On close to scalar families for fractional evolution equations: Zero-one law

Laura R. Gambera[1]

São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences (Ibilce), Departamento de Matemática, R. Cristóvâo Colombo, 2265 - Jardim Nazareth, São José do Rio Preto - SP, 15054-000, Brazil.

Carlos Lizama[2]

Universidad de Santiago de Chile (Usach), Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Las Sophoras 173, Estación Central, CP 9160000, Santiago, Chile.

Andrea Prokopczyk[1]

São Paulo State University (Unesp), Institute of Biosciences, Humanities and Exact Sciences (Ibilce), Departamento de Matemática, R. Cristóvâo Colombo, 2265 - Jardim Nazareth, São José do Rio Preto - SP, 15054-000, Brazil.

Abstract

For $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$ being a strongly continuous (α,β)-resolvent family on a Banach space, we show that the assumption $\sup_{t>0}\|\frac{1}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})}R_{\alpha,\beta}(t)-I\| =: \theta < 1$ yields that $R_{\alpha,\beta}(t)=t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})$ for all $t>0$ and $\lambda \geq 0$, where $E_{\alpha,\beta}(s)$ denotes the Mittag-Leffler function with parameters $\alpha,\beta>0$. This implication is known as the zero-one law. In particular, we provide new insights on the structural properties of the theories of C_0-semigroups, strongly continuous cosine families, β-times integrated semigroups and α-resolvent families, among others. For β-times integrated semigroups, an example shows that in the sector $\{(\alpha,\beta) : 0<\alpha\leq 1;\ \alpha\leq\beta\}$ the requirement $\theta<1$ is optimal: for $\theta=1$ the result is false.

Keywords: One parameter families of bounded operators, cosine families, C_0-semigroups, β-times integrated semigroups, α-resolvent families, one-zero law.
MSC2010: 47D09; 47D62; 47D06; 34G10.

1. Introduction

Let $\alpha>0, \beta>0$ be given. Let A be a closed linear operator with domain $D(A)$ defined on a complex Banach space X. In this paper, we will study an intriguing structural property of a class of strongly continuous family of bounded and linear operators $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$, which commute with A on $D(A)$, and satisfy the following resolvent equation

$$R_{\alpha,\beta}(t)x = g_{\beta}(t)x + A\int_0^t g_{\alpha}(t-s)R_{\alpha,\beta}(s)xds, \quad x\in X, \quad t\geq 0, \tag{1.1}$$

where $g_{\gamma}(t) := \frac{t^{\gamma-1}}{\Gamma(\gamma)}$, $\gamma>0$, and Γ denotes the Gamma function. This family contains several important classes of well-known subfamilies. Some of them are: Strongly continuous semigroups ($\alpha=\beta=1$) [2, Section 3.1]; Strongly continuous cosine families ($\alpha=2$ and $\beta=1$)[2, Section 3.4]; β-times integrated semigroups and cosine families ($\alpha=1$, $\beta>1$ and $\alpha=2$, $\beta>1$, respectively) [2, Section 3.2]; α-times resolvent families ($\beta=1$ and $\alpha>0$)[32] (also called fractional resolvent families [19] or solution families

Email addresses: `laura.rgambera@gmail.com; Tel. 55-17-99169-3155` (Laura R. Gambera), `carlos.lizama@usach.cl; Tel: 56-2-27181903` (Carlos Lizama), `andreacp@ibilce.unesp.br; Tel: 55-17-99126-6649` (Andrea Prokopczyk)

[1]Partially supported by CAPES.
[2]Corresponding author. Partially supported by Concurso Apoyo Asistencia Eventos 2017, DICYT, Universidad de Santiago de Chile, USACH, and FONDECYT grant 1180041.

[20]), among other intermediate classes of current interest. Although several qualitative properties are well known for the class of semigroups and cosine families, much less has been reported in the setting of integrated semigroups and, specially, α-times resolvent families.

The concept of α-times resolvent families, or solution operators, plays an important role in the theory of fractional abstract Cauchy problems, that models several physical phenomena. One example is the fractional diffusion-wave equation

$$D_t^{\alpha} u(x,t) = k^2 u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha < 2, \quad k \in \mathbb{R}, \tag{1.2}$$

with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$ (the last one only when $1 < \alpha < 2$). As seen in [4, Example 3.6], the explicit form of the α-times resolvent family is

$$R_{\alpha,1}(t)f(x) = \frac{1}{2|k|t^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} \phi_{\frac{\alpha}{2}}\left(\frac{|s|}{|k|t^{\frac{\alpha}{2}}}\right) f(x-s)ds,$$

where $\phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}$ is a function of Wright type.

The fractional diffusion equation (1.2) ($\alpha \in (0,1)$) has been introduced by Nigmatullin in [28] to describe diffusion in special types of porous media. Mainardi [24] has shown that the fractional wave equation (1.2) ($\alpha \in (1,2)$) governs the propagation of mechanical diffusive waves in viscoelastic media.

Some general properties are well-known. For example, to relate different α-resolvent families and its generators, Li, Chen and Li [19] shown that if the operator A generates a bounded α-times resolvent family then, with some suitable β, $-A^{\beta}$ also generates an α-times resolvent family. There also exists a principle of subordination (see [4, Chapter 3]). For instance, if an operator A generates a cosine operator family then it also generates an α-resolvent family for any $0 < \alpha < 2$, but the converse is not true. Moreover, if we consider for any $\alpha \in (0,2)$ and $\theta \in [0,\pi)$ the differential operator $B_\theta = e^{i\theta}\partial_{xx}^2$ with $D(B_\theta) = \{g \in W^{2,2}(0,1), g(0) = g(1) = 0\}$ on $X = L^2(0,1)$, then we have that B_θ generates a bounded α-resolvent family if and only if $|\theta| \le \left(1 - \frac{\alpha}{2}\right)\pi$. However, for $\frac{\pi}{2} < |\theta| \le \left(1 - \frac{\alpha}{2}\right)\pi$, the operator B_θ does not generates any C_0-semigroup [4, Section 2.2].

Concerning the differences of structure among the various subfamilies $\{R_{\alpha,\beta}(t)\}_{t\ge 0}$, it has been recently proven that, if the set of all bounded strongly continuous cosine families is treated as a metric space under the metric of the uniform convergence associated with the operator norm on the space $\mathcal{B}(X)$ of all bounded linear operators on X, then the isolated points of this set are precisely the scalar cosine families [7]. By definition, a scalar cosine family is a family whose members are all scalar multiples of the identity operator. Remarkably, this picture changes dramatically when we turn our attention to semigroups of operators. In such case, the isolated points constitute only a small fraction of the set of all scalar semigroups. The proof of this and related properties relies on the fact that if the distance between cosine families (resp. semigroups) and their scalar counterparts is less than a certain bound, say γ, then the cosine family (resp. semigroup) must be scalar. They are called $0-\gamma$ Laws. We observe that only recently the problem to determine the optimal bound for cosine families was solved, obtaining $\gamma = \frac{8}{3\sqrt{3}}$ [5, 9, 10, 12, 13, 35]. It is surprising that a corresponding result for integrated semigroups and sine families ($\alpha = \beta = 2$) has just been discovered in 2017 [6]. Motivated by the above earlier works, we want to know if similar phenomena can also happen when considering more general classes of families of bounded operators. Then it is natural to ask the following:

(Q) Can we find $0-\gamma$ Laws for the class of strongly continuous families of bounded operators defined by (1.1)?.

In this paper we will answer this question in the affirmative at least for the case $\gamma = 1$. More precisely, we prove:

Theorem 1.1. *Let $0 < \alpha \le 2$ and $\beta > 0$ be given and let $(R_{\alpha,\beta}(t))_{t\ge 0}$ be an (α,β)- resolvent family generated by A. If*

$$\sup_{t>0} \left\| \frac{1}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})} R_{\alpha,\beta}(t) - I \right\| =: \theta < 1,$$

then $R_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})I$ for all $t > 0$ and $\lambda \ge 0$.

We recall that the special functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \quad \alpha, \beta \in \mathbb{C}, \quad \Re\alpha > 0, \; \Re\beta > 0, \; z \in \mathbb{C}, \tag{1.3}$$

are called Mittag-Leffler functions. The former was introduced by Mittag-Leffler in connection with his method of summation of some divergent series. The main properties of these functions are given in the book by Erdélyi [11, Section 18.1]. The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems.

As a second contribution of this paper, we will give an example to show that the optimal bound, say θ, in case of $\beta \geq \alpha$ and $0 < \alpha \leq 1$ is strictly less than 1, which interpretes the fact that, roughly speaking, β-times integrated α-resolvent families have more regularity at $t = 0$ than (α, β)-resolvent families, when they are over the diagonal $\beta = \alpha$, and in this way widely improving a recent result for the very special case $\alpha = 1, \beta = 2$ [6]. See example 3.6 below.

It should be noted that the method used in this paper comes at the cost of losing best constants. The optimal bound for arbitrary (α, β)-resolvent families is a very much difficult task and is left open.

2. Preliminaries

Let Y be a Banach space. For a vector-valued function $f : \mathbb{R}_+ \to Y$ we recall that the Riemann-Liouville fractional integral of order $\beta \geq 0$ is defined by

$$J_t^{\beta} f(t) = (g_\beta * f)(t) := \int_0^t g_\beta(t-s) f(s) ds,$$

where $g_0(t) := \delta(t)$, the Dirac delta, and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $t > 0$. We begin with a purely algebraic notion of the theory of (α, β)-resolvents of bounded and linear operators [23].

Definition 2.1. Let X be a Banach space and $\alpha > 0, \beta > 0$. A one parameter family $\{R_{\alpha,\beta}(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called an (α, β)-resolvent family if the following conditions are satisfied:

(a) $\lim_{t \to 0} t^{1-\beta} R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)} I$ if $0 < \beta < 1$, $R_{\alpha,1}(0) = I$ and $R_{\alpha,\beta}(0) = 0$ if $\beta > 1$.

(b) $R_{\alpha,\beta}(s) R_{\alpha,\beta}(t) = R_{\alpha,\beta}(t) R_{\alpha,\beta}(s)$ for all $s, t > 0$;

(c) The functional equation

$$R_{\alpha,\beta}(s) J_t^{\alpha} R_{\alpha,\beta}(t) - J_s^{\alpha} R_{\alpha,\beta}(s) R_{\alpha,\beta}(t) = g_\beta(s) J_t^{\alpha} R_{\alpha,\beta}(t) - g_\beta(t) J_s^{\alpha} R_{\alpha,\beta}(s),$$

holds for all $t, s > 0$.

The integrals in (c) are understood strongly in the sense of Bochner. We observe the remarkable fact that in the scalar case, i.e. $X = \mathbb{C}$, and taking $\beta = 1$, we have that the Mittag-Leffler function $E_{\alpha,1}(zt^{\alpha}), z \in \mathbb{C}$, satisfies the functional equation (c). See [23, Example 3.10]. In particular, it shows that the functional equation (c) is a proper generalization of Cauchy's functional equation (corresponding to the case $\alpha = \beta = 1$) and D'Alembert functional equation (corresponding to the case $\alpha = 2$ and $\beta = 1$) because for $\alpha = 1$ we have $E_{1,1}(zt) = e^{zt}$ and for $\alpha = 2$ we have $E_{2,1}(zt^2) = \cosh(\sqrt{z}\, t)$, $z \in \mathbb{C}$.

Remark 2.2. For $0 < \alpha < 1$ and $\beta = 1$ an equivalent functional equation to (c) was proposed by Peng and Li [29, 30]. See [23, Remark 3.11]. Other equivalent representation involving the sum $R_{\alpha,1}(t + s)$ appears in [26].

We observe that $(\alpha,1)$-resolvent families are called *α-resolvent families,* or solution operator, or fractional resolvent family/operator in the current literature. For $0<\alpha=\beta<1$ the above definition was studied by Li and Peng [21]. We note that this concept was introduced earlier [1], but without reference to the condition near to zero given in (a).

The linear operator A defined by

$$D(A) := \{x \in X \,:\, \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ exists } \}$$

and

$$Ax := \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ for } x \in D(A)$$

is called the *generator* of the (α,β)-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.

For example, if A is a bounded operator, then

$$R_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} g_{\alpha n+\beta}(t)A^n = t^{\beta-1}\sum_{n=0}^{\infty}\frac{A^n t^{\alpha n}}{\Gamma(\alpha n+\beta)} = t^{\beta-1}E_{\alpha,\beta}(At^\alpha), \quad t>0,$$

defines a uniformly continuous (α,β)-resolvent family. Given $\beta>1$, observe that the family $\{R_{\alpha,\beta}(t)\}_{t>0}$ is $(\beta-1)$-times integrated with respect to $\{R_{\alpha,1}(t)\}_{t\geq 0}$ as the identity

$$R_{\alpha,\beta}(t) = (g_{\beta-1} * R_{\alpha,1})(t) = J_t^{\beta-1}R_{\alpha,1}(t), \quad t>0, \tag{2.1}$$

holds. The following characterization is often used as definition.

Theorem 2.3. *[23, Theorem 3.1 and Theorem 4.3] Let $\alpha>0$ and $\beta>0$ be given. A strongly continuous family $\{R_{\alpha,\beta}(t)\}_{t>0}\subset\mathcal{B}(X)$ of bounded linear operators in X is an (α,β)-resolvent family generated by A if and only if the following conditions are satisfied*

(i) $\lim_{t\to 0} t^{1-\beta}R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)}I$ *if* $0<\beta<1$, $R_{\alpha,1}(0)=I$ *and* $R_{\alpha,\beta}(0)=0$ *if* $\beta>1$.

(ii) $R_{\alpha,\beta}(t)x\in D(A)$ *and* $R_{\alpha,\beta}(t)Ax = AR_{\alpha,\beta}(t)x$ *for all* $x\in D(A)$ *and* $t\geq 0$;

(iii) $R_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AR_{\alpha,\beta}(s)x ds, \quad t\geq 0, \quad x\in D(A).$

For $\beta>1$, we have that $D(A)$ is closed, but not necessarily densely defined [8, Proposition 3.10]. In the diagonal case $\alpha=\beta$ this notion appears by the first time in [1, Definition 2.3]. If $0<\alpha=\beta<1$ then A must be densely defined [21, Theorem 3.1].

Roughly speaking, the notion of $(\alpha,1)$-resolvent families is associated with the Caputo fractional derivative, whereas the notion of (α,α)-resolvent family is linked with the Riemann-Liouville fractional derivative. Other relevant cases are $(\alpha,\gamma+(1-\gamma)\alpha)$-resolvent families with $0<\alpha<1,\ 0\leq\gamma\leq 1$, see [16], and $(\alpha,\alpha+\gamma(2-\alpha))$-resolvent families with $1<\alpha<2,\ 0\leq\gamma\leq 1$, see [25], because they are related with the notion of Hilfer fractional derivative that interpolates between the Caputo and Riemann-Liouville fractional derivative (for $0<\alpha<1$ take $\gamma=1$ and $\gamma=0$, respectively).

Assuming that A is the generator of an γ-times integrated semigroup $(\gamma\geq 0)$, i.e. an $(1,\gamma+1)$-resolvent family, then $(\alpha,\alpha\gamma+1)$-resolvent families and $(\alpha,\alpha(\gamma+1))$-resolvent families are important for $0<\alpha<1$ because these are the key for the treatment of existence, regularity and representation of fractional diffusion equations, see [18]. The same happens with $(\alpha,\frac{\alpha\gamma}{2}+1)$-resolvent families and $(\alpha,\alpha(\frac{\gamma}{2}+1))$-resolvent families for $1\leq\alpha\leq 2$ because these are present in the theoretical analysis of fractional wave equations, see [17].

3. Main results

In what follows we denote by $E_{\alpha,\beta}(z)$ the Mittag-Leffler function with parameters $\alpha,\beta>0$. Some simple examples are: $E_{1,1}(z)=e^z,\ E_{2,1}(z)=\cosh(\sqrt{z}),\ E_{1,2}(z)=\dfrac{e^z-1}{z}$ and $E_{2,2}(z)=\frac{\sinh\sqrt{z}}{\sqrt{z}}$. The following is the main result of this paper.

Theorem 3.1. *Let $0 < \alpha \le 2$ and $\beta > 0$ be given and let $(R_{\alpha,\beta}(t))_{t\ge 0}$ be an (α,β)- resolvent family generated by A. If*

$$\sup_{t>0} \| \frac{1}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)} R_{\alpha,\beta}(t) - I \| =: \theta < 1, \tag{3.1}$$

then $R_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)I$ for all $t > 0$ and $\lambda \ge 0$.

Proof. For all $t \ge 0$, $x \in X$ and $\lambda \ge 0$ as given in the hypothesis, we define $B(t)x := \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)R_{\alpha,\beta}(\tau)x d\tau$. From [22, Theorem 5.3], we have that $B(t)x \in D(A)$ and

$$(\lambda - A)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)R_{\alpha,\beta}(\tau)x d\tau = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x - R_{\alpha,\beta}(t)x. \tag{3.2}$$

Denote

$$b_{\alpha,\beta}(\lambda,t) := \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)E_{\alpha,\beta}(\lambda\tau^\alpha)d\tau.$$

Since $\lambda \ge 0$ we obtain from (1.3) that $b_{\alpha,\beta}(\lambda,t) > 0$ for $t > 0$ and we have the following estimate

$$\begin{aligned}
&\left\| x - \frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t)x \right\| \\
&= \left\| \frac{1}{b_{\alpha,\beta}(\lambda,t)} \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)(\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)x - R_{\alpha,\beta}(\tau)x)d\tau \right\| \\
&= \left\| \frac{1}{b_{\alpha,\beta}(\lambda,t)} \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)\left[x - \frac{R_{\alpha,\beta}(\tau)x}{\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)}\right]d\tau \right\| \\
&\le \frac{1}{b_{\alpha,\beta}(\lambda,t)} \sup_{\tau\ge 0}\left\| I - \frac{R_{\alpha,\beta}(\tau)}{\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)}\right\| \|x\| \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)E_{\alpha,\beta}(\lambda\tau^\alpha)d\tau \\
&\le \theta\|x\|.
\end{aligned}$$

Thus, since $\theta < 1$, it follows that the operator $\frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t)$ is boundedly invertible for all $t > 0$ and

$$\left\| b_{\alpha,\beta}(\lambda,t)(B(t))^{-1} \right\| \le \sum_{k=0}^{\infty} \|I - \frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t)\|^k \le \sum_{k=0}^{\infty}\theta^k = \frac{1}{1-\theta},$$

which implies that

$$\|(B(t))^{-1}\| \le \frac{1}{(1-\theta)b_{\alpha,\beta}(\lambda,t)}, \tag{3.3}$$

in the norm of $\mathcal{B}(X)$. From (3.1) we have that

$$\begin{aligned}
\|(\lambda - A)B(t)x\| &= \|R_{\alpha,\beta}(t)x - t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x\| = \|t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)[\frac{R_{\alpha,\beta}(t)}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)} - I]x\| \\
&\le \theta t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\|x\|,
\end{aligned}$$

for all $t > 0$ and $x \in X$. Hence, by (3.3) we obtain for each $x \in D(A)$

$$\|(\lambda - A)x\| = \|(\lambda - A)B(t)(B(t)^{-1}x)\| \le \frac{\theta\, t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)}{(1-\theta)b_{\alpha,\beta}(\lambda,t)}\|x\|, \ \forall t > 0. \tag{3.4}$$

Now, we observe that

$$b_{\alpha,\beta}(\lambda,t) = t^{\alpha+\beta-1}\frac{d}{dt}E_{\alpha,\beta}(\lambda t^\alpha), \tag{3.5}$$

which can be checked using the Laplace transform in both sides of the above equality and taking into account the following identity [31]:

$$\int_0^\infty e^{-\mu t}t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{k!\mu^{\alpha-\beta}}{(\mu^\alpha \mp \omega)^{k+1}}, \quad Re(\mu) > |\omega|^{1/\alpha}. \tag{3.6}$$

Inserting (3.5) in (3.4) we obtain

$$\|(\lambda - A)x\| \leq \frac{\theta\, E_{\alpha,\beta}(\lambda t^\alpha)}{(1-\theta)t^\alpha E'_{\alpha,\beta}(\lambda t^\alpha)}\|x\|, \qquad \forall t > 0. \tag{3.7}$$

We notice that the asymptotic expansion 18.1(22) p. 210 in [11] shows that

$$E_{\alpha,\beta}(z) \sim \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} \quad |z| \to \infty, \quad |\arg z| < \frac{\alpha\pi}{2}.$$

Moreover, from [15] we have

$$E'_{\alpha,\beta}(z) = \frac{1}{\alpha z} E_{\alpha,\beta-1}(z) - (\beta - 1)E_{\alpha,\beta}(z).$$

Hence $E'_{\alpha,\beta}(z) \sim \frac{1}{\alpha^2} z^{\frac{1-\beta-\alpha}{\alpha}} e^{z^{\frac{1}{\alpha}}}[(1-\beta) + z^{\frac{1}{\alpha}}]$. In particular, we deduce that

$$\frac{E_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)} \sim \frac{\alpha z}{(1-\beta) + z^{\frac{1}{\alpha}}} \quad |z| \to \infty, \quad |\arg z| < \frac{\alpha\pi}{2}.$$

Consequently,

$$\frac{E_{\alpha,\beta}(\lambda t^\alpha)}{t^\alpha E'_{\alpha,\beta}(\lambda t^\alpha)} \sim \frac{\alpha\lambda}{(1-\beta) + \lambda^{\frac{1}{\alpha}} t}, \quad t \to \infty.$$

We conclude that by taking $t \to \infty$ in (3.7), we get $Ax = \lambda x$ for all $x \in D(A)$. Finally, because for each $x \in X$ and $t > 0$ we have $B(t)x \in D(A)$ we deduce from (3.2) that

$$R_{\alpha,\beta}(t)x = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x, \quad x \in X, \quad t \geq 0.$$

□

Considering the special case $\lambda = 0$, we obtain the following important consequence that we state as a Theorem.

Theorem 3.2. *Let $0 < \alpha \leq 2$ and $\beta > 0$ be given and let $(R_{\alpha,\beta}(t))_{t\geq 0}$ be an (α,β)- resolvent family generated by A. If*

$$\sup_{t\geq 0} \|\frac{1}{g_\beta(t)} R_{\alpha,\beta}(t) - I\| < 1, \tag{3.8}$$

then $R_{\alpha,\beta}(t) = g_\beta(t)I$ for all $t > 0$.

Observe that the result is only dependent of $\beta > 0$, which interpretes the fact that one-zero laws takes into account the regularizing effect of the parameter β on the family of operators near to zero.

As a corollary of Theorem 3.2 we retrieve the zero-one law for C_0-semigroups (*i.e.* $\alpha = \beta = 1$) that can be found for example in [35, Theorem 3.2]. See also [36, Remark 3.1.4].

Corollary 3.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A. Suppose that*

$$\sup_{t>0} \|T(t) - I\| < 1.$$

Then $T(t) = I$ for all $t \geq 0$.

From the literature, a zero-two law [35] for cosine families, and also a $0 - \frac{3}{2}$ law [3, Theorem 1.1] for cosine families on general Banach spaces was proved without considering strong continuity. See also the reference [10]. In [9], Chojnacki gives an extension of the results from [35] in the case of cosine families, not necessarily continuous, in a normed algebra.

We observe that a zero-one law for strongly continuous cosine families was proved in [34, Theorem 1.1]. As a consequence of our Theorem 3.2, we retrieve such zero-one law for cosine families (*i.e.* $\alpha = 2, \beta = 1$) as follows.

Corollary 3.4. *Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family generated by A. Suppose that*

$$\sup_{t>0}\|C(t)-I\|<1.$$

Then $C(t)=I$ for all $t\geq 0$.

Remark 3.5. The case $\alpha=1$ and $\beta=2$ in Theorem 3.2 is of particular interest. In such case the family $(R_{1,2}(t))_{t\geq 0}$ corresponds to an integrated semigroup and hence Theorem 3.2 coincides with recent results of Bobrowski [6, Theorem 2.3]. We note that the findings of Bobrowski are stated in an arbitrary unital Banach Algebra.

It is interesting to observe that for the range $\beta\geq\alpha$ and $0<\alpha\leq 1$ in Theorem 3.2, the bound $\theta=1$ is optimal. The following example inspired in [6, Example 2.4] shows this fact:

Example 3.6. For $\beta\geq\alpha$, $0<\alpha\leq 1$ and $\lambda>0$ we consider the scalar (α,β)-resolvent family defined by

$$R_{\alpha,\beta}(t):=\int_0^t g_{\beta-\alpha}(t-s)E_{\alpha,1}(-\lambda s^{\alpha})ds=t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha}),$$

where the last identity can be proved using (3.6). In other words, $R_{\alpha,\beta}(t)$ is an (α,β)-resolvent family with generator $-\lambda$. For all $t>0$ we have

$$\begin{aligned}|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1|&=|\frac{1}{g_\beta(t)}\int_0^t g_{\beta-\alpha}(t-s)E_{\alpha,1}(-\lambda s^{\alpha})ds-1|\\&=|\frac{1}{g_\beta(t)}t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})-1|.\end{aligned}$$

Now, we notice from [33] that the function $x\to E_{\alpha,\beta}(-x)$ is completely monotone for all $0<\alpha\leq 1$ and $\beta\geq\alpha$. Therefore $E_{\alpha,\beta}(-\lambda t^{\alpha})\geq 0$ for all $t\geq 0$. In particular, $R_{\alpha,\beta}(t)\geq 0$.

From the identity

$$R_{\alpha,\beta}(t)=g_\beta(t)-\lambda\int_0^t g_\alpha(t-s)R_{\alpha,\beta}(s)ds,$$

we deduce $\frac{1}{g_\beta(t)}t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})-1=-\frac{\lambda}{g_\beta(t)}\int_0^t g_\alpha(t-s)R_{\alpha,\beta}(s)ds\leq 0$. Consequently

$$|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1|=1-\frac{1}{g_\beta(t)}t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\leq 1.$$

Then, $\sup_{t>0}|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1|\leq 1$.

Conversely, from [31, Theorem 1.6, p.35] we have that for any $0<\alpha<2$ and $\beta>0$ there exists a constant $C>0$ such that, for each $\frac{\alpha\pi}{2}<\mu<\min\{\pi,\alpha\pi\}$ the estimate

$$|E_{\alpha,\beta}(z)|\leq\frac{C}{1+|z|}\qquad \mu<|\arg(z)|\leq\pi,$$

holds. This shows that $\lim_{t\to\infty}E_{\alpha,\beta}(-\lambda t^{\alpha})=0$. Hence

$$\begin{aligned}\sup_{t>0}|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1|&=\sup_{t>0}|\Gamma(\beta)E_{\alpha,\beta}(-\lambda t^{\alpha})-1|\\&\geq\lim_{t\to\infty}|\Gamma(\beta)E_{\alpha,\beta}(-\lambda t^{\alpha})-1|=1.\end{aligned}$$

Therefore,

$$\sup_{t>0}|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1|=1,$$

while $R_{\alpha,\beta}(t)\neq g_\beta(t)$, proving that $\theta<1$ is optimal.

Remark 3.7. The question of whether we could extend the validity of the example to the range $\beta \geq \alpha$ and $1 < \alpha \leq 2$, is left as an open problem.

Consider α-resolvent families $S_\alpha(t)$ (*i.e.* $0 < \alpha \leq 2, \beta = 1$). As a simple corollary we obtain the following zero-one law, which constitutes a completely new result.

Corollary 3.8. *Let* $0 < \alpha \leq 2$ *and* $(S_\alpha(t))_{t\geq 0}$ *be an* α*-resolvent family generated by* A*. Suppose that*

$$\sup_{t>0} \|S_\alpha(t) - I\| < 1.$$

Then $S_\alpha(t) = I$ *for all* $t \geq 0$*.*

In what follows we suppose $\lambda \neq 0$. We will highlight some of consequences of Theorem 3.1. We begin with the following extension of Corollary 3.3, which seems to be new in the present form although is also an easy consequence of Wallen's formula, see [7, Lemma 10].

Corollary 3.9. *Let* $(T(t))_{t\geq 0}$ *be a* C_0*-semigroup generated by* A*. Let* $\lambda \geq 0$ *be given and suppose that*

$$\sup_{t>0} \|e^{-\lambda t}T(t) - I\| < 1.$$

Then $T(t) = e^{\lambda t}I$ *for all* $t \geq 0$*.*

In the following corollary, we show a new result about the zero-one law for cosine families (see [34]).

Corollary 3.10. *Let* $(C(t))_{t\geq 0}$ *be a strongly continuous cosine family generated by* A*, and* $\lambda > 0$ *be given. Suppose that*

$$\sup_{t\geq 0} \|\frac{1}{\cosh(\sqrt{\lambda}t)}C(t) - I\| < 1.$$

Then $C(t) = \cosh(\sqrt{\lambda}t)I$ *for all* $t \geq 0$*.*

The following is an extension of the zero-one law to α-resolvent families. This result is also new.

Corollary 3.11. *Let* $0 < \alpha \leq 2$ *and* $(S_\alpha(t))_{t\geq 0}$ *be a* α*-resolvent family generated by* A*. Given* $\lambda > 0$ *suppose that*

$$\sup_{t\geq 0} \|\frac{1}{E_{\alpha,1}(\lambda t^\alpha)}S_\alpha(t) - I\| < 1.$$

Then $S_\alpha(t) = E_{\alpha,1}(\lambda t^\alpha)I$ *for all* $t \geq 0$*. Moreover,* $\theta < 1$ *is optimal in the range* $0 < \alpha \leq 1$*.*

References

[1] D. Araya and C. Lizama. *Almost automorphic mild solutions to fractional differential equations.* Nonlinear Analysis, Ser. A, TMA, 69 (2008), 3692-3705.

[2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. Vector-valued Laplace transforms and Cauchy problems. Second edition. Monographs in Mathematics, 96. Birkhäuser/Springer Basel AG, Basel, 2011.

[3] W. Arendt. *A* $0 - 3/2$ *- Law for Cosine Functions.* Ulmer Seminare, Funktionalanalysis und Evolutionsgleichungen, 17 (2012), 349-350.

[4] E. Bazhlekova. *Fractional evolution equations in Banach spaces*, PhD thesis, University Press Facilities, Eindhoven University of Technology, 2001.

[5] A. Bobrowski, W. Chojnacki and A. Gregosiewicz. *On close-to-scalar one-parameter cosine families.* J. Math. Anal. Appl. 429 (1) (2015), 383–394

[6] A. Bobrowski. *On Wallen-type formulae for integrated semigroups and sine functions.* Arch. Math. (Basel) 108 (4) (2017), 395–404.

[7] A. Bobrowski and W. Chojnacki. *Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space.* Studia Math. 217 (3) (2013), 219–241.

[8] C. Chen and M. Li. *On fractional resolvent operator functions.* Semigroup Forum, 80 (2010), 121–142.

[9] W. Chojnacki. *Around Schwenninger and Zwart's zero-two law for cosine families.* Arch. Math. (Basel) 106 (6) (2016), 561–571.

[10] W. Chojnacki. *On cosine families close to scalar cosine families.* J. Aust. Math. Soc. 99 (2) (2015), 166–174.

[11] A. Erdelyi. Higher Transcendental Functions. Vol III. McGraw-Hill, New-York, 1953.

[12] J. Esterle. *Bounded cosine functions close to continuous scalar bounded cosine functions.* Integral Equations Operator Theory 85 (3) (2016), 347–357.

[13] J. Esterle. *A short proof of the zero-two law for cosine functions.* Arch. Math. (Basel) 105 (4) (2015), 381–387.

[14] Y. Fujita, *Integrodifferential equations which interpolate the heat equation and the wave equation.* Osaka J. Math. 27 (1990), 30

[15] R. Gorenflo, J. Loutchko, Y. Luchko. *Computation of the Mittag-Leffler function* $E_{\alpha,\beta}(z)$ *and its derivative* Fract. Calc. Appl. Anal., 5(4):491518, 2002.

[16] H. Gou and Li. *Existence of mild solutions for fractional nonautonomous evolution equations of Sobolev type with delay.* J. Inequal. Appl. 252 (2017), 1–20.

[17] V. Keyantuo, C. Lizama and M. Warma. *Existence, regularity and representation of solutions of time fractional wave equations.* Electr. J. Diff. Equ. 222 (2017), 1–42.

[18] V. Keyantuo, C. Lizama and M. Warma. *Existence, regularity and representation of solutions of time fractional diffusion equations.* Adv. Differential Equations 21 (9-10) (2016), 837–886.

[19] M. Li, C. Chen and F.B. Li. *On fractional powers of generators of fractional resolvent families.* J. Funct. Anal. 259 (10) (2010), 2702–2726.

[20] K. Li, J. Peng and J. Jia. *Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives.* J. Funct. Anal. 263 (2012), 476–510.

[21] K. Li and J. Peng. *Fractional resolvents and fractional evolution equations.* Appl. Math. Lett. 25 (5) (2012), 808–812.

[22] C. Lizama and H. Prado. *On duality and spetral properties of (a,k)-regularized resolvents*, Proc. Royal Soc. Edimburgh: Section A Mathematics 139 (2009), 505–517.

[23] C. Lizama and F. Poblete. *On a functional equation associated with* (a, k)*-regularized resolvent families.* Abstr. Appl. Anal., vol. 2012, Article ID 495487, 23 pages, 2012. doi:10.1155/2012/495487.

[24] F. Mainardi. *Fractional diffusive waves in viscoelastic solids.* in: J.L. Wegner, F.R. Norwood (Eds.), IUTAM Symposium - Nonlinear Waves in Solids, ASME/AMR, Fairfeld NJ (1995), 93-97.

[25] Z. D. Mei, J. G. Peng and J. H. Gao. *General fractional differential equations of order* $\alpha \in (1, 2)$ *and type* $\beta \in [0, 1]$ *in Banach spaces.* Semigroup Forum 94 (3) (2017), 712–737.

[26] Z. D. Mei, J. G. Peng and Y. Zhang. *A characteristic of fractional resolvents.* Fract. Calc. Appl. Anal. 16 (4) (2013), 777-790.

[27] K. S. Miller and S. G. Samko. *A note on the complete monotonicity of the generalized Mittag-Leffler function.* Real Analysis Exchange. 23 (2) (1997/1998), 753–755.

[28] R. R. Nigmatullin. *The realization of the generalized transfer equation in a medium with fractal geometry.* Phys. Stat. Sol. B 133 (1986), 425-430.

[29] J. Peng and K. Li. *A note on property of the Mittag-Leffler function.* J.Math. Anal. Appl. 370 (2010), 635-638.

[30] J. Peng and K. Li. *A novel characteristic of solution operator for the fractional abstract Cauchy problem.* J. Math. Anal. Appl. 385 (2) (2012), 786–796.

[31] I. Podlubny. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

[32] J. Prüss. Evolutionary Integral Equations and Applications. Birkhäuser/Springer Basel AG, Basel, 2012.

[33] W. R. Schneider. *Completely monotone generalized Mittag- Leffler functions.* Expo. Math. 14 (1996), 3–16.

[34] F. L. Schwenninger and H. Zwart. *Less than one implies zero.* Studia Math. 229 (2) (2015), 181-188.

[35] F. L. Schwenninger and H. Zwart. *Zero-two law for cosine families.* J. Evol. Equ. 15 (2015), 559-569.

[36] O. Staffans. Well-Posed Linear Systems. Encyclopedia of Mathematics and its Applications 103. Cambridge University Press, Cambridge, 2005.