PERIODIC SOLUTIONS OF ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

HERNÁN R. HENRÍQUEZ AND CARLOS LIZAMA

Abstract. We characterize the existence of periodic solutions for a class of abstract retarded functional differential equation with infinite delay. We apply our results to the nonlinear equation

$$x'(t) = Ax(t) + L(x_t) + N(x_t) + f(t), \quad t \in \mathbb{R},$$

where $A : D(A) \subset X \to X$ is a closed operator defined on a Banach space X, L is a bounded linear map, $N : \mathcal{B} \to X$ is a continuous function defined on an appropriate phase space $\mathcal{B}$ and $f \in L^p(\mathbb{T}, X)$.

1. Introduction

Motivated by the fact that abstract retarded functional differential equations (abbreviated, ARFDE) with infinite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic, almost periodic and asymptotically almost periodic solutions has been considered by several authors. We refer the reader to the books [6, 7], and to the papers [11, 12, 13, 3, 15, 4, 10, 2] and the references listed therein for information on this subject.

In this work we are concerned with the existence of periodic solutions for a class of linear and semi-linear abstract retarded functional differential equations with infinite delay.

Let X be a Banach space endowed with a norm $\|\cdot\|$. In this paper we study the existence of periodic solutions for the class of abstract functional differential equations described in the form

$$x'(t) = Ax(t) + L(x_t) + f(t), \; t \in \mathbb{R}. \tag{1.1}$$

In this description $x(t) \in X$ and the history $x_t : (-\infty, 0] \to X$, given by $x_t(\theta) = x(t+\theta)$ for $\theta \leq 0$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. We will assume that $L : \mathcal{B} \to X$ is a bounded linear map and $A : D(A) \subseteq X \to X$ is a closed linear operator. Moreover, $f : \mathbb{R} \to X$ is a locally p-integrable and 2π-periodic function for $1 \leq p < \infty$.

As usual, we represent by $[D(A)]$ the Banach space $D(A)$ endowed with the graph norm

$$\|x\|_{[D(A)]} = \|x\| + \|Ax\|.$$

A similar problem for equations with finite delay has been considered in [8]. In this note we will show that we can extend the Theorem 3.4 in [8] to include the infinite delay case.

Next we recall the basic concepts necessary to obtain our results. Let Y, Z be Banach spaces. In what follows, we denote by $\mathcal{L}(Y, Z)$ the Banach space of bounded linear operators from Y into Z, and we abbreviate this notation to $\mathcal{L}(Y)$ in the case $Y = Z$. We begin with the concept of $\mathcal{R}$-boundedness (see [1, Definition 3.1]).

Key words and phrases. Periodic solutions; abstract functional differential equations; UMD spaces.
The first author was supported in part by CONICYT under Grant FONDECYT 1090009.
The second author is partially financed by FONDECYT Grant number 1100485.
2010 *Mathematics subject classification.* Primary: 34K13; Secondary: 34K30; 35R10.

Definition 1.1. *A family of operators $\mathcal{T} = \{T_i : i \in I\} \subseteq \mathcal{L}(Y, Z)$ is said to be $\mathcal{R}$-bounded if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each finite set $J \subseteq I$, $T_i \in \mathcal{T}$, $y_i \in Y$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality*

$$\| \sum_{i \in J} \varepsilon_i T_i y_i \|_{L^p(\Omega, Z)} \leq C \| \sum_{i \in J} \varepsilon_i y_i \|_{L^p(\Omega, Y)}$$

is verified. The smallest of the constant C is called $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

To complete these concepts we define the UMD spaces. But, since we just will use some results from the literature, it is enough for us to present a simple definition of UMD space. A Banach space Z is said to be UMD if the Hilbert transform is bounded on $L^p(\mathbb{R}, Z)$ for some (and then for all) $1 < p < \infty$.

Next we denote $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. Specifically, in what follows for $1 \leq p < \infty$ we denote by $L^p(\mathbb{T}, Y)$ the space of 2π-periodic p-integrable functions from $\mathbb{R}$ into Y. Similarly, the notation $W^{1,p}(\mathbb{T}, Y)$ stands for the Sobolev space of 2π-periodic functions $f : \mathbb{R} \to Y$ such that $f' \in L^p(\mathbb{T}, Y)$. Moreover, q will be used to denote the conjugate exponent of p.

2. Existence of periodic solutions of ARFDE with finite delay.

In this section we study the existence of periodic solutions of a linear ARFDE with finite delay 2π. Specifically, we will be concerned with the equation

$$x'(t) = Ax(t) + F(x_t) + f(t), \; t \in \mathbb{R}, \tag{2.1}$$

where the function $x_t : [-2\pi, 0] \to X$ is given by $x_t(\theta) = x(t+\theta)$ for $-2\pi \leq \theta \leq 0$, $A : D(A) \subseteq X \to X$ is a closed linear operator and $f : \mathbb{R} \to X$ is a locally p-integrable and 2π-periodic function, for $1 < p < \infty$.

The existence of periodic solutions for this equation when $F : L^p([-2\pi, 0], X) \to X$ is a bounded linear map was studied in [8]. In this paper we use the following concept of solution.

Definition 2.1. *Let $1 \leq p < \infty$ and let $f : \mathbb{R} \to X$ be a locally p-integrable function. We say that $x : \mathbb{R} \to X$ is a strong L^p-solution of equation (2.1) if $x(\cdot) \in C(\mathbb{R}, [D(A)]) \cap W^{1,p}_{loc}(\mathbb{R}, X)$ and (2.1) holds a.e. for $t \in \mathbb{R}$.*

The equation (2.1) is usually studied on the space $C([-2\pi, 0], X)$. For this reason, we assume that F is a bounded linear map from $C([-2\pi, 0], X)$ into X that can be extended to a bounded linear map $\widehat{F} : L^p([-2\pi, 0], X) \to X$. Hence, we can consider the equation

$$x'(t) = Ax(t) + \widehat{F}(x_t) + f(t), \; t \in \mathbb{R}, \tag{2.2}$$

It is clear that a 2π-periodic solution of (2.2) is also a 2π-periodic solution of (2.1). Moreover, if $e_k(\theta) = e^{ik\theta}$ for $-2\pi \leq \theta \leq 0$ and $k \in \mathbb{Z}$, then

$$B_k x = \widehat{F}(e_k x) = F(e_k x).$$

We denote $\sigma_{\mathbb{Z}}(\Delta) = \{k \in \mathbb{Z} : ikI - A - B_k \text{ has no inverse}\}$. We have the following result.

Theorem 2.1. *Let X be a UMD space and $1 < p < \infty$. Assume that F can be extended as a bounded linear map on $L^p([-2\pi, 0], X)$. Then the following conditions are equivalent:*

(a) For every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution of (2.1).

(b) $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - B_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded.

Proof. Assume that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution of (2.1), say $x(\cdot)$. Then, by the assumption on F, we have that $x(\cdot)$ is also the unique locally p-integrable and 2π-periodic strong solution of (2.2). Hence, (b) follows from [8, Theorem 3.4]. Conversely, assume that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - B_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded. From [8, Theorem 3.4] we obtain that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution $x(\cdot)$ of (2.2). Since $\hat{F}$ is an extension of F, we conclude that $x(\cdot)$ is the unique solution of (2.1), proving (a). □

3. Existence of periodic solutions of ARFDE with infinite delay.

In this section we apply the Theorem 2.1 to study the existence of periodic solutions of equation (1.1) with infinite delay. To get our objective we need an appropriate phase space.

We use an axiomatic definition of the phase space $\mathcal{B}$, which is similar to the one used in [5]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$.
  (ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.
  (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,
  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Usually the definition of phase space includes the axiom

(A1) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

However because we are studying the existence of strong solutions for the equation (1.1) this axiom is not relevant and we can replace it by a weaker one.

Example 1. *The phase space $C_g(X)$.*

Let g be a positive continuous function on $(-\infty, 0]$. Let $\mathcal{B} = C_g(X)$ be the space consisting of all continuous functions $\varphi : (-\infty, 0] \to X$ such that $\dfrac{\|\varphi(\theta)\|}{g(\theta)}$ is bounded on $(-\infty, 0]$. We assume that g satisfies conditions (g-1) and (g-2) in the terminology of [5]. This means that

(g-1) The function $G(t) = \sup_{\theta \leq -t} \dfrac{g(t + \theta)}{g(\theta)}$ is locally bounded for $t \geq 0$.

(g-2) $g(\theta) \to \infty$ as $\theta \to -\infty$.

With the norm in $\mathcal{B}$ defined by

$$\|\varphi\|_g = \sup_{\theta \leq 0} \frac{\|\varphi(\theta)\|}{g(\theta)}$$

the space $\mathcal{B}$ is a phase space that satisfies (A1) ([5, Theorem 1.3.2]).

Example 2. *The phase space $C_r \times L^p(\rho, X)$*

Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [5]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho \parallel \varphi \parallel^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\parallel \varphi \parallel_{\mathcal{B}} = \sup\{\parallel \varphi(\theta) \parallel : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta) \parallel \varphi(\theta) \parallel^p \, d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho, X)$ satisfies axioms (A), (B) and (A1). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta) \, d\theta \right)^{1/2}$, for $t \geq 0$. See [5, Theorem 1.3.8] for details.

To establish our results, we will need additional properties of the phase space $\mathcal{B}$. In this work we discuss two possibilities.

We next denote by $C_{00}(X)$ the space of continuous functions from $(-\infty, 0]$ to X with compact support. We consider the following axiom for the phase space $\mathcal{B}$ ([5]).

(C2) If a uniformly bounded sequence $(\varphi^n)_n$ in $C_{00}(X)$ converges to a function φ in the compact-open topology, then φ belongs to $\mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

It is easy to see ([5]) that if the axiom (C2) holds, then the space of continuous and bounded functions $C_b((-\infty, 0], X)$ is continuously included in $\mathcal{B}$. Thus, there exists a constant $Q > 0$ such that

$$\|\varphi\|_{\mathcal{B}} \leq Q \|\varphi\|_\infty, \quad \varphi \in C_b((-\infty, 0], X).$$

Furthermore, in this case ([5], Proposition 7.1.5) the function $K(\cdot)$ involved in axiom (A) can be chosen as the constant Q.

As an example, we mention that if the function g satisfies the conditions (g-1) and (g-2), then the space $C_g(X)$ defined in Example 1 satisfies the axiom (C2) ([5, Theorem 1.3.2]). In a similar way, if $\int_{-\infty}^{-r} \rho(\theta) d\theta < \infty$, then the space $C_r \times L^p(\rho, X)$ also satisfies the axiom (C2).

Assume now that $\mathcal{B}$ satisfies the axiom (C2). Let $L : \mathcal{B} \to X$ be a bounded linear map. Let

$$C_{2\pi}([-2\pi, 0], X) = \{\varphi \in C([-2\pi, 0], X) : \varphi(0) = \varphi(-2\pi)\}.$$

For $\varphi \in C_{2\pi}([-2\pi, 0], X)$ we write $\tilde{\varphi}$ for the 2π-periodic extension of φ to $(-\infty, 0]$. Since $\tilde{\varphi} \in \mathcal{B}$, we can define $F : C_{2\pi}([-2\pi, 0], X) \to X$ by

$$F(\varphi) = L(\tilde{\varphi}). \tag{3.1}$$

We consider the following condition.

(H1) There is a bounded linear map $\widehat{F} : L^p([-2\pi, 0], X) \to X$ that is an extension of F.

We can establish the following result of existence.

Theorem 3.1. *Let X be a UMD space and $1 < p < \infty$. Assume that the phase space $\mathcal{B}$ satisfies axioms (A1) and (C2) and that the condition (H1) holds. Then the following conditions are equivalent:*

(a) For every function $f \in L^p(\mathbb{T}, X)$ there exists a unique 2π-periodic L^p-strong solution of (1.1).

(b) $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - L_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded.

Proof. Assume that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique 2π-periodic L^p-strong solution $x(\cdot)$ of (1.1). Since $\mathcal{B}$ satisfies the axiom (C2) we can define $F : C_{2\pi}([-2\pi, 0], X) \to X$ according to (3.1). It is clear that F is a bounded linear map. By hypothesis (H1), there is a bounded linear map $\widehat{F} : L^p([-2\pi, 0], X) \to X$ that is an extension of F. We consider now the following functional differential equation with finite delay 2π,

$$y'(t) = Ay(t) + \widehat{F}(y_t) + f(t), \; t \in \mathbb{R}, \tag{3.2}$$

where for this equation y_t denotes the function $y_t(\theta) = y(t+\theta)$ for $-2\pi \le \theta \le 0$.

It is immediate that $x(\cdot)$ is a 2π-periodic solution of (1.1) if and only if it is a 2π-periodic solution of (3.2). Proceeding as in the Section 2, for $k \in \mathbb{Z}$, we define $e_k(\theta) = e^{ik\theta}$ for $\theta \le 0$, and $L_k x = L(e_k x)$. It is clear that $L_k x = B_k x$. Hence, applying Theorem 2.1 to the equation (3.2) we obtain (b). Conversely, assume (b). By Theorem 2.1, for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution $y(\cdot)$ of (3.2) and, hence, it is also a solution of (1.1). This proves (a). $\square$

Example 3. *In the space $\mathcal{B} = C_g(X)$, where g satisfies the conditions (g-1) and (g-2), let $L : \mathcal{B} \to X$ given by*

$$L(\varphi) = \int_{-\infty}^{0} a(\theta)\varphi(\theta)d\theta,$$

where $a : (-\infty, 0] \to \mathbb{C}$ is a continuous function such that $\displaystyle\int_{-\infty}^{0} |a(\theta)|g(\theta)d\theta < \infty$. Then L is a bounded linear map. Moreover, if $\varphi \in C_{2\pi}([-2\pi, 0], X)$, then

$$\begin{aligned} F(\varphi) &= \int_{-\infty}^{0} a(\theta)\tilde{\varphi}(\theta)d\theta \\ &= \sum_{k=0}^{\infty} \int_{-2(k+1)\pi}^{-2k\pi} a(\theta)\tilde{\varphi}(\theta)d\theta \\ &= \sum_{k=0}^{\infty} \int_{-2\pi}^{0} a(\xi - 2k\pi)\tilde{\varphi}(\xi - 2k\pi)d\xi \\ &= \sum_{k=0}^{\infty} \int_{-2\pi}^{0} a(\xi - 2k\pi)\varphi(\xi)d\xi. \end{aligned}$$

Let $1 < p < \infty$ and assume that

$$\sum_{k=0}^{\infty} \Big(\int_{-2\pi}^{0} |a(\xi - 2k\pi)|^q d\xi \Big)^{1/q} < \infty,$$

then the function $b(\xi) = \sum_{k=0}^{\infty} a(\xi - 2k\pi) \in L^q([-2\pi, 0])$ ([9, Proposition 4.2.4]*). Consequently,*

$$F(\varphi) = \int_{-2\pi}^{0} b(\xi)\varphi(\xi)d\xi, \;\; \varphi \in C_{2\pi}([-2\pi, 0], X).$$

Since $b \in L^q([-2\pi, 0])$, the map F admits a bounded linear extension $\widehat{F} : L^p([-2\pi, 0], X) \to X$ given by

$$\widehat{F}(\varphi) = \int_{-2\pi}^{0} b(\xi)\varphi(\xi)d\xi, \;\; \varphi \in L^p([-2\pi, 0], X).$$

In this case, the operator B_k is given by

$$B_k x = \int_{-2\pi}^{0} b(\theta)e^{ik\theta}x d\theta.$$

Example 4. *In the space $\mathcal{B} = C_0 \times L^p(\rho, X)$, $1 < p < \infty$, assume that $\rho(\cdot)$ is a bounded function such that $\int_{-\infty}^0 \rho(\theta)d\theta < \infty$. For instance, the function $\rho(\theta) = e^{\gamma\theta}$ for $\gamma > 0$ satisfies these conditions. Let $L : \mathcal{B} \to X$ given by*

$$L(\varphi) = \int_{-\infty}^0 a(\theta)\varphi(\theta)d\theta,$$

where $a : (-\infty, 0] \to \mathbb{C}$ is a continuous function such that $\sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} \frac{|a(\theta)|^q}{\rho(\theta)^{q/p}} d\theta \Big)^{1/q} < \infty$. Since

$$\int_{-2(k+1)\pi}^{-2k\pi} \frac{|a(\theta)|^q}{\rho(\theta)^{q/p}} d\theta \le \Big(\int_{-2(k+1)\pi}^{-2k\pi} \frac{|a(\theta)|^q}{\rho(\theta)^{q/p}} d\theta \Big)^{1/q}$$

for k large enough, then

$$\int_{-\infty}^0 \frac{|a(\theta)|^q}{\rho(\theta)^{q/p}} d\theta < \infty$$

which implies that L is a bounded linear map. Moreover,

$$\begin{aligned} \sum_{k=0}^{\infty} \Big(\int_{-2\pi}^0 |a(\xi - 2k\pi)|^q d\xi \Big)^{1/q} &\le \sup_{\theta \le 0} \rho(\theta)^{1/p} \sum_{k=0}^{\infty} \Big(\int_{-2\pi}^0 \frac{|a(\xi - 2k\pi)|^q}{\rho(\xi - 2k\pi)^{q/p}} d\xi \Big)^{1/q} \\ &\le \sup_{\theta \le 0} \rho(\theta)^{1/p} \sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} \frac{|a(\theta)|^q}{\rho(\theta)^{q/p}} d\theta \Big)^{1/q} \\ &< \infty, \end{aligned}$$

which allows us to repeat our assertions in the Example 3.

To avoid the assumption (H1), first we introduce the appropriate space of Stepanov bounded functions.

Definition 3.1. *Let $1 \le p < \infty$. We denote by $BS^p((-\infty, 0], X)$ the space consisting of all measurable functions $\varphi : (-\infty, 0] \to X$ such that $\sup_{\theta \le 0} \int_{\theta - 2\pi}^{\theta} \|\varphi(\xi)\|^p d\xi < \infty$. The space $BS^p((-\infty, 0], X)$ endowed with the norm*

$$\|\varphi\|_{S^p} = \sup_{\theta \le 0} \Big(\int_{\theta - 2\pi}^{\theta} \|\varphi(\xi)\|^p d\xi \Big)^{1/p}$$

is a Banach space.

We denote $X \times BS^p((-\infty, 0], X)$ the space of functions $\varphi : (-\infty, 0] \to X$ included in $BS^p((-\infty, 0], X)$ endowed with the product norm

$$\|\varphi\| = \|\varphi(0)\| + \|\varphi\|_{S^p}.$$

We consider the following axiom for the phase space $\mathcal{B}$.

(S-p) The space $X \times BS^p((-\infty, 0], X)$ is continuously included in $\mathcal{B}$.

The space $\mathcal{B} = X \times BS^p((-\infty, 0], X)$ satisfies axioms (A), (B) and (S-p). A more general situation is exhibited in the following example.

Example 5. *Let $\mathcal{B} = C_0 \times L^1(\rho, X)$ the space constructed in Example 2. Assume further that $1 < p < \infty$, q is the conjugate exponent of p and $\sum_{k=-\infty}^{0} \left(\int_{2\pi(k-1)}^{2\pi k} \rho(\theta)^q d\theta \right)^{1/q} < \infty$. Then $X \times BS^p((-\infty, 0], X)$*

is continuously included in $\mathcal{B}$. In fact, if $\varphi \in X \times BS^p((-\infty,0],X)$, then

$$
\begin{aligned}
\|\varphi\|_{\mathcal{B}} &= \|\varphi(0)\| + \int_{-\infty}^{0} \rho(\theta)\|\varphi(\theta)\| d\theta \\
&= \|\varphi(0)\| + \sum_{k=-\infty}^{0} \int_{2\pi(k-1)}^{2\pi k} \rho(\theta)\|\varphi(\theta)\| d\theta \\
&\le \|\varphi(0)\| + \sum_{k=-\infty}^{0} \left(\int_{2\pi(k-1)}^{2\pi k} \rho(\theta)^q d\theta \right)^{1/q} \left(\int_{2\pi(k-1)}^{2\pi k} \|\varphi(\theta)\|^p d\theta \right)^{1/p} \\
&\le \|\varphi(0)\| + \|\varphi\|_{S^p} \sum_{k=-\infty}^{0} \left(\int_{2\pi(k-1)}^{2\pi k} \rho(\theta)^q d\theta \right)^{1/q} \\
&\le C\|\varphi\|_{X\times BS^p((-\infty,0],X)},
\end{aligned}
$$

where $C = \max\left\{ 1, \sum_{k=-\infty}^{0} \left(\int_{2\pi(k-1)}^{2\pi k} \rho(\theta)^q d\theta \right)^{1/q} \right\}$.

It is worth to point out that, in general, $\mathcal{B}$ does not satisfy the axiom (A1). Actually, even in the scalar case $X = \mathbb{R}$, we can take

$$
\varphi(\theta) = \begin{cases} k, & -2\pi k \le \theta \le -2\pi k + \frac{1}{k} \\ 0, & \text{otherwise } \theta \le 0. \end{cases}
$$

Let $x(t) = \varphi(t)$ for $t \le 0$ and $x(t) = 0$ for $t \ge 0$. For $0 < t < 2\pi$ and $k \in \mathbb{N}$, $\frac{1}{k} < t$, we have that

$$
\begin{aligned}
\|x_t - \varphi\|_{S^p}^p &\ge \int_{-2\pi k - t}^{-2\pi k + \frac{1}{k}} \|x(t+\theta) - \varphi(\theta)\|^p d\theta \\
&= \int_{-2\pi k - t}^{-2\pi k - t + \frac{1}{k}} k^p d\theta + \int_{-2\pi k}^{-2\pi k + \frac{1}{k}} k^p d\theta \\
&= 2k^{p-1},
\end{aligned}
$$

which shows that the function $t \mapsto x_t$ is not continuous at $t = 0$.

Next we assume that $\mathcal{B}$ satisfies the axiom (S-p). Let $L : \mathcal{B} \to X$ be a bounded linear map. We consider the following condition

(H2) Let $x : \mathbb{R} \to X$ be a continuous function that is p-Stepanov bounded on $(-\infty, 0]$. Then the function $\mathbb{R} \to X$, $t \mapsto L(x_t)$, is continuous.

For $\varphi \in L^p([-2\pi, 0], X)$ we write $\tilde{\varphi}$ for the 2π-periodic extension of φ to $(-\infty, 0]$. Since $\tilde{\varphi} \in BS^p((-\infty,0],X) \subset \mathcal{B}$, we can define $F : L^p([-2\pi,0],X) \to X$ by

$$
F(\varphi) = L(\tilde{\varphi}). \tag{3.3}
$$

Proceeding as in the previous paragraph, applying the Theorem 2.1 to the equation (3.4), we can establish the following result.

Theorem 3.2. *Let X be a UMD space and $1 < p < \infty$. Assume that $\mathcal{B}$ satisfies the axiom (S-p) and that the condition (H2) holds. Then the following conditions are equivalent :*

(a) For every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution of (1.1).

(b) $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\{ik(ikI - A - L_k)^{-1} : k \in \mathbb{Z}\}$ is R-bounded.

Proof. Suppose that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution $x(\cdot)$ of (1.1). Since $\mathcal{B}$ satisfies the axiom (S-p) we can define F from the Lebesgue vector-valued space $L^p([-2\pi, 0], X)$ to X by the relation (3.3). It is clear that F is a bounded linear map. We can introduce now the functional differential equation with delay 2π,

$$y'(t) = Ay(t) + F(y_t) + f(t),\ t \in \mathbb{R}. \tag{3.4}$$

Observe that $x(\cdot)$ is a 2π-periodic solution of (1.1) if and only if it is a 2π-periodic solution of (3.4). For $k \in \mathbb{Z}$, we define $e_k(\theta) = e^{ik\theta}$ for $\theta \leq 0$ and $L_k x = L(e_k x)$. Note that $L_k x = B_k x$. Hence, applying Theorem 2.1 to the equation (3.4) we deduce (b). Conversely, assuming (b), we have by Theorem 2.1 that for every function $f \in L^p(\mathbb{T}, X)$ there exists a unique locally p-integrable and 2π-periodic strong solution $y(\cdot)$ of (3.4). But then $y(\cdot)$ is also a solution of (1.1), proving (a).

□

Example 6. *Let $\mathcal{B} = X \times BS^p((-\infty, 0], X)$, $1 < p < \infty$. Let $L : \mathcal{B} \to X$ be given by*

$$L(\varphi) = \int_{-\infty}^{0} a(\theta)\varphi(\theta)d\theta,$$

where $a : (-\infty, 0] \to \mathbb{C}$ is a continuous function such that $\sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta)|^q d\theta\Big)^{1/q} < \infty$ and

$$\sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta - |h|) - a(\theta)|^q d\theta\Big)^{1/q} \to 0, \quad h \to 0.$$

Since

$$\begin{aligned}
\|L(\varphi)\| &\leq \sum_{k=0}^{\infty} \int_{-2(k+1)\pi}^{-2k\pi} |a(\theta)| \|\varphi(\theta)\| d\theta \\
&\leq \sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta)|^q d\theta\Big)^{1/q} \Big(\int_{-2(k+1)\pi}^{-2k\pi} \|\varphi(\theta)\|^p d\theta\Big)^{1/p} \\
&\leq \sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta)|^q d\theta\Big)^{1/q} \|\varphi\|_{S^p}
\end{aligned}$$

which implies that L is a bounded linear map. Moreover, let $x : \mathbb{R} \to X$ be a continuous function that is p-Stepanov bounded on $(-\infty, 0]$. Then, for $h \geq 0$, we have

$$\begin{aligned}
L(x_{t-h} - x_t) &= \int_{-\infty}^{0} a(\theta)(x(t - h + \theta) - x(t + \theta))d\theta \\
&= \int_{-\infty}^{0} [a(\theta - h) - a(\theta)]x(t + \theta)d\theta + \int_{0}^{h} a(s - h)x(t + s)ds
\end{aligned}$$

which implies that

$$\begin{aligned}
\|L(x_{t-h} - x_t)\| &\leq \sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta - h) - a(\theta)|^q d\theta\Big)^{1/q} \Big(\int_{-2(k+1)\pi}^{-2k\pi} \|x(t + \theta)\|^p d\theta\Big)^{1/p} \\
&\quad + \|\int_{0}^{h} a(s - h)x(t + s)ds\| \\
&\leq \sum_{k=0}^{\infty} \Big(\int_{-2(k+1)\pi}^{-2k\pi} |a(\theta - h) - a(\theta)|^q d\theta\Big)^{1/q} \|x_t\|_{S^p} + \|\int_{0}^{h} a(s - h)x(t + s)ds\|
\end{aligned}$$

converges to zero as $h \to 0$. A similar estimates holds for $h \le 0$. Consequently, L also verifies the condition (H2). In this case, we can also represent B_k as

$$B_k x = \int_{-2\pi}^{0} b(\theta) e^{ik\theta} x d\theta$$

for an appropriate function $b(\cdot)$.

As an application of our results, in what follows we consider a nonlinear problem. We assume that the assumptions of Theorem 3.1 are fulfilled, so that the linear equation (1.1) has a unique 2π-periodic strong L^p-solution for all $f \in L^p(\mathbb{T}, X)$. We introduce the space $E = C(\mathbb{T}, [D(A)]) \cap W^{1,p}(\mathbb{T}, X)$ endowed with the norm

$$\|u\|_E = \|u\|_\infty + \|Au\|_\infty + \|u'\|_p,$$

where $\|\cdot\|_\infty$ denotes the norm of uniform convergence and $\|\cdot\|_p$ is the norm in $L^p(\mathbb{T}, X)$. It is clear that E is a Banach space.

Let us denote by $P : L^p(\mathbb{T}, X) \to E$ to the map defined by $P(f) = u$, where u is the 2π-periodic strong L^p-solution of (1.1). It is clear that P is linear. Moreover, if the sequence $(u_n)_n \in E$ converges to u for the norm in E, then $u_{n,t} \to u_t$ as $n \to \infty$ for the norm in $\mathcal{B}$ and for all $t \in \mathbb{R}$. This implies that P is closed. Hence, by the closed graph theorem, P is continuous.

We next consider the nonlinear equation

$$x'(t) = Ax(t) + L(x_t) + N(x_t) + f(t),\ t \in \mathbb{R}, \tag{3.5}$$

where $N : \mathcal{B} \to X$ is a continuous function and $f \in L^p(\mathbb{T}, X)$. We can establish the following immediate consequence.

Corollary 3.1. *Assume that the hypotheses of Theorem 3.1 are fulfilled. Suppose in addition that N satisfies the Lipschitz condition*

$$\|N(\varphi) - N(\psi)\| \le k\|\varphi - \psi\|_{\mathcal{B}}, \quad \varphi, \psi \in \mathcal{B},$$

for a positive constant k such that $\|P\|k(2\pi)^{1/p}Q < 1$. Then the equation (3.5) has a unique 2π-periodic strong L^p-solution.

Proof. We define the map $\Gamma : E \to E$ by

$$\Gamma z(t) = P(f)(t) + P(\widetilde{N}(z))(t),$$

where $\widetilde{N}(z)$ is the function given by

$$\widetilde{N}(z)(t) = N(z_t),\ t \in \mathbb{R}.$$

It is easy to see that $\widetilde{N} : E \to L^p(\mathbb{T}, X)$, which implies that Γ is well defined. Furthermore,

$$\begin{aligned}
\|\Gamma z - \Gamma y\|_E &= \|P(\widetilde{N}(z)) - P(\widetilde{N}(y))\|_E \\
&\le \|P\|\|\widetilde{N}(z) - \widetilde{N}(y)\|_{L^p(\mathbb{T},X)} \\
&= \|P\| \left(\int_0^{2\pi} \|\widetilde{N}(z)(t) - \widetilde{N}(y)(t)\|^p dt \right)^{1/p} \\
&\le \|P\| \left(\int_0^{2\pi} \|N(z_t) - N(y_t)\|^p dt \right)^{1/p} \\
&\le \|P\|k \left(\int_0^{2\pi} \|z_t - y_t\|_{\mathcal{B}}^p dt \right)^{1/p} \\
&\le \|P\|k(2\pi)^{1/p}Q\|z - y\|_E
\end{aligned}$$

Hence Γ is a contraction. The fixed point x of Γ is the 2π-periodic strong L^p-solution of the equation (3.5). $\square$

References

[1] Denk, R., Hieber, M., Prüss, J., $\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type, Memoirs of the American Mathematical Society, Volume 166, Providence, 2003.

[2] Ezzinbi, K., Fatajou, S., N'Guérékata, G. M., Pseudo almost automorphic solutions for some partial functional differential equations with infinite delay, Applicable Analysis 87 (5) (2008), 591-605.

[3] Henríquez, H. R., Vásquez, C. H., Almost periodic solutions of abstract retarded functional differential equations with unbounded delay, Acta Appl. Math. 57 (1999), 105-132.

[4] Henríquez, H. R., Asymptotically almost periodic solutions of abstract retarded functional differential equations of first order, Nonlinear Analysis: Real World Applications 10 (4) (2009), 2441-2454.

[5] Hino, Y; Murakami, S., Naito, T., *Functional Differential Equations with Infinite Delay*, Lect. Notes in Maths. Vol. 1473. Springer-Verlag, Berlin, 1991.

[6] Hino, Y., Naito, T., Van Minh, N., Shin, J. Son, *Almost Periodic Solutions of Differential Equations in Banach Spaces.* Taylor and Francis, London, 2002.

[7] Liu, J., N'Guérékata, G. M., Van Minh, N., *Topics on Stability and Periodicity in Abstract Differential Equations*, Series on Concrete and Applicable Mathematics, Vol. 6, World Scientific Publ. Co., Singapore, 2008.

[8] Lizama, C., Fourier multipliers and periodic solutions of delay equations in Banach spaces J. Math. Anal. Appl. 324 (2006), 921-933.

[9] Marle, C-M., *Mesures et Probabilités*, Hermann, Paris, 1974.

[10] Meng, X., Chen, L. Periodic solution and almost periodic solution for a nonautonomous Lotka - Volterra dispersal system with infinite delay, J. Math. Anal. Appl. 339 (2008), 125-145.

[11] Ruess, W. M., Compactness and asymptotic stability for solutions of functional differential equations with infinite delay. In *Evolution Equations.* (G. Ferreyra, G. Ruiz Goldstein and F. Neubrander Eds.) Lect. Notes Pure and Appl. Math. 168, pp. 361-374 (1995). Marcel Dekker.

[12] Ruess, W. M., Existence of solutions to partial functional differential equations with delay. In *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type.* (A. G. Kartsatos Ed.) Lect. Notes Pure and Appl. Math. 178, pp. 259-288 (1996). Marcel Dekker.

[13] Ruess, W. M., Summers, W. H., Almost periodicity and stability for solutions to functional differential equations with infinite delay, Diff. and Integral Equations 9 (1996), 1225-1252.

[14] Saperstone, S. H., *Semidynamical Systems in Infinite Dimensional Spaces*, Springer-Verlag, New York, 1981.

[15] Zhang, L., Xu, Y., Existence of pseudo almost periodic solutions of functional differential equations with infinite delay, Applicable Analysis 88 (12) (2009), 1713-1726.

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile.

E-mail address: `hernan.henriquez@usach.cl`

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile.

E-mail address: `carlos.lizama@usach.cl`