in [0, \pi)$ and let the operator A_p on $L^p(\mathbb{R}^N)$ be given by $A_p := e^{i\theta}\Delta_p$. It follows from (4.7) that for $\lambda e^{-i\theta} \in \Sigma_{\pi-\varepsilon}$,

$$\|(\lambda - A_p)^{-1}\| = \|(\lambda - e^{i\theta}\Delta_p)^{-1}\| = \|(\lambda e^{-i\theta} - \Delta_p)^{-1}\| \le \frac{C}{|\lambda|}.$$

Therefore, if $\frac{\pi}{2} < \theta < \left(1 - \frac{\alpha}{2}\right)\pi$, then $\rho(A_p) \supset \Sigma_{\alpha\pi}$ and

$$\|(\lambda - A_p)^{-1}\| \le \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\alpha\pi}. \tag{4.8}$$

By [4, Corollary 2.16] or [6, Proposition 3.1], the estimate (4.8) implies that A_p generates an $(\alpha, 1)$-resolvent family on $L^p(\mathbb{R}^N)$. Hence, by Proposition 4.9(b), A_p generates an $(\alpha, 1)^\beta$-resolvent family on $L^p(\mathbb{R}^N)$ for any $\beta \ge 0$. Such problems are also treated in [5]. Moreover, the technique for constructing the solution operators through a contour integral representation is adapted in the recent paper [40] to handle the case of almost sectorial operators. But if $0 < \alpha < \frac{1}{2}$, by inspecting the location of the spectrum of A_p we see that A_p does not generate an exponentially bounded $(1, 1)^\beta$-resolvent family, that is a β-times integrated semigroup on $L^p(\mathbb{R}^N)$, for any $\beta \ge 0$.

Remark 4.11. In view of the asymptotic expansion (2.17) of the Wright function (see [17, Theorems 2.1.1, 2.1.3 and 2.1.4] and [42]), for a locally integrable function $f : [0, \infty) \to X$ which is exponentially bounded at infinity, and for any $0 < \sigma < 1$, the integral $\int_0^\infty \Phi_\sigma(\tau) f(\tau)\, d\tau$ converges. This property will be frequently used in the following without further mention.

Concerning subordination of resolvent families we have the following preliminary result.

Lemma 4.12. *Let A be a closed linear operator on a Banach space X. Let $0 < \alpha \le 1$, $\beta \ge 0$. Then the following assertions hold.*

(a) *Assume that A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β. Let $0 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and set*

$$P(t)x := \sigma t^{\sigma-1} \int_0^\infty s\Phi_\sigma(s) P_\alpha^\beta(st^\sigma) x ds, \ t > 0, \, x \in X. \tag{4.9}$$

Then $(g_1 * P)(t)x$ is exponentially bounded. Moreover, $(g_1 * P)(t)x = \mathbb{P}(t)x$ where

$$\mathbb{P}(t)x := \int_0^\infty \frac{\sigma s}{t^{\sigma+1}} \Phi_\sigma(st^{-\sigma})(g_{\frac{1}{\sigma}} * P_\alpha^\beta)(s)x ds, \; t > 0, \, x \in X. \tag{4.10}$$

(b) *Assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β. Let $0 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and set*

$$\mathbb{S}(t)x := \int_0^\infty \frac{1}{t^\sigma} \Phi_\alpha(st^{-\sigma})(g_{\frac{1}{\sigma}} * S_\alpha^\beta)(s)x ds, \; t > 0, \, x \in X. \tag{4.11}$$

Then $\mathbb{S}$ is exponentially bounded. Moreover, $\mathbb{S}(t)x = (g_1 * S)(t)x$ where

$$S(t)x = \int_0^\infty \Phi_\sigma(s) \mathbb{S}_\alpha^\beta(st^\sigma)x \, ds, \quad \forall \, t \geq 0, \; x \in X. \tag{4.12}$$

Proof. Let A, α and β be as in the statement of the lemma.

(a) Assume that A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β and let $0 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and $x \in X$. Let $P(t)$ be given by (4.9). By hypothesis, there exist $M, \omega \geq 0$ such that $\|(g_1 * P_\alpha^\beta)(t)x\| \leq Me^{\omega t}\|x\|$ for every $x \in X$. We show that there exist some constants $M_1, \omega_1 \geq 0$ such that for every $x \in X$, $\|(g_1 * P)(t)x\| \leq M_1 e^{\omega_1 t}\|x\|$, $t \geq 0$. Using (4.9), Fubini's theorem, (2.20), (2.9) and (2.15), we get that for every $t \geq 0$, after a change of variable,

$$\begin{aligned}
&\Big\| \int_0^t P(\tau)x \, d\tau \Big\| \leq \int_0^\infty \Phi_\sigma(s) \Big\| \int_0^{st^\sigma} P_\alpha^\beta(\tau)x \, d\tau \Big\| \, ds \\
&\leq M\|x\| \int_0^\infty \Phi_\sigma(s) e^{\omega s t^\sigma} \, ds = M\|x\| \sum_{n=0}^\infty \frac{(\omega t^\sigma)^n}{n!} \int_0^\infty \Phi_\sigma(s) s^n \, ds \\
&\leq M\|x\| \sum_{n=0}^\infty \frac{(\omega t^\sigma)^n}{n!} \frac{\Gamma(n+1)}{\Gamma(\sigma n + 1)} = \leq M_1 e^{t\omega^{\frac{1}{\sigma}}} \|x\|,
\end{aligned}$$

for some constant $M_1 \geq 0$, proving the claim. Taking the Laplace transform by using (2.18) and Fubini's theorem, we have that for Re $> \omega$ and $x \in X$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} \mathbb{P}(t)x \, dt &= \int_0^\infty e^{-\lambda t} \int_0^\infty \frac{\sigma s}{t^{\sigma+1}} \Phi_\sigma(st^{-\sigma})(g_{\frac{1}{\sigma}} * P_\alpha^\beta)(s)x ds \, dt \\
&= \int_0^\infty e^{-\lambda^\sigma s}(g_{\frac{1}{\sigma}} * P_\alpha^\beta)(s)x ds = \lambda^{-1}\lambda^{-\alpha'\beta}(\lambda^{\alpha'} - A)^{-1}x.
\end{aligned}$$

Similarly, we have that for Re $> \omega$ and $x \in X$,

$$\int_0^\infty e^{-\lambda t}(g_1 * P)(t)x\,dt = \lambda^{-1}\int_0^\infty e^{-\lambda t}P(t)x\,dt$$
$$= \lambda^{-1}\int_0^\infty P_\alpha^\beta(\tau)x\int_0^\infty e^{-\lambda t}\frac{\sigma\tau}{t^{\sigma+1}}\Phi_\sigma(\tau t^{-\sigma})\,dt\,d\tau = \lambda^{-1}\int_0^\infty e^{-\tau\lambda^\sigma}P_\alpha^\beta(\tau)x\,d\tau$$
$$= \lambda^{-1}\lambda^{-\alpha'\beta}(\lambda^{\alpha'} - A)^{-1}x.$$

By the uniqueness theorem for the Laplace transform and by continuity, we have that $(g_1 * P)(t)x = \mathbb{P}(t)x$ for all $t > 0$ and $x \in X$ and this completes the proof of part (a).

(b) Assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β and let $0 < \alpha' < \alpha$, $\sigma := \frac{\alpha'}{\alpha}$ and $x \in X$. Then there exist $M, \omega \geq 0$ such that $\|(g_1 * S_\alpha^\beta)(t)x\| \leq Me^{\omega t}\|x\|$, $x \in X$, $t \geq 0$. Since $\frac{1}{\sigma} > 1$, it follows from Lemma 4.3 that there exist some constants $M_1, \omega_1 \geq 0$ such that for every $x \in X$, $t \geq 0$,

$$\|(g_{\frac{1}{\sigma}} * S_\alpha^\beta)(t)x\| \leq M_1 e^{\omega_1 t}\|x\|. \tag{4.13}$$

Using (4.11), (2.20), (4.13), (2.9), (2.15) ans proceeding as in part (a), we have that

$$\|S(t)x\| \leq M_1\|x\|\sum_{n=0}^\infty \frac{(\omega_1 t^\sigma)^n}{\Gamma(\sigma n + 1)} = M_1 E_{\sigma,1}(\omega_1 t^\sigma)\|x\| \leq Me^{t\omega_1^{\frac{1}{\sigma}}}\|x\|,$$

for some constant $M \geq 0$ and this completes the proof of the lemma. □

Next, we present the principle of subordination of the families S_α^β and P_α^β in terms of the parameter α.

Theorem 4.13. *Let A be a closed linear operator on a Banach space X and let $0 < \alpha \leq 1, \beta \geq 0$. Then the following assertions hold.*

(a) *If A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β, then it generates an $(\alpha', \alpha')^\beta$-resolvent family $P_{\alpha'}^\beta$ for each $0 < \alpha' < \alpha$ and for $x \in X$,*

$$P_{\alpha'}^\beta(t)x = \sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)P_\alpha^\beta(st^\sigma)x\,ds, \quad \forall\, t > 0, \quad \textit{where} \quad \sigma := \frac{\alpha'}{\alpha}. \tag{4.14}$$

(b) *If A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β, then it generates an $(\alpha', 1)^\beta$-resolvent family $S_{\alpha'}^\beta$ for each $0 < \alpha' < \alpha$ and for every $x \in X$,*

$$S_{\alpha'}^\beta(t)x = \int_0^\infty \Phi_\sigma(s)S_\alpha^\beta(st^\sigma)x\,ds, \quad \forall\, t \geq 0, \quad \textit{where} \quad \sigma := \frac{\alpha'}{\alpha}. \tag{4.15}$$

Proof. Let A be a closed linear operator on a Banach space X and let $0<\alpha\le 1, \beta\ge 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family P_α^β. Let $0<\alpha'<\alpha$ and let $P_{\alpha'}^\beta$ be given by (4.14). Then it is clear that $P_{\alpha'}^\beta$ is strongly continuous from $(0,\infty)$ into $\mathcal{L}(X)$. We show that $P_{\alpha'}^\beta(t)$ is strongly continuous at 0 if $\alpha'(\beta+1)\ge 1$. Since $P_\alpha^\beta(t)\simeq g_{\alpha(\beta+1)}(t)=\frac{1}{\Gamma(\alpha(\beta+1)}t^{\alpha(\beta+1)-1}$ as $t\to 0$, we get from (4.14) that

$$P_{\alpha'}^\beta(t)\simeq t^{\frac{\alpha'}{\alpha}-1}t^{\frac{\alpha'}{\alpha}\alpha(\beta+1)-\frac{\alpha'}{\alpha}}=t^{\alpha'(\beta+1)-1}\quad \text{as } t\to 0.$$

We have shown that $P_{\alpha'}^\beta(t)$ is strongly continuous at 0 if $\alpha'(\beta+1)\ge 1$. By Lemma 4.12 there exist some constants $M,\omega\ge 0$ such that $\|(g_1*P_{\alpha'}^\beta)(t)x\|\le Me^{\omega t}\|x\|$ for every $x\in X$ and $t>0$. Now, it follows from (4.6) and (2.18) that $\{\lambda^{\alpha'}:\ \mathrm{Re}(\lambda)>\omega\}\subset\rho(A)$ and for $\mathrm{Re}(\lambda)>\omega$, $x\in X$,

$$\begin{aligned}&(\lambda^{\alpha'}-A)^{-1}x=\lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda^\sigma t}P_\alpha^\beta(t)xdt\\ &=\lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda t}\sigma t^{\sigma-1}\int_0^\infty s\Phi_\sigma(s)P_\alpha^\beta(st^\sigma)xds\, dt=\lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda t}P_{\alpha'}^\beta(t)x\, dt.\end{aligned}$$

Hence, A generates an $(\alpha',\alpha')^\beta$-resolvent family $P_{\alpha'}^{\beta'}$ given by (4.14) and we have shown part (a).

(b) Now assume that A generates an $(\alpha,1)^\beta$-resolvent family S_α^β. Then by definition, there exists $\omega\ge 0$ such that $\{\lambda^\alpha:\ \mathrm{Re}(\lambda)>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}x=\lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}S_\alpha^\beta(t)x\, dt,\ \mathrm{Re}(\lambda)>\omega,\ \forall\, x\in X. \tag{4.16}$$

Let $0<\alpha'<\alpha$ and let $S_{\alpha'}^\beta$ be given by (4.15). Then it is clear that $S_{\alpha'}^\beta$ is strongly continuous from $[0,\infty)$ into $\mathcal{L}(X)$. By Lemma 4.12 there exist some constants $M,\omega\ge 0$ such that $\|(g_1*S_{\alpha'}^\beta)(t)x\|\le Me^{\omega t}\|x\|$ for every $x\in X$ and $t>0$. It follows from (4.16) and (2.19) that $\{\lambda^{\alpha'}:\ \mathrm{Re}(\lambda)>\omega\}\subset\rho(A)$ and for every $x\in X$ and $\mathrm{Re}(\lambda)>\omega$,

$$\begin{aligned}&\lambda^{\alpha'-1}(\lambda^{\alpha'}-A)^{-1}x=\lambda^{\alpha'\beta}\lambda^{\sigma-1}\int_0^\infty e^{-\lambda^\sigma t}S_\alpha^\beta(t)xdt\\ &=\lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda t}\int_0^\infty\Phi_\sigma(s)S_\alpha^\beta(st^\sigma)xds\, dt=\lambda^{\alpha'\beta}\int_0^\infty e^{-\lambda t}S_{\alpha'}^\beta(t)x\, dt.\end{aligned}$$

Hence, A generates an $(\alpha',1)^\beta$-resolvent family $S_{\alpha'}^{\beta'}$ given by (4.15). The proof of the theorem is finished. $\square$

We have the following result as a direct consequence of Theorem 4.13.

Corollary 4.14. *Let $0 < \alpha \leq 1$, $\beta \geq 0$ and let A be a closed linear operator on a Banach space X. If A generates an exponentially bounded $\beta-$times integrated semigroup $(T_\beta(t))$, then A generates an $(\alpha,\alpha)^\beta$-resolvent family $(P_\alpha^\beta(t))$ given by*

$$P_\alpha^\beta(t)x = \alpha \int_0^\infty \frac{s}{t^{\alpha+1}} \Phi_\alpha(st^{-\alpha}) T_\beta(s)x ds \tag{4.17}$$
$$= \alpha \int_0^\infty \frac{\tau}{t^{1-\alpha}} \Phi_\alpha(\tau) T_\beta(\tau t^\alpha) x d\tau,\ t > 0,\ x \in X,$$

and is exponentially bounded away from 0.

Let $(S_\alpha^\beta(t))$ be the associated $(\alpha,1)^\beta$-resolvent family generated by A which exists by Remark 4.4 (a). *Then*

$$S_\alpha^\beta(t)x = \int_0^\infty t^{-\alpha} \Phi_\alpha(st^{-\alpha}) T_\beta(s)x ds = \int_0^\infty \Phi_\alpha(\tau) T_\beta(\tau t^\alpha) x d\tau, \tag{4.18}$$

$t > 0$, $x \in X$, and is exponentially bounded . In particular, it follows from (4.17) *and* (4.18) *that $(P_\alpha^\beta(t))$ and $(S_\alpha^\beta(t))$ are analytic for $t > 0$.*

Proof. We just have to show that P_α^β is exponentially bounded at ∞ and that S_α^β is exponentially bounded. By hypothesis, there exist some constants $M, \omega \geq 0$ such that $\|T_\beta(t)x\| \leq Me^{\omega t}\|x\|$ for every $t > 0$ and $x \in X$. Therefore, using (4.18), (2.20), (2.9), (2.15) and using the same procedure as in the proof of Lemma 4.12, we have that for every $t > 0$ and $x \in X$,

$$\|S_\alpha^\beta(t)x\| \leq M_1 e^{t\omega^{\frac{1}{\alpha}}} \|x\|,$$

for some constant $M_1 \geq 0$ and we have shown that S_α^β is exponentially bounded. Now, let $t_0 > 0$ be fixed. Similarly, using (4.18), (2.20), (2.9) and (2.15), we have that for every $t \geq t_0$ and $x \in X$,

$$\|P_\alpha^\beta(t)x\| \leq M\|x\| \frac{1}{t^{1-\alpha}} E_{\alpha,\alpha}(\omega t^\alpha) \leq M_1 e^{t\omega^{\frac{1}{\alpha}}} \|x\|,$$

for some constant $M_1 \geq 0$. Hence, P_α^β is exponentially bounded at ∞ and the proof is finished. □

Operators with polynomially bounded resolvent in a right-half plane generate integrated semigroups. Namely, if A satisfies (1.5), then A generates an exponentially bounded integrated semigroup (see [2, Theorem 3.2.8] and [31]).

We notice that if $\beta = 0$, that is, A is the generator of a C_0-semigroup, then the representations (4.17) and (4.18) have been obtained in [4, 21].

Remark 4.15. We notice the following facts.

(a) As we have mentioned in Remark 4.7, in general, generators of integrated semigroups are not densely defined [2, Remark 3.2.3, p.123]. We refer to [2, Chapers 3 and 6] and [26, 27] for some examples.

(b) In general generators of resolvent families even in the case $\beta = 0$ are not stable under bounded perturbations. In the case $\beta = 0$, an example in [4, Example 2.24] shows that they need not be stable by perturbations by scalar multiples of the identity (that is, cI where $c \in \mathbb{C}$). Therefore the resolvent families obtained through Corollary 4.14 are of special interest since they are stable under the perturbations by multiple of the identities. Other admissible perturbations have been studied, see e.g. [2, p.232], [18, Theorems 3.1 and 3.3], [22] and the references therein.

We have the following result which can be viewed as an extension of Lemma 4.12. It may also be of interest in its own right.

Lemma 4.16. *Let A be a closed linear operator on a Banach space X. Let $0 < \alpha \leq 1$, $\beta \geq 0$ and $\mu > 0$. Then the following assertions hold.*

(a) *Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family P_α^β. Let $0 < \alpha' < \alpha \leq 1$, $\sigma := \frac{\alpha'}{\alpha}$ and let $P_{\alpha'}^\beta$ be the $(\alpha',\alpha')^\beta$-resolvent family generated by A. Then*

$$\int_0^\infty \frac{\sigma s}{t^{\sigma+1}} \Phi_\sigma(st^{-\sigma})(g_\mu * P_\alpha^\beta)(s)xds = (g_{\mu\sigma} * P_{\alpha'}^\beta)(t)x, \; t > 0, \, x \in X. \quad (4.19)$$

(b) *Assume that A generates an $(\alpha,1)^\beta$-resolvent family S_α^β. Let $0 < \alpha' < \alpha \leq 1$, $\sigma := \frac{\alpha'}{\alpha}$ and let $S_{\alpha'}^\beta$ be the $(\alpha',1)^\beta$-resolvent family generated by A. Then*

$$\int_0^\infty \frac{1}{t^\sigma} \Phi_\alpha(st^{-\sigma})(g_\mu * S_\alpha^\beta)(s)xds = (g_{\mu\sigma} * S_{\alpha'}^\beta)(t), \; t \geq 0, \, x \in X. \quad (4.20)$$

Proof. Let A, α, β be as in the statement of the lemma, $x \in X$ and $\mu > 0$.

(a) Assume that A generates an $(\alpha,\alpha)^\beta$-resolvent family P_α^β. Let ω be the real number from the definition of P_α^β. Let $0 < \alpha' < \alpha$. Taking the Laplace transform, we have that for $\mathrm{Re}(\lambda) > \omega$ and $x \in X$,

$$(\widehat{g_{\mu\sigma} * P_{\alpha'}^\beta})(\lambda)x = \lambda^{-\mu\sigma}\lambda^{-\alpha'\beta}(\lambda^{\alpha'} - A)^{-1}x = \lambda^{-\mu\sigma-\alpha'\beta}(\lambda^{\alpha'} - A)^{-1}x. \quad (4.21)$$

On the other hand, using (2.18) and Fubini's theorem, we obtain that for $\mathrm{Re}(\lambda) > \omega$ and $x \in X$,

$$\int_0^\infty e^{-\lambda t} \int_0^\infty \frac{\sigma s}{t^{\sigma+1}} \Phi_\sigma(st^{-\sigma})(g_\mu * P_\alpha^\beta)(s)x ds\, dt = \int_0^\infty e^{-\lambda^\sigma s}(g_\mu * P_\alpha^\beta)(s)x ds$$
$$= \lambda^{-\sigma(\mu+\alpha\beta)}(\lambda^{\alpha\sigma} - A)^{-1}x = \lambda^{-\sigma\mu-\alpha'\beta}(\lambda^{\alpha'} - A)^{-1}x. \tag{4.22}$$

Using (4.21) and (4.22), the equality (4.19) follows from the uniqueness theorem for the Laplace transform and by continuity.

(b) Similarly, for $\mathrm{Re}(\lambda) > \omega$ and $x \in X$,

$$\widehat{(g_{\sigma\mu} * S_{\alpha'}^\beta)}(\lambda)x = \lambda^{-\sigma\mu-\alpha'\beta}\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x. \tag{4.23}$$

Using (2.19) and Fubini's theorem, we obtain for $\mathrm{Re}(\lambda) > \omega$ and $x \in X$,

$$\int_0^\infty e^{-\lambda t} \int_0^\infty \frac{1}{t^\sigma} \Phi_\alpha(st^{-\sigma})(g_\mu * S_\alpha^\beta)(s)x ds\, dt \tag{4.24}$$
$$= \lambda^{\sigma-1} \int_0^\infty e^{-\lambda^\sigma t}(g_{\alpha\mu} * S_\alpha^\beta)(s)x ds$$
$$= \lambda^{\sigma-1}\lambda^{-\mu\sigma-\alpha'\beta)}\lambda^{\alpha'-\sigma)}(\lambda^{\alpha'} - A)^{-1}x = \lambda^{-\sigma\mu-\alpha'\beta}\lambda^{\alpha'-1}(\lambda^{\alpha'} - A)^{-1}x.$$

Using (4.23) and (4.24), the equality (4.20) also follows from the uniqueness theorem for the Laplace transform and by continuity. □

The following result on the regularity properties of S_α^β is crucial and will be used several times in the subsequent sections to obtain our main results.

Lemma 4.17. *Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $0 < \alpha \le 1, \beta \ge 0$, $k := \lceil \alpha\beta \rceil$, $n := \lceil \beta \rceil$ and assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β. Then the following properties hold:*

(a) *Let $m \in \mathbb{N} \cup \{0\}$. Then for every $x \in D(A^{m+1})$ and $\forall\, t \ge 0$,*

$$S_\alpha^\beta(t)x = \sum_{j=0}^{m} g_{\alpha(\beta+j)+1}(t)A^j x + \int_0^t g_{\alpha(m+1)}(t-s)S_\alpha^\beta(s)A^{m+1}x\, ds. \tag{4.25}$$

(b) *For every $x \in D(A^{n+1})$, the mapping $t \mapsto (g_{k-\alpha\beta} * S_\alpha^\beta)(t)x$ belongs to $C^k([0,\infty); D(A))$ and*

$$\frac{d^k}{dt^k}\left[(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x\right] = \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^n x. \tag{4.26}$$

Moreover, for $j = 0, 1, \dots, k-1$,

$$\frac{d^j}{dt^j}[g_{k-\alpha\beta} * S_\alpha^\beta](0)x = 0, \quad \text{and} \quad \frac{d^k}{dt^k}[g_{k-\alpha\beta} * S_\alpha^\beta](0)x = x. \tag{4.27}$$

(c) *In general, for every* $x \in D(A^{n+1-i})$, $i = 0, 1, \dots, n$, *the mapping* $t \mapsto (g_{k-\alpha\beta} * g_{\alpha i} * S_\alpha^\beta)(t)x$ *belongs to* $C^k([0,\infty); D(A))$ *and*

$$\begin{aligned} &\frac{d^k}{dt^k}\left[(g_{k-\alpha\beta} * g_{\alpha i} * S_\alpha^\beta)(t)x\right] \\ &= \sum_{j=0}^{n-i} g_{\alpha j+1+\alpha i}(t)A^j x + (g_{\alpha(n-\beta)} * g_\alpha * S_\alpha^\beta)(t)A^{n+1-i}x. \end{aligned} \tag{4.28}$$

(d) *For every* $x \in D(A^n)$, *the mapping* $t \mapsto (g_{k-\alpha\beta} * S_\alpha^\beta)(t)x$ *belongs to* $C^k([0,\infty); X)$ *and the equalities* (4.26), (4.27) *hold.*

(e) *In general, for every* $x \in D(A^{n-i})$, $i = 0, 1, \dots, n$, *the mapping* $t \mapsto (g_{k-\alpha\beta} * g_{\alpha i} * S_\alpha^\beta)(t)x$ *belongs to* $C^k([0,\infty); X)$ *and*

$$\begin{aligned} &\frac{d^k}{dt^k}\left[(g_{k-\alpha\beta} * g_{\alpha i} * S_\alpha^\beta)(t)x\right] \\ &= \sum_{j=0}^{n-i} g_{\alpha j+1+\alpha i}(t)A^j x + A(g_{\alpha(n-\beta)} * g_\alpha * S_\alpha^\beta)(t)A^{n-i}x. \end{aligned} \tag{4.29}$$

Proof. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let $0 < \alpha \le 1, \beta \ge 0$ and set $k := \lceil \alpha\beta \rceil$, $n := \lceil \beta \rceil$. Assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β.

(a) We prove (4.25) by induction. If $m = 0$, then for every $x \in D(A)$, the equality (4.25) reads

$$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + \int_0^t g_\alpha(t-s)S_\alpha^\beta(s)Ax\, ds, \ \ \forall\, t \ge 0$$

which is given by Lemma 4.5(b). Assume that (4.25) holds for $m-1$ for some $m \in \mathbb{N}$. Now, let $x \in D(A^{m+1}) \subset D(A^m)$. Then using Lemma 4.5(a)-(b), we have that

$$\begin{aligned} S_\alpha^\beta(t)x &= \sum_{j=0}^{m-1} g_{\alpha(\beta+j)+1}(t)A^j x + (g_{\alpha m} * S_\alpha^\beta)(t)A^m x \\ &= \sum_{j=0}^{m-1} g_{\alpha(\beta+j)+1}(t)A^j x + A^m g_{\alpha m} * \Big(g_{\alpha\beta+1}x + g_\alpha * S_\alpha^\beta Ax\Big)(t) \end{aligned}$$

$$= \sum_{j=0}^{m} g_{\alpha(\beta+j)+1}(t)A^j x + (g_{\alpha(m+1)} * S_\alpha^\beta)(t)A^{m+1}x.$$

We conclude that (4.25) holds and this completes the proof of part (a).

(b) Let $x \in D(A^{n+1})$. Then using (4.25) with $m = n$ and (2.1) we get that for every $t \geq 0$,

$$(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x = \sum_{j=0}^{n} g_{k+\alpha j+1}(t)A^j x + (g_{\alpha(n+1)+k-\alpha\beta} * S_\alpha^\beta)(t)A^{n+1}x.$$

Therefore, using (2.6) and Lemma 4.5(b) we have that for all $t \geq 0$,

$$\frac{d^k}{dt^k}\Big[(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x\Big] = \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n+1)-\alpha\beta} * S_\alpha^\beta)(t)A^{n+1}x$$

$$= \sum_{j=0}^{n} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\beta)} * A^n(S_\alpha^\beta - g_{\alpha\beta+1}))(t)x$$

$$= \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^n x,$$

and we have shown (4.26). Since $A^n x \in D(A)$, it follows from (4.26) and Lemma 4.5 that $\frac{d^k}{dt^k}(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x \in C([0,\infty); D(A))$. Hence, $(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x \in C^k([0,\infty); D(A))$. Since $g_1(0) = 1$ and $g_{\alpha j+1}(0) = 0$ for every $j = 1, 2, \ldots, n-1$, the identities in (4.27) follow from (4.26) and this completes the proof of the assertion (b).

(c) Let $x \in D(A^{n+1-i})$, $i = 0, 1, \ldots, n$. Proceeding as in the proof of part (b), we obtain that

$$(g_{\alpha i} * g_{k-\alpha\beta} * S_\alpha^\beta)(t)x = \sum_{j=0}^{n-1-i} g_{k+\alpha j+\alpha i+1}(t)A^j x + (g_{k+\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n-i}x.$$

Using Lemma 4.5(b) and (2.6), the preceding equality implies that

$$\frac{d^k}{dt^k}[(g_{\alpha i} * g_{k-\alpha\beta} * S_\alpha^\beta)(t)x] = \sum_{j=0}^{n-1-i} g_{\alpha j+\alpha i+1}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n-i}x$$

$$= \sum_{j=0}^{n-i} g_{\alpha j+\alpha i+1}(t)A^j x + (g_{\alpha(n-\beta)} * g_\alpha * S_\alpha^\beta)(t)A^{n+1-i}x.$$

Hence, $g_{\alpha i} * g_{k-\alpha\beta} * S_\alpha^\beta)(t)x \in C^k([0,\infty); D(A)$ and one has (4.28).

(d) Let $x \in D(A^n)$. Proceeding as in part (b), we also get (4.26) and this implies that $(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x \in C^k([0,\infty);X)$ and (4.27) holds.

(e) Let $x \in D(A^{n-i})$, $i = 0,1,\dots,n$. Proceeding as in part (c), we obtain that $(g_{k-\alpha\beta} * g_{\alpha i} * S_\alpha^\beta)(t)x \in C^k([0,\infty);X)$ and (4.29) holds and this completes the proof of the lemma. □

5. Regularized abstract Cauchy problem

In this section we show that the existence of the above defined resolvent family S_α^β is necessary and sufficient for the well-posedness of the regularized abstract Cauchy problem

$$\begin{cases} \mathbb{D}_t^\alpha v(t) = Av(t) + g_{\alpha\beta+1}(t)x, \quad t > 0 \\ v(0) = 0, \end{cases} \tag{5.1}$$

where A is a closed linear operator with domain $D(A)$ defined in a Banach space X, that we assume throughout this section without any mention.

The following is the main result of this section.

Theorem 5.1. *Let $0 < \alpha \le 1$ and $\beta \ge 0$. Then the following assertions are equivalent.*

(i) *The operator A generates an $(\alpha,1)^\beta$-resolvent family S_α^β on X.*
(ii) *For all $x \in X$, there exists a unique classical solution v of Problem (5.1) such that $(g_{1-\alpha} * v)(t)$ is exponentially bounded. That is, $v \in C([0,\infty);D(A))$, $g_{1-\alpha} * v \in C^1([0,\infty);X)$, $(g_{1-\alpha} * v)(t)$ is exponentially bounded, and (5.1) is satisfied.*

Proof. Let A, α and β be as in the statement of the theorem.

(i) ⇒ (ii): Assume that A generates an $(\alpha,1)^\beta$-resolvent family S_α^β on X and let $x \in X$. Define

$$v(t) := (g_\alpha * S_\alpha^\beta)(t)x = \int_0^t g_\alpha(t-s)S_\alpha^\beta(s)x\,ds, \;\; t \ge 0.$$

Then $v(0) = 0$ and by Lemma 4.5(c) we have that $v \in C([0,\infty);D(A))$. Since for every $t \ge 0$,

$$(g_{1-\alpha} * v)(t) = (g_{1-\alpha} * g_\alpha * S_\alpha^\beta)(t)x = (g_1 * S_\alpha^\beta)(t)x = \int_0^t S_\alpha^\beta(s)x\,ds,$$

it follows that $g_{1-\alpha} * v \in C^1([0,\infty);X)$. Using (2.7) and Lemma 4.5(c), we get that for every $t \ge 0$,

$$\mathbb{D}_t^\alpha v(t) = (g_{1-\alpha} * v')(t) = \frac{d}{dt}\left(g_{1-\alpha} * v\right)(t) = \frac{d}{dt}\left[(g_1 * S_\alpha^\beta)(t)x\Big)\right] = S_\alpha^\beta(t)x$$

$$= A(g_\alpha * S_\alpha^\beta)(t)x + g_{\alpha\beta+1}(t)x = Av(t) + g_{\alpha\beta+1}(t).$$

Hence, v is a classical solution of (5.1). Since $(g_1 * S_\alpha^\beta)(t)$ is exponentially bounded and

$$(g_{1-\alpha} * v)(t) = (g_{1-\alpha} * g_\alpha * S_\alpha^\beta)(t)x = (g_1 * S_\alpha^\beta)(t)x,$$

it follows that $(g_{1-\alpha} * v)(t)$ is exponentially bounded. Assume that (5.1) has two classical solutions v_1 and v_2 and set $V := v_1 - v_2$. Then $V \in C([0,\infty); D(A))$, $V(0) = 0$, $(g_{1-\alpha} * V) \in C^1([0,\infty); X)$ and $\mathbb{D}_t^\alpha V(t) = AV(t)$ for every $t \geq 0$. Taking the Laplace transform on both sides of this equality, we get that for $\mathrm{Re}(\lambda) > \omega$ (where ω is the real number from the above mentioned exponential bound), $(\lambda^\alpha - A)\widehat{V}(\lambda) = 0$. Since $(\lambda^\alpha - A)$ is invertible, we have that $\widehat{V}(\lambda) = 0$. By the uniqueness theorem for the Laplace transform and by continuity, we get that $V(t) = 0$ for every $t \geq 0$. We have shown uniqueness of solutions and this completes the proof of part (ii).

(ii) $\Rightarrow$ (i): For $x \in X$, we let $S_{\alpha,\beta}(t)x := \mathbb{D}_t^\alpha v(t,x)$ where $v(t,x)$ is the unique solution of (5.1). By the closed graph theorem, $S_{\alpha,\beta}(t) \in \mathcal{L}(X)$ for every $t \geq 0$ (see the proof of Theorem 3.2.13 in [2]). Moreover, $(S_{\alpha,\beta}(t))$ is strongly continuous. Using (2.3) and the fact that $v(0) = 0$ we get that

$$(g_\alpha * S_{\alpha,\beta})(t)x = (g_\alpha * \mathbb{D}_t^\alpha v)(t) = v(t,x) - v(0,x) = v(t,x).$$

Hence, $(g_\alpha * S_{\alpha,\beta})(t)x \in D(A)$ for every $x \in X$ and one has the identity

$$A(g_\alpha * S_{\alpha,\beta})x + g_{\alpha\beta+1}(t)x = Av(t,x) + g_{\alpha\beta+1}(t)x = S_{\alpha,\beta}(t)x. \tag{5.2}$$

Since by assumption $(g_{1-\alpha} * v)(t)$ is exponentially bounded and

$$(g_1 * S_{\alpha,\beta})(t)x = (g_1 * g_{1-\alpha} * v')(t) = (g_{1-\alpha} * g_1 * v')(t) = (g_{1-\alpha} * v)(t),$$

we have that $(g_1 * S_{\alpha,\beta})(t)x$ is exponentially bounded. By the uniform exponential boundedness principle [2, Lemma 3.2.14], there exist some constants $M, \omega \geq 0$ such that

$$\|(g_{1-\alpha} * v)(t)\| = \|(g_1 * S_{\alpha,\beta})(t)x\| \leq Me^{\omega t}\|x\|,\ x \in X,\ t \geq 0. \tag{5.3}$$

Taking the Laplace transform on both sides of the identity (5.2), we get that for $\mathrm{Re}(\lambda) > \omega$,

$$A\lambda^{-\alpha}\widehat{S_{\alpha,\beta}}(\lambda)x - \widehat{S_{\alpha,\beta}}(\lambda)x = -\lambda^{-\alpha\beta-1}x.$$

After multiplying both sides of the preceding identity by λ^α, we get that

$$(\lambda^\alpha - A)\widehat{S_{\alpha,\beta}}(\lambda)x = \lambda^{-\alpha\beta-1+\alpha}x.$$

The preceding equality implies that $(\lambda^\alpha - A)$ is surjective for $\mathrm{Re}(\lambda) > \omega$. We show that it is injective as well. To this end, suppose $(\lambda^\alpha - A)x = 0$

for some $x \in D(A)$ and $\mathrm{Re}(\lambda) > \omega$, that is $Ax = \lambda^\alpha x$ for $\mathrm{Re}(\lambda) > \omega$. It is enough to consider that $Ax = \lambda^\alpha x$ for λ real and $\lambda > \omega$. Then setting $v(t) = (g_{\alpha\beta+1} * \tilde{E})(t)x$ where $\tilde{E}(t)x = t^{\alpha-1}E_{\alpha,\alpha}(\lambda^\alpha t^\alpha)x$, we prove that v is a solution of Equation (5.1). Obviously $v \in C([0,\infty); D(A))$ and $(g_{1-\alpha} * v) \in C^1([0,\infty); X)$. Using (2.14), we have that for every $t > 0$,

$$\begin{aligned}\mathbb{D}_t^\alpha v(t) &= (g_{1-\alpha} * g_{\alpha\beta} * \tilde{E})(t)x \\ &= (g_{\alpha\beta+1-\alpha} * \tilde{E})(t)x = g_{\alpha\beta+1-\alpha} * (g_\alpha + \lambda^\alpha g_\alpha * \tilde{E}))(t)x \\ &= g_{\alpha\beta+1}(t)x + A(g_{\alpha\beta+1} * \tilde{E})(t)x = g_{\alpha\beta+1}(t)x + Av(t).\end{aligned}$$

We have shown that v is a solution of Equation (5.1). Since all the solutions v of Equation (5.1) satisfy (5.3), we must have this estimate for the solution $v(t) = (g_{\alpha\beta+1} * \tilde{E})(t)x$ just found. But using (2.9) we have that

$$\tilde{E}(t) = t^{\alpha-1}\sum_{n=0}^{\infty}\frac{\lambda^{\alpha n}t^{\alpha n}}{\Gamma(\alpha(n+1))} = \sum_{n=0}^{\infty}\frac{\lambda^{\alpha n}t^{\alpha(n+1)-1}}{\Gamma(\alpha(n+1))}$$

which gives

$$(g_{1-\alpha} * v)(t) = (g_{\alpha\beta+2-\alpha} * \tilde{E})(t)x = t^{\alpha\beta+1}E_{\alpha,\alpha\beta+2}(\lambda^\alpha t^\alpha)x,$$

and hence by (2.15), $\|(g_{1-\alpha} * v)(t)\| \leq Me^{\lambda t}\|x\|$ and this estimate is sharp. Therefore, we can only have (5.3) if $x = 0$. Therefore, $(\lambda^\alpha - A)$ is injective, hence is invertible and

$$\widehat{S_{\alpha,\beta}}(\lambda)x = \lambda^{-\alpha\beta}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x,$$

that is, for every $x \in X$ and $\mathrm{Re}(\lambda) > \omega$,

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{\alpha\beta}\int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)x\,dt.$$

Hence, A generates an $(\alpha,1)^\beta$-resolvent family S_α^β and by the uniqueness theorem for the Laplace transform and by continuity we have that $S_\alpha^\beta(t)x = S_{\alpha,\beta}(t)x$ for every $t \geq 0$ and $x \in X$. We have shown the assertion (i) and the proof of the theorem is finished. □

We observe that if the family S_α^β is exponentially bounded, then the solution v in Theorem 5.1 is exponentially bounded as well.

Remark 5.2. We note that in Theorem 5.1, the assertion $(g_{1-\alpha} * v)(t)$ is exponentially bounded agrees with the limiting case $\alpha = 1$ in which the conclusion reads $v(t)$ is exponentially bounded (see e.g. [2, Theorem 3.2.13]). An example showing that the exponential boundedness assumption cannot be omitted is included in [2, Remark 3.2.15(b)] for the limiting case $\alpha = 1$.

6. The homogeneous abstract Cauchy problem

In this section we use the above defined resolvent families to investigate the existence and the representation of solutions of homogeneous abstract Cauchy problems of fractional order. More precisely, we consider the problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t) = Au(t), \quad t > 0,\ 0 < \alpha \leq 1, \\ u(0) = x, \end{cases} \tag{6.1}$$

where A is a closed linear operator with domain $D(A)$ defined in a Banach space X and x is a given vector in X.

Definition 6.1. *A function $u \in C([0,\infty); D(A))$ is said to be a classical solution of Problem (6.1) if $g_{1-\alpha} * (u - u(0)) \in C^1([0,\infty); X)$ and (6.1) is satisfied.*

We adopt the following definition of mild solutions.

Definition 6.2. *A function $u \in C([0,\infty); X)$ is said to be a mild solution of Problem (6.1) if $I_t^\alpha u(t) := (g_\alpha * u)(t) \in D(A)$ for every $t \geq 0$, and*

$$u(t) = x + A \int_0^t g_\alpha(t-s) u(s)\, ds, \quad \forall\, t \geq 0.$$

Throughout this section we assume that A is a closed linear operator with domain $D(A)$ defined in a Banach space X. First, we show the uniqueness of mild solutions and hence, of classical solutions.

Proposition 6.3. *Let $0 < \alpha \leq 1$. Then the following assertions hold.*

(a) *If u is a classical solution of (6.1), then it is a mild solution of (6.1).*
(b) *If $(\lambda^\alpha - A)$ is invertible for $Re(\lambda)$ large enough, and if a mild solution u exists and $(g_1 * u)(t)$ is exponentially bounded, then it is unique.*

Proof. Let $0 < \alpha \leq 1$ and let A be a closed linear operator with domain $D(A)$ defined on a Banach space X.

(a) Let u be a classical solution of (6.1). Since $u \in C([0,\infty); D(A))$ we have that $(g_\alpha * u)(t) \in C([0,\infty); D(A))$. Since $\mathbb{D}_t^\alpha u(t) = Au(t)$, that is, $(g_{1-\alpha} * u')(t) = Au(t)$, we have that $(g_\alpha * g_{1-\alpha} * u')(t) = A(g_\alpha * u)(t)$, i.e., $(g_1 * u')(t) = A(g_\alpha * u)(t)$. Hence, $u(t) - u(0) = A(g_\alpha * u)(t)$ for every $t \geq 0$ and we have shown that u is a mild solution of (6.1).

(b) Assume that (6.1) has two mild solutions u and v and set $U := u - v$. Then $U \in C([0,\infty); X)$, $(g_\alpha * U)(t) \in D(A)$ for every $t \geq 0$ and $U(t) = A(g_\alpha * U)(t)$. Taking the Laplace transform, we get that $(I - \lambda^{-\alpha} A)\widehat{U}(\lambda) = 0$ for $\mathrm{Re}(\lambda) > \omega$ (where $\omega \geq 0$ is the real number from the exponential boundedness of $(g_1 * u)(t)$). Since by assumption $(I - \lambda^{-\alpha} A)$ is invertible, we have that $\widehat{U}(\lambda) = 0$. By the uniqueness theorem for the Laplace transform and by continuity, we get that $U(t) = 0$ for every $t \geq 0$. Hence, $u(t) = v(t)$ for every $t \geq 0$. The proof is finished. □

Remark 6.4. We notice that in order to prove the existence of solutions of Problem (6.1), we proceed by direct construction and make minimal use of the Laplace transform.

The following theorem is the main result of this section.

Theorem 6.5. *Let $0 < \alpha \leq 1, \beta \geq 0$ and set $k := \lceil \alpha\beta \rceil$, $n := \lceil \beta \rceil$. Assume that A generates an $(\alpha, 1)^\beta$-resolvent family S_α^β. Then the following assertions hold.*

(a) *For every $x \in D(A^{n+1})$, the function $u(t) := D_t^{\alpha\beta} S_\alpha^\beta(t)x$ is the unique classical solution of the abstract Cauchy problem* (6.1).

(b) *For every $x \in D(A^n)$, the function $u(t) := D_t^{\alpha\beta} S_\alpha^\beta(t)x$ is the unique mild solution of the abstract Cauchy problem* (6.1).

Proof. Let A, α, β, n, k and S_α^β be as in the statement of the theorem. First we prove existence of mild and classical solutions.

(a) Let $x \in D(A^{n+1})$. By (4.26) in Lemma 4.17, we have for every $t \geq 0$,

$$\begin{aligned} u(t) : &= D_t^{\alpha\beta} S_\alpha^\beta(t)x = \frac{d^k}{dt^k} \left(g_{k-\alpha\beta} * S_\alpha^\beta \right)(t)x \\ &= \sum_{j=0}^{n-1} g_{\alpha j+1}(t) A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t) A^n x. \end{aligned} \tag{6.2}$$

It follows from (6.2) and Lemma 4.17 that $u \in C([0, \infty); D(A))$ and $u(0) = x$. Using (6.2) and Lemma 4.5, we get that for every $t \geq 0$,

$$\begin{aligned} & g_{1-\alpha} * (u - u(0))(t) = (g_{1-\alpha} * u)(t) - g_{2-\alpha}(t)x \\ &= g_{1-\alpha} * \Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t) A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t) A^n x \Big] - g_{2-\alpha}(t)x \\ &= \sum_{j=1}^{n-1} g_{\alpha j+2-\alpha}(t) A^j x + g_{\alpha(n-\beta)+1-\alpha} * \Big[g_{\alpha\beta+1}(t) A^n x + (g_\alpha * S_\alpha^\beta)(t) A^{n+1} x \Big] \\ &= \sum_{j=1}^{n} g_{\alpha j+2-\alpha}(t) A^j x + (g_{\alpha(n-\beta)+1} * S_\alpha^\beta)(t) A^{n+1} x. \end{aligned} \tag{6.3}$$

Using (6.3) we get that for every $t \geq 0$,

$$\frac{d}{dt}\Big[g_{1-\alpha} * (u - u(0))(t)\Big] = \sum_{j=1}^{n} g_{\alpha j+1-\alpha}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n+1}x \in C([0,\infty); X).$$

Hence, $g_{1-\alpha} * (u - u(0)) \in C^1([0,\infty); X)$. It remains to show that u satisfies (6.1). Using (6.2), (2.3), (2.7) and Lemma 4.5(b), we have that for $t \geq 0$,

$$\begin{aligned}
\mathbb{D}_t^\alpha u(t) &= \mathbb{D}_t^\alpha D_t^{\alpha\beta} S_\alpha^\beta(t)x = \mathbb{D}_t^\alpha\Big[\frac{d^k}{dt^k}\big(g_{k-\alpha\beta} * S_\alpha^\beta\big)\Big](t)x \qquad (6.4)\\
&= g_{1-\alpha} * \Big[\frac{d^{k+1}}{dt^{k+1}}\big(g_{k-\alpha\beta} * S_\alpha^\beta\big)\Big](t)x\\
&= g_{1-\alpha} * \frac{d}{dt}\Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^n x\Big]\\
&= \sum_{j=1}^{n-1} g_{\alpha j+1-\alpha}(t)A^j x + \frac{d}{dt}\Big[(g_{1-\alpha} * g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^n x\Big]\\
&= \sum_{j=1}^{n-1} g_{\alpha j+1-\alpha}(t)A^j x + g_{\alpha n+1-\alpha}(t)A^n x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n+1}x\\
&= \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^{j+1}x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n+1}x\\
&= AD_t^{\alpha\beta} S_\alpha^\beta(t)x = Au(t)
\end{aligned}$$

and this completes the proof of the existence part in assertion (a).

(b) Let $x \in D(A^n)$ and set

$$u(t) := D_t^{\alpha\beta} S_\alpha^\beta(t)x = \frac{d^k}{dt^k}\Big[(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x\Big].$$

It follows from Lemma 4.17(d) that $u \in C([0,\infty); X)$ and $u(0) = x$. Since (by (4.26) in Lemma 4.17)

$$\begin{aligned}
I_t^\alpha u(t) :&= (g_\alpha * D_t^{\alpha\beta} S_\alpha^\beta)(t)x = g_\alpha * \Big[\frac{d^k}{dt^k}(g_{k-\alpha\beta} * S_\alpha^\beta)\Big](t)x\\
&= \sum_{j=0}^{n-1} g_{\alpha j+1+\alpha}(t)A^j x + (g_{\alpha(n-\beta)} * g_\alpha * S_\alpha^\beta)(t)A^n x,
\end{aligned}$$

it follows from Lemma 4.5 that $I_t^\alpha u(t) \in D(A)$ for every $t \geq 0$. Using Lemma 4.17 and Lemma 4.5, we have that for every $t \geq 0$,

$$\begin{aligned} u(t) &= \sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^n x \qquad (6.5)\\ &= x + A\Big[\sum_{j=1}^{n-1} g_{\alpha j+1}(t)A^{j-1}x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n-1}x\Big]\\ &= x + A\Big[\sum_{j=1}^{n} g_{\alpha j+1}(t)A^{j-1}x + (g_{\alpha(n-\beta)} * g_\alpha * S_\alpha^\beta)(t)A^{n}x\Big]\\ &= x + Ag_\alpha * \Big[\sum_{j=0}^{n-1} g_{\alpha j+1}(t)A^{j}x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n}x\Big] = x + A(g_\alpha * u)(t) \end{aligned}$$

Hence, u is a mild solution of (6.1) and this completes the proof of the existence part in assertion (b).

It remains to show the uniqueness of solutions. Let $x \in D(A^n)$ and let u be a mild solution. We just have to show that $(g_1 * u)(t)$ is exponentially bounded. Using (6.2), we have that for every $t \geq 0$,

$$(g_1 * u)(t) = \sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j x + (g_{\alpha(n-\beta)+1} * S_\alpha^\beta)(t)A^n x.$$

Using Lemma 4.3 we get from the preceding equality that there exist some constants $M, \omega \geq 0$ such that for every $t \geq 0$,

$$\|(g_1 * u)(t)\| \leq Me^{\omega t}\sum_{j=0}^{n} \|A^j x\|.$$

Hence, $(g_1 * u)(t)$ is exponentially bounded. Now, Proposition 6.3 implies the uniqueness of mild and classical solutions. The proof is finished. □

Remark 6.6. We note the following facts regarding Theorem 6.5.

(a) First, we observe that although in (6.1) we have the Caputo fractional derivative $\mathbb{D}_t^\alpha$, the solution is given by the Riemann-Liouville derivative $D_t^{\alpha\beta} S_\alpha^\beta(t)x$. If $\alpha\beta$ is not an integer, then the function $\mathbb{D}_t^{\alpha\beta} S_\alpha^\beta(t)x$ is not a solution of (6.1), unless $x = 0$.

(b) Second, we have that for every $x \in D(A^{n+1})$, $(g_{k-\alpha\beta} * S_\alpha^\beta)(t)x \in C^{k+1}((0,\infty); X)$, that is, $D_t^{\alpha\beta} S_\alpha^\beta(t)x$ is differentiable with value in X on the open interval $(0,\infty)$. But, $D_t^{\alpha\beta}S_\alpha^\beta(t)x$ is not differentiable at $t=0^+$. Indeed, it is well-known that for every function $v\in C([0,\infty);X)$ and $0<\alpha\le 1$,

$$\lim_{t\downarrow 0}\frac{I_t^\alpha v(t)}{g_{\alpha+1}(t)}=\lim_{t\downarrow 0}\frac{(g_\alpha * v)(t)}{g_{\alpha+1}(t)}=v(0). \tag{6.6}$$

Since $D_t^{\alpha\beta}S_\alpha^\beta(0)x=x$ and $D_t^{\alpha\beta}S_\alpha^\beta(t)x$ is a solution of (6.1), we have that

$$\lim_{t\downarrow 0}\frac{D_t^{\alpha\beta}S_\alpha^\beta(t)x-x}{g_{\alpha+1}(t)}=\lim_{t\downarrow 0}\frac{I_t^\alpha\mathbb{D}_t^\alpha D_t^{\alpha\beta}S_\alpha^\beta(t)x}{g_{\alpha+1}(t)}=\lim_{t\downarrow 0}\frac{I_t^\alpha D_t^{\alpha\beta}S_\alpha^\beta(t)Ax}{g_{\alpha+1}(t)}=Ax. \tag{6.7}$$

The identity (6.7) shows that the mapping $t\mapsto D_t^{\alpha\beta}S_\alpha^\beta(t)x$ is not differentiable at $t=0^+$ if $Ax\neq 0$.

(c) Finally, although $D_t^{\alpha\beta}S_\alpha^\beta(t)x$ is not differentiable at $t=0^+$, we have that $g_{1-\alpha}*\frac{d}{dt}\left[D_t^{\alpha\beta}S_\alpha^\beta(t)x\right]$ is continuous at $t=0$, that is, $\mathbb{D}_t^\alpha D_t^{\alpha\beta}S_\alpha^\beta(t)x$ is continuous at $t=0$. Indeed, since (by (6.3)),

$$g_{1-\alpha}*\frac{d}{dt}\left[D_t^{\alpha\beta}S_\alpha^\beta(t)x\right]=\sum_{j=0}^{n-1}g_{\alpha j+1}(t)A^{j+1}x+(g_{\alpha(n-\beta)}*S_\alpha^\beta)(t)A^{n+1}x,$$

it follows that

$$\lim_{t\to 0}\left(g_{1-\alpha}*\frac{d}{dt}\left[D_t^{\alpha\beta}S_\alpha^\beta(t)x\right]\right)=Ax=\lim_{t\to 0}\mathbb{D}_t^\alpha D_t^{\alpha\beta}S_\alpha^\beta(t)x=D_t^{\alpha\beta}S_\alpha^\beta(0)Ax. \tag{6.8}$$

More precisely, we have the following result.

Lemma 6.7. *Let $0<\alpha\le 1,\beta\ge 0$ and set $k:=\lceil\alpha\beta\rceil$, $n:=\lceil\beta\rceil$. Assume that A generates an $(\alpha,1)^\beta$-resolvent family S_α^β. Then for every $x\in D(A^{n+1})$,*

$$\lim_{t\to 0}\frac{S_\alpha^\beta(t)x-g_{\alpha\beta+1}(t)x}{g_{\alpha\beta+1+\alpha}(t)}=Ax=\lim_{t\to 0}\frac{D_t^{\alpha\beta}S_\alpha^\beta(t)x-x}{g_{1+\alpha}(t)}=\lim_{t\to 0}\mathbb{D}_t^\alpha D_t^{\alpha\beta}S_\alpha^\beta(t)x. \tag{6.9}$$

Proof. We have to prove the first two equalities, the last one corresponds to (6.8). Let $x\in D(A^{n+1})$. Using Lemma 4.17 we get that, for every $t>0$,

$$\begin{aligned}\frac{S_\alpha^\beta(t)x-g_{\alpha\beta+1}(t)x}{g_{\alpha\beta+1+\alpha}(t)}&=\frac{\sum_{j=1}^n g_{\alpha(\beta+j)+1}A^jx}{g_{\alpha\beta+1+\alpha}(t)}+\frac{(g_{\alpha(n+1)}*S_\alpha^\beta)(t)A^{n+1}x}{g_{\alpha\beta+1+\alpha}(t)}\\&=\sum_{j=1}^n t^{\alpha(j-1)}A^jx+\frac{I_t^\alpha(g_{\alpha n}*S_\alpha^\beta)(t)A^{n+1}x}{g_{\alpha\beta+1+\alpha}(t)}.\end{aligned}$$

Taking the limit of the preceding equality as $t \to 0$ and using (6.6), we obtain

$$\begin{aligned}
&\lim_{t\to 0} \frac{S_\alpha^\beta(t)x - g_{\alpha\beta+1}(t)x}{g_{\alpha\beta+1+\alpha}(t)} \\
&= \lim_{t\to 0} \sum_{j=1}^{n} t^{\alpha(j-1)} A^j x + \lim_{t\to 0} \frac{(I_t^{\alpha(n+1)} S_\alpha^\beta)(t) A^{n+1} x}{g_{\alpha(n+1)+1}(t)} \frac{g_{\alpha(n+1)+1}(t)}{g_{\alpha\beta+1+\alpha}(t)} \\
&= \lim_{t\to 0} \sum_{j=1}^{n} t^{\alpha(j-1)} A^j x + \lim_{t\to 0} \frac{(I_t^{\alpha(n+1)} S_\alpha^\beta)(t) A^{n+1} x}{g_{\alpha(n+1)+1}(t)} t^{\alpha(n-\beta)} \\
&= Ax.
\end{aligned}$$

We have shown the first equality in (6.9). Finally, using (4.26) in Lemma 4.17 we get that for all $t > 0$,

$$\frac{D_t^{\alpha\beta} S_\alpha^\beta(t)x - x}{g_{1+\alpha}(t)} = \frac{\sum_{j=1}^n g_{\alpha j+1}(t) A^j x}{g_{\alpha+1}(t)} + \frac{I_t^\alpha (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t) A^{n+1} x}{g_{\alpha+1}(t)}.$$

Taking the limit of the preceding equality as $t \to 0$ and using (6.6), we get

$$\begin{aligned}
&\lim_{t\to 0} \frac{D_t^{\alpha\beta} S_\alpha^\beta(t)x - x}{g_{1+\alpha}(t)} \\
&= \lim_{t\to 0} \frac{\sum_{j=1}^n g_{\alpha j+1}(t) A^j x}{g_{\alpha+1}(t)} + \lim_{t\to 0} \frac{I_t^\alpha (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t) A^{n+1} x}{g_{\alpha+1}(t)} = Ax,
\end{aligned}$$

and this completes the proof of the lemma. □

We have the following description of the generator A of the resolvent family S_α^β. We refer to [2, Lemma 3.2.2] for related results in the case of integrated semigroups and [2, Proposition 3.14.5] in the case of cosine families.

Proposition 6.8. *Let $0 < \alpha \leq 1$, $\beta \geq 0$ and assume that A generates an $(\alpha, 1)^\beta$- resolvent family S_α^β. Then*

$$A = \{(x, y) \in X \times X,\ S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + (g_\alpha * S_\alpha^\beta)(t)y,\ \forall\, t > 0\}. \quad (6.10)$$

Proof. First we notice that since the $(\alpha, 1)^\beta$- resolvent family S_α^β is non-degenerate, the right hand side of (6.10) defines a single-valued operator. Next, let $x, y \in X$. We have to show that $x \in D(A)$ and $Ax = y$ if and only if

$$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + (g_\alpha * S_\alpha^\beta)(t)y,\ \forall\, t > 0. \quad (6.11)$$

Indeed, let $x \in D(A)$ and assume that $Ax = y$. Since A generates an $(\alpha, 1)^\beta$- resolvent family S_α^β and $Ax = y$, then (6.11) follows from Lemma 4.5. Conversely, let $x, y \in X$ and assume that (6.11) holds. Taking the Laplace transform on both sides of (6.11), we get that for $\mathrm{Re}(\lambda) > \omega$ (where $\omega \geq 0$ is the real number from the definition of S_α^β),

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \lambda^{-1}x + \lambda^{-1}(\lambda^\alpha - A)^{-1}y. \tag{6.12}$$

The identity (6.12) implies that $x \in D(A)$. Applying $(\lambda^\alpha - A)$ to both sides of (6.12), we get that

$$\lambda^{\alpha-1}x = \lambda^{-1}(\lambda^\alpha - A)x + \lambda^{-1}y = \lambda^{\alpha-1}x - \lambda^{-1}Ax + \lambda^{-1}y,$$

and this implies that $Ax = y$. The proof is finished. □

7. The inhomogeneous Cauchy problem

In this section we study the solvability of inhomogeneous fractional order abstract Cauchy problems. More precisely, we investigate the existence, uniqueness and the representation of solutions of the following fractional order abstract Cauchy problem:

$$\begin{cases} \mathbb{D}_t^\alpha u(t) = Au(t) + f(t), & t > 0,\ 0 < \alpha \leq 1, \\ u(0) = x, \end{cases} \tag{7.1}$$

where A is a closed linear operator with domain $D(A)$ in a Banach space X (that we assume throughout the section without any mention), $f:\ [0, \infty) \to X$ is a given function and x is a given vector in X.

Definition 7.1. *A function $u \in C([0, \infty); D(A))$ is said to be a classical solution of (7.1) if $g_{1-\alpha} * (u - u(0)) \in C^1([0, \infty); X)$ and (7.1) is satisfied.*

We adopt the following definition of mild solutions.

Definition 7.2. *A function $u \in C([0, \infty); X)$ is said to be a mild solution of (7.1) if $I_t^\alpha u(t) := (g_\alpha * u)(t) \in D(A)$ for every $t \geq 0$, and*

$$u(t) = x + A \int_0^t g_\alpha(t-s)u(s)\, ds + \int_0^t g_\alpha(t-s)f(s)\, ds,\ \ \forall\, t \geq 0.$$

As for the homogeneous problem in Section 6, we have the following uniqueness result. Its proof runs similar to the proof of Proposition 6.3.

Proposition 7.3. *Let $0 < \alpha \leq 1$. Then the following assertions hold.*

(a) *If u is a classical solution of (7.1), then it is a mild solution of (7.1).*

(b) *If* $(\lambda^\alpha - A)$ *is invertible for* $Re(\lambda)$ *large enough, and if a mild solution* u *exists and* $(g_1 * u)(t)$ *is exponentially bounded, then it is unique.*

Remark 7.4. As for the homogeneous equation in Section 6, to prove the existence of mild and classical solutions of Problem (7.1), we proceed by a direct method without the use of the Laplace transform.

We have the following result of existence and representation of classical and mild solutions of Problem (7.1) which is the main result of this section.

Theorem 7.5. *Let* $0 < \alpha \le 1, \beta \ge 0$ *and set* $n := \lceil \beta \rceil$, $k := \lceil \alpha\beta \rceil$. *Assume that* A *generates an* $(\alpha,1)^\beta$*-resolvent family* S_α^β. *Then the following assertions hold.*

(a) *For every* $f \in C^{k+1}([0,\infty);X)$, $f^{(i)}(0) \in D(A^{n+1-i})$, $i = 0,1,\dots,k$, $\mathbb{D}_t^{\alpha\beta} f(t) := (g_{k-\alpha\beta} * f^{(k)})(t)$ *is exponentially bounded and* $x \in D(A^{n+1})$, *Problem* (7.1) *has a unique classical solution* u *given by*

$$u(t) = D_t^{\alpha\beta} S_\alpha^\beta(t)x + D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t), \;\; t \ge 0. \tag{7.2}$$

(b) *For* $f \in C^k([0,\infty);X)$, $f^{(i)}(0) \in D(A^{n-i})$, $i = 0,1,\dots,k-1$, $\mathbb{D}_t^{\alpha\beta} f(t) := (g_{k-\alpha\beta} * f^{(k)})(t)$ *is exponentially bounded and* $x \in D(A^n)$, *Problem* (7.1) *has a unique mild solution* u *given by* (7.2).

Proof. Let A, α, β, n and k be as in the statement of the theorem. First we show existence of solutions.

(a) Let $x \in D(A^{n+1})$. By Lemma 4.17, $D_t^{\alpha\beta} S_\alpha^\beta(t)x \in C([0,\infty);D(A))$ and $D_t^{\alpha\beta} S_\alpha^\beta(0)x = x$. By the proof of Theorem 6.5(a) we have that $g_{1-\alpha} * (D_t^{\alpha\beta} S_\alpha^\beta(t)x - x) \in C^1([0,\infty);X)$. Now, assume that f satisfies the hypothesis in the statement of the theorem. Using Lemma 4.17, Lemma 4.5(b), the equality (2.8), the fact that $f \in C^{k+1}([0,\infty);X)$ and proceeding as in the proof of Theorem 6.5, we get that for every $t \ge 0$,

$$\begin{aligned} D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) &= \frac{d^k}{dt^k}\Big[g_{k-\alpha\beta} * (g_\alpha * \frac{d}{dt}(S_\alpha^\beta * f)(t)\Big] \qquad (7.3)\\ &= \sum_{i=0}^{k}\sum_{j=1}^{n-i} g_{\alpha j+i+1}A^{j-1}(t)f^{(i)}(0) + \sum_{i=0}^{k}(g_{\alpha(n-\beta)+i(1-\alpha)+\alpha} * S_\alpha^\beta)(t)A^{n-i}f^{(i)}(0)\\ &\quad + (g_{k-\alpha\beta} * g_\alpha * S_\alpha^\beta * f^{(k+1)})(t). \end{aligned}$$

It follows from (7.3) and Lemma 4.5 that

$$D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) \in C([0,\infty);D(A)).$$

Let u be given by (7.2). We have shown that $u \in C([0,\infty);D(A))$. Using (6.3) and (7.3), we get that for every $t \geq 0$,

$$
\begin{aligned}
&(g_{1-\alpha} * (u - u(0)))(t) \\
&= \sum_{j=1}^{n} g_{\alpha j+2-\alpha}(t)A^j x + (g_{\alpha(n-\beta)+1} * S_\alpha^\beta)(t)A^{n+1}x \\
&+ \sum_{i=0}^{k}\sum_{j=1}^{n-i} g_{\alpha(j-1)+i+2}(t)A^{j-1}f^{(i)}(0) \\
&+ \sum_{i=0}^{k}(g_{\alpha(n-\beta)+i(1-\alpha)+1} * S_\alpha^\beta)(t)A^{n-i}f^{(i)}(0) + (g_{k-\alpha\beta} * g_1 * S_\alpha^\beta * f^{(k+1)})(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\left[(g_{1-\alpha} * (u - u(0))(t)\right] &= \sum_{j=1}^{n} g_{\alpha j+1-\alpha}(t)A^j x + (g_{\alpha(n-\beta)} * S_\alpha^\beta)(t)A^{n+1}x \\
+ \sum_{i=0}^{k}\sum_{j=1}^{n-i} g_{\alpha(j-1)+i+1}(t)A^{j-1}f^{(i)}(0) &+ \sum_{i=0}^{k}(g_{\alpha(n-\beta)+i(1-\alpha)} * S_\alpha^\beta)(t)A^{n-i}f^{(i)}(0) \\
&+ (g_{k-\alpha\beta} * S_\alpha^\beta * f^{(k+1)})(t).
\end{aligned}
\tag{7.4}
$$

It follows from (7.4), Lemma 4.5 and Lemma 4.17 that $g_{1-\alpha} * (u - u(0)) \in C^1([0,\infty);X)$. By the proof of Theorem 6.5(a) we have that

$$\mathbb{D}_t^\alpha D_t^{\alpha\beta} S_\alpha^\beta(t)x = AD_t^{\alpha\beta} S_\alpha^\beta(t)x. \tag{7.5}$$

Using (7.3), (2.7) and Lemma 4.5, we get that for every $t \geq 0$,

$$
\begin{aligned}
&\mathbb{D}_t^\alpha D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) = g_{1-\alpha} * \frac{d}{dt}\left[\frac{d^k}{dt^k}\Big(g_{k-\alpha\beta} * g_\alpha * \frac{d}{dt}(S_\alpha^\beta * f)\Big)(t)\right] \\
&= g_{1-\alpha} * \frac{d^{k+2}}{dt^{k+2}}\left[(g_{k+1+\alpha} * f)(t) + (g_{k-\alpha\beta+\alpha} * Ag_\alpha * S_\alpha^\beta * f)(t)\right] \\
&= \frac{d^{k+2}}{dt^{k+2}}\left[(g_{k+2} * f)(t) + (g_{k+1-\alpha\beta} * Ag_\alpha * S_\alpha^\beta * f)(t)\right] \\
&= f(t) + \frac{d^{k+1}}{dt^{k+1}}\left[(g_{k-\alpha\beta} * Ag_\alpha * S_\alpha^\beta * f)(t)\right] \\
&= f(t) + AD_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t).
\end{aligned}
\tag{7.6}
$$

Combining (7.5) and (7.6), we have that for every $t \geq 0$,

$$\mathbb{D}_t^\alpha u(t) = \mathbb{D}_t^\alpha D_t^{\alpha\beta} S_\alpha^\beta(t)x + \mathbb{D}_t^\alpha D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t)$$

$$= AD_t^{\alpha\beta} S_\alpha^\beta(t)x + f(t) + AD_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t)$$
$$= A\left[D_t^{\alpha\beta} S_\alpha^\beta(t)x + D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t)\right] + f(t) = Au(t) + f(t).$$

Hence, u is a classical solution of (7.1) and this completes the proof of existence part in assertion (a).

(b) Let $x \in D(A^n)$. By Lemma 4.17, $D_t^{\alpha\beta} S_\alpha^\beta(t)x \in C([0,\infty);X)$. It follows from the proof of Theorem 6.5(b) that

$$I_t^\alpha D_t^{\alpha\beta} S_\alpha^\beta(t)x := (g_\alpha * D_t^{\alpha\beta} S_\alpha^\beta)(t)x \in D(A)$$

for every $t \geq 0$. Assume that f satisfies the hypothesis in the statement of the theorem. Proceeding as in (7.3) we get that for every $t \geq 0$.

$$I_t^\alpha D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) = g_\alpha * D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t)$$
$$= \sum_{i=0}^{k-1}\sum_{j=1}^{n-i} g_{\alpha j+i+1}(t)A^{j-1}f^{(i)}(0) + \sum_{i=0}^{k-1}(g_{\alpha(n-\beta)+i(1-\alpha)+\alpha} * S_\alpha^\beta)(t)A^{n-i}f^{(i)}(0)$$
$$+ (g_{k-\alpha\beta} * g_\alpha * S_\alpha^\beta * f^{(k)})(t). \tag{7.7}$$

It follows from (7.7), Lemma 4.17 and Lemma 4.5 that

$$I_t^\alpha D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) \in D(A)$$

for every $t \geq 0$. We have shown that

$$I_t^\alpha u(t) = I_t^\alpha D_t^{\alpha\beta} S_\alpha^\beta(t)x + I_t^\alpha D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) \in D(A)$$

for every $t \geq 0$. It follows from (6.5) in the proof of Theorem 6.5(b) that for every $t \geq 0$,

$$D_t^{\alpha\beta} S_\alpha^\beta(t)x = x + A(g_\alpha * D_t^{\alpha\beta} S_\alpha^\beta)(t)x. \tag{7.8}$$

Proceeding as in (7.6) and using Lemma 4.5, we have that for every $t \geq 0$,

$$D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) = \frac{d^k}{dt^k}\left[\left(g_{k-\alpha\beta} * g_\alpha * \frac{d}{dt}(S_\alpha^\beta * f)\right)(t)\right]$$
$$= \frac{d^{k+1}}{dt^{k+1}}\left[(g_{k+1+\alpha} * f)(t) + Ag_\alpha * (g_{k-\alpha\beta} * g_\alpha * S_\alpha^\beta * f)(t)\right]$$
$$= (g_\alpha * f)(t) + A\Big(g_\alpha * \frac{d^{k+1}}{dt^{k+1}}[(g_{k-\alpha\beta} * g_\alpha * S_\alpha^\beta * f)]\Big)(t)$$
$$= (g_\alpha * f)(t) + A\big(g_\alpha * D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)\big)(t). \tag{7.9}$$

Combining (7.8) and (7.9) we get that for every $t \geq 0$,

$$u(t) = D_t^{\alpha\beta} S_\alpha^\beta(t)x + D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}\big(S_\alpha^\beta * f\big)(t)$$

$$= x + A(g_\alpha * D_t^{\alpha\beta} S_\alpha^\beta)(t)x + (g_\alpha * f)(t) + A\big(g_\alpha * D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)\big)(t)$$
$$= x + (g_\alpha * f)(t) + Ag_\alpha * \Big[D_t^{\alpha\beta} S_\alpha^\beta(t)x + D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t)\Big]$$
$$= x + A(g_\alpha * u)(t) + (g_\alpha * f)(t).$$

Hence, u is a mild solution of (7.1) and this completes the proof of the existence pat in assertion (b).

It remains to show the uniqueness of solutions. Let $x \in D(A^n)$ and let f satisfy the assumptions in part (b) of the theorem. Let u be a mild solution. It follows from (6.2) and (7.3) that for every $t \geq 0$,

$$\begin{aligned}(g_1 * u)(t) = &\sum_{j=0}^{n-1} g_{\alpha j+2}(t)A^j x + (g_{\alpha(n-\beta)+1} * S_\alpha^\beta)(t)A^n x\\ &+ \sum_{i=0}^{k-1}\sum_{j=1}^{n-1-i} g_{\alpha j+i+2}(t)A^{j-1}f^{(i)}(0)\\ &+ \sum_{i=0}^{k-1}(g_{\alpha(n-\beta)+i(1-\alpha)+\alpha+1} * S_\alpha^\beta)(t)A^{n-1-i}f^{(i)}(0)\\ &+ (g_{k-\alpha\beta} * g_{\alpha+1} * S_\alpha^\beta * f^{(k)})(t). \end{aligned} \tag{7.10}$$

Since by assumption there exists some constants $M_1, \omega_1 \geq 0$ such that $\|(g_{k-\alpha\beta} * f^{(k)})(t)\| \leq M_1 e^{\omega_1 t}$, using Lemma 4.3, the equality (7.10) implies that there exist some constants $M, \omega \geq 0$ such that

$$\begin{aligned}(g_1 * u)(t) \leq Me^{\omega t}\Big[&\sum_{j=0}^{n}\|A^j x\| + \sum_{i=0}^{k-1}\sum_{j=1}^{n-1-i}\|A^{j-1}f^{(i)}(0)\|\\ &+ \sum_{i=0}^{k-1}\|A^{n-1-i}f^{(i)}(0)\| + M_1 e^{\omega_1 t}\Big].\end{aligned}$$

We have shown that $(g_1 * u)(t)$ is exponentially bounded. Now, the uniqueness of mild and classical solutions follows from Proposition 7.3 and this completes the proof of the theorem. □

As a corollary of Theorem 7.5, we have the following representation of solutions in case A also generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β.

Corollary 7.6. *Let $0 < \alpha \le 1, \beta \ge 0$ and set $n := \lceil \beta \rceil$, $k := \lceil \alpha\beta \rceil$. Assume that A generates an $(\alpha, \alpha)^\beta$-resolvent family P_α^β. Let S_α^β be the $(\alpha, 1)^\beta$-resolvent family generated by A (which exists by Remark 4.4(a)). Then the following assertions hold.*

(a) *For every $f \in C^{k+1}([0,\infty); X)$, $f^{(j)}(0) \in D(A^{n+1-j})$, $j = 0, 1, \dots, k$, $\mathbb{D}_t^{\alpha\beta} f(t) := (g_{k-\alpha\beta} * f^{(k)})(t)$ is exponentially bounded, and for every $x \in D(A^{n+1})$, the unique classical solution u of (7.1) is given by*

$$u(t) = D_t^{\alpha\beta}\Big(S_\alpha^\beta(t)x + \int_0^t P_\alpha^\beta(t-s) f(s)\, ds\Big). \tag{7.11}$$

(b) *For every $f \in C^{k}([0,\infty); X)$, $f^{(j)}(0) \in D(A^{n-j})$, $j = 0, 1, \dots, k-1$, $\mathbb{D}_t^{\alpha\beta} f(t) := (g_{k-\alpha\beta} * f^{(k)})(t)$ is exponentially bounded, and for every $x \in D(A^{n})$, the unique mild solution u of (7.1) is given by (7.11).*

Proof. Using Remark 4.4 we have that for every $t \ge 0$,

$$\begin{aligned} D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t) &= D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(g_{1-\alpha} * P_\alpha^\beta * f)(t) \\ &= D_t^{\alpha\beta} g_\alpha * \frac{d}{dt}(g_{1-\alpha} * P_\alpha^\beta * f)(t) \\ &= D_t^{\alpha\beta}\frac{d}{dt}\Big[(g_1 * P_\alpha^\beta * f)(t)\Big] = D_t^{\alpha\beta}(P_\alpha^\beta * f)(t), \end{aligned}$$

and this completes the proof. □

8. Applications

In this section we give several examples where the results of the previous sections are applied.

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is an open set with Lipschitz continuous boundary $\partial\Omega$. Let the real valued coefficients satisfy, $a_{ij} \in L^\infty(\Omega)$, $b_j, c_j, d \in L^\infty(\Omega)$, $i, j = 1, 2, \dots, N$ and $\gamma \in L^\infty(\partial\Omega)$. We assume that $\gamma(z) \ge 0$ for σ-a.e. $z \in \partial\Omega$, where σ is the Lebesgue surface measure, $\gamma \in L^\infty(\partial\Omega)$ and that there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N, \tag{8.1}$$

for a.e. $x \in \Omega$. Let the operator A be given formally by

$$Au = \sum_{j=1}^N D_j\Big(\sum_{i=1}^N a_{i,j} D_i u + b_j u\Big) - \Big(\sum_{i=1}^N c_i D_i u + du\Big) \tag{8.2}$$

and let

$$\frac{\partial u}{\partial \nu_A} := \sum_{j=1}^{N} \Big(\sum_{i=1}^{N} a_{ij} D_i u + b_j u \Big) \cdot \nu_j,$$

where ν denotes the outer normal vector of Ω at the boundary $\partial\Omega$.

Example 8.1 (Non-homogeneous Neumann and Robin boundary conditions on L^p). Let A be the elliptic operator given in (8.2) and let $1 \le p < \infty$. For convenience we assume that $\Omega \subset \mathbb{R}^N$ is connected, otherwise we could consider each connected component separately. For $0 < \alpha \le 1$, we consider the fractional order inhomogeneous diffusion equation with non-homogeneous boundary conditions

$$\begin{cases} \mathbb{D}_t^{\alpha} u(t,x) = Au(t,x) + f(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u(t,z)}{\partial \nu_A} + \gamma(z) u(t,z) = g(t,z), & t > 0,\ z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{8.3}$$

Here, $u_0 \in L^p(\Omega)$, $f \in C([0,\infty); L^p(\Omega))$, $g \in C([0,\infty); L^p(\partial\Omega))$ are given functions. We consider the first order Sobolev space

$$H^1(\Omega) := \Big\{ u \in L^2(\Omega),\ \int_{\Omega} |\nabla u|^2 \, dx < \infty \Big\}$$

endowed with the norm

$$\|u\|_{H^1(\Omega)} := \Big(\int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \Big)^{1/2}.$$

Let $\mathcal{A}_\gamma$ be the bilinear closed form in $L^2(\Omega)$ with domain $H^1(\Omega)$ defined for $u, v \in H^1(\Omega)$ by

$$\begin{aligned} \mathcal{A}_\gamma(u,v) : = &\int_{\Omega} \sum_{j=1}^{N} \Big(\sum_{i=1}^{N} a_{ij} D_i u + b_j u \Big) D_j v \, dx \\ &+ \int_{\Omega} \Big(\sum_{j=1}^{N} c_j D_j u + du \Big) v \, dx + \int_{\partial\Omega} \gamma u v \, d\sigma. \end{aligned}$$

We define the linear closed operator $A_{2,\gamma}$ on the product space $L^2(\Omega) \times L^2(\partial\Omega)$ as follows:

$$\begin{cases} D(A_{2,\gamma}) := \{(u,0) :\ u \in H^1(\Omega),\ \ Au \in L^2(\Omega),\ \frac{\partial u}{\partial \nu_A} \in L^2(\partial\Omega)\} \\ A_{2,\gamma}(u,0) = \Big(Au, -\frac{\partial u}{\partial \nu_A} - \gamma u \Big). \end{cases} \tag{8.4}$$

Let $u \in H^1(\Omega)$. It is easy to check that $(u,0) \in D(A_{2,\gamma})$ and $-A_{2,\gamma}(u,0) = (f,g)$ if and only if for every $v \in H^1(\Omega)$,

$$\mathcal{A}_\gamma(u,v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,d\sigma. \tag{8.5}$$

We notice $D(A_{2,\gamma})$ is not dense in $L^2(\Omega) \times L^2(\partial\Omega)$ unless $\frac{\partial u}{\partial \nu_A} + \gamma u = 0$ on $\partial\Omega$. Applying Sobolev and Hölder inequalities we have that there exists a constant $\omega \geq 0$ such that for every function $u \in H^1(\Omega)$,

$$\mathcal{A}_\gamma(u,u) \geq \frac{\mu}{2}\int_\Omega |\nabla u|^2\,dx - \omega\int_\Omega |u|^2\,dx. \tag{8.6}$$

We define the zero boundary conditions operator $A^0_{2,\gamma}$ on $L^2(\Omega)$ as follows:

$$\begin{cases} D(A^0_{2,\gamma}) = \{u \in H^1(\Omega) :\ Au \in L^2(\Omega),\ \frac{\partial u}{\partial \nu_A} + \gamma u = 0\} \\ A^0_{2,\gamma}u = Au. \end{cases} \tag{8.7}$$

The operator $A^0_{2,\gamma}$ is a realization of the operator A in $L^2(\Omega)$ with Robin boundary conditions and Neumann boundary conditions if $\gamma \equiv 0$.

By [3], $-A^0_{2,\gamma}$ generates an analytic C_0-semigroup $(T_0(t))_{t\geq 0}$ on $L^2(\Omega)$. Moreover, the semigroup T_0 interpolates on $L^p(\Omega)$, $1 \leq p \leq \infty$. Hence, there are consistent semigroups on $L^p(\Omega)$ for every $p \in [1,\infty]$ which are strongly continuous if $p \in [1,\infty)$. We denote the semigroups on $L^p(\Omega)$, $1 \leq p < \infty$, by $T_{0,p}$ so that the semigroup T_0 on $L^2(\Omega)$ coincides with $T_{0,2}$.

Next, for $1 \leq p < \infty$, we let $\mathbb{X}^p(\overline{\Omega}) := L^p(\Omega) \times L^p(\partial\Omega)$. Let $A_{p,\gamma}$ be the operator defined in $\mathbb{X}^p(\overline{\Omega})$ as follows:

$$\begin{cases} D(A_{p,\gamma}) := \Big\{(u,0) \in D(A_{2,\gamma}) :\ Au \in L^p(\Omega),\ \frac{\partial u}{\partial \nu_A} \in L^p(\partial\Omega)\Big\} \\ A_{p,\gamma}(u,0) = \Big(Au, -\frac{\partial u}{\partial \nu_A} - \gamma u|_{\partial\Omega}\Big). \end{cases} \tag{8.8}$$

Let $u \in H^1(\Omega)$. As in the case $p = 2$, we have that $(u,0) \in D(A_{p,\gamma})$ if and only if there exists $(f,g) \in \mathbb{X}^p(\overline{\Omega})$ such that u is a weak solution of

$$\begin{cases} -Au = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu_A} + \gamma u = g & \text{on } \partial\Omega. \end{cases}$$

That is, $u \in H^1(\Omega)$ and for every $v \in H^1(\Omega)$

$$\mathcal{A}_\gamma(u,v) = \int_\Omega fv\,dx + \int_{\partial\Omega} gv\,d\sigma. \tag{8.9}$$

We claim that the operator $A_{p,\gamma}$ generates a once-integrated semigroup $(T_1^p(t))$ on $\mathbb{X}^p(\overline{\Omega})$ given for every $(f,g) \in \mathbb{X}^p(\overline{\Omega})$ and $\lambda > 0$ by

$$T_1^p(t)(f,g) = \lambda \int_0^t \widetilde{T}_{0,p}(s)(\lambda - A)^{-1}(f,g)\, ds + [I - \widetilde{T}_{0,p}(t)](\lambda - A)^{-1}(f,g). \tag{8.10}$$

Here $\widetilde{T}_{0,p}(t)(u,0) = (T_{0,p}(t),0)$, where $(T_{0,p}(t))$ is the strongly continuous semigroup on $L^p(\Omega)$ introduced above.

Indeed, first, let $A_{2,\gamma}$ be the operator on $\mathbb{X}^2(\overline{\Omega})$ defined in (8.4). Let ω be as in (8.6) and fix $\lambda > \omega$. Then, for all $u \in H^1(\Omega)$,

$$\lambda \int_\Omega |u|^2\, dx + \mathcal{A}_\gamma(u,u) \geq \min\left\{\lambda - \omega, \frac{\mu}{2}\right\} \|u\|^2_{H^1(\Omega)}, \tag{8.11}$$

where $\mu > 0$ is given in (8.1). It follows from the Lax-Milgram Theorem [13, Section 5.8] that for every $(f,g) \in \mathbb{X}^2(\overline{\Omega})$, there exists a unique $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\lambda \int_\Omega uv\, dx + \mathcal{A}_\gamma(u,v) = \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, d\sigma. \tag{8.12}$$

That is, there exists a function $u \in H^1(\Omega)$ with $(u,0) \in D(A_{2,\gamma})$ and

$$(\lambda - A_{2,\gamma})(u,0) = (\lambda u, 0) - A_{2,\gamma}(u,0) = (f,g).$$

We have shown that $\lambda - A_{2,\gamma} :\ D(A_{2,\gamma}) \to \mathbb{X}^2(\overline{\Omega})$ is a bijection for $\lambda > \omega$. Assume now that $f \geq 0$ and $g \leq 0$. Let $(u,0) := (\lambda - A_{2,\gamma})^{-1}(f,g)$ and set $v := u^+$. Then using (8.12), we get that

$$\begin{aligned} 0 \geq \int_\Omega fv\, dx + \int_{\partial\Omega} gv\, d\sigma &= \lambda \int_\Omega uv\, dx + \mathcal{A}_\gamma(u,v) \\ &= \lambda \int_\Omega |v|^2\, dx + \mathcal{A}_\gamma(v,v) \geq 0. \end{aligned}$$

Using (8.11), the preceding estimate implies that $v = 0$, that is, $u \leq 0$ a.e. on Ω. We have shown that $(\lambda - A_{2,\gamma})^{-1}$ is a positive operator. Since every positive operator is continuous, we obtain that $\lambda - A_{2,\gamma}$ is invertible. In particular, we have also proved that the operator $A_{2,\gamma}$ is closed. Hence, $D(A_{2,\gamma})$ is a Banach space for the graph norm of $A_{2,\gamma}$, and by definition of $A_{2,\gamma}$ we have that $D(A_{2,\gamma}) \subset H^1(\Omega) \times \{0\}$. Since both of these spaces are continuously embedded into $\mathbb{X}^2(\overline{\Omega})$, we deduce from the closed graph theorem that $D(A_{2,\gamma})$ is continuously embedded into $H^1(\Omega) \times \{0\}$. We have shown that the operator $A_{2,\gamma}$ is resolvent positive. Since $\mathbb{X}^2(\overline{\Omega})$ is a Banach lattice with order continuous norm, it follows from [2, Theorem 3.11.7], that the operator $A_{2,\gamma}$ generates a once-integrated semigroup T_1^2 on $\mathbb{X}^2(\overline{\Omega})$.

Second, let $2 \le p < \infty$ and let $A_{p,\gamma}$ be the operator on $\mathbb{X}^p(\overline{\Omega})$ defined in (8.8). Let $(u,0) \in D(A_{p,\gamma})$. Then $u \in H^1(\Omega)$ and satisfies (8.9). If $p > N/2 \ge (N-1)/2$, then elliptic regularity (see e.g. [41, Corollary 2.9]) shows that $u \in C(\overline{\Omega})$. Hence, $D(A_{p,\gamma}) \subset C(\overline{\Omega}) \times \{0\}$ and in particular we have that $D(A_{p,\gamma}) \subset \mathbb{X}^p(\overline{\Omega})$. If $2 \le p < N$, then using again elliptic elliptic regularity (see e.g. [8, Theorem 4.1 and Corollary 4.2]), we get that that $u \in L^{p^\star}(\Omega)$ and $u|_{\partial\Omega} \in L^{p_\star}(\partial\Omega)$ where $p^\star := pN/(N-p) \ge p$ and $p_\star := p(N-1)/(N-p) \ge p$. In any case, we have that $D(A_{p,\gamma}) \subset \mathbb{X}^p(\overline{\Omega})$. Hence, $A_{p,\gamma}$ is the part of the resolvent positive operator $A_{2,\gamma}$ in $\mathbb{X}^p(\overline{\Omega})$, and hence, is also resolvent positive. It follows from [2, Theorem 3.11.7] that the operator $A_{p,\gamma}$ generates a once integrated semigroup T_1^p on $\mathbb{X}_p(\overline{\Omega})$. Since $\overline{D(A_{p,\gamma})}^{\mathbb{X}^p(\overline{\Omega})} = L^p(\Omega) \times \{0\}$, we have that the representation (8.10) follows from the abstract result contained in [26, Proposition 2.4].

Finally, if $1 \le p < 2$, then we obtain the result by using the consistency properties of the semigroups $(T_{0,p}(t))$ introduced above. The proof of the claim is finished.

Using the above defined operator $A_{p,\gamma}$, we have that the fractional order Problem (8.3) can be rewritten as an abstract Cauchy problem on the Banach space $L^p(\Omega) \times L^p(\partial\Omega) = \mathbb{X}^p(\overline{\Omega})$:

$$\begin{cases} \mathbb{D}_t^\alpha \,(u(t),0) = A_{p,\gamma}(u(t),0) + (f(t),g(t)), \quad t > 0, \\ (u(0),0) = (u_0,0). \end{cases} \tag{8.13}$$

Then all the results in Theorem 7.5 hold for Problem (8.13) and hence, for Problem 8.3 with here $n = k = 1$.

Next, we consider the space of continuous functions.

Example 8.2 (Non-homogeneous Neumann and Robin boundary conditions on the space of continuous functions). Let Ω be as in Example 8.1. We notice that the space $C(\overline{\Omega})$ does not have an order continuous norm. Let A be the elliptic operator given in (8.2). Here, for $0 < \alpha \le 1$, we consider the fractional diffusion equation

$$\begin{cases} \mathbb{D}_t^\alpha u(t,x) = Au(t,x) + f(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u(t,z)}{\partial \nu_A} + \gamma(z)u(t,z) = g(t,z), & t > 0,\ z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{8.14}$$

where $u_0 \in C(\overline{\Omega})$, $f \in C([0,\infty]; C(\overline{\Omega}))$ and $g \in C([0,\infty]; C(\partial\Omega))$ are given functions. This problem in the case $\alpha = 1$ has been investigated in [34]. We define the operator $A_{\infty,\gamma}$ on $\mathbb{X}^\infty(\overline{\Omega}) = C(\overline{\Omega}) \times C(\partial\Omega)$ as follows:

$$\begin{cases} D(A_{\infty,\gamma}) := \Big\{(u,0) \in D(A_{2,\gamma}) \cap \mathbb{X}^\infty(\Omega) : \; Au \in C(\overline{\Omega}), \; \frac{\partial u}{\partial \nu_A} \in C(\partial\Omega)\Big\} \\ A_{\infty,\beta}(u,0) = \Big(Au, -\frac{\partial u}{\partial \nu_A} - \gamma u|_{\partial\Omega}\Big), \end{cases}$$

where $A_{2,\gamma}$ is the operator on $\mathbb{X}^2(\overline{\Omega})$ defined in (8.4). The operator $A_{\infty,\gamma}$ is a realization of the operator A with non-homogeneous Robin boundary conditions and non-homogeneous Neumann boundary conditions if $\gamma \equiv 0$. By definition, $D(A_{\infty,\gamma}) \subset C(\overline{\Omega}) \times \{0\}$. By [33] (see also [41] for the case of the Laplace operator), the space $C(\overline{\Omega}) \times \{0\}$ is invariant under the resolvent of $A_{\infty,\gamma}$ and its part $A^0_{\infty,\gamma}$ in $C(\overline{\Omega}) \times \{0\}$ generates a strongly continuous semigroup $(T_\infty(t))_{t\geq 0}$ on $C(\overline{\Omega}) \times \{0\}$. Hence, $A_{\infty,\gamma}$ generates a once-integrated semigroup $(T_1^\infty(t))_{t\geq 0}$ on $\mathbb{X}^\infty(\overline{\Omega})$ given for $(f,g) \in \mathbb{X}^\infty(\overline{\Omega})$ and $\lambda > 0$ by

$$T_1^\infty(t)(f,g) = \lambda \int_0^t T_\infty(s)(\lambda - A)^{-1}(f,g)\, ds + [I - T_\infty(t)](\lambda - A)^{-1}(f,g).$$

Using $A_{\infty,\gamma}$, we have that (8.14) can be rewritten as an abstract Cauchy problem on the Banach space $C(\overline{\Omega}) \times C(\partial\Omega) = \mathbb{X}^\infty(\overline{\Omega})$:

$$\begin{cases} \mathbb{D}_t^\alpha \,(u(t),0) = A_{\infty,\gamma}(u(t),0) + (f(t), g(t)), \quad t > 0, \\ (u(0),0) = (u_0, 0). \end{cases} \tag{8.15}$$

Hence, then the results in Theorem 7.5 hold for Problem (8.14) and hence, for Problem (8.15) with $n = k = 1$.

Next, we investigate the non-homogeneous Dirichlet boundary conditions.

Example 8.3 (**Non-homogeneous Dirichlet boundary conditions on the space of continuous functions).** For simplicity we assume that Ω is as in Example 8.1. To investigate the non-homogeneous Dirichlet boundary conditions in a space of continuous functions, one has to work directly with a realization A_∞ of A with non-homogeneous Dirichlet boundary conditions in a space of continuous functions because the L^p-regularity conditions on the boundary data are not sufficient to obtain continuous solutions. For $0 < \alpha \leq 1$, we consider the following fractional order diffusion equation

$$\begin{cases} \mathbb{D}_t^\alpha u(t,x) = Au(t,x) + f(t,x), & t > 0, \; x \in \Omega, \\ u(t,z) = g(t,z), & t > 0, \; z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{8.16}$$

where $u_0 \in C(\overline{\Omega})$, $f \in C([0,\infty];C(\overline{\Omega}))$ and $g \in C([0,\infty];C(\partial\Omega))$ are given functions and A is the operator given in (8.2). Problem (8.16) in the case $\alpha = 1$ and $A = \Delta$ (the Laplace operator) has been investigated in [2, Chapter 6]. We recall that $C(\overline{\Omega})$ does not have an order continuous norm. We define a realization A_∞ of the operator A with non-homogeneous Dirichlet boundary conditions on $\mathbb{X}^\infty(\overline{\Omega}) := C(\overline{\Omega}) \times C(\partial\Omega)$ by

$$\begin{cases} D(A_\infty) := \left\{(u, u|_{\partial\Omega}) \in \mathbb{X}^\infty(\overline{\Omega}) :\ u \in H^1(\Omega),\ Au \in C(\overline{\Omega})\right\}, \\ A_\infty(u, u|_{\partial\Omega}) = (Au, -u|_{\partial\Omega}). \end{cases}$$

Then (8.16) can be rewritten as a Cauchy problem on $\mathbb{X}^\infty(\overline{\Omega})$ as follows:

$$\begin{cases} \mathbb{D}_t^\alpha\,(u(t),0) = A_\infty(u(t),0) + (f(t),g(t)), \quad t > 0, \\ (u(0),0) = (u_0,0). \end{cases} \tag{8.17}$$

It is well-known (see e.g. [2, Chapter 6]) that the operator A_∞ does not generate a once-integrated semigroup on $\mathbb{X}^\infty(\overline{\Omega})$. It turns out that A_∞ generates a twice integrated semigroup $(T_2^\infty(t))_{t\geq 0}$ on $\mathbb{X}^\infty(\overline{\Omega})$ (see [2, Chapter 3 and Chapter 6]). Hence, the results in Theorem 7.5 hold for Problem (8.17) and hence, for Problem (8.16) with $n = 2$ and $k = 1$ if $0 < \alpha \leq 1/2$ and $n = k = 2$ if $1/2 < \alpha \leq 2$.

To conclude the paper, we consider the time fractional order Schrödinger equation.

Example 8.4 (**The time fractional order Schrödinger equation).** First we consider the problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t,x) = i\Delta_p u(t,x) + f(t,x), \ \ t > 0,\ x \in \mathbb{R}^N, \ \ 0 < \alpha \leq 1, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^N, \end{cases} \tag{8.18}$$

where Δ_p is a realization of the Laplace operator on $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. It is well-known (see [2, Theorem 8.3.9]) that the operator $i\Delta_p$ generates a β_p-times integrated semigroup on $L^p(\Omega)$ with $\beta_p := N\left|\frac{1}{2} - \frac{1}{p}\right|$ if $1 < p < \infty$ and $\beta_1 > \frac{N}{2}$ if $p = 1$. This shows in particular that $i\Delta_2$ generates a strongly continuous semigroup on $L^2(\Omega)$. Then the results in Theorem 7.5 hold for Problem (8.18) with $n = \lceil\beta_p\rceil$ and $k = \lceil\alpha\beta_p\rceil$ if $1 \leq p < \infty$, $p \neq 2$ and $n = k = 0$ if $p = 2$.

Next, we consider the problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t,x) = e^{i\theta}\Delta_p u(t,x) + f(t,x), \ \ t > 0,\ x \in \mathbb{R}^N, \ \ 0 < \alpha < 1, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}^N, \end{cases} \tag{8.19}$$

where the angle θ satisfies $\frac{\pi}{2} < \theta < \left(1 - \frac{\alpha}{2}\right)\pi$. Let the operator A_p be given by $A_p := e^{i\theta}\Delta_p$. Then $D(A_p) = W^{2,p}(\mathbb{R}^N)$. We have shown in Example 4.10 that if $0 < \alpha < \frac{1}{2}$, then A_p does not generates a β-times integrated semigroup in $L^p(\mathbb{R}^N)$ for any $\beta \geq 0$, but A_p generates an $(\alpha, 1)$-resolvent family S_α on $L^p(\mathbb{R}^N)$ for every $0 < \alpha < 1$. Therefore, if $0 < \alpha < \frac{1}{2}$, then one cannot use the concept of β-times integrated semigroup to solve Problem (8.19). With the technique of resolvent families we are able to solve Problem (8.19). More precisely, using Theorem 7.5, we have the following result.

- For every $u_0 \in W^{2,p}(\mathbb{R}^N)$ and $f \in W^{1,1}([0,\infty); L^p(\mathbb{R}^N))$, Problem (8.19) has a classical solution u.
- For every $u_0 \in L^p(\mathbb{R}^N)$ and $f \in L^1_{loc}([0,\infty); L^p(\mathbb{R}^N))$, Problem (8.19) has a mild solution.

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