GLOBALLY ATTRACTIVE MILD SOLUTIONS FOR NON-LOCAL IN TIME SUBDIFFUSION EQUATIONS OF NEUTRAL TYPE

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Abstract. We prove the existence of at least one globally attractive mild solution to the equation

$$\partial_t(b * [x - h(\cdot, x(\cdot))])(t) + A(x(t) - h(t, x(t))) = f(t, x(t)), \quad t \geq 0,$$

under the assumption, among other hypothesis, that A is an almost sectorial operator on a Banach space X and the kernel b belongs to a large class, which covers many relevant cases from physics applications, in particular the important case of time-fractional evolution equations of neutral type.

1. Introduction

Existence of globally attractive solutions for mathematical models is a very challenging topic that is drawing the attention of many researchers in the last decade. For instance, Alzabut and Abdeljawad in [1] studied the existence of a globally attractive periodic solutions of an impulsive delay logarithmic population model. Bartuccelli, Deane and Gentile [3] analyzed globally and locally attractive solutions for quasi-periodically forced systems, and Li and Cheng [8] established conditions for the existence of globally attractive periodic solutions of a perturbed functional differential equation. In general, the attracting character of the solutions can be deduced by different methods. For instance, using a result due to Tang [14], or via the measure of noncompactness due to Banás [2]. In this paper, we will take this last approach.

In a paper of Liang et.al. [9] the existence of globally attractive mild solutions to the Cauchy problem, for fractional evolution equations of neutral type in the form

$$\partial_t^q(x(t) - h(t, (x(t))) + A(x(t) - h(t, x(t)) = f(t, x(t)), \quad t > 0, \quad 0 < q < 1, \tag{1}$$

was proved. Here, the fractional derivative is understood in the Caputo sense and A is an almost sectorial operator.

Since the fractional derivative of order $\alpha > 0$ is defined by means of the finite convolution of a given function and the special kernel

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,$$

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where Γ denotes the Gamma function, it is natural to ask if it is possible to replace this particular kernel g_α for a more general kernel $b(t)$. This approach has been taken for some authors in the last time in order to relax the properties of the memory kernel g_α, which is the main responsible of the behavior and qualitative properties of fractional models. This leads to the analysis of equations in the form:

$$\partial_t(b * [x - x_0])(t) + Ax(t) = f(t, x(t)), \tag{2}$$

with initial datum $x(0) = x_0$. In this way, the non-local time term on the left hand side includes such classical cases as the Riemann-Liouville and the Caputo fractional derivatives, respectively, for convenient choices of b. For instance, Kemppainen et. al. [6] and Vergara and Zacher [15] studied decay estimates for non-local in time subdiffusion equations in the form (2) when $A = -\Delta$, the Laplace operator on $\mathbb{R}^N$. It should be noted that in [6] and [15] the kernel b satisfies the following remarkable property:

($\mathcal{PC}$) $b \in L^1_{loc}(\mathbb{R}_+)$ is nonnegative, and nonincreasing and there exists a kernel $a \in L^1_{loc}(\mathbb{R}_+)$ such that $a * b \equiv 1$ in $(0, \infty)$.

In particular, if b satisfies condition ($\mathcal{PC}$) then a is completely positive cf. [4, Theorem 2.2] or [13, Remarks on p.326]. For example, for $b(t) = g_{1-\alpha}(t)$ we have $a(t) = g_\alpha(t)$ where $0 < \alpha < 1$. Another very interesting example is given by

$$b(t) = \int_0^1 g_\beta(t)d\beta \qquad \text{and} \qquad a(t) = \int_0^\infty \frac{e^{-st}}{1+s}ds. \tag{3}$$

In this case the operator $\partial_t(b * \cdot)$ is a so-called operator of distributed order, see e.g. [7]. More examples are discussed in [15, Section 6].

This paper is inspired by the papers [9] and [6, 15] and on the equations (1) and (2), respectively. Since one of the outcomes of [9] was an existence result of attractive mild solutions, we wish to extend this analysis to the following class of equations

$$\partial_t(b * [x - h(\cdot, x(\cdot))])(t) + A(x(t) - h(t, x(t)) = f(t, x(t)), \quad t \geq 0.$$

Our initial observation is that under the hypothesis of the condition ($\mathcal{PC}$) the above equation is equivalent to the following class of abstract integral Volterra equations

$$\begin{cases} x(t) - h(t, x(t)) & + \displaystyle\int_0^t a(t-s)A[x(s) - h(s, x(s))]ds \\ & = \int_0^t a(t-s)f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) = x_0, & \end{cases} \tag{4}$$

Motivated by this observation, we ask ourselves: Under which conditions on a general kernel $a(t)$ there exists globally attractive mild solutions for the abstract model (4)?

Our purpose in this paper is to provide an answer to this question. Roughly speaking, we find the following class of kernels:

($\mathcal{K}$) $a \in C((0, \infty)) \cap L^1_{loc}(\mathbb{R}_+)$ is completely monotonic.

We notice that both kernels $a(t) = g_q(t)$, $0 < q < 1$ and $a(t)$ given in (3) satisfy the property ($\mathcal{K}$).

We recall that abstract integral Volterra equations of the form (4) appears in several fields of wide interest. For instance, in the theory of viscoelastic materials [12], and Navier-Stokes equations with memory [5], among others.

This paper is organized as follows: In Section 2, we briefly recall some results on the measure of noncompactness and a fixed point theorem of Darbo type. We also recall the notion of almost sectorial operator, deeply studied by Periago and Straub [10]. Then, the definitions of resolvent and integral resolvent are presented. We refer to the monograph of J. Prüss [12] for further information on the subject of resolvent families of operators. It is worthwhile to note that the notion of integral resolvent that we employ in this paper is slightly different than those considered in Prüss's book. We finish the section showing a connection between both definitions.

Section 3 contain the main result of this paper, namely Theorem 3.5. Under the condition that A is an almost sectorial operator and a set of six hypothesis involving the non-linear term f, and the integral resolvent family, we conclude the existence of at least one globally attractive mild solution of (4) in the space of continuous and bounded functions with values in the space $X_\alpha := D(A^\alpha)$, endowed with an appropriate norm. This main result is complemented with Proposition 3.6 that shows a practical criteria in order to satisfy one of the more striking hypothesis of the main Theorem. Finally, Section 4 shows as application, how the main result obtained in [9] can be deduced from our findings.

2. Preliminaries

Let X be a complex Banach space with norm $\|\cdot\|$. By $B(x, r)$ we denote the closed ball centered at x with radius r and by $\mathcal{M}_X$ we denote the family of all nonempty and bounded subsets of X. The subfamily consisting of all relatively compact sets is denoted by $\mathcal{N}_X$. As usual, for a linear operator A, we denote by $D(A)$ the domain of A, and by $R(z; A) := (zI - A)^{-1}$, $z \in \rho(A)$ the resolvent operator of A. Moreover, we denote by $L(X)$ the space of all bounded linear operators from X to X with the usual operator norm $\|\cdot\|_{L(X)}$.

Definition 2.1. [2] A function $\mu : \mathcal{M}_X \to \mathbb{R}_+$ is said to be a measure of noncompactness in X if it satisfies the following conditions:

(1) The set $Ker\mu := \{\Omega \in \mathcal{M}_X : \mu(\Omega) = 0\}$ is nonempty and $Ker\mu \subset \mathcal{N}_X$;
(2) $\Omega \subset \Omega_0$ implies $\mu(\Omega) \leq \mu(\Omega_0)$, for each $\Omega, \Omega_0 \in \mathcal{M}_X$;
(3) $\mu(Conv(\Omega)) = \mu(\Omega)$, where $Conv(\Omega)$ denotes the convex hull of Ω;
(4) $\mu(\overline{\Omega}) = \mu(\Omega)$, where $\overline{\Omega}$ denotes the closure of $\Omega \in \mathcal{M}_X$;
(5) $\mu(\lambda\Omega + (1-\lambda)\Omega_0) \leq \lambda\mu(\Omega) + (1-\lambda)\mu(\Omega_0)$, for $\lambda \in [0, 1]$ and $\Omega, \Omega_0 \in \mathcal{M}_X$;
(6) If $\{\Omega_n\}$ is a sequence of sets in $\mathcal{M}_X$ such that $\Omega_{n+1} \subset \Omega_n$, $\overline{\Omega_n} = \Omega_n$, with $n = 1, 2, ...$, and if $\lim_{n\to\infty} \mu(\Omega_n) = 0$, then the intersection $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty.

The following is a fixed point theorem of Darbo type for measures of noncompactness.

Lemma 2.2. [2] *Let $\mathfrak{M}$ be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $H : \mathfrak{M} \to \mathfrak{M}$ be a continuous mapping. Assume that there exists a* *constant* $k \in [0,1)$, *such that:*

$$\mu(H(\Omega)) \leq k\mu(\Omega),$$

for any nonempty subset Ω *of* $\mathfrak{M}$. *Then* H *has a fixed point in* $\mathfrak{M}$.

Next, we present a particular measure of noncompactness that will be useful to us in this paper.
For any nonempty a bounded subset Y of the space $BC(\mathbb{R}_+, X)$ and a positive number T, we denote by $\omega^T(x,\epsilon)$ the modulus of continuity of a function x on the interval $[0,T]$, where $x \in Y$ and $\epsilon \geq 0$. Namely:

$$\omega^T(x,\epsilon) = \sup\{||x(t) - x(s)|| : t, s \in [0,T], |t-s| \leq \epsilon\}.$$

We then write additionally

$$\begin{aligned} &\omega^T(Y,\epsilon) = \sup\{\omega^T(x,\epsilon) : x \in Y\},\\ &\omega_0^T(Y) = \lim_{\epsilon\to 0}\omega^T(Y,\epsilon),\\ &\omega_0(Y) = \lim_{T\to\infty}\omega_0^T(Y), \end{aligned}$$

and

$$\operatorname{diam}(Y) = \sup\{||x(t) - y(t)|| : x, y \in Y\}.$$

Finally, consider the function μ defined on the family $\mathcal{M}_{BC(\mathbb{R}_+,X)}$ by the formula:

$$\mu(Y) = \omega_0(Y) + \limsup_{t\to\infty} \operatorname{diam}(Y). \tag{5}$$

It is known that μ is a measure of noncompactness [2].

Let S_μ^0 with $0 < \mu < \pi$ be the open sector:

$$S_\mu^0 = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu\}$$

and S_μ be its closure, that is:

$$S_\mu = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \mu\} \cup \{0\}. \tag{6}$$

Definition 2.3. [10] Let $-1 < \gamma < 0$ and $0 \leq \omega < \pi$ be given. By $\Theta_\omega^\gamma(X)$ we denote the family of all linear and closed operators $A : D(A) \subseteq X \to X$, which satisfy:

1. $\sigma(A) \subseteq S_\omega$, and
2. for every $\omega < \mu < \pi$ there exists a constant C_μ such that:

$$||R(z;A)||_{L(X)} \leq C_\mu |z|^\gamma, \quad \text{for all} \quad z \in \mathbb{C} \setminus S_\mu.$$

Definition 2.4. A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^\gamma(X)$.

Given $A \in \Theta_\omega^\gamma(X)$, we denote

$$X_\alpha := D(A^\alpha),$$

and $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$. The existence of the complex power A^α is a consequence of the functional calculus developed in [10, Section 2]. See also [10, Theorem 3.2 and Proposition 3.3] for their main properties. In contrast to the case of sectorial operators, having $0 \in \rho(A)$ does not imply that the complex powers $A^{-\alpha}$ with $\operatorname{Re}\alpha > 0$ are bounded. However, the operator $A^{-\alpha}$ belongs to $L(X)$ whenever $\operatorname{Re}\alpha > 1 + \gamma$. See [10, Proposition 3.4].

Consider $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. We denote for $t \in S^0_{\frac{\pi}{2}-\omega}$,

$$T(t) := \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz}R(z;A)dz, \tag{7}$$

where $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$ and Γ_θ denotes the path

$$\{re^{-i\theta} \ : r > 0\} \cup \{re^{i\theta} \ : \ r > 0\},$$

oriented such that S^0_θ lies to the left of Γ_θ. We have the following properties on $T(t)$. For a proof, see [10, Theorem 3.9].

(i) $T(t)$ forms an analytic semigroup in $S^0_{\frac{\pi}{2}-\omega}$ and

$$\frac{d^n}{dt^n}T(t) = (-A)^nT(t), \quad \text{for all } \ t \in S^0_{\frac{\pi}{2}-\omega}.$$

(ii) There exists a constant $C_0 = C_0(\gamma) > 0$ such that

$$||T(t)||_{L(X)} \le C_0t^{-\gamma-1}, \quad \text{for all } \ t > 0.$$

(iii) The range $R(T(t))$ of $T(t)$ for each $t \in S^0_{\frac{\pi}{2}-\omega}$ is contained in $D(A^\infty)$. In particular for all $\beta \in \mathbb{C}$ with $Re\beta > 0, R(T(t)) \subset D(A^\beta)$ and

$$A^\beta T(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz}R(z;A)xdz, \quad \text{for all } \ x \in X,$$

and hence there exists a constant $C' = C'(\gamma,\beta) > 0$ such that:

$$||A^\beta T(t)||_{L(X)} \le C't^{-\gamma-Re\beta-1}, \quad \text{for all } \ t > 0.$$

(iv) If $\beta > 1+\gamma$, then $D(A^\beta) \subset \Sigma_T$, where Σ_T is the continuity set of the semigroup $\{T(t)\}_{t\ge0}$. That is:

$$\Sigma_T = \{x \in X : \lim_{t\to0} T(t)x = x\}.$$

The relation between the resolvent operators of A and the semigroup $T(t)$ is characterized by the following Lemma:

Lemma 2.5. [10, Theorem 3.13] *Let $A \in \Theta^\gamma_\omega(X)$, with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Then, for every $\lambda \in \mathbb{C}$, with $Re\lambda > 0$, we have:*

$$R(\lambda;-A) = \int_0^\infty e^{-\lambda t}T(t)dt. \tag{8}$$

In other words, if $A \in \Theta^\gamma_\omega(X)$ then $-A$ is the generator of an analytic semigroup of growth order $\gamma + 1$.

Definition 2.6. [12] Let B be a closed linear operator with domain $D(B) \subset X$ and $a \in L^1_{loc}(\mathbb{R}_+)$. A family $\{S(t)\}_{t\ge0}$ of bounded and linear operators in X is called a resolvent with generator B if the following conditions are satisfied:

(1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
(2) $S(t)$ commutes with B, which means that $S(t)D(B) \subset D(B)$ and $BS(t)x = S(t)Bx$, for all $x \in D(B)$ and $t \ge 0$;

(3) the resolvent equation holds:

$$S(t)x = x + \int_0^t a(t-s)BS(s)xds \quad \text{for all} \quad x \in D(B),\ t \geq 0. \tag{9}$$

Definition 2.7. Let B be a closed linear operator with domain $D(B) \subset X$ and $a \in C(\mathbb{R}_+)$. A strongly continuous family $\{P(t)\}_{t\geq 0}$ of bounded linear operators in X is called an integral resolvent with generator B if the following conditions are satisfied:

(1) $P(0) = a(0)I$;
(2) $P(t)$ commutes with B;
(3) the integral resolvent equation holds:

$$P(t)x = a(t)x + \int_0^t a(t-s)BP(s)xds \quad \text{for all} \quad x \in D(B), t \geq 0. \tag{10}$$

We recall that the finite convolution of two functions f and g is denoted by:

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Definition 2.8. A resolvent $S(t)$ (resp. an integral resolvent $P(t)$) is called analytic if the function $S(\cdot) : \mathbb{R}_+ \to B(X)$ (resp. $P(\cdot) : \mathbb{R}_+ \to B(X)$) admits a analytic extension to a sector $\Sigma(0,\theta) := \{z \in \mathbb{C} : |arg(z)| < \theta, \quad \theta \in (0, \frac{\pi}{2})\}$

An analytic resolvent $S(t)$ (resp. an integral resolvent $P(t)$) is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is $M = M(\omega, \theta)$ such that:

$$\|S(z)\| \leq Me^{\omega Rez} \quad (\text{resp.} \|P(z)\| \leq Me^{\omega Rez}).$$

Directly from [12, Theorem 0.1, p.5] we have the following result.

Proposition 2.9. *Let $P(t)$ be an analytic integral resolvent of type (ω_0, θ_0) and $a(t)$ of exponential growth. Let $\widehat{P}(t)$ be denote the Laplace tranform of $P(t)$. Then, for each $\omega > \omega_0$ and $\theta < \theta_0$:*

$$\|\widehat{P}(\lambda)\| \leq \frac{C}{|\omega - \lambda|}, \qquad \lambda \in \Sigma\left(\omega, \theta + \frac{\pi}{2}\right), \ \textit{for some } C = C(\omega, \theta) > 0. \tag{11}$$

The relation between resolvent and integral resolvent families is given in the following proposition.

Proposition 2.10. *Suppose that $a \in C^1(\mathbb{R}_+)$, then*

$$\int_0^t P(s)xds = \int_0^t a(t-s)S(s)xds, \quad t > 0, \quad x \in X. \tag{12}$$

In particular, if B is the generator of a resolvent family, then B is also the generator of a integral resolvent family, given by the formula

$$P(t)x = a(0)S(t)x + \int_0^t a'(t-s)S(s)xds, \quad t > 0, \quad x \in X. \tag{13}$$

Proof. Using the identities (9) and (10), we have in view of the commutativity of the convolution

$$\begin{aligned}(S*P)(t) &= (I*P)(t) + B(a*S*P)(t) = (I*P)(t) + S*(B(a*P))(t)\\ &= (I*P)(t) + (S*[P-a])(t) = (I*P)(t) + (S*P)(t) - (S*a)(t).\end{aligned}$$

Therefore, we obtain (12). Differentiating (12) with respect to t, we obtain (13). □

3. Main result

We first consider the linear problem

(14)
$$\begin{cases} x(t) - h(t) = \int_0^t a(t-s)(-A)[x(s)-h(s)]ds + \int_0^t a(t-s)f(s)ds + x_0 - h(0),\\ x(0) = x_0,\end{cases}$$

where $a \in L^1_{loc}(\mathbb{R}_+)$, A is a closed linear operator defined on a Banach space X, $f : \mathbb{R}_+ \to X$ and $h : \mathbb{R}_+ \to [D(A)]$ are given functions and $x_0 \in X$. By a strong solution of (14) we mean a function $x \in C(\mathbb{R}_+; [D(A)])$ that satisfies (14).

Definition 3.1. Let $a \in C^1(\mathbb{R}_+)$ be given. Suppose that $-A$ is the generator of a resolvent $\{S(t)\}_{t\geq 0}$. Given functions $f : \mathbb{R}_+ \times X \to X$ $h : \mathbb{R}_+ \times X \to [D(A)]$ and $x_0 \in X$, we call the continuous function $x : \mathbb{R}_+ \to X$ given by

$$x(t) = S(t)(x_0 - h(0)) + h(t) + \int_0^t P(t-s)f(s)ds, \quad t \geq 0, \tag{15}$$

a mild solution of (14), where S and P are given by (9) and (10) respectively.

Proposition 3.2. *Let $f : \mathbb{R}_+ \times X \to [D(A)]$ be given. If $x_0 \in D(A)$ then each mild solution is a strong solution.*

Proof. Define $B = -A$ and $x_1 := x_0 - h(0)$. Since $x_0 \in D(A)$, where A is closed, and $f(t) \in D(A)$ for all $t \geq 0$, we have $S(t)x_1 \in D(A)$ and $(P*f)(t) \in D(A)$ for all $t \geq 0$, respectively. Therefore $x(t) \in D(A)$ for all $t \geq 0$. Then by (17), properties of the convolution, Definition 2.6 and Definition 2.7 we obtain

$$\begin{aligned}(a*B(x-h))(t) &= (a*B[Sx_1 + P*f])(t) = (a*BS)(t)x_1 + (a*BP*f)(t)\\ &= S(t)x_1 - x_1 + ((P-a)*f)(t) = x(t) - h(t) - x_1 - (a*f)(t)\end{aligned}$$

which proves the proposition. □

Consider the following nonlinear abstract integral equation:

$$\begin{cases} x(t) - h(t,x(t)) &= \int_0^t a(t-s)(-A)[x(s)-h(s,x(s))]ds\\ &\quad + \int_0^t a(t-s)f(s,x(s))ds + x_0 - h(0,x_0),\\ x(0) &= x_0,\end{cases} \tag{16}$$

where $a \in L^1_{loc}(\mathbb{R}_+)$, A is an almost sectorial operator on a complex Banach space X, $f : \mathbb{R}_+ \times X \to X$ and $h : \mathbb{R}_+ \times X \to [D(A)]$ are given functions and $x_0 \in X$. By a strong solution of (16) we mean a function $x \in C(\mathbb{R}_+; [D(A)])$ that satisfies (16).

Given and almost sectorial operator A there exists kernels $a \in C^1(\mathbb{R}_+)$ such that $B := -A$ is the generator of simultaneously a resolvent $S(t)$ and an integral resolvent $P(t)$. We denote the set of all such kernels by $\mathcal{K}$.

For example, if A is the generator on an almost sectorial operator we know that $-A$ is the generator of an analytic semigroup $T(t)$, and is explicitly given by the formula (7). Then, by [12, Corollary 2.4, p.56] we obtain that the class of kernels $a \in C((0, \infty)) \cap L^1_{loc}(\mathbb{R}_+)$ that are completely monotonic, belongs to $\mathcal{K}$.

Now, we give the definition of mild solution to the equation (16).

Definition 3.3. Let A be an almost sectorial operator and $a \in \mathcal{K}$. Given functions $f : \mathbb{R}_+ \times X \to X$ $h : \mathbb{R}_+ \times X \to [D(A)]$ and $x_0 \in X$, we say that a continuous function $x : \mathbb{R}_+ \to X$ that satisfies the equation

$$x(t) = S(t)(x_0 - h(0, x_0)) + h(t, x(t)) + \int_0^t P(t - s)f(s, x(s))ds, \quad t \geq 0, \tag{17}$$

is a mild solution of (16), where S and P are given by (9) and (10) respectively.

Definition 3.4. A mild solution $x(t)$ of (16) is said to be globally attractive if:

$$\lim_{t\to\infty} (x(t) - y(t)) = 0,$$

for any mild solution $y(t)$ of (16).

Let $BC(\mathbb{R}_+, X_\alpha)$ be denote the Banach space consisting of all real functions defined as bounded and continuous from $\mathbb{R}_+$ to X_α with the norm $\|x\|_\infty = \sup_{t\geq 0} \|x(t)\|_\alpha$. Recall that $X_\alpha = D(A^\alpha)$. The main result of this paper is the following theorem.

Theorem 3.5. *Let $A \in \Theta^\gamma_\omega(X)$, with $-1 < \alpha + \gamma < 0$, $0 < \alpha < 1$ and $0 < \omega < \frac{\pi}{2}$ and $a \in \mathcal{K}$. Assume that:*

- (H1) *$f : \mathbb{R}_+ \times X_\alpha \to X$ is continuous, and there exists a positive function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

$$\begin{cases} ||f(t, x)|| \leq \nu(t) \text{ for all } x \in X; \\ \text{the function } \quad s \to ||A^\alpha P(t - s)||\nu(s) \text{ belongs to } L^1([0, t[, \mathbb{R}_+), \text{ and} \end{cases} \tag{18}$$

$$\lim_{t\to\infty} \eta(t) := \lim_{t\to\infty} \int_0^t ||A^\alpha P(t - s)||\nu(s)ds = 0. \tag{19}$$

- (H2) *$h : \mathbb{R}_+ \times X_\alpha \to X_\alpha$ is bounded, continuous and there exist a constant $L \in (0, 1)$ such that:*

$$||h(t_1, x(t_1)) - h(t_2, x(t_2))||_\alpha \leq L(|t_1 - t_2| + ||x(t_1) - x(t_2)||_\alpha),$$

for all $x \in BC(\mathbb{R}_+, X_\alpha)$.

(H3) *For each nonempty, bounded set* $D \subset BC(\mathbb{R}_+, X_\alpha)$*, the family of functions*

$$\{t \to h(t, \varphi(t)) : \varphi \in D\}$$

is equicontinuous.

(H4) *The operator* $\tau \to A^\alpha P(\tau)x$ *is bounded.*

(H5) *The resolvent family* $\{S(t)\}_{t\geq 0}$ *is uniformly continuous for* $t > 0$ *and*

$$\sup_{t\geq 0} ||A^\alpha S(t)x|| < \infty, \qquad \text{for all } x \in X.$$

(H6)

$$\lim_{\tau\to t} \int_0^t ||A^\alpha (P(\tau - s) - P(t - s))||\nu(s)ds = 0, \quad \forall t \geq 0.$$

Then:

(1) *The problem* (16) *has at least a mild solution on* $BC(\mathbb{R}_+, X_\alpha)$*.*

(2) *Mild solutions of* (16) *are globally attractive.*

Proof. Consider the operator H defined as follows:

$$(Hx)(t) = S(t)(x_0 - h(0, x_0)) + h(t, x(t)) + \int_0^t P(t - s)f(s, x(s))ds, \quad t \geq 0. \tag{20}$$

Step 1: We prove that there exists a ball

$$B_r = \{x \in BC(\mathbb{R}_+, X_\alpha) : ||x||_\infty \leq r\}$$

with radius r and centered at 0, such that $H(B_r) \subset B_r$. In fact, for any $r > 0$ and $x \in B_r$, in view of (H2),

$$||h(t, x(t))||_\alpha \;\leq\; ||h(t, x(t)) - h(t, 0)||_\alpha + ||h(t, 0)||_\alpha \leq Lr + M_1, \tag{21}$$

where $M_1 := \sup_{t\in\mathbb{R}_+} ||h(t, 0)||_\alpha < \infty$ since h is bounded. Moreover, by (19) in (H2), we get $\sup\limits_{t\in\mathbb{R}_+} \eta(t) \leq K$ for a positive constant K. Let $x \in B_r$ be arbitrary, then by (21) and (H5)

$$\begin{aligned} ||H(x)(t)||_\alpha &\leq ||S(t)(x_0 - h(0, x_0))||_\alpha + ||h(t, x(t))||_\alpha + \int_0^t ||P(t - s)f(s, x(s))||_\alpha ds \\ &\leq ||A^\alpha S(t)(x_0 - h(0, x_0))|| + Lr + M_1 + \int_0^t ||A^\alpha P(t - s)f(s, x(s))||ds \\ &\leq \sup_{t\geq 0} ||A^\alpha S(t)(x_0 - h(0, x_0)|| + Lr + M_1 + \sup_{t\geq 0} \int_0^t ||A^\alpha P(t - s)\nu(s)||ds. \end{aligned}$$

Choose r such that:

$$r \geq \frac{\sup\limits_{t\geq 0} ||A^\alpha S(t)(x_0 - h(0, x_0))|| + M_1 + K}{1 - L},$$

then:

$$||(Hx)(t)||_\alpha \leq r,$$

that is, $H(B_r) \subset B_r$.

Step 2: We prove that the operator H is continuous on B_r. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B_r, such that $x_n \to x \in B_r$, as $n \to \infty$. Then:

$$\|f(s, x_n(s)) - f(s, x(s))\| \to 0, \quad n \to \infty, \tag{22}$$

because the function f is continuous on $\mathbb{R}_+ \times X_\alpha$. Given $T > 0$ and for every $t \in [0, T]$ fixed, using (H2) and (H4) we obtain:

$$\begin{aligned} ||H(x_n)(t) - H(x)(t)||_\alpha &\leq ||h(t, x_n(t)) - h(t, x(t))||_\alpha \\ &+ \int_0^t ||P(t-s)(f(s, x_n(s)) - f(s, x(s)))||_\alpha ds \\ &\leq L||x_n - x||_\infty + \int_0^t ||A^\alpha P(t-s)|| \, ||(f(s, x_n(s)) - f(s, x(s)))||ds. \end{aligned} \tag{23}$$

Now, consider $g_n(s) = ||A^\alpha P(t-s)|| \, ||(f(s, x_n(s)) - f(s, x(s))||$. By (22), we have:

$$\lim_{n\to\infty} g_n(s) = 0,$$

and by (H1)

$$g_n(s) \leq 2||A^\alpha P(t-s)||\nu(s) \in L^1([0, t[, \mathbb{R}_+)$$

which follows by (18). By the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} \int_0^t ||A^\alpha P(t-s)|| \, ||(f(s, x_n(s)) - f(s, x(s))||ds = 0,$$

and clearly $\lim_{n\to\infty} L||x_n - x||_\infty = 0$. Therefore, by (23) we have:

$$\lim_{n\to\infty} ||H(x_n)(t) - H(x)(t)||_\alpha = 0.$$

This proves that H is continuous en B_r.

Step 3: Let Ω be an arbitrary nonempty subset of B_r. We prove that:

$$\mu(H(\Omega)) \leq L\mu(\Omega).$$

Indeed, let us choose $x \in \Omega$ and t_1, t_2 with $|t_1 - t_2| \leq \epsilon$. For $0 < t_1 < t_2 \leq T$, we have:

$$\begin{aligned} ||H(x)(t_2) - H(x)(t_1)||_\alpha \ &\leq \ ||(S(t_2) - S(t_1))(x_0 - h(0, x_0))||_\alpha \\ &\quad + ||h(t, x(t_2)) - h(t, x(t_1))||_\alpha \\ &\quad + ||\int_0^{t_1} (P(t_2 - s) - P(t_1 - s))f(s, x(s))ds||_\alpha \\ &\quad + \int_{t_1}^{t_2} ||A^\alpha P(t_2 - s)||\nu(s)ds \\ &= \ I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{24}$$

As a consequence of the continuity of $\{S(t)\}_{t\geq 0}$ in the operator topology for $t > 0$, we have that

$$I_1 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

By (H3), we see that:

$$I_2 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

For I_3, we have:

$$||\int_0^{t_1} (P(t_2-s) - P(t_1-s))f(s,x(s))ds||_\alpha \leq \int_0^{t_1} ||A^\alpha(P(t_2-s) - P(t_1-s))||\nu(s)ds$$

and by (H6), we have that:

$$I_3 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

Finally, by (H1), (18), and by continuity of the integral, we have:

$$I_4 \leq \int_{t_1}^{t_2} ||A^\alpha P(t-s)||\nu(s)ds \to 0 \quad \text{as} \quad t_2 \to t_1.$$

Thus, we obtain:

$$\omega_0^T(H(\Omega)) = 0.$$

Consequently, we have:

(25) $$\omega_0(H(\Omega)) = 0.$$

Now, by our assumptions, for arbitrary fixed $t \in \mathbb{R}_+$ and $x, y \in \Omega$ we deduce that:

$$\begin{aligned} & ||Hx(t) - Hy(t)||_\alpha \\ \leq\ & ||h(t,x(t)) - h(t,y(t))||_\alpha + \int_0^t ||P(t-s)(f(s,x(s)) - f(s,y(s))ds||_\alpha ds \\ \leq\ & L||x(t) - y(t)||_\alpha + 2M_2\eta(t). \end{aligned}$$

By (19), we have:

(26) $$\limsup_{t\to\infty} \operatorname{diam}(H(\Omega))(t) \leq L \limsup_{t\to\infty} \operatorname{diam}\Omega(t).$$

Therefore, using the measure of noncompactness μ defined in (5) and using (25) and (26), we obtain:

(27) $$\mu(H(\Omega)) \leq L\mu(\Omega).$$

Step 4: We prove that the conclusion (1) is true. In fact, since $0 < L < 1$, in view of (27) and Lemma 2.2, we deduce that the operator H has a fixed point x in the ball B_r. Hence, equation (16) has at least one mild solution $x(t)$.

Step 5: We prove that the conclusion (2) is true.

Indeed, for any other mild solution $y(t)$ of equation (16), we have:

$$\begin{aligned} ||x(t) - y(t)||_\alpha &\leq ||Hx(t) - Hy(t)||_\alpha \\ &\leq L||x(t) - y(t)||_\alpha + 2M_2\eta(t). \end{aligned}$$

Then, by (19), we have:

$$\lim_{t\to\infty} ||x(t) - y(t)||_\alpha \leq \frac{2M_2}{1-L} \lim_{t\to\infty} \eta(t) = 0.$$

That is, mild solutions of (16) are globally atracttive.

□

We finish this section with the following practical criteria in order to verify condition (H6).

Proposition 3.6. *Let $A \in \Theta_\omega^\gamma(X)$, with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$, $\omega > 0$ be fixed, $a \in C^1(\mathbb{R}_+)$ such that $a(0) = 0$ and A the generator of an analytic integral resolvent $P(t)$. Then:*

$$\lim_{\tau \to t} \int_0^t ||A^\alpha(P(\tau - s) - P(t-s)||\nu(s)ds = 0, \quad \forall t > 0, \quad \alpha \in (0,1).$$

Proof. Since $A^\alpha = A^{-(1-\alpha)}A$, where $A^{-(1-\alpha)} \in L(X)$, for $\alpha \in (0,1)$, it is enough to prove that:

$$\lim_{\tau \to t} \int_0^t ||A(P(\tau - s) - P(t-s))||\nu(s)ds = 0, \quad \forall t > 0.$$

We first note the identity

$$A(P(r) - P(s)) = A\int_s^r P'(\tau)d\tau = \int_s^r AP'(\tau)d\tau.$$

By means of Lemma 2.1 in [12] we can obtain the following estimate

$$|\frac{1}{\hat{a}(\lambda)}| \leq e^{c(|\lambda - \omega|^\varsigma)} \tag{28}$$

on a sector $\Sigma(\omega, \frac{\pi}{2} + \theta)$ where $\omega > \omega_0$, $\theta < \theta_0$ are fixed but arbitrary and where $\varsigma := \frac{\pi}{\pi + 2\theta'}$ with $\theta < \theta' < \theta_0$. See also [12, Formula (2.19) p. 58]. Note that

$$AP'(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} A\widehat{P'}(\lambda)d\lambda = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} A\lambda\widehat{P}(\lambda)d\lambda, \tag{29}$$

because $a(0) = 0$, and being $\Gamma = \Gamma_1 \cup \Gamma_2$, defined by:

$$\Gamma_1 = re^{i(\frac{\pi}{2} + \theta)} + \omega, \quad \Gamma_2 = Re^{i\varphi} + \omega,$$

with $R = t^{-(1-\varsigma)^{-1}}, r > R$ and $\varphi \in [0, \frac{\pi}{2} + \theta]$. Taking Laplace's transform in both sides of (10), we obtain:

$$\widehat{P}(\lambda) = \left(\frac{1}{\widehat{a}(\lambda)} + A\right)^{-1}.$$

Using the identity $I = \left(\frac{1}{\widehat{a}(\lambda)} + A\right)\widehat{P}(\lambda)$ we have $A\lambda\widehat{P}(\lambda) = \lambda - \frac{\lambda}{\widehat{a}(\lambda)}\widehat{P}(\lambda)$. Therefore, from (29) and Cauchy's theorem, we get

$$\begin{aligned} AP'(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} A\lambda\widehat{P}(\lambda)d\lambda = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\left[\frac{\lambda}{\widehat{a}(\lambda)}\widehat{P}(\lambda) - \lambda\right]d\lambda \\ &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{\lambda}{\widehat{a}(\lambda)}\widehat{P}(\lambda)d\lambda. \end{aligned}$$

Now, by (11) and (28) we conclude that

$$\begin{aligned}
\|AP'(t)\| &= \frac{1}{2\pi}\int_{\Gamma}|e^{\lambda t}|\frac{|\lambda|}{|\widehat{a}(\lambda)|}\|\widehat{P}(\lambda)\||d\lambda| \\
&\leq \frac{C}{2\pi}\int_{\Gamma}e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^{\varsigma}}|d\lambda| \\
&= \frac{C}{2\pi}\int_{\Gamma_1}e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^{\varsigma}}|d\lambda| + \frac{C}{2\pi i}\int_{\Gamma_2}e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^{\varsigma}}|d\lambda| \\
&= \frac{C}{2\pi}\int_{R}^{\infty}e^{\omega t}e^{-rt\sin\theta}\frac{|\omega+r|}{r}e^{cr^{\varsigma}}dr + \frac{C}{2\pi i}\int_{0}^{\frac{\pi}{2}+\theta}e^{\omega t}e^{-Rt\cos\varphi}\frac{\omega+R}{R}e^{cR^{\varsigma}}Rd\varphi \\
&\leq \frac{Ce^{\omega t}}{2\pi}\Bigg[|\omega|\int_{R}^{\infty}e^{-rt\sin\theta}\, e^{cr^{\varsigma}}\frac{dr}{r} + \int_{R}^{\infty}e^{-rt\sin\theta}e^{cr^{\varsigma}}dr \\
&\qquad + R\int_{0}^{\frac{\pi}{2}+\theta}e^{Rt\cos\varphi}e^{cR^{\varsigma}}d\varphi + |\omega|\int_{0}^{\frac{\pi}{2}+\theta}e^{Rt\cos\varphi}e^{cR^{\varsigma}}d\varphi\Bigg] \\
&=: \frac{Ce^{\omega t}}{2\pi}[(I)+(II)+(III)+(IV)].
\end{aligned}$$

We observe that the term (I) appeared in [12], page 59 and can be estimated by $t^{\frac{\pi}{2\theta'}}$. For (II), we use the change of variable $b = t\sin\theta, s = rb$, and the identity 12 on page 710 of [11], together with the fact that $0 < \varsigma < 1$. We deduce that the integral is finite. Consequently, we can define, for all $t > 0$:

$$\begin{aligned}
\Psi(t) &:= \frac{Ce^{\omega t}}{2\pi i}\Bigg[|\omega|\int_{R}^{\infty}e^{-rt\sin\theta}\frac{e^{cr^{\varsigma}}}{r}dr + \int_{R}^{\infty}e^{-rt\sin\theta}e^{cr^{\varsigma}}dr \\
&\qquad + R\int_{0}^{\frac{\pi}{2}+\theta}e^{Rt\cos\varphi}e^{cR^{\varsigma}}d\varphi + |\omega|\int_{0}^{\frac{\pi}{2}+\theta}e^{Rt\cos\varphi}e^{cR^{\varsigma}}d\varphi\Bigg].
\end{aligned}$$

Therefore Ψ is a positive continuous function satisfying $\|AP'(t)\| \leq \Psi(t)$ for all $t > 0$. In this way, we conclude that:

$$\begin{aligned}
||A(P(\tau-s)-P(t-s))|| &= ||\int_{t-s}^{\tau-s}AP'(r)dr|| \leq \int_{t-s}^{\tau-s}\|AP'(r)\|dr \\
&\leq \int_{t-s}^{\tau-s}\Psi(r)dr =: \Phi(\tau).
\end{aligned}$$

where Φ is positive, increasing and $\Phi(\tau) \to 0$ as $\tau \to t$. Therefore, by the monotone convergence theorem, we obtain:

$$\begin{aligned}
\lim_{\tau\to t}\int_{0}^{t}||A(P(\tau-s)-P(t-s))||\nu(s)ds &\leq \lim_{\tau\to t}\int_{0}^{t}\int_{t-s}^{\tau-s}\Psi(r)\nu(s)drds \\
&= \int_{0}^{t}\left[\lim_{\tau\to t}\int_{t-s}^{\tau-s}\Psi(r)\right]\nu(s)drds \\
&= \int_{0}^{t}\left[\lim_{\tau\to t}\Phi(\tau)\right]\nu(s)drds = 0.
\end{aligned}$$

□

4. An application

For $0 < q < 1$, we consider the problem

$$\begin{cases} D_t^q[x(t) - h(t, x(t))] = (-A)[x(t) - h(t, x(t))] + f(t, x(t)) \\ x(0) = x_0, \end{cases} \tag{30}$$

where D_t^q denotes the fractional derivative of order $q > 0$ in the sense of Caputo and $A \in \Theta_\omega^\gamma(X)$.

Recall that, by definition, we have for any differentiable function f that $D_t^q f(t) := (g_{1-q} * f')(t)$ where $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $\beta > 0$. Then, we may convolve with g_q both sides in (30) and obtain the equivalent problem

$$\begin{cases} x(t) - h(t, x(t)) &= \int_0^t g_q(t-s)(-A)[x(s) - h(s, x(s))]ds \\ &\quad + \int_0^t g_q(t-s) f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) = x_0, \end{cases} \tag{31}$$

which takes the form of the abstract model (16) with $a(t) = g_q(t)$. Therefore, our first application retrieve the main result in [9].

Corollary 4.1. *Let $-1 < \alpha + \gamma < 0$ and $0 < \alpha < \beta < 1$ be given. Assume that:*

(F1) *$f : \mathbb{R}_+ \times X_\alpha \to X$ is continuous, and there exists a positive function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

$$\begin{cases} ||f(t, x)|| \leq \nu(t), \\ \text{the function} \quad s \to \frac{\nu(s)}{(t-s)^{1+q(\gamma+q)}} \ \text{belongs to } L^1([0, t[, \mathbb{R}_+), \end{cases} \tag{32}$$

$$\lim_{t\to\infty} \eta(t) := \lim_{t\to\infty} \int_0^t \frac{\nu(s)}{(t-s)^{1+q(\gamma+\alpha)}} ds = 0 \tag{33}$$

(F2) *The function $h : \mathbb{R}_+ \times X_\alpha \to X_\alpha$ is bounded, continuous and there exists a constant $L \in (0, 1)$ such that:*

$$||h(t_1, x(t_1)) - h(t_2, x(t_2))||_\alpha \leq L(|t_1 - t_2| + ||x(t_1) - x(t_2)||_\alpha).$$

(F3) *For each nonempty, bounded set $D \subset BC(\mathbb{R}_+, X_\alpha)$, the family of functions*

$$\{t \to h(t, \varphi(t)) : \varphi \in D\}$$

is equicontinuous.

Then:

1. *For every $x_0 \in D(A^\beta)$ with $\beta > 1 + \gamma$, the problem (30) has at least mild solution on $BC(\mathbb{R}_+, X_\alpha)$.*
2. *All solution are globally attractive.*

Proof. Since $A \in \Theta_\omega^\gamma(X)$, the operator A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$. Let $P(t)$ and $S(t)$ be operators defined by:

$$P(t) := t^{q-1}\mathcal{P}_q(t)$$
$$S(t) := \mathcal{S}_q(t),$$

where:

$$\mathcal{S}_q(t)x = \int_0^\infty \Psi_q(\sigma)T(\sigma t^q)x d\sigma, \quad t \in S^0_{\frac{\pi}{2}-\omega}, \quad x \in X,$$

$$\mathcal{P}_q(t)x = \int_0^\infty q\sigma\Psi_q(\sigma)T(\sigma t^q)x d\sigma, \quad t \in S^0_{\frac{\pi}{2}-\omega}, \quad x \in X,$$

and $\Psi_q(\sigma)$ is the function of Wright type given by

$$\Psi_q(z) \;:=\; \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-qn+1-q)} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{(n-1)!}\Gamma(nq)\sin(n\pi q), \quad z \in \mathbb{C}.$$

Define $a(t) = g_q(t)$. It is not difficult to see that $S(t)$ is a resolvent family and $P(t)$ is an integral family, both generated by $-A$. We recall from [9] the following properties:

(a) For $-1 < r < \infty$, with $q > 0$ and $-1 < \alpha + \gamma < 0$, we have:
   (1) $\Psi_q(t) \geq 0, \quad t > 0,$
   (2) $\int_0^\infty \Psi_q(t)t^r dt = \frac{\Gamma(1+r)}{\Gamma(1+qr)}$, and the following estimates hold:

$$(34)\qquad \begin{aligned} ||\mathcal{S}_q(t)x|| &\leq \tfrac{C_0\Gamma(-\gamma)}{\Gamma(1-q(1+\gamma))}t^{-q(1+\gamma)}||x||,\\ ||\mathcal{P}_q(t)x|| &\leq \tfrac{qC_0\Gamma(1-\gamma)}{\Gamma(1-q\gamma)}t^{-q(1+\gamma)}||x||,\\ ||A^\alpha\mathcal{P}_q(t)x|| &\leq \tfrac{qC'\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))}t^{-q(1+\gamma+\alpha)}||x||, \end{aligned}$$

(b) For $t > 0$, $\mathcal{P}_q(t)$ and $\mathcal{S}_q(t)$ are continuous in the uniform operator topology.

With this, the mild solution of equation (30) is a solution of the problem:

$$(35)\quad x(t) = \mathcal{S}_q(t)(x_0 - h(0,x_0)) + h(t,x(t)) + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)f(s,x(s))ds, \quad t \geq 0,$$

We will prove that, with these ingredients, all the hypothesis of Theorem 3.5 are satisfied. Indeed:

(i) For (H1), we prove that the function $s \to ||A^\alpha\mathcal{P}_q(t-s)||\nu(s)$ belongs to $L^1([0,t[,\mathbb{R}_+)$. In fact, by (34), we have:

$$||A^\alpha(t-s)^{q-1}\mathcal{P}_q(t-s)||\nu(s) \leq C\frac{\nu(s)}{(t-s)^{1+q(\gamma+q)}},$$

and in view of (32), the claim is proved. Also, by (32) and (33), we have:

$$\lim_{t\to\infty}\int_0^t ||A^\alpha P(t-s)||\nu(s)ds \leq \lim_{t\to\infty}\int_0^t \frac{\nu(s)}{(t-s)^{1+q(\gamma+\alpha)}}ds = 0.$$

(ii) The hypotheses (H2) and (H3) are the same than (F2) and (F3).

(iii) The hypothesis (H4) is a consequence of the estimates (34).

(iv) We prove (H5). Indeed, by the property a), we obtain

$$\begin{aligned} ||A^\alpha\mathcal{S}_q(t)x|| &\leq \int_0^\infty \Psi_q(\sigma)||A^\alpha T(\sigma t^q)x||d\sigma\\ &\leq K\int_0^\infty \Psi_q(\sigma)(\sigma t^q)^{-\gamma-\alpha-1}d\sigma\\ &\leq K'\int_0^\infty \Psi_q(\sigma)\sigma^{-\gamma-\alpha-1}d\sigma, \end{aligned}$$

with $r = -\gamma - \alpha - 1 > -1$, by hypothesis. Then, by property a)(2):

$$\sup_{t\geq 0} ||A^\alpha \mathcal{S}_q(t)x|| < \infty.$$

(v) We will prove (H6). In fact, we have

$$\begin{aligned}
& \int_0^{t_1} ||A^\alpha((t_2-s)^{q-1}\mathcal{P}_q(t_2-s) - (t_1-s)^{q-1}\mathcal{P}_q(t_1-s))||\nu(s)ds \\
= & \int_0^{t_1} ||(t_2-s)^{q-1}\mathcal{P}_q(t_2-s) - (t_1-s)^{q-1}\mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds \\
\leq & \int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1} - (t_1-s)^{q-1}]||_\alpha \nu(s)ds \\
+ & \int_0^{t_1} ||(t_1-s)^{q-1}[\mathcal{P}_q(t_2-s) - \mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds.
\end{aligned}$$

By (34), we obtain:

$$\begin{aligned}
& \int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1} - (t_1-s)^{q-1}]||_\alpha \nu(s)ds \\
\leq & \; qC' \tfrac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha)} \int_0^{t_1} \frac{|(t_2-s)^{q-1} - (t_1-s)^{q-1}|}{(t_2-s)^{q-1}} \frac{\nu(s)}{(t_2-s)^{1+q(\gamma+\alpha)}} ds.
\end{aligned}$$

Therefore, by (32), we get:

$$\int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1} - (t_1-s)^{q-1}]||_\alpha \nu(s)ds \to 0, \quad \text{as} \quad t_2 \to t_1.$$

Furthermore, for $\epsilon > 0$, small enough, we obtain:

$$\begin{aligned}
& \int_0^{t_1} ||(t_1-s)^{q-1}[\mathcal{P}_q(t_2-s) - \mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds \\
\leq & \; q\int_0^{t_1}\int_0^\infty \sigma\Psi_q(\sigma)(t_1-s)^{q-1}||T((t_2-s)^q\sigma) - T((t_1-s)^q\sigma||_\alpha \nu(s)d\sigma ds \\
\leq & \; q\int_0^{t_1-2\epsilon}\int_0^\infty \sigma\Psi_q(\sigma)(t_1-s)^{q-1}||T((t_2-s)^q\sigma - \epsilon^q\sigma) - T((t_1-s)^q\sigma - \epsilon^q\sigma)||\times \\
& \qquad \times||A^\alpha T(\epsilon^q\sigma)||\nu(s)d\sigma ds \\
& +M_2\int_{t_1-2\epsilon}^{t_1}\left(\frac{(t_1-s)^{q-1}}{(t_1-s)^{q(\alpha+\gamma+1)}} + \frac{(t_1-s)^{q-1}}{(t_2-s)^{q(\alpha+\gamma+1)}}\right)\nu(s)ds \\
\leq & \; \frac{qC'}{\epsilon^{q(\gamma+q+1)}}\int_0^{t_1-2\epsilon}\int_0^\infty \sigma^{-(\gamma+q)}\Psi_q(\sigma)||T((t_2-s)^q\sigma - \epsilon^q\sigma) - T((t_1-s)^q\sigma - \epsilon^q\sigma)||\times \\
& \qquad \times\tfrac{\nu(s)}{(t_1-s)^{1-q}}d\sigma ds \\
& +M_2\int_{t_1-2\epsilon}^{t_1}\left(\frac{(t_1-s)^{q-1}}{(t_1-s)^{q(\alpha+\gamma+1)}} + \frac{(t_1-s)^{q-1}}{(t_2-s)^{q(\alpha+\gamma+1)}}\right)\nu(s)ds \\
= & \; J_1 + J_2.
\end{aligned}$$

The continuity of the function $t \to ||T(t)||$ for $t \in (0, T)$ implies that:

$$J_1 \to 0 \quad \text{as} \quad t_2 \to t_1.$$

Furthermore, it is easy to see that:

$$J_2 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Therefore:

$$\lim_{t_2\to t_1}\int_0^{t_1} ||A^\alpha((t_2-s)^{\alpha-1}\mathcal{P}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathcal{P}_q(t_1-s))||\nu(s)ds=0.$$

□

References

[1] J. O. Alzabut and T. Abdeljawad. On existence of a globally attractive periodic solution of impulsive delay logarithmic population model. *Appl. Math. Comput.* 198 (1) (2008), 463–469.

[2] J. Banaś and K. Goebel. Measures of Noncompactness in Banach space, Lecture Notes in Pure and Applied Mathematics, vol 60, Marcel-Dekker, New York, 1980.

[3] M. V. Bartuccelli, J. H. B. Deane and G. Gentile. Guido Globally and locally attractive solutions for quasi-periodically forced systems. *J. Math. Anal. Appl.* 328 (1) (2007), 699–714.

[4] Ph. Clément and J. A. Nohel. Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal.* 12 (1981), 514–534.

[5] B. de Andrade and A. Viana. Abstract Volterra integrodifferential equations with applications to parabolic models with memory. *Math. Ann.* 369 (3-4) (2017), 1131–1175.

[6] J. Kemppainen, J. Siljander, V. Vergara and R. Zacher. Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}$. *Math. Ann.* 366 (3-4) (2016), 941–979.

[7] A.N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* 340 (2008), 252–281.

[8] J.W. Li and S. Cheng. Globally attractive periodic solution of a perturbed functional differential equation. *J. Comput. Appl. Math.* 193 (2) (2006), 652–657.

[9] J. Liang, S-H. Yan, F. Li, T-W. Huang. On the existence of mild solutions to the Cauchy problem for a class of fractional evolution equation. *Adv. Diff. Equ.* 2012 (2012), 1–27.

[10] F. Periago, B. Straub. A functional calculus for almost sectorial operators an applications to abstract evolution equations. *J.Evol. Equ.* 2 (2002), 41–68.

[11] A. P. Prudnikov, Y. A. Brychkov, and O.I. Marichev. Integrals and Series. Vol 1. Taylor and Francis, London, 2002.

[12] J. Prüss. Evolutionary Integral Equations and Applications. Birkhäuser, Basel, 1993.

[13] J. Prüss. Positivity and regularity of hyperbolic Volterra equations in Banach spaces. *Math. Ann.* 279 (2) (1987), 317–344.

[14] X. H. Tang. Asymptotic behavior of delay differential equations with instantaneously terms. *J. Math. Anal. Appl.* 302 (2) (2005), 342–359.

[15] V. Vergara, R. Zacher. Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. *SIAM J. Math. Anal.* 47 (1) (2015), 210–239.

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