COMPACT ALMOST AUTOMORPHIC SOLUTIONS TO INTEGRAL EQUATIONS WITH INFINITE DELAY

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Abstract. Given $a \in L^1(\mathbb{R})$ and the generator A of an L^1-integrable resolvent family of linear bounded operators defined on a Banach space X, we prove the existence of compact almost automorphic solutions of the semilinear integral equation $u(t) = \int_{-\infty}^t a(t-s)[Au(s) + f(s, u(s))]ds$ for each $f : \mathbb{R} \times X \to X$ compact almost automorphic in t, for each $x \in X$, and satisfying Lipschitz and Hölder type conditions. In the scalar linear case, we prove that $a \in L^1(\mathbb{R})$ positive, nonincreasing and log-convex is sufficient to obtain existence of compact almost automorphic solutions.

1. Introduction

We study in this paper the existence and regularity of compact almost automorphic solutions for linear and semilinear integral equations with infinite delay of the form

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))]ds, \quad t \in \mathbb{R}, \tag{1.1}$$

where $a \in L^1(\mathbb{R}_+)$, $A : D(A) \subseteq X \to X$ is the generator of an integral resolvent family defined on a complex Banach space X, and $f : \mathbb{R} \times X \to X$ is a compact almost automorphic function in the first variable and satisfying suitable Lipschitz or Hölder type conditions in the second variable.

Equations of type (1.1) arise in the study of heat flow in materials of fading memory type (see [4] and [23]). In contrast with other papers dealing with this subject, we do not assume that the history of the system is known from $-\infty$ to some instant t_0. Instead we study solutions of (1.1) existing for all $t \in \mathbb{R}$ and subject to some global condition.

In the linear case, and under the assumptions that $a \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is completely positive and the first moment of a exists, it was proved in [4] that problem (1.1) is equivalent to

$$u(t) + \frac{d}{dt}(\alpha u(t) + \int_{-\infty}^{t} k(t-s)u(s)ds) = (\int_0^\infty a(\tau)d\tau)(Au(t) + f(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

for some $\alpha > 0$ and $k \in L^1(\mathbb{R}_+)$ nonnegative and nonincreasing. This linear integrodifferential equation was studied in [5] obtaining existence and regularity of solutions when

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A generates a contraction semigroup on X. Under the stronger assumption that A generates an analytic semigroup, it was studied in [4] where regularity of solutions in spaces $L^p(X), BUC(X), W^{\alpha,p}(X)$, and the space of α-Hölder continuous and bounded functions $BC^\alpha(X)$, for $0 < \alpha < 1$ and $1 \leq p < \infty$, was established. These maximal regularity results were then applied to study existence and regularity of solutions for (1.1) in case of the nonlinearity $g(t, u(t)) := \phi(u(t))$.

A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic (resp. compact almost automorphic) if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n),$$

for each $t \in \mathbb{R}$ (resp. uniformly on compact subsets of $\mathbb{R}$).

The existence of almost automorphic solutions is one of the most attractive topics in the qualitative theory of evolution equations due to their applications in physical science, mathematical biology and many others subjects (see [3, 8, 9, 11, 14, 15, 16, 26]). Recently, to deal with delay equations and related topics, the concept of compact almost automorphic functions emerged (see [7, 17] and [10]). Whereas the regularity of solutions for (1.1) in the space of almost automorphic functions was considered in [6], the existence of *compact* almost automorphic solutions for abstract integral equations with infinite delay remains an untreated topic in the literature.

Our plan in this paper is as follows: In section 2, we give some preliminaries concerning the definition and main properties of integral resolvent families. In section 3, we prove our first regularity result on compact almost automorphic solutions for the linear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{1.3}$$

provided A is the generator of an integral resolvent family. Then we exhibit a wide class of kernels $a(t)$ such that the scalar version of (1.3) has a compact almost automorphic solution (see Corollary 3.5). Section 4 is devoted to the nonlinear case. We first state two results on existence of compact almost automorphic solutions for (1.1) when the function f is compact almost automorphic in the first variable and satisfies conditions of Lipschitz type in the second variable (cf. Theorem 4.3 and Theorem 4.5). Then we give a general result (Theorem 4.9) which includes the situation in that f satisfies a condition of Hölder type in the second variable (cf. Corollary 4.4).

2. Preliminaries

We introduce some notations. We denote by $BC(X)$ the space consisting of continuous and bounded functions $f : \mathbb{R} \to X$ endowed with the norm of uniform convergence

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

We set $AP(X)$, $AA(X)$, and $AA_c(X)$ for the closed subspaces formed by the almost periodic functions, the almost automorphic functions, and the compact almost automorphic functions, respectively. It is well known that

$$AP(X) \subseteq AA_c(X) \subseteq AA(X) \subseteq BC(X).$$

We recall that the Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is given by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

where the integral is absolutely convergent for $Re\lambda > \omega$. Furthermore, we denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators, and the notation $\rho(A)$ stands for the resolvent set of A. In order to establish an operator theoretical approach to equation (1.1), we consider the following definition (cf. [22]).

Definition 2.1. Let A be a closed linear operator with domain $D(A) \subseteq X$. We say that A is the generator of an *integral resolvent* if there exists $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{1/\hat{a}(\lambda) : Re\lambda > \omega\} \subseteq \rho(A)$ and

$$(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The concept of integral resolvent, as defined above, is closely related with the concept of resolvent family (see Prüss [24, Chapter I]). A closed but weaker definition was formulated by Prüss [24, Definition 1.6]. The book of Gripenberg, Londen and Staffans [13] contains an overview of the theory for the scalar case.

Because of the uniqueness of the Laplace transform, an integral resolvent family with $a(t) \equiv 1$ is the same as a C_0-semigroup whereas that an integral resolvent family with $a(t) = t$ corresponds to the concept of sine family, see [2, Section 3.15].

We note that integral resolvent families are a particular case of (a, k)-regularized families introduced in [19]. These are studied in a series of several papers in recent years (see [20], [21], [25]). According to [19] an integral resolvent family $S(t)$ corresponds to a (a, a)-regularized family.

In a similar way as occurs for C_0-semigroups, we can establish several relations between the integral resolvent family and its generator. The following result is a direct consequence of [19, Proposition 3.1 and Lemma 2.2].

Proposition 2.2. *Let $S(t)$ be the integral resolvent family on X with generator A. Then the following properties hold:*

(a) $S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(b) Let $x \in D(A)$ and $t \geq 0$. Then

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds. \tag{2.1}$$

(c) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$ and

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)xds.$$

In particular, $S(0) = a(0)I$.

If an operator A with domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function which satisfies $\limsup_{t\to 0^+} \frac{||S(t)||}{a(t)} < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - a(t)x}{(a * a)(t)}.$$

For instance, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup and $a(t) = t$ actually corresponds to the generator of a sine family. We refer the reader to [21, Theorem 2.1] for these properties. Furthermore, a characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for C_0 semigroups, can be directly deduced from [19, Theorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be also deduced from the more general context of (a, k) regularized resolvents (see [20, 21] and [25]).

3. Compact Almost Automorphic Solutions: The linear case

In this section we consider the existence and uniqueness of compact almost automorphic solutions to the inhomogeneous linear evolution equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{3.1}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R}_+)$.

We introduce the following integrability assumption for strongly continuous functions $S : [0, \infty) \to \mathcal{B}(X)$.

(INT) There exists $\phi \in L^1(\mathbb{R}_+)$ such that $\|S(t)\| \le \phi(t)$ for all $t \in \mathbb{R}_+$.

The following property of convolution is needed to establish our next results.

Lemma 3.1. *Let $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded linear operators that satisfies assumption (INT). If $f : \mathbb{R} \to X$ is a compact almost automorphic function, and $w(t)$ is given by*

$$w(t) = \int_{-\infty}^{t} S(t-s)f(s)\,ds,$$

then $w \in AA_c(X)$.

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. There exist a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous functions $v \in BC(X)$ such that $f(t + s_n)$ converges to $v(t)$ and $v(t - s_n)$ converges to $f(t)$ uniformly on compact subsets of $\mathbb{R}$.

Since

$$w(t + s_n) = \int_{-\infty}^{t+s_n} S(t + s_n - s)f(s)ds = \int_{-\infty}^{t} S(t-s)f(s + s_n)ds,$$

using the Lebesgue dominated convergence theorem, we obtain that $w(t + s_n)$ converges to $z(t) = \int_{-\infty}^{t} S(t-s)v(s)ds$ as $n \to \infty$ for each $t \in \mathbb{R}$.

Furthermore, the preceding convergence is uniform on compact subsets of $\mathbb{R}$. To show this assertion, we take a compact set $K = [-a, a]$. For $\varepsilon > 0$, we choose $L_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} \int_{L_\varepsilon}^{\infty} \phi(s)ds &\leq \varepsilon, \\ \|f(s+s_n) - v(s)\| &\leq \varepsilon, \;\; n \geq N_\varepsilon, \;\; s \in [-L, L], \end{aligned}$$

where $L = L_\varepsilon + a$. For $t \in K$, we now can estimate

$$\begin{aligned} \|w(t+s_n) - z(t)\| &\leq \int_{-\infty}^{t} \phi(t-s)\|f(s+s_n) - v(s)\|ds \\ &\leq \int_{-\infty}^{-L} \phi(t-s)\|f(s+s_n) - v(s)\|ds \\ &\quad + \int_{-L}^{t} \phi(t-s)\|f(s+s_n) - v(s)\|ds \\ &\leq 2\|f\|_\infty \int_{t+L}^{\infty} \phi(s)ds + \varepsilon \int_0^{\infty} \phi(s)ds \\ &\leq \varepsilon \left(2\|f\|_\infty + \int_0^{\infty} \phi(s)ds \right), \end{aligned}$$

which proves that the convergence is independent of $t \in K$.

Repeating this argument, one can show that $z(t - s_n)$ converges to $w(t)$ as $n \to \infty$ uniformly for t in compact subsets of $\mathbb{R}$. This completes the proof. □

The following is the main result of this section.

Theorem 3.2. *Let $a \in L^1(\mathbb{R}_+)$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying assumption (INT). If f is a compact almost automorphic function with values in $D(A)$, then the unique bounded solution of equation*

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{3.2}$$

is compact almost automorphic.

Proof. Let $u(t)$ be the function given by

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Since the values $f(t) \in D(A)$, it follows that $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see e.g. [24, Proposition 1.2]). Using (2.1) and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{-\infty}^{t} a(t-s)Au(s)ds &= \int_{-\infty}^{t} a(t-s)A\int_{-\infty}^{s} S(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{-\infty}^{s} a(t-s)AS(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{\tau}^{t} a(t-s)AS(s-\tau)f(\tau)ds d\tau \\
&= \int_{-\infty}^{t}\int_{0}^{t-\tau} a(t-\tau-p)AS(p)dpf(\tau)d\tau \\
&= \int_{-\infty}^{t} (S(t-\tau)f(\tau)-a(t-\tau)f(\tau))d\tau \\
&= u(t) - \int_{-\infty}^{t} a(t-\tau)f(\tau)d\tau
\end{aligned}$$

which establishes that $u(\cdot)$ is the solution of equation (3.2). Applying Lemma 3.1 we infer that u is compact almost automorphic. □

Remark 3.3. Related with assumption (INT) and the preceding results, it is important to mention that a complete discussion about integral resolvent families upper bounded by an integrable function is given in [24, Chapter 3, section 10].

Taking $X = \mathbb{R}$, we obtain the following result for the scalar case.

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compact almost automorphic function, $a \in L^1(\mathbb{R}_+)$, and let $\rho > 0$ be a real number. If the solution $S_\rho(t)$ of the one dimensional equation*

$$S_\rho(t) = a(t) - \rho\int_0^t a(t-s)S_\rho(s)ds, \tag{3.3}$$

satisfy $|S_\rho(t)| \le \phi_\rho(t)$, with $\phi_\rho \in L^1(\mathbb{R}_+)$, then the equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{3.4}$$

has a compact almost automorphic solution given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s)f(s)ds, \quad t \in \mathbb{R}, \tag{3.5}$$

It is worthwhile to point out that the following result provide a wide class of kernels $a(t)$ such that the condition $|S_\rho(t)| \le \phi_\rho(t)$, with $\phi_\rho \in L^1(\mathbb{R}_+)$, used in Corollary 3.4 holds.

Corollary 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a compact almost automorphic function and let $\rho > 0$ be a real number. Suppose $a \in L^1(\mathbb{R}_+)$ is positive, nonincreasing and log-convex, then the following properties are fulfilled:*

a) There is $S_\rho \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ such that equation (3.3) is satisfied.

b) The equation (3.4) has a compact almost automorphic solution given by (3.5)

Proof. Assertion a) follows from [24, Lemma 4.1, p. 98] and, assertion b) is a consequence of a) and Corollary 3.4. □

Remark 3.6. We note that if $a \in L^1(\mathbb{R}_+)$ is positive, nonincreasing and log-convex, then a is completely positive (see [4]).

4. Semilinear Integral Equations on the Line

In this section we are concerned with the study of existence of compact almost automorphic solutions for equation (1.1). We begin by establishing the terminology. Let Y be a Banach space with norm $||\cdot||_Y$. A continuous function $F : \mathbb{R}\times Y \to X$ is said to be compact almost automorphic in the first variable if for each $y \in Y$, and for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subseteq (s'_n)_{n\in\mathbb{N}}$ such that

$$G(t, y) := \lim_{n\to\infty} F(t + s_n, y)$$

is well defined for each $t \in \mathbb{R}$, and

$$F(t, y) = \lim_{n\to\infty} G(t - s_n, y),$$

and the convergence in these limits is uniform for t in compact subsets of $\mathbb{R}$. The space consisting of such functions F will be denoted by $AA_c(Y;X)$.

The results of this section are based on the following lemma. A proof can be found in [10, Lemma 2.1].

Lemma 4.1. *Let $f \in AA_c(Y;X)$ and let $u \in AA_c(Y)$. Assume that f satisfies the Lipschitz condition*

$$||f(t, y_2) - f(t, y_1)|| \le L||y_2 - y_1||_Y, \quad \forall t \in \mathbb{R},\ \forall y_1, y_2 \in Y. \tag{4.1}$$

Then the X-valued function G defined by $G(t) := f(t, u(t))$ belongs to $AA_c(X)$.

Definition 4.2. Let A be the generator of an integral resolvent family $\{S(t)\}_{t\ge0}$. A continuous function $u : \mathbb{R} \to X$ satisfying the integral equation

$$u(t) = \int_{-\infty}^{t} S(t - s)f(s, u(s))ds,\ \ \forall t \in \mathbb{R}, \tag{4.2}$$

is called a *mild* solution on $\mathbb{R}$ to the equation (1.1).

Theorem 4.3. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\ge0}$ that satisfies the assumption (INT). Let $f : \mathbb{R}\times X \to X$ be a compact almost automorphic in the first variable function that satisfies the Lipschitz condition (4.1). Assume that $L < ||\phi||_1^{-1}$. Then equation (1.1) has a unique compact almost automorphic mild solution.*

Proof. We define the operator $F : AA_c(X) \to AA_c(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t - s)f(s, \varphi(s))\, ds, \quad t \in \mathbb{R}. \tag{4.3}$$

In view of Lemma 4.1 and Lemma 3.1 the map F is well defined. Moreover, for $\varphi_1, \varphi_2 \in AA_c(X)$ we have:

$$\begin{aligned} \|F\varphi_1 - F\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\ &\leq L \sup_{t\in\mathbb{R}} \int_0^\infty \|S(\tau)\| \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\| d\tau \\ &\leq L\|\varphi_1 - \varphi_2\|_\infty \int_0^\infty \phi(\tau)d\tau. \end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA_c(X)$ such that $Fu = u$. It is clear that u is a compact almost automorphic mild solution of equation (1.1). □

An immediate consequence of Theorem 4.3 and Corollary 3.5 is the following result for the scalar equation.

Corollary 4.4. *Let $\rho > 0$ be a real number. Suppose $a \in L^1(\mathbb{R})$ is a positive, nonincreasing and log-convex function, and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be compact almost automorphic in the first variable function that satisfies the Lipschitz condition*

$$|f(t,x) - f(t,y)| \leq L|x-y|, \forall t, x, y \in \mathbb{R}.$$

Then there is $S_\rho \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ satisfying the linear equation (3.3). Moreover, if $L < ||S_\rho||_1^{-1}$, then the semilinear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s,u(s))]ds, \quad t \in \mathbb{R},$$

has a unique compact almost automorphic mild solution.

A different Lipschitz condition is considered in the following result. An integral resolvent family $\{S(t)\}_{t\geq 0}$ is said to be uniformly bounded if there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Theorem 4.5. *Assume that A generates a uniformly bounded integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption (INT). Let $f : \mathbb{R} \times X \to X$ be a compact almost automorphic in the first variable function that satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \forall t \in \mathbb{R}, x, y \in X, \tag{4.4}$$

where $L \in L^1(\mathbb{R}) \cap BC(\mathbb{R})$. Then equation (1.1) has a unique compact almost automorphic mild solution.

Proof. We define the operator F as in (4.3). Let φ_1, φ_2 be in $AA_c(X)$. We can estimate

$$\begin{aligned} ||(F\varphi_1)(t) - (F\varphi_2)(t)|| &= \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\ &\leq M \int_{-\infty}^{t} L(s)\|\varphi_1(s) - \varphi_2(s)\| ds \end{aligned}$$

Repeating the argument, we get

$$\begin{aligned}
&||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| \\
\leq\ & M^n \int_{-\infty}^{t}\int_{-\infty}^{s}\cdots\int_{-\infty}^{s_{n-2}} L(s)L(s_1)\cdots L(s_{n-1})||\varphi_1(s_{n-1})-\varphi_2(s_{n-1})||ds_{n-1}\cdots ds_1 ds \\
\leq\ & \frac{M^n}{n!}\left(\int_{-\infty}^{t} L(\tau)d\tau\right)^n ||\varphi_1-\varphi_2||_\infty \\
\leq\ & \frac{(M||L||_1)^n}{n!}||\varphi_1-\varphi_2||_\infty .
\end{aligned}$$

Since $\frac{(M||L||_1)^n}{n!} < 1$ for n sufficiently large, applying the contraction principle we conclude that F has a unique fixed point $u \in AA_c(X)$ which completes the proof. □

We next study the existence of compact almost automorphic mild solutions of equation (1.1) when the function f is not Lipschitz continuous. To avoid the Lipschitz conditions considered in the previous results, we need assume that f satisfies appropriate compactness conditions. To abridge the text, we begin by introducing the following assumption.

(UAA) Let $f : \mathbb{R}\times X \to X$ be a continuous function that satisfies the following conditions:

(a) For each $x \in X$, function $f(\cdot, x) \in AA_c(X)$, and this property is uniform for x in compact subsets of X. This means that for every sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a continuous function $g : \mathbb{R} \times X \to X$ such that $f(t+s_n, x) \to g(t, x)$ and $g(t-s_n, x) \to f(t, x)$ as $n \to \infty$, and the convergence is uniform for t in compact subsets of $\mathbb{R}$ and x in compacts subsets of X.

(b) The family of functions $\{f(t, \cdot) : t \in \mathbb{R}\}$ is equicontinuous.

Remark 4.6. If g is the function involved in part (a) of assumption (UAA), from the decomposition

$$\begin{aligned}
g(t, x+y) - g(t, x) &= g(t, x+y) - f(t+s_n, x+y) \\
&\quad + f(t+s_n, x+y) - f(t+s_n, x) \\
&\quad + f(t+s_n, x) - g(t, x),
\end{aligned}$$

it follows that the family of functions $\{g(t, \cdot) : t \in \mathbb{R}\}$ is equicontinuous.

We can now establish a slight extension of Lemma 4.1. We denote by $\mathcal{R}(u)$ the image of a function u.

Lemma 4.7. *Let $u \in AA_c(X)$. Assume that f satisfies the assumption (UAA). Then the X-valued function $t \mapsto f(t, u(t))$ belongs to $AA_c(X)$.*

Proof. For a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, there are a subsequence $(s_n)_{n\in\mathbb{N}}$, a continuous functions $v \in BC(X)$, and a continuous function $g : \mathbb{R} \times X \to X$ such that $u(t + s_n) \to v(t)$, $v(t - s_n) \to u(t)$, $f(t + s_n, x) \to g(t, x)$, and $g(t - s_n, x) \to f(t, x)$ as $n \to \infty$. Moreover, since $\overline{\mathcal{R}(u)}$ is a compact subset of X, the convergence considered in these limits is uniform with respect to $x \in \overline{\mathcal{R}(u)}$ and t in a compact subset of $\mathbb{R}$. Consequently, from the decomposition,

$$\begin{aligned} f(t+s_n, u(t+s_n)) - g(t, v(t)) &= f(t+s_n, u(t+s_n)) - f(t+s_n, v(t)) \\ &\quad + f(t+s_n, v(t)) - g(t, v(t)), \end{aligned}$$

and using condition (b) of assumption (UAA), we can show that $f(t+s_n, u(t+s_n)) - g(t, v(t)) \to 0$ as $n \to \infty$ uniformly for t in compact subsets of $\mathbb{R}$. Similarly, using Remark 4.6, we can prove that $g(t-s_n, v(t-s_n))$ converges to $f(t, u(t))$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$. □

To establish our result we will consider functions f that satisfies the following boundedness condition.

(B) There exist a continuous nondecreasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|f(t,x)\| \le W(\|x\|)$ for all $t \in \mathbb{R}$ and $x \in X$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(t) \ge 1$ for all $t \in \mathbb{R}$, and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$C_h(X) = \{u \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} \frac{u(t)}{h(t)} = 0\}$$

endowed with the norm

$$\|u\|_h = \sup_{t\in\mathbb{R}} \frac{\|u(t)\|}{h(t)}.$$

It is clear that $C_h(X)$ is a Banach space isometrically isomorphic with the space $C_0(\mathbb{R}, X)$ consisting of functions that vanish at infinity. Consequently, $K \subseteq C_h(X)$ is a relatively compact set if verifies the following conditions:

(c-1) The set $K(t) = \{u(t) : u \in K\}$ is relatively compact in X for each $t \in \mathbb{R}$.

(c-2) The set K is equicontinuous.

(c-3) For each $\varepsilon > 0$ there exists $L > 0$ such that $\|u(t)\| \le \varepsilon h(t)$ for all $u \in K$ and all $|t| > L$.

We are in a position to establish the following result of existence of solutions. This result is based on the Leray-Schauder alternative theorem ([12, Theorem 6.5.4]) that follows.

Lemma 4.8. *Let D be a closed convex subset of a Banach space X such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

Theorem 4.9. *Assume that A generates a uniformly bounded integral resolvent family $\{S(t)\}_{t\ge0}$ that satisfies assumption (INT). Let $f : \mathbb{R} \times X \to X$ be a function that satisfies assumptions (UAA) and (B), and the following conditions:*

(a) For each $C \geq 0$, the function $t \mapsto \int_{-\infty}^{t} \phi(t-s)W(Ch(s))ds$ is included in $BC(\mathbb{R})$. We set

$$\beta(C) = \| \int_{-\infty}^{t} \phi(t-s)W(Ch(s))ds\|_h.$$

(b) For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|v-u\|_h \leq \delta$ implies that $\int_{-\infty}^{t} \phi(t-s)\|f(s,v(s)) - f(s,u(s))\|ds \leq \varepsilon$ for all $t \in \mathbb{R}$.

(c) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(d) For all $a, b \in \mathbb{R}$, $a < b$, and $r > 0$, the set $\{f(s,x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ is relatively compact in X.

Then equation (1.1) has a compact almost automorphic mild solution.

Proof. We define the operator F on $C_h(X)$ as in (4.3). We will show that F has a fixed point in $AA_c(X)$. We divide the proof in several steps.
(i) For $u \in C_h(X)$, we have that

$$\begin{aligned} \|Fu(t)\| &\leq \int_{-\infty}^{t} \phi(t-s)W(\|u(s)\|)ds \\ &\leq \int_{-\infty}^{t} \phi(t-s)W(\|u\|_h h(s))ds. \end{aligned} \tag{4.5}$$

It follows from condition (a) that $F : C_h(X) \to C_h(X)$.
(ii) The map F is continuous. In fact, for $\varepsilon > 0$, we take δ involved in condition (b). If $u, v \in C_h(X)$ and $\|u - v\|_h \leq \delta$, then

$$\begin{aligned} \|Fu(t) - Fv(t)\| &\leq \int_{-\infty}^{t} \phi(t-s)\|f(s,u(s)) - f(s,v(s))\|ds \\ &\leq \varepsilon, \end{aligned}$$

which shows the assertion.
(iii) We will show that F is completely continuous. To abbreviate the text, we will set $B_r(Z)$ for the closed ball with center at 0 and radius r in a space Z. Let $V = F(B_r(C_h(X)))$ and $v = F(u)$ for $u \in B_r(C_h(X))$.

Initially, we will prove that $V(t)$ is a relatively compact subset of X for each $t \in \mathbb{R}$. It follows from condition (a) that the function $s \mapsto \phi(s)W(rh(t-s))$ is integrable on $[0,\infty)$. Hence, for $\varepsilon > 0$, we can choose $a \geq 0$ such that $\int_a^{\infty} \phi(s)W(rh(t-s))ds \leq \varepsilon$. Since

$$v(t) = \int_0^a S(s)f(t-s,u(t-s))ds + \int_a^{\infty} S(s)f(t-s,u(t-s))ds$$

and

$$\| \int_a^{\infty} S(s)f(t-s,u(t-s))ds\| \leq \int_a^{\infty} \phi(s)dsW(r) \leq \varepsilon,$$

we get

$$v(t) \in \overline{ac(\{S(s)f(\xi,x) : 0 \leq s \leq a, t-a \leq \xi \leq t, \|x\| \leq r\})} + B_\varepsilon(X),$$

where $c(K)$ denotes the convex hull of K. Using that $S(\cdot)$ is strongly continuous and the property (d) of f, we infer that $K = \{S(s)f(\xi,x) : 0 \leq s \leq a, t-a \leq \xi \leq t, \|x\| \leq r\}$ is a relatively compact set, and $V(t) \subseteq \overline{ac(K)} + B_\varepsilon(X)$, which establishes our assertion.

We next will show that the set V is equicontinuous. In fact, proceeding as above, we can decompose

$$\begin{aligned} v(t+s)-v(t) &= \int_0^s S(\xi)f(t+s-\xi,u(t+s-\xi))d\xi \\ &\quad + \int_0^a (S(\xi+s)-S(\xi))f(t-\xi,u(t-\xi))d\xi \\ &\quad + \int_a^\infty (S(\xi+s)-S(\xi))f(t-\xi,u(t-\xi))d\xi. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $a > 0$ and $\delta_1 > 0$ such that

$$\begin{aligned} &\| \int_0^s S(\xi)f(t+s-\xi,u(t+s-\xi))d\xi + \int_a^\infty (S(\xi+s)-S(\xi))f(t-\xi,u(t-\xi))d\xi\| \\ \leq &\int_0^s \phi(\xi)W(rh(t+s-\xi))d\xi + \int_a^\infty (\phi(\xi+s)+\phi(\xi))W(rh(t-\xi))d\xi \\ \leq &\ \varepsilon/2 \end{aligned}$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\xi,u(t-\xi)) : 0 \leq \xi \leq a, u \in B_r(C_h(X))\}$ is a relatively compact set and $S(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ such that $\|(S(\xi + s) - S(\xi))f(t-\xi,u(t-\xi))\| \leq \varepsilon/(2a)$ for $s \leq \delta_2$. Combining these estimate, we get $\|v(t+s)-v(t)\| \leq \varepsilon$ for s enough small and independent of $u \in B_r(C_h(X))$.

Finally, applying condition (a), we can show that

$$\frac{v(t)}{h(t)} \leq \frac{1}{h(t)} \int_{-\infty}^t \phi(t-s)W(rh(s))ds \to 0, |t| \to \infty,$$

and this convergence is independent of $u \in B_r(C_h(X))$.

Hence V satisfies conditions (c-1), (c-2) and (c-3), which completes the proof that V is a relatively compact set in $C_h(X)$.

(iv) If $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda F(u^\lambda)$ for some $0 < \lambda < 1$, from the estimate

$$\begin{aligned} \|u^\lambda(t)\| &= \lambda\| \int_{-\infty}^t S(t-s)f(s,u^\lambda(s))ds\| \\ &\leq \int_{-\infty}^t \phi(t-s)W(\|u^\lambda(s)\|)ds \\ &\leq \int_{-\infty}^t \phi(t-s)W(\|u^\lambda\|_h h(s))ds \\ &\leq \beta(\|u^\lambda\|_h)h(t), \end{aligned}$$

we get

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \leq 1$$

and, combining with condition (c), we conclude that the set $\{u^\lambda : u^\lambda = \lambda F(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(v) It follows from Lemma 4.7 that $F(\overline{AA_c(X)}) \subseteq \overline{AA_c(X)}$ and, consequently, we can consider $F : \overline{AA_c(X)} \to \overline{AA_c(X)}$. Using properties (i)-(iv) we have that this map is completely continuous. Applying Lemma 4.8 we infer that F has a fixed point $u \in \overline{AA_c(X)}$.

Let $(u_n)_n$ be a sequence in $AA_c(X)$ that converges to u. Using (4.5) we see that $(Fu_n)_n$ converges to $Fu = u$ uniformly on $\mathbb{R}$. This implies that $u \in AA_c(X)$, and completes the proof. □

Corollary 4.10. *Assume that A generates a uniformly bounded integral resolvent family $\{S(t)\}_{t\geq 0}$ that satisfies assumption (INT). Let $f : \mathbb{R} \times X \to X$ be a function that satisfies assumptions (UAA) and the Hölder type condition*

$$\|f(t,y) - f(t,x)\| \leq C_1\|y - x\|^{\alpha}, \quad 0 < \alpha < 1, \tag{4.6}$$

for all $x, y \in X$, where $C_1 > 0$ is a constant. Moreover, assume the following conditions:
(i) $f(t,0) = q$.
(ii) For all $a, b \in \mathbb{R}$, $a < b$, and $r > 0$, the set $\{f(s,x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ is relatively compact in X.
Then equation (1.1) has a compact almost automorphic mild solution.

Proof. Let $C_0 = \|q\|$. We take $W(\xi) = C_0 + C_1\xi^{\alpha}$. Then condition (B) is satisfied. We choose a function h such that $\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \phi(t-s)h(s)^{\alpha}ds := C_2 < \infty$. Then it is not difficult to see that function f satisfies (a) in Theorem 4.9. To verify (b), note that for each $\varepsilon > 0$ there is $0 < \delta^{\alpha} < \frac{\epsilon}{C_1C_2}$ such that for every $u, v \in C_h(X)$, $\|v - u\|_h \leq \delta$ implies that $\int_{-\infty}^{t} \phi(t-s)\|f(s,v(s)) - f(s,u(s))\|ds \leq \varepsilon$ for all $t \in \mathbb{R}$. On the other hand, the hypothesis (c) in the statement of Theorem 4.9 can be easily verified using the definition of W. This completes the proof.

□

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