THE FRACTIONAL FOURIER TRANSFORM AND QUADRATIC–FIELD MAGNETIC RESONANCE IMAGING

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Abstract. The fractional Fourier transform (FrFT) is revisited in the framework of strongly continuous semigroups to restate known results and to explore new properties of the FrFT. We then show how the FrFT can be used to reconstruct Magnetic Resonance (MR) images acquired under the presence of quadratic field inhomogeneity. Particularly, we prove that the order of the FrFT is a measure of the distortion in the reconstructed signal. Moreover, we give a dynamic interpretation to the order as time evolution of a function. We also introduce the notion of ρ–α space as an extension of the Fourier or k-space in MR, and we use it to study the distortions introduced in two common MR acquisition strategies. We formulate the reconstruction problem in the context of the FrFT and show how the semigroup theory allows us to find new reconstruction formulae for discrete sampled signals. Finally, the theoretical results are supplemented with numerical examples that show how it performs in a standard 1D MR signal reconstruction. Since this new reconstruction technique takes into account the quadratic component of the inhomogeneity, it allows for better reconstruction with existing scanners, or alternatively, allows for more relaxed requirements for the design of the main coil.

Key words. Fractional Fourier transform; strongly continuous semigroups; magnetic resonance imaging; quadratic magnetic fields;

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1. Introduction. The fractional Fourier transform (FrFT) is an extension of the Fourier transform first developed by Kober [12] in the late thirties and rediscovered by Namias [18] in the late seventies. Namias used the FrFT in the context of quantum mechanics as a way to solve certain problems involving quantum harmonic oscillators. He not only stated the standard definition for the FrFT, but also developed an operational calculus for this new transform. The general idea is to consider the eigenvalue decomposition of the Fourier transform in terms of the Hermite (or Gauss–Hermite [21, Chapter 2, §5.2]) functions. Then, by continuously interpolating the eigenvalues from 1 to themselves, a family of operators that deform the identity operator into the Fourier transform is obtained. In the particular case of the FrFT, this interpolation is performed by replacing the n–th eigenvalue by its a–th power, for a between 0 and 1. This value is called the *transform order*. Since this interpolation depends on a single parameter, the interpolating operators form a one–parameter family. Later Kerr [11] exploited this fact to prove that this family is indeed a strongly continuous unitary group. However, the periodic nature of this family was apparently overlooked.

Starting in the eighties and continuing until today, much research has been done on applications of the FrFT to problems in applied sciences and physics. Particularly, in optics [1], [16], [19], [20] the FrFT fits nicely in the context of *Linear Canonical Transforms* [4], itself an extension of the Fourier transform that has been the subject

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of extensive research. This allows to construct optical devices that physically produce the fractional Fourier transform of optical signals [14], [22]. In turn, it is possible to extend certain quantities to define their fractional counterparts, as is the case for the fractional correlation [17], [21, Chapter 11, §2]. A remarkable aspect of these applications is that the transform order has a natural physical interpretation in terms of the distance between the lenses of the device.

In the last years, these developments have crossed the boundaries of optics into the field of signal processing, as the FrFT has found several applications in that context [2], [21, Chapters 10, 11]. In these cases, the transform order is thought as a fixed parameter that allows us to extract time–frequency information from the signal. However, the physical interpretation depends heavily on the application being considered. As in optics, several concepts defined in the context of time–frequency analysis have been extended to their fractional counterparts, such as fractional uncertainty relations [9], [24], [25], fractional convolutions [3] and fractional bandlimited signals [23], [26] among others. The notion of a *discrete* FrFT (DFrFT) is crucial in digital signal processing. Concretely, in discrete signal processing there are a discrete and finite number of measurements of a signal on a fractional domain. Using the DFrFT it is possible to find a good approximation of the original signal in the time domain in the same way that an inverse DFT is performed on discrete frequency measurements. Unfortunately, to this day there is no consensus as to which discrete transform *is* the DFrFT. There are two possible approaches to this transform. The first consists in constructing a set of discrete eigenvectors to reproduce the approach outlined in the first paragraph. This method is commonly known as the DFrFT [21, Chapter 6, §2]. The second approach consists in considering the signal as fractional bandlimited, so that the integrals can be truncated and approximated by Riemann sums. These can be calculated using the traditional FFT algorithm [21, Chapter 6, §3].

In this work, we show a new application for the FrFT to Magnetic Resonance Imaging (MRI). In MRI, the object or a patient is placed in a strong, homogeneous and constant magnetic field, which interacts with the nuclear spin of the elements composing it, causing the magnetization of the object. These nuclei are excited by means of an electromagnetic radio–frequency (RF) pulse tuned to the resonance frequency of the element that one wants to image, typically hydrogen. Then, as the excited spins decay to their minimal energy state, they emit electromagnetic radiation that is measured as a voltage induced in the acquisition coils. To resolve the spatial location of the nuclei, spatial gradients are used to make the frequency and phase of the decay dependent on the position of the nucleus. The relation between the measured signal, called the Magnetic Resonace (MR) signal, and the magnetization of the object (which is proportional to the proton density) is known as the *signal equation* and the problem of recovering the magnetization from this signal is usually understood as a *reconstruction problem*. The remarkable theoretical property of this process is that the MR signal is the Fourier transform of the magnetization of the underlying object.

In MRI, the signal intensity increases with the external field strength and, on the other hand, the reconstruction quality strongly depends on the homogeneity of the external field. To achieve uniform and high fields simultaneously is a difficult technical challenge. Part of the high cost of the main coil comes from this requirement. In general, the homogeneity of the field is easier to achieve with lengthier coils, but this conflicts with the desire of having shorter bores, for patient accessibility and comfort.

Manufacturers must trade off an acceptable field inhomogeneity, typically in the order of one part per million, and the bore usability. The final consequence is that there is a remaining inhomogeneity that includes linear, quadratic and higher order terms. In this work we show how the FrFT can be used to take into account the quadratic terms of this inhomogeneity.

How to avoid or compensate the image distortions coming from the field inhomogeneity is an active field of research in MRI. The case of constant and linear inhomogeneities are well understood and easily compensated: in many situations the field can be approximated by a hyper plane as shown by [10]. For more complex shaped inhomogeneities there are only heuristic approaches, and the ability to compensate in a way that the resulting image has no distortions depends on the specific kind of inhomogeneity. However, when only quadratic field distortions are considered, the modifications arising in the signal equation induce a quadratic–phase term on the Fourier transform of the magnetization. Thus, the signal equation in this case resembles closely the FrFT of the magnetization. Consequently, the FrFT provides a framework to study and correct images formed in the presence of quadratic fields. We remark that this also allows to approximate more general field inhomogeneities by means of their Taylor approximation of order two.

The purpose of the present work is two–fold. Firstly, the authors believe that the periodicity of the underlying group is one of the fundamental properties of the FrFT. Consequently, we propose to consistently reformulate the FrFT in the framework of strongly continuous semigroups. This allows us to state several new and interesting identities involving the FrFT. Secondly, we establish the connection between the reconstruction problem of MRI in presence of quadratic fields with the FrFT. As an application of our work, we use some of the deduced identities to propose a new approach for the reconstruction problem in MRI. Since the FrFT is itself an extension of the Fourier transform, we show that these methods can be used to explore the reconstruction problem even in absence of inhomogeneities.

The present document is structured as follows. In §2 we reformulate the FrFT in the framework of strongly continuous semigroups. This allows us to explore new properties of the FrFT which are a consequence of this theory. In §3 we explore how the FrFT fits in the framework of MRI. Particularly, we show that the order of the FrFT is a measure of the distortion introduced by quadratic field inhomogeneities in the MR signal. We also introduce the notion of *ρ–α space* and we use it to study the distortions introduced in two common MRI acquisition strategies. In §4 we formulate the reconstruction problem in the context of the FrFT. We also show how the semigroup theory allows us to find new reconstruction formulae for discrete sampled signals. Finally, in §5 we show a numerical implementation of this method and show how it performs in a standard 1D MR signal reconstruction.

2. A reformulation of the fractional Fourier transform. This section reformulates the fractional Fourier transform (FrFT) within the context of semigroup theory. In turn, formulas of approximation arising in this theory will allow us to propose new methods to reconstruct the magnetization of the object from the Magnetic Resonance (MR) signal. To begin, consider $L^2(\mathbb{R})$, the Hilbert space which consists of all complex–valued, Lebesgue measurable functions with domain $\mathbb{R}$ and such that:

$$\|f\|^2 := \int_{-\infty}^{\infty} |f(u)|^2 \, du < \infty.$$

The FrFT on $L^2(\mathbb{R})$ was originally described by Kober [12] and was later rediscovered by Namias in the context of quantum mechanics [18]. One equivalent definition is as follows (see [21, Definition F, p.132]):

$$\mathcal{F}^a = \mathrm{e}^{-i\frac{\pi}{2}a\mathcal{H}}, \tag{2.1}$$

where $\mathcal{H}$ is proportional to the Hamiltonian of the quantum mechanical harmonic oscillator. This is a heuristic definition, because if (2.1) is expanded as a power series, delicate questions of convergence arise. In fact, this is a well studied matter in the theory of strongly continuous semigroups [7]. In an interesting work, Kerr [11] proved that the FrFT constitutes a strongly continuous unitary group of operators on $L^2(\mathbb{R})$. However, the structure of $L^2(\mathbb{R})$ as a Hilbert space was not used to its full extent, since Kerr's approach considers the FrFT as defined by an integral formula (cf. Corollary 2.5 below). On the other hand, the striking fact that the underlying group is periodic seems to have been overlooked. In this section, we formalize (2.1) by means of the theory of semigroups of linear operators, introducing the definition of the FrFT from a new perspective.

To begin, recall that, for $n \in \mathbb{N}_0$, the Gauss–Hermite functions are defined as:

$$\psi_n(u) = \frac{1}{\sqrt{2^n n! \sqrt{2}}} H_n\left(\sqrt{2\pi}u\right) \mathrm{e}^{-\pi u^2},$$

where:

$$H_n(u) = (-1)^n \mathrm{e}^{u^2} \frac{d^n}{du^n} \mathrm{e}^{-u^2}, \tag{2.2}$$

is the n–th Hermite polynomial. It is a well known fact that the Gauss–Hermite functions form an orthonormal system in $L^2(\mathbb{R})$. For $n \in \mathbb{Z}$, we define the operators $P_n : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ as:

$$(P_n f)(u) := \langle f, \psi_n \rangle \, \psi_n(u).$$

Define now the unbounded linear operator $\mathcal{H} : D(\mathcal{H}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ by:

$$\mathcal{H}f(u) := \frac{i}{2}\left(\frac{1}{4}f''(u) - \pi^2 u^2 f(u) + \frac{\pi}{2} f(u)\right), \tag{2.3}$$

with (maximal) domain given by:

$$D(\mathcal{H}) = \left\{ f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R}) \text{ and } M_{u^2} f \in L^2(\mathbb{R}) \right\},$$

where M_{u^2} denotes the multiplication by u^2 operator, i.e. $M_{u^2} f(u) := u^2 f(u)$. We begin our considerations with the following summary of some important properties of the sequence of operators $\{P_n\}_{n=0}^{\infty}$ that will be used throughout this work.

Proposition 2.1. *The following assertions are valid:*

(i) $\{P_n\}_{n=0}^{\infty}$ *is a sequence of bounded and linear projections on* $L^2(\mathbb{R})$*, that is* $P_n P_m = 0$ *if* $m \neq n$ *and* $P_n^2 = P_n$.

(ii) *For all* $n \in \mathbf{N}_0$, $\|P_n\| \leq 1$.

(iii) *For all* $n \in \mathbf{N}_0$, $P_n \mathcal{H} = \mathcal{H} P_n = -in\frac{\pi}{2} P_n$.

(iv) $\sum_{n=0}^{\infty} P_n f = f$ *for all* $f \in L^2(\mathbb{R})$.

(v) *The domain of* $\mathcal{H}$ *is dense in* $L^2(\mathbb{R})$.

We now turn our attention to dynamical systems (or semigroups) on infinite dimensional spaces. Let X be a complex Banach space and let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X endowed with the operator norm. A family $\{T(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ is called a group on X if it satisfies Abel's functional equation:

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \in \mathbb{R}, \\ T(0) = I, \end{cases}$$

and the orbit maps $t \to T(t)f$ are continuous from $\mathbb{R}$ into X for every $f \in X$. The *infinitesimal generator* of the group $\{T(t)\}_{t\in\mathbb{R}}$ is defined by:

$$Af := \lim_{t\to 0} \frac{T(t)f - f}{t},$$

whenever $f \in D(A)$. The following is the main result of this section.

THEOREM 2.2. *For all $f \in L^2(\mathbb{R})$ the family $\{T(t)\}_{t\in\mathbb{R}}$ given by:*

$$T(t)f = \sum_{n=0}^{\infty} \mathrm{e}^{-in\frac{\pi}{2}t} P_n f,$$

defines an strongly continuous group in $L^2(\mathbb{R})$. Moreover:
(i) *$T(t)$ is periodic with period 4.*
(ii) *$T(1) = \mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform.*

We remark the simplicity of the argument used to prove strong continuity in Appendix B compared with [11, Theorem 2.2] (see also [15, Theorem 3.3]) and even the simpler way to derive Abel's functional equation compared with [15, Theorem 4.1].

We introduce the FrFT by means of the following definition.

DEFINITION 2.3. *We define the fractional Fourier transform of order $a \in \mathbb{R}$ for a function $f \in L^2(\mathbb{R})$ as:*

$$\mathcal{F}^a(f)(\rho) := [T(a)f](\rho), \quad \rho \in \mathbb{R}.$$

The group $\{T(a)\}_{a\in\mathbb{R}}$ will also be called the Fourier group.

The following lemma is the key to rigorously show that $\{T(a)\}_{a\in\mathbb{R}}$ is, in fact, the same as the usual definition of the FrFT of order a given by means of an integral formula. The identity stated below can be found in the literature, e.g. in [21]. However, for the sake of completeness, we include a direct proof in Appendix C.

LEMMA 2.4. *For all $\rho, u \in \mathbb{R}$ and all $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$ we have:*

$$\sum_{n=0}^{\infty} \psi_n(\rho)\, \mathrm{e}^{-in\alpha} \psi_n(u) = \sqrt{1 - i\cot\alpha}\; \mathrm{e}^{i\pi\left(\rho^2\cot\alpha - 2\rho u \csc\alpha + u^2\cot\alpha\right)}.$$

We remark that to prove this lemma, we put a restriction to the values of α. This is because the first term provides a definition of the fractional Fourier transform for arguments $f \in L^2(\mathbb{R})$ whereas the second term is valid for arguments $f \in L^1(\mathbb{R})$. Hence, in between, the orthogonality of the Gauss–Hermite functions on $L^2(\mathbb{R})$ is lost.

As announced, from Lemma 2.4 and Theorem 2.2 we deduce the following result, which shows the well–known and equivalent definition of the FrFT in terms of a Riemann integral.

Corollary 2.5. *Given $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$ we denote $a := \frac{2}{\pi}\alpha$. We have for all $\rho \in \mathbb{R}$,*

$$[T(a)f](\rho) = \sqrt{1 - i\cot\alpha} \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{i\pi\left(\rho^2 \cot\alpha - 2\rho u \csc\alpha + u^2 \cot\alpha\right)} du. \tag{2.4}$$

In practice, the restriction on the values of α has a number of consequences for the fractional Fourier transform as defined in (2.4). For instance, the behavior of the fractional Fourier transform in the border cases $\alpha \in \pi\mathbb{Z}$ is not obvious. Moreover, we need to specify which branch of $\sqrt{1 - i\cot\alpha}$ is being considered. These problems were treated in [15], first by confining the values of α to the interval $(-\pi, \pi)$ and then by treating the values on the boundary separately. In contrast, our approach by means of continuous groups produces a definition for the fractional Fourier transform valid for any value of $a \in \mathbb{R}$.

In the next theorem, we summarize some of our previous remarks and we derive new properties and identities using the framework of strongly continuous groups. As is standard, we denote by $\sigma(\mathcal{H})$ the spectrum and by $\sigma_p(\mathcal{H})$ the point spectrum of $\mathcal{H}$.

Theorem 2.6. *The following assertions are valid:*

(i) *The infinitesimal generator of the Fourier group is $\mathcal{H}$.*
(ii) *The Fourier group is unitary, that is $T(t)^* = T(-t)$.*
(iii) $\sigma(\mathcal{H}) = \sigma_p(\mathcal{H}) \subset i\frac{\pi}{2}\mathbb{Z}$.
(iv) *The following identity holds:*

$$P_n f = \frac{1}{4} \int_0^4 \mathrm{e}^{-in\frac{\pi}{2}s}\, [T(s)f]\, ds.$$

In other words, $P_n f$ is the n–th Fourier coefficient of the fractional Fourier transform $T(t)$.

(v) *For all $\lambda \notin i\frac{\pi}{2}\mathbb{Z}$, we have:*

$$(\lambda I - \mathcal{H})^{-1} f = \frac{1}{1 - \mathrm{e}^{-4\lambda}} \int_0^4 \mathrm{e}^{-\lambda s}\, [T(s)f]\, ds.$$

(vi) *For all $f \in D(\mathcal{H})$, $\mathcal{H}f = \sum_{n=1}^{\infty} -in\frac{\pi}{2} P_n f$.*

To finish this section, we take advantage of the variety of approximation formulae of the theory of strongly continuous semigroups to obtain the following result. It will be the basis for the description of a new formula to reconstruct the magnetization from the MR signal given in the next section. For a proof, see Appendix E.

Theorem 2.7. *For all $f \in L^2(\mathbb{R})$ we have in the L^2 norm:*

$$T(t)f = \lim_{q \to \infty} \mathrm{e}^{-tq} \sum_{m=0}^{\infty} \frac{t^m q^m}{m!} T\left(\frac{m}{q}\right) f.$$

3. The connection between the fractional Fourier transform and Magnetic Resonance Imaging. As seen in the previous section, the fractional Fourier transform offers a framework that extends the classical Fourier transform through the transform order a. However, it is not always clear how to assign a physical interpretation to this variable. In optics, for example, there are natural interpretations of the order and the transform itself [16], [19], [20]. In this section, we will show that in Magnetic Resonance Imaging (MRI), the order is related to the distortion introduced by quadratic field inhomogeneities. This connection shows that the FrFT extends the framework of MRI, from homogeneous fields to quadratic ones. This is remarkable, since one of the technical difficulties of MRI is to produce magnetic fields that are simultaneously high and homogenous. Thus, by extending the framework of MRI, the FrFT could eventually allow technical breakthroughs in this field.

As we described in §1, the cornerstone of MRI is that the Magnetic Resonance (MR) signal $s(t)$ is related to the magnetization of the object through its Fourier transform. To clarify the exposition, we will ignore T_1 and T_2 relaxations and we will consider that a Larmor frequency demodulation has been performed [13]. Consequently, the equation of the MR signal generated by a 1D object in an uniform magnetic field $B(u) = B_0$ is:

$$s(t) = \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{-2\pi i k(t) u}\, du. \tag{3.1}$$

where $k(t)$ is a predetermined function that represents how the measurements are performed, i.e. the *trajectory* traversed on the Fourier domain (for a discussion of some common choices for $k(t)$, see below). Equation 3.1 is called *the signal equation*. It essentially tells us that the Fourier transform of f evaluated at $k(t)$ equals $s(t)$. Now consider a field inhomogeneity, i.e. let $B(u) = B_0 + p(u)$ be a non–constant function of space. In this case, (3.1) becomes:

$$s(t) = \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{-2\pi i (k(t)u + p(u)t)}\, du.$$

If this inhomogeneity $p(u)$ corresponds to a second order polynomial $p(u) = p_2u^2 + p_1u + p_0$, then the signal equation becomes:

$$\begin{aligned} s(t) &= \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{-2\pi i\left(k(t)u + (p_2u^2 + p_1u + p_0)t\right)}\, du \\ &= \mathrm{e}^{-2\pi i p_0 t} \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{i\pi\left(-2p_2tu^2 - 2(k(t)+p_1t)u\right)}\, du. \end{aligned} \tag{3.2}$$

From Corollary 2.5, the fractional Fourier transform $\mathcal{F}^a$ $(0 < a < 2)$ of a function $f(u)$ can be written as:

$$\mathcal{F}^a(f)(\rho) = C_\alpha(\rho) \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{i\pi\left(u^2\cot\alpha - 2\rho u \csc\alpha\right)}\, du, \tag{3.3}$$

where:

$$C_\alpha(\rho) \equiv \mathrm{e}^{i\pi\rho^2\cot\alpha}\sqrt{1 - i\cot\alpha}, \qquad \alpha \equiv a\pi/2. \tag{3.4}$$

FIG. 3.1. *Example of the trajectories described on the ρ–α space. Figure 3.1a shows a one dimensional constant gradient trajectory whereas Figure 3.1b shows a one dimensional 2DFT trajectory.*

Consider the following change of variables:

$$\alpha(t) \equiv \cot^{-1}(-2p_2 t), \tag{3.5}$$

$$\rho(t) \equiv \frac{k(t) + p_1 t}{\sqrt{1 + 4p_2^2 t^2}} = \frac{k(t) + p_1 t}{\csc \alpha(t)}. \tag{3.6}$$

It is clear that by means of these relations, we can transform (3.2) into the FrFT of variable order of f. We deduce the following result.

THEOREM 3.1. *For quadratic field inhomogeneities, the signal $s(t)$ multiplied by a weight factor depending on t are the samples of the fractional Fourier transform of variable order.*

For a proof, see Appendix F. It is important to note that normally $k(t) = 0$ for $t < 0$, that is, the system is causal. Thus, if we choose the branch of $\cot^{-1}$ such that $\alpha \in (0, \pi)$, then $\alpha(0) = \pi/2$ $(a = 1)$ and $\csc \alpha(t) > 1$. Consequently, for finite values of t we are far from $a = 0$ or $a = 2$, which could eventually pose problems with the integral definition of the FrFT in (3.3).

The definition of α and ρ in (3.5) and (3.6) provides a parametric trajectory $(\rho(t), \alpha(t))$ in a space we call the ρ–α space. Since α is an angle, this space is conveniently represented in polar coordinates. Assuming that $p_2 < 0$ (see §3.1 below) and by virtue of the previous discussion, the trajectory in the ρ–α space starts immediately after the excitation $(t = 0)$ in the *frequency axis* or *Fourier axis* $(\alpha = \pi/2$ or $a = 1)$ and as time passes it curves toward the *object axis* $(\alpha = a = 0)$.

In the following paragraphs we analyze some common trajectories $k(t)$ when distorted by a quadratic field. Particularly, we analyze this distortion using the ρ–α space trajectories described.

3.1. Constant gradient. One common trajectory assumes that $k(t) = 0$ for $t < 0$ and $k(t) = G_0 t$ for $t \geq 0$, for G_0 constant. Physically, this corresponds to a constant read–out gradient as would be the case in a projection reconstruction sequence. For simplicity, assume that the inhomogeneity is purely quadratic, i.e.

$p(u) = p_2 u^2$. Additionally, assume that $p_2 < 0$, which resembles the typical case in that the intensity of the B_0 field is greater in the center of a magnet. Linear and constant terms can be ignored without loss of generality, because the first is equivalent to a change in G_0 and is easily corrected [10], whereas the second can be corrected by an appropriate modulation of the MR signal. Then the trajectory in ρ–α space is:

$$\alpha(t) = \cot^{-1}(-2p_2 t) = \cot^{-1}(2|p_2|t),$$
$$\rho(t) = k(t)\sin\alpha(t) = \begin{cases} 0 & t < 0, \\ G_0 t \sin\alpha(t) & t \geq 0. \end{cases}$$

Note that:

$$G_0 t \sin\alpha(t) = \frac{G_0}{2|p_2|}\cot\alpha(t)\sin\alpha(t) = \frac{G_0}{2|p_2|}\cos\alpha(t).$$

Consequently, the ρ–α trajectory is a circumference centered at $(G_0/4|p_2|, 0)$. Figure 3.1a shows this trajectory starting in $t = 0$ at the origin and asymptotically approaching the object axis as t grows.

For small values of t, the trajectory departs little from the frequency axis, and therefore distortions due to field variations are small. This is consistent with the general knowledge that short read–outs are less sensitive to inhomogeneity. Moreover, as p_2 tends to zero the field inhomogeneity vanishes and the center of the circumference located at $G_0/4|p_2|$ tends to infinity. Consequently, the ρ–α trajectory becomes a straight line in the frequency direction:

$$\alpha(t) \equiv \frac{\pi}{2},$$
$$\rho(t) = \begin{cases} 0 & t < 0, \\ G_0 t & t \geq 0. \end{cases}$$

3.2. Standard 2DFT readout. Another commonly used trajectory assumes that $k(t)$ linearly decreases until a given time t_0 is reached and then linearly increases passing through the origin, a standard procedure in 2DFT readouts [13]. Concretely:

$$k(t) = \begin{cases} 0 & t < 0, \\ -G_0 t & 0 \leq t < t_0, \\ G_0(t - 2t_0) & t_0 \leq t, \end{cases}$$

where G_0 is a constant representing the magnitude of the read–out gradient and t_0 is the instant where its sign is changed. Assuming once more that the inhomogeneity is $p(u) = p_2 u^2$ with $p_2 < 0$, the trajectory in ρ–α space is given by:

$$\alpha(t) = \cot^{-1}(-2p_2 t),$$
$$\rho(t) = \frac{1}{\sqrt{1 + 4p_2^2 t^2}} \times \begin{cases} 0 & t < 0, \\ -G_0 t & 0 \leq t < t_0, \\ G_0(t - 2t_0) & t_0 \leq t, \end{cases}$$

It is not difficult to see that:

$$\rho(t) = \begin{cases} 0 & t < 0, \\ -G_0 t \sin\alpha(t) & 0 \le t < t_0, \\ G_0(t - 2t_0)\sin\alpha(t) & t_0 \le t. \end{cases}$$
$$= \begin{cases} 0 & t < 0, \\ -\frac{G_0}{2|p_2|}\cos\alpha(t) & 0 \le t < t_0, \\ \frac{G_0}{2|p_2|}\left(\cos\alpha(t) - 2\cot\alpha_0 \sin\alpha(t)\right) & t_0 \le t, \end{cases}$$

where $\alpha_0 = \alpha(t_0)$. This trajectory is formed by two pieces of circumferences, one for the negative frequencies centered at $(-G_0/4|p_2|, 0)$ which continues with another one centered at $(G_0/4|p_2|, -G_0 t_0)$, as shown in Figure 3.1b.

To summarize these analysis, note that in the homogeneous field case, the measurements on the magnetization of the object are always performed in the Fourier domain. In constrast, the ρ–α space analysis shows that quadratic fields induce a time–evolution on the domain where the measurements are performed. This evolution can be studied through the time–evolution of the transform order a. This remarkable connection shows that not only the FrFT allows us to handle MRI with quadratic fields, but also induces a physical interpretation to the transform order a.

4. Reconstruction from samples acquired in fractional domains. In the previous sections we described how the fractional Fourier transform lends itself to the study of the signal equation that results when quadratic fields are being considered. In this section, we deal with the problem of reconstructing the magnetization of the object from this measured data. Notice that in the presence of quadratic field inhomogeneities, the measured data will not correspond to a specific frequency k as in the undistorted Fourier case, but to a fractional frequency u with transform order a. Formally, this problem can be described as follows. Let $f \in L^2(\mathbb{R})$ be the underlying magnetization. Let M be a fixed positive integer. Our problem is to find an approximation $\tilde{f} \in L^2(\mathbb{R})$ from the samples:

$$S^{a_m}(\rho_m) = \mathcal{F}^{a_m}(f)(\rho_m),$$

where $\{(\rho_m, a_m)\}_{m=-M}^{M}$ is a sequence of fractional frequencies ρ associated to their transform orders a_m. In this section we will use the theoretical framework introduced in §2 to propose a new method to solve this problem. We begin with the following result:

Theorem 4.1. *For all $f \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$ fixed, we have:*

$$f = \sum_{n=0}^{\infty} \langle \mathcal{F}^a(f), \psi_n \rangle \; \mathrm{e}^{in\frac{\pi}{2}a}\, \psi_n, \tag{4.1}$$

where the equality is to be understood in the L^2–norm.

For a proof, see Appendix G. It is clear from (4.1) that the discrete sequence of samples $\{S^{a_m}(\rho_m)\}_{m=-M}^{M}$ does not provide enough information to reconstruct a function f using the inverse FrFT. As a result, the reconstruction is not unique and it is in fact an interpolation problem. The standard approach is to consider that the measured data is enough to approximate the inner products in (4.1) as:

$$\langle \mathcal{F}^a(f),\, \psi_n \rangle \approx \sum_{m=-M}^{M} S^{a_m}(\rho_m)\psi_n(\rho_m)\Delta\rho_m, \quad n \in \mathbb{N}_0, \tag{4.2}$$

where we have replaced the fractional frequencies ρ and transform order a by the values of the sequence $\{S^{a_m}(\rho_m)\}_{m=-M}^{M}$ and added $\Delta\rho_m$ as a suitable finite approximation for the differential $d\rho$. Concretely, we approximate the integrals in the inner products by means of Riemann sums. This technique is commonly referenced as "assuming that the unmeasured data is equal to zero". However, this statement has no formal interpretation in our framework. This is because (4.1) is to be understood in the $L^2(\mathbb{R})$ sense and consequently there is no consistent way to interpret the point–value of a function. Nevertheless, (4.2) is to be understood as a definition of the approximation of the inner products using Riemann sums, allowing us to overcome this conceptual problem. For simplicity, in what follows we will assume that $\Delta\rho_m \equiv 1$ for the relevant values of m.

In what follows, we denote (cf. Lemma 2.4):

$$K_\rho^\alpha(u) := \sqrt{1 - i\cot\alpha}\ \mathrm{e}^{i\pi(\rho^2\cot\alpha - 2\rho u\csc\alpha + u^2\cot\alpha)}. \tag{4.3}$$

From Theorem 4.1, we deduce the following reconstruction formula:

$$f(u) \approx \tilde{f}(u) = \sum_{m=-M}^{M} S^{a_m}(\rho_m) K_{\rho_m}^{-\frac{\pi}{2}a_m}(u). \tag{4.4}$$

In fact, putting (4.2) in (4.1) we obtain:

$$\begin{aligned} f(u) &\approx \sum_{n=0}^{\infty}\left(\sum_{m=-M}^{M} S^{a_m}(\rho_m)\psi_k(\rho_m)\right)\mathrm{e}^{in\frac{\pi}{2}a_m}\psi_n(u) \\ &= \sum_{m=-M}^{M} S^{a_m}(u_m)\left(\sum_{n=0}^{\infty}\psi_n(\rho_m)\,\mathrm{e}^{in\frac{\pi}{2}a_m}\psi_n(u)\right), \end{aligned}$$

and hence (4.4) follows from Lemma 2.4.

Clearly, the reconstruction $\tilde{f}$ given in (4.4) is not identical to the true image f. Note that:

$$\tilde{f}(u) = \sum_{m=-M}^{M} S^{a_m}(\rho_m)\sqrt{1 + i\cot\alpha_m}\ \mathrm{e}^{-i\pi\left(\rho_m^2\cot\alpha_m - 2\rho_m u\csc\alpha_m + u^2\cot\alpha_m\right)}, \tag{4.5}$$

where $\alpha_m = \frac{\pi}{2}a_m$. The above formula coincides with the approach to the discrete fractional Fourier transform given by Ozaktas et al. [21, p.218-219]. On the other hand, it corresponds to a direct extension of the windowed Fourier reconstruction method in MRI (see [13, p.195]) which is recovered in case $\alpha_m = \pi/2$. In fact, in this case we have $\rho_m = k_m$ and $a_m \equiv 1$ so that:

$$f(u) \approx \tilde{f}(u) = \sum_{m=-M}^{M} S(k_m)\ \mathrm{e}^{2\pi i k_m u}, \tag{4.6}$$

where we denote $S(k_m) \equiv S^1(k_m)$.

The interesting point now is that, using approximation formulae for semigroups, we can obtain other reconstruction methods. Among all possible approximation formulae arising from this theory, we use as an example that of Theorem 2.7 to deduce the following result.

THEOREM 4.2. *For all $f \in L^2(\mathbb{R})$ and $a \in \mathbb{R}$ we have,*

$$f = \lim_{q\to\infty} \mathrm{e}^{aq} \sum_{n,m=0}^{\infty} \frac{(-aq)^n}{n!} \mathrm{e}^{-i\frac{n}{q}m} \langle \psi_m, \mathcal{F}^a f \rangle \ \psi_m,$$

where the equality is understood in the L^2–norm.

For a proof, see Appendix H. Assume again (4.2). From Theorem 4.2 we deduce the following reconstruction formula:

$$\tilde{f}(u) = \lim_{q\to\infty} \mathrm{e}^{a_m q} \sum_{m=-M}^{M} \sum_{n=0}^{\infty} \frac{(-a_m q)^n}{n!} S^{a_m}(u_m) K_{u_m}^{\frac{n}{q}}(u).$$

Observe that in the particular Fourier case $a_m = 1$ and $\rho_m = k_m$, we obtain a new reconstruction formula only in terms of the Fourier transform samples:

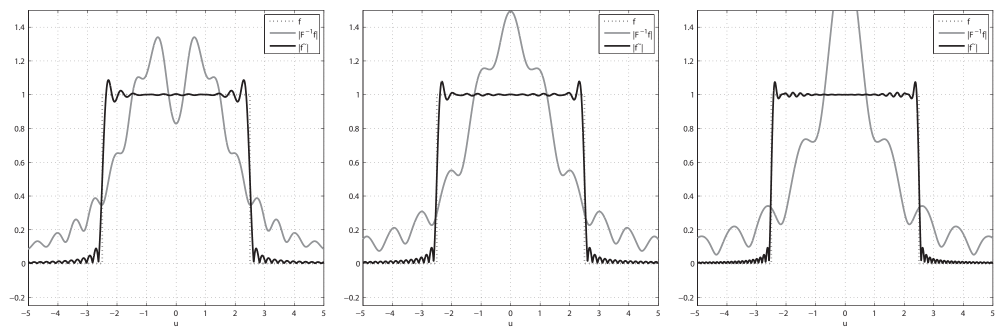
$$\tilde{f}(u) = \lim_{q\to\infty} \mathrm{e}^{q} \sum_{m=-M}^{M} \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} S(k_m) K_{k_m}^{\frac{n}{q}}(u), \tag{4.7}$$

which in the customary form reads:

$$\tilde{f}(u) = \lim_{q\to\infty} \mathrm{e}^{q} \sum_{m=-M}^{M} \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} S(k_m)\, \mathrm{e}^{-i\pi\left(k_m^2 \cot\left(\frac{n}{q}\right) - 2k_m u \csc\left(\frac{n}{q}\right) + u^2 \cot\left(\frac{n}{q}\right)\right)}.$$

To implement the reconstruction formula defined in Theorem 4.2, we must consider a finite value for q. In this case, the approximation has two interesting features. Firstly, in windowed Fourier reconstructions the samples are multiplied by a set of scalars to suppress the high–frequency content in the reconstruction. This set of scalars is normally chosen manually. In the proposed approximation these coefficients arise naturally and are independent of the number of samples, sampling frequency or the frequency content of the original signal. Secondly, in windowed Fourier reconstruction the *samples* are multiplied by a set of scalars, whereas in our method the exponentials are multiplied by a set of scalars. It is also interesting to note that exponentials corresponding to different transform orders are mixed to produce a suitable reconstruction that, by virtue of Theorem 4.2, converges to the original signal. This suggests that the approximation formulas arising in strongly continuous group theory allow us to explore new frameworks for FrFT reconstructions *and* windowed Fourier reconstructions.

5. Numerical experiments. In this section we will show some numerical experiments that represent a generic MR reconstruction. Concretely, we will consider a 1D object of 5 cm with a constant magnetization equal to 1 centered at the origin. A



(a) (b) (c)

FIG. 5.1. *Comparison between the original function (dotted line), its standard Fourier transform reconstruction (solid grey line) and the fractional Fourier transform reconstruction (solid line). Figures 5.1a, 5.1b and 5.1c show the cases when the data was acquired on fractional domains corresponding to transform orders a equal to* 2/3*,* 1/2 *and* 1/3.

FOV (*Field of View*, i.e. the size of the interval over which the signal will be reconstructed) of 10 cm and 257 samples will be acquired, i.e. $M = 128$. In a standard MRI application, this means that the acquisition step is $\Delta\rho_m = 5 \times 10^{-2}$ and $\rho_M = 6.4$. The measurements were performed at transform orders a equal to 1, 2/3, 1/2 and 1/3. In each case, the magnetization was reconstructed using the standard Fourier reconstruction and the FrFT reconstruction. The comparison between the reconstructions using the FrFT (4.5) and the standard Fourier reconstruction (4.6) can be seen in Figure 5.1 for a equal to 2/3, 1/2 and 1/3. Note that, as expected, the FrFT reconstruction recovers very well the original signal, whereas the standard Fourier reconstruction fails. Note also that as a decreases, the standard Fourier reconstruction becomes increasingly inaccurate. On the other hand, the FrFT reconstruction slightly improves its accuracy as a decreases.

To study the advantages of the reconstruction method proposed in Theorem 4.2, we restate first the reconstruction formula as:

$$\tilde{f}(u) = \lim_{q\to\infty} \sum_{m=-M}^{M} \sum_{n=0}^{\infty} \frac{q^n}{n!}\, \mathrm{e}^{-q} S^{a_m}(\rho_m) K_{\rho_m}^{-\frac{n}{q}a_m}(u). \tag{5.1}$$

This formula involves an infinite series and a limit. However, the weighting factors, which correspond to the coefficients of the exponential series, decay rapidly to zero and thus only a fraction of the coefficients in the series contribute in a significant way to the reconstruction. The drawback is that these weights involve factorials which become increasingly difficult to accurately calculate as n increases. To overcome these difficulties, we begin by approximating these factors by a Gaussian function. We use the following lemma:

LEMMA 5.1. *For sufficiently large q, we have:*

$$\frac{q^n}{n!}\, \mathrm{e}^{-q} \approx \frac{1}{\sqrt{2\pi q}}\, \mathrm{e}^{-\frac{1}{2}\frac{(n-q)^2}{q}}. \tag{5.2}$$

For a proof, see Appendix I. The proposed approximation becomes more accurate as q increases, which is exactly the kind of behavior in which we are interested. Also, the approximation allows us to estimate the error produced in truncating the series by means of the standard deviation of said approximation, which in turn depends on the value of q. Define:

$$c_q(n) = \frac{1}{\sqrt{2\pi q}}\, \mathrm{e}^{-\frac{1}{2}\frac{(n-q)^2}{q}}. \tag{5.3}$$

Since the summation index is independent of the samples $S^\alpha(\rho_m)$, we can interchange the sums in (5.1). Define the functions:

$$\phi_q^\alpha(\rho, u) = \sum_{n=0}^{\infty} c_q(n)\, K_\rho^{-\frac{n}{q}\alpha}(u) \tag{5.4}$$

Using this notation, we can construct a one–parameter family of reconstructions for f given by:

$$\tilde{f}_q(u) = \sum_{m=-M}^{M} S^{\alpha_m}(\rho_m)\, \phi_q^{-\alpha_m}(\rho_m, u). \tag{5.5}$$

The reconstruction defined in (5.1) can be though as a suitable smoothing of the set of kernels $K_\rho^a(u)$. This contrasts with the standard approach of smoothing the *data*, i.e. multiplying the samples $S^{a_m}(\rho_m)$ by a suitable set of scalars to obtain better reconstructions. To show the benefits of the proposed approximation formula, we are going to show the results for $a = 1$ and for q equal to 250, 2500 and 25000. In this case, we are using essentially (4.6) and (4.7) to obtain the reconstructions. The results can be seen in Figure 5.2. The parameter q controls the smoothness of the reconstruction. For low values of q, the approximation $\tilde{f}_q$ is smooth and consequently does not reconstruct the discontinuities well enough. As q increases, the discontinuities become more evident, as does the Gibbs phenomenon.

6. Discussion. In this work we formulated the fractional Fourier transform (FrFT) within the context of strongly continuous semigroups. As stated in the introduction, the FrFT constitutes a one–parameter family of operators that continuously produce interpolations between the identity operator I and the Fourier transform $\mathcal{F}$. The proposed framework shows that this family of operators is also endowed with an functional analytic structure, namely that of a strongly continuous semigroup. As a consequence of the the uniqueness of the solution to the associated abstract Cauchy problem of first order, interpolations that share the same algebraic structure are equal, up to renormalization, to the FrFT. Additionally, the proposed framework suggests a different interpretation to the operator $\mathcal{F}^a$ as a function of the transform order a. Traditionally, in the signal processing literature the transform order has been thought as a fixed quantity. The proposed framework gives a *dynamic* interpretation of the order, as it can represent time evolution of a function. Also, this framework allows us to exploit approximation results from this theory to propose new methods for reconstruction problems.

Our work also shows the natural connection between the FrFT and Magnetic Resonance Imaging (MRI) done with quadratic fields. Not only the FrFT appears naturally in this context, but it also endows the transform order a with a natural physical interpretation which is consistent with the one suggested by the semigroup framework. Concretely, quadratic fields induce the time–evolution of the domain where the measurements on the magnetization of the object are being performed, from the Fourier

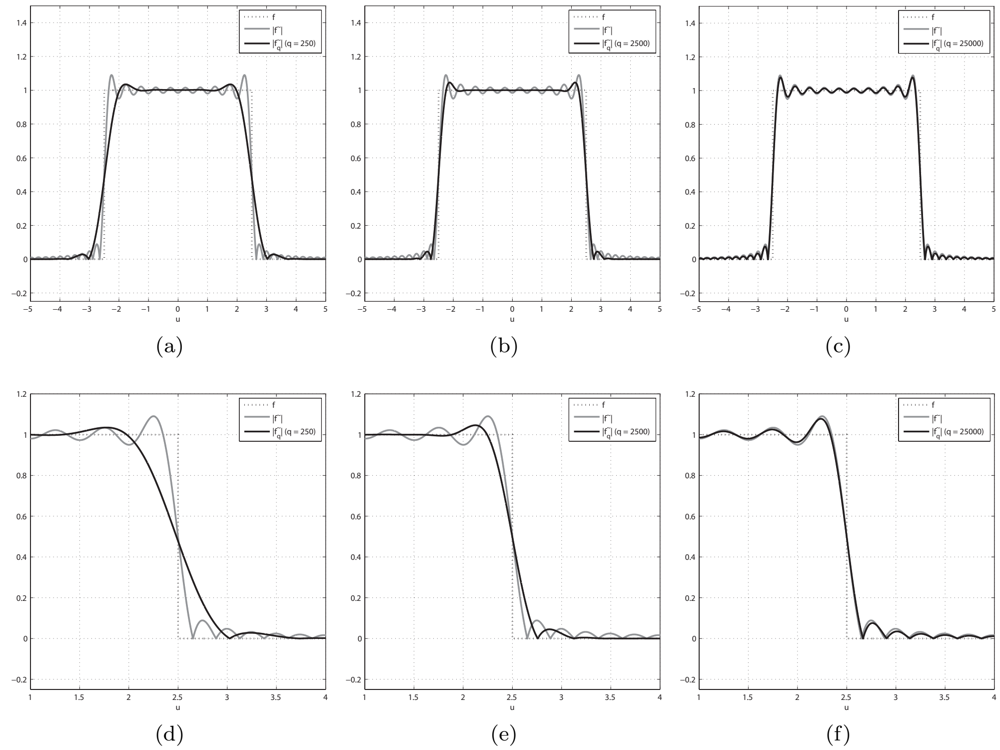


FIG. 5.2. *Comparison between the original function (dotted line), its standard FrFT reconstruction (solid grey line) and the proposed approximation formula (solid line). Figures 5.2a, 5.2b and 5.2c show both reconstructions and the original function for q equal to 250, 2500 and 25000 respectively. Figures 5.2d, 5.2e and 5.2f show a detail of the above plots near the discontinuity of the original signal, showing how the value of q controls the smoothness of the reconstruction.*

domain to fractional domains. This contrasts with MRI performed with homogeneous fields, where the measurements are always performed on the Fourier domain. We believe that this interpretation, by itself, is a significant conceptual breakthrough. This connection also has some important practical consequences. Concretely, the FrFT allows one to recover a signal from a signal equation produced with quadratic fields and consequently, it is possible to go beyond MRI performed with homogeneous field to MRI performed with quadratic fields. This implies several technical advantages such as being able to construct Magnetic Resonance (MR) devices with coils of reduced longitude.

Appendix A. Proof of Proposition 2.1.

Let $f \in L^2(\mathbb{R})$. Since clearly $\psi_n \in L^2(\mathbb{R})$ it follows that $P_n f \in L^2(\mathbb{R})$ and we have $P_m(P_n f) = \langle P_n f, \psi_m \rangle \, \psi_m = \langle f, \psi_n \rangle \, \langle \psi_n, \psi_m \rangle \, \psi_m = \psi_m \, \langle \psi_n, \psi_m \rangle \, P_n f$. Hence, assertion (i) follows from the orthogonality of the set $\{\psi_n\}_{n=0}^{\infty}$. Using the Cauchy–Schwarz inequality, we deduce $\|P_n f\|^2 = |\,\langle f, \psi_n \rangle\,| \|\psi_n\|^2 \leq \|f\|^2 \|\psi_n\|^2 = \|f\|^2$, which implies assertion (ii). To prove assertion (iii), we use the identity [21, Table 2.8 p. 41]:

$$\psi_n''(u) + 4\pi^2 \left(\frac{2n+1}{2\pi} - u^2 \right) \psi_n(u) = 0,$$

which is equivalent to:

$$-in\frac{\pi}{2}\psi_n(u) = \frac{i}{2}\left(\frac{1}{4}\psi_n''(u) - \pi^2 u^2\psi_n(u) + \frac{\pi}{2}\psi_n(u)\right) = \mathcal{H}\psi_n(u).$$

Hence:

$$P_n\mathcal{H}f = \psi_n\left\langle \mathcal{H}f,\, \psi_n\right\rangle = \psi_n\left\langle f,\, -\mathcal{H}\psi_n\right\rangle = \psi_n\left\langle f,\, in\frac{\pi}{2}\psi_n\right\rangle = -in\frac{\pi}{2}P_nf,$$

and:

$$\mathcal{H}P_nf = \left\langle f,\, \psi_n\right\rangle \mathcal{H}\psi_n = -in\frac{\pi}{2}P_nf.$$

Assertion (iv) follows directly from the fact that $\{\psi_n\}_{n=0}^{\infty}$ is a complete orthogonal system in the Hilbert space $L^2(\mathbb{R})$. Finally, assertion (v) follows from classical results in semigroup theory (see e.g. [5, Chapter 1, §4 and Chapter II, Theorem 1.4]). □

Appendix B. Proof of Proposition 2.2.

We first prove that for all $f \in L^2(\mathbb{R})$ we have $T(t)f \in L^2(\mathbb{R})$. In fact, for $f \in L^2(\mathbb{R})$ we have:

$$\begin{aligned}\|T(t)f\|^2 &= \left\langle \sum_{n=0}^{\infty} \mathrm{e}^{-in\frac{\pi}{2}t}P_nf,\, \sum_{m=0}^{\infty} \mathrm{e}^{-im\frac{\pi}{2}t}P_mf\right\rangle \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathrm{e}^{-i(n-m)t}\left\langle P_mP_nf,\, f\right\rangle \\ &= \sum_{n=0}^{\infty}\left\langle P_nf,\, f\right\rangle = \sum_{n=0}^{\infty}\left|\left\langle \psi_n,\, f\right\rangle\right|^2 = \|f\|^2 . \end{aligned} \tag{B.1}$$

The above identity implies the claim and that:

$$\|T(t)\| \leq 1, \tag{B.2}$$

for all $t \in \mathbb{R}$. From Proposition 2.1 (i) and (iv) we deduce the group property $T(t+s) = T(t)T(s)$ and $T(0) = I$. In fact, we have:

$$\begin{aligned}T(t)T(s)f &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathrm{e}^{-in\frac{\pi}{2}t}\,\mathrm{e}^{-im\frac{\pi}{2}s}P_nP_mf \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-in\frac{\pi}{2}(t+s)}P_nf = T(t+s)f.\end{aligned} \tag{B.3}$$

Finally, we claim that $T(t)f \to f$ as $t \to 0$ which shows that $T(t)$ is strongly continuous. In fact, since the domain of $\mathcal{H}$ is dense in $L^2(\mathbb{R})$, it is sufficient to prove the claim for $f \in D(\mathcal{H})$. Let $g = \mathcal{H}f$. We have by Proposition 2.1 (iv) and (ii):

$$\begin{aligned}
\|T(t)f - f\|^2 &= \left\| \sum_{n=0}^{\infty} \left(\mathrm{e}^{-in\frac{\pi}{2}t} - 1 \right) P_n f \right\|^2 \\
&= \left\| \frac{2i}{\pi} \sum_{n=1}^{\infty} \left(\mathrm{e}^{-in\frac{\pi}{2}t} - 1 \right) \frac{1}{n} P_n g \right\|^2 \\
&= \frac{4}{\pi^2} \sum_{n=1}^{\infty} | \, \mathrm{e}^{-in\frac{\pi}{2}t} - 1|^2 \frac{1}{n^2} \|P_n g\|^2 \\
&\leq \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{2}{n^2} - 2\frac{\cos(n\pi t/2)}{n^2} \right) \|g\|^2 \\
&= \frac{4}{\pi^2} \left(\frac{\pi^2 t}{2} + \frac{\pi^2 t^2}{8} \right),
\end{aligned} \tag{B.4}$$

where in the last equality we have used the identity:

$$\sum_{k=1}^{\infty} \frac{\cos ku}{k^2} = \frac{\pi^2}{6} - \frac{\pi u}{2} + \frac{u^2}{4},$$

valid for $x \in [0, 2\pi]$ (see [8, Formula 1.443(3)]). This proves the claim. Finally, it is clear from the definition that $T(t)$ has period 4. That $T(1)$ is the Fourier transform of f follows from their representation in terms of the Gauss–Hermite functions (see e.g. [21]). ∎

Appendix C. Proof of Lemma 2.4.

All the integrals in the proof are to be taken over $\mathbb{R}$. We will need the following two identities:

$$\mathrm{e}^{2us-s^2} = \sum_{n=0}^{\infty} H_n(u) \frac{s^n}{n!}, \tag{C.1}$$

$$\mathrm{e}^{-u^2} = \frac{1}{\sqrt{\pi}} \int \mathrm{e}^{2isu-s^2} ds. \tag{C.2}$$

The first one corresponds to the exponential generating function of the Hermite polynomials [8, Formula 8.957(1)], which is valid for $u, s \in \mathbb{C}$, whereas the second corresponds to the the Gaussian integral [8, Formulas 9.241(1) and 9.251], which is valid for $u \in \mathbb{C}$. Using the Rodrigues formula for the Hermite polynomials (cf. (2.2)) we obtain:

$$H_n(u) = \frac{(-2i)^n}{\sqrt{\pi}} \mathrm{e}^{u^2} \int s^n \, \mathrm{e}^{2ius-s^2} ds.$$

Hence, by the definition of ψ_n we have:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \psi_n(\rho)\, \mathrm{e}^{-in\alpha} \psi_n(u) \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-in\alpha} \psi_n(u) \frac{2^{1/4}}{\sqrt{2^n n!}} H_n\left(\sqrt{2\pi}\rho\right) \mathrm{e}^{-\pi\rho^2} \\
&= 2^{1/4}\, \mathrm{e}^{-\pi\rho^2} \sum_{n=0}^{\infty} \mathrm{e}^{-in\alpha} \psi_n(u) \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{2\pi}\rho\right) \\
&= 2^{1/4}\, \mathrm{e}^{-\pi\rho^2} \sum_{n=0}^{\infty} \mathrm{e}^{-in\alpha} \psi_n(u) \frac{1}{\sqrt{2^n n!}} \frac{(-2i)^n}{\sqrt{\pi}}\, \mathrm{e}^{2\pi\rho^2} \int s^n\, \mathrm{e}^{2i\sqrt{2\pi}\rho s - s^2} ds \\
&= 2^{1/4}\, \mathrm{e}^{-\pi\rho^2} \sum_{n=0}^{\infty} \mathrm{e}^{-in\alpha} \frac{2^{1/4}}{\sqrt{2^n n!}} H_n\left(\sqrt{2\pi}u\right) \mathrm{e}^{-\pi u^2} \frac{(-2i)^n}{\sqrt{\pi}\sqrt{2^n n!}}\, \mathrm{e}^{2\pi\rho^2} \int s^n\, \mathrm{e}^{2i\sqrt{2\pi}\rho s - s^2} ds \\
&= 2^{1/2}\, \mathrm{e}^{\pi\rho^2}\, \mathrm{e}^{-\pi u^2} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{-in\alpha} \frac{(-i)^n}{n!} H_n\left(\sqrt{2\pi}u\right) \int s^n\, \mathrm{e}^{2i\sqrt{2\pi}\rho s - s^2} ds \\
&= \mathrm{e}^{\pi\rho^2}\, \mathrm{e}^{-\pi u^2} \sqrt{\frac{2}{\pi}} \int \left(\sum_{n=0}^{\infty} \frac{\left(-is\, \mathrm{e}^{-i\alpha}\right)^n}{n!} H_n\left(\sqrt{2\pi}u\right) \right) \mathrm{e}^{2i\sqrt{2\pi}\rho s - s^2} ds.
\end{aligned}
$$

We must verify the summation under the integral sign in the last step. Define:

$$
S(n,s) := \frac{\left(-is\, \mathrm{e}^{-i\alpha}\right)^n}{n!} H_n\left(\sqrt{2\pi}u\right) \mathrm{e}^{2i\sqrt{2\pi}\rho s - s^2}.
$$

Note that:

$$
\begin{aligned}
|S(n,s)| &\leq \frac{|s|^n}{n!} \left| H_n\left(\sqrt{2\pi}u\right) \right| \mathrm{e}^{-s^2} \\
&\leq c_H\, \mathrm{e}^{-\pi u^2} \sqrt{\frac{2^n}{n!}} |s|^n\, \mathrm{e}^{-s^2} \\
&\leq c_H\, \mathrm{e}^{-\pi u^2} 2^n \frac{|s|^n}{n!}\, \mathrm{e}^{-s^2},
\end{aligned}
$$

where $c_H \approx 1.086435$ [8, Formula 8954(2)]. The bounding function is integrable over both n and s. Concretely, performing the sum over n first results in an exponential function. The subsequent integration over s is convergent because of the Gaussian function. Conversely, performing the integration over s first produces a term proportional to $2^{-n/2}(n-1)!!$, where !! denotes de double factorial, i.e. the absolute moment of a Gaussian distribution of mean zero and standard deviation $2^{-1/2}$. The resulting series is convergent by Hadamard's test. This justifies the exchange between the integral over s and the sum over n. Combining this with (C.1) we obtain:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \psi_n(\rho)\,\mathrm{e}^{-in\alpha}\psi_n(u) \\
&= \mathrm{e}^{\pi\rho^2}\,\mathrm{e}^{-\pi u^2}\sqrt{\frac{2}{\pi}}\int\left(\sum_{n=0}^{\infty}\frac{\left(-is\,\mathrm{e}^{-i\alpha}\right)^n}{n!}H_n(\sqrt{2\pi}u)\right)\mathrm{e}^{2i\sqrt{2\pi}\rho s-s^2}ds \\
&= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{2}{\pi}}\int \mathrm{e}^{2\sqrt{2\pi}u\left(-is\,\mathrm{e}^{-i\alpha}\right)-\left(-is\,\mathrm{e}^{-i\alpha}\right)^2}\,\mathrm{e}^{2i\sqrt{2\pi}\rho s-s^2}ds \\
&= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{2}{\pi}}\int \mathrm{e}^{-2i\sqrt{2\pi}us\,\mathrm{e}^{-i\alpha}+s^2\,\mathrm{e}^{-i2\alpha}}\,\mathrm{e}^{2i\sqrt{2\pi}\rho s-s^2}ds \\
&= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{2}{\pi}}\int \mathrm{e}^{-s^2\left(1-\mathrm{e}^{-i2\alpha}\right)}\,\mathrm{e}^{2i\sqrt{2\pi}s\left(\rho-u\,\mathrm{e}^{-i\alpha}\right)}ds.
\end{aligned}$$

Using the change of variables $\tau = s\sqrt{1-\mathrm{e}^{-i2\alpha}}$ on the last integral:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \psi_n(\rho)\,\mathrm{e}^{-in\alpha}\psi_n(u) \\
&= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{2}{\pi\left(1-\mathrm{e}^{-i2\alpha}\right)}}\int \mathrm{e}^{-\tau^2}\,\mathrm{e}^{2i\sqrt{2\pi}\frac{\left(\rho-u\,\mathrm{e}^{-i\alpha}\right)\tau}{\sqrt{1-\mathrm{e}^{-i2\alpha}}}}d\tau.
\end{aligned}$$

Comparing with (C.2) we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} \psi_n(\rho)\,\mathrm{e}^{-in\alpha}\psi_n(u) &= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{2}{1-\mathrm{e}^{-i2\alpha}}}\,\mathrm{e}^{-2\pi\frac{\left(\rho-u\,\mathrm{e}^{-i\alpha}\right)^2}{1-\mathrm{e}^{-i2\alpha}}} \\
&= \mathrm{e}^{\pi\left(\rho^2-u^2\right)}\sqrt{\frac{\mathrm{e}^{i\alpha}}{i\sin\alpha}}\,\mathrm{e}^{-\frac{\pi\,\mathrm{e}^{i\alpha}}{i\sin\alpha}\left(\rho-u\,\mathrm{e}^{-i\alpha}\right)^2} \\
&= \sqrt{1-i\cot\alpha}\,\mathrm{e}^{-2\pi i\left(-\frac{1}{2}\rho^2\cot\alpha+\rho u\csc\alpha-\frac{1}{2}u^2\cot\alpha\right)},
\end{aligned}$$

proving the lemma. □

Appendix D. Proof of Theorem 2.6.

To prove assertion (i), let $\mathcal{A}$ be the infinitesimal generator of $T(t)$. Then, by the definition of $\mathcal{A}$, $T(t)$ and Proposition 2.1 (iii) we have:

$$\mathcal{A}\psi_m = T'(t)|_{t=0}\,\psi_m = \sum_{n=0}^{\infty} -in\frac{\pi}{2}P_n\psi_m = \sum_{n=0}^{\infty}\mathcal{H}P_n\psi_m = \mathcal{H}\psi_m,$$

which shows that $\mathcal{A} = \mathcal{H}$. Since $\mathcal{H}^* = -\mathcal{H}$, i.e. $\mathcal{H}$ is skew–adjoint, assertion (ii) follows from [5, Chapter 1, §3.5]. The remaining assertions are consequence of the periodicity of $T(t)$ (see [5, Chapter IV, §2 (c)]). □

Appendix E. Proof of Theorem 2.7.

Since $\{T(t)\}_{t\in\mathbb{R}}$ constitutes a strongly continuous semigroup with $\|T(t)\| \le 1$, from [5, p.216] we have:

$$T(t)f = \lim_{q\to\infty}\mathrm{e}^{-tq}\,\mathrm{e}^{tqT(1/n)}f, \qquad t \ge 0,$$

and hence the result follows from the expansion in Taylor series of the exponential function, and the group property. □

Appendix F. Proof of Theorem 3.1.

Consider $\alpha(t)$ and $\rho(t)$ as defined by (3.5) and (3.6). If we use the branch of $\cot^{-1}$ such that $\alpha(t) \in [0, \pi]$, then $\csc\alpha(t) > 0$ for all values of t and consequently we can write:

$$\begin{aligned} -2p_2 t &= \cot\alpha(t), \\ -2\left(k(t) + p_1 t\right) &= -2\rho(t)\csc\alpha(t). \end{aligned}$$

By replacing these expressions on (3.2) we have

$$s(t) = \mathrm{e}^{-i2\pi p_0 t} \int_{-\infty}^{\infty} f(u)\, \mathrm{e}^{i\pi(u^2\cot\alpha - 2\rho u \csc\alpha)} du, \tag{F.1}$$

where we have dropped the dependence of the variable t for simplicity. Using (3.3) and (3.4) on (F.1) yields the relation:

$$s(t) = \mathrm{e}^{-i2\pi p_0 t} C_\alpha(\rho)^{-1} \mathcal{F}^\alpha(f)(\rho),$$

or:

$$\mathcal{F}^{\alpha(t)}(f)(\rho(t)) = \mathrm{e}^{i2\pi p_0 t} C_{\alpha(t)}(\rho(t))\, s(t),$$

proving the theorem. □

Appendix G. Proof of Theorem 4.1.

Let $a \in \mathbb{R}$ be given. For all $g \in L^2(\mathbb{R})$ we have by Proposition 2.1:

$$g = \sum_{n=0}^{\infty} P_n g = \sum_{n=0}^{\infty} \langle g, \psi_n \rangle\, \psi_n.$$

Setting $g = \mathcal{F}^a(f)$ we obtain:

$$\mathcal{F}^a(f) = \sum_{n=0}^{\infty} \langle \mathcal{F}^a(f), \psi_n \rangle\, \psi_n.$$

Using the fact that $\mathcal{F}^a \equiv T(a) \in \mathcal{B}(L^2(\mathbb{R}))$, we get:

$$f = \sum_{n=0}^{\infty} \langle \mathcal{F}^a(f), \psi_n \rangle\, \mathcal{F}^{-a}(\psi_n).$$

Since $P_n\psi_m = \psi_n$ if $n = m$ and 0 otherwise, we deduce that $\mathcal{F}^{-a}(\psi_n) = \mathrm{e}^{in\frac{\pi}{2}a}\psi_n$, and hence:

$$f = \sum_{n=0}^{\infty} \langle \mathcal{F}^a(f), \psi_n \rangle\, \mathrm{e}^{in\frac{\pi}{2}a}\psi_n. \tag{G.1}$$

□

Appendix H. Proof of Theorem 4.2.

Let $a \in \mathbb{R}$ be given. Using the semigroup notation, define $g := T(a)f \in L^2(\mathbb{R})$. From Theorem 2.7 and Definition 2.3 we have:

$$T(-a)g = \lim_{q\to\infty} \mathrm{e}^{aq} \sum_{n=0}^{\infty} \frac{(-aq)^n}{n!} T\left(\frac{n}{q}\right) g,$$

Remark that:

$$T\left(\frac{n}{q}\right) g = \sum_{m=0}^{\infty} \mathrm{e}^{-i\frac{n}{q}m} P_m g = \sum_{m=0}^{\infty} \mathrm{e}^{-i\frac{n}{q}m} \langle \psi_m, g \rangle \, \psi_m.$$

Hence:

$$\begin{aligned} f = T(-a)g &= \lim_{q\to\infty} \mathrm{e}^{aq} \sum_{n,m=0}^{\infty} \frac{(-aq)^n}{n!} \mathrm{e}^{-i\frac{n}{q}m} \langle \psi_m, g \rangle \; \psi_m \\ &= \lim_{q\to\infty} \mathrm{e}^{aq} \sum_{n,m=0}^{\infty} \frac{(-aq)^n}{n!} \mathrm{e}^{-i\frac{n}{q}m} \langle \psi_m, T(a)f \rangle \; \psi_m. \end{aligned}$$

Replacing $T(a)f = \mathcal{F}^a(f)$ we obtain the desired result. □

Appendix I. Proof of Lemma 5.1.

This approximation is an application of the central limit theorem to a Poisson distributed random variable. Let $\lambda \in \mathbb{R}_0^+$ and let $\{X_n\}_{n\in\mathbb{N}_0}$ be a sequence of independent and identically distributed Poisson random variables with parameter λ. Note that $\mathbb{E}\{X_n\} \equiv \lambda$ and $\mathbb{V}\{X_n\} \equiv \lambda$. Let $N \in \mathbb{N}_0$. Define:

$$S_N := \sum_{n=1}^{N} X_n.$$

The central limit theorem tells us that:

$$\frac{S_N - N\lambda}{\sqrt{N\lambda}} \to \mathcal{N}(0,1)$$

in distribution as $N \to \infty$. Thus, for N sufficiently large:

$$S_N \sim \mathcal{N}(N\lambda, \sqrt{N\lambda}).$$

But the sum of independent Poisson random variables distributes as a Poisson random variable. In other words, $S_N \sim \text{Poisson}\,(N\lambda)$. Setting $q = N\lambda$ concludes the proof. Note that the Poisson distribution has finite third moment. Consequently, by the Barry–Esseen theorem [6, §XVI.5], the convergence ratio is at least of the order of $N^{-1/2}$. □

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