Existence and monotonicity of nonlocal boundary value problems: The one dimensional case

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Abstract

We consider nonlocal equations of the general form

$$\big(a * u''\big)(\cdot) + \lambda f\big(\cdot, u(\cdot)\big) = 0.$$

By developing a Green's function representation for the solution of the boundary value problem we study existence, uniqueness, and qualitative properties (e.g., positivity or monotonicity) of solutions to these problems. We apply our methods to fractional order differential equations. We also demonstrate an application of our methodology both to convolution equations with nonlocal boundary conditions as well as those with a nonlocal term in the convolution equation itself.

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Convolution, monotonicity, nonlocal differential equation, fractional differential equation.
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1. Introduction

Our main concern in this article is the study of existence of solutions and qualitative properties for the nonlocal equation

$$\big(a * u''\big)(\cdot) + \lambda f\big(\cdot, u(\cdot)\big) = 0. \tag{1.1}$$

Problems of this kind appear in a variety of contexts. The convolution in the left hand side in equation (1.2) gives a nonlocal character to the model, which means that what happens in a point depends on an average over an interval that contains that value. We reveal in this paper results that admits a dual interpretation. First, we may consider u'' as the one-dimensional Laplacian: Δu. In such case, the equation we study reads:

$$(a * \Delta u)(x) + \lambda f\big(x, u(x)\big) = 0, \quad x \in (0, 1). \tag{1.2}$$

For instance, if $a = \delta_0$ the Dirac delta with mass concentrated at zero, we have the elliptic boundary value problem:

$$\Delta u(x) + \lambda f\big(x, u(x)\big) = 0, \quad x \in (0, 1). \tag{1.3}$$

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Such equations, typically arises in the analysis of continuous mechanical systems. The equilibrium equations of one-dimensional continuum mechanics – bars, beams, strings and the like – are formulated as boundary value problems like (1.3). In the case of equation (1.2), the nonlocal effects in space correspond to long-range interactions (many spatial scales).

A second interpretation for (1.1) is to consider the time variable instead the spatial variable, and the equation under study reads:

$$\left(a * u''\right)(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1). \tag{1.4}$$

For instance, in case $a(t) = \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}, 1 < \alpha \leq 2$, the left hand side corresponds to the Caputo fractional order time-derivative [43] and we have the nonlinear fractional evolution equation:

$$D_t^\alpha u(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1), \quad 1 < \alpha \leq 2. \tag{1.5}$$

In this case, as well as in equation (1.4), the nonlocal effects in time suppose memory or delay effects (many temporal scales).

These phenomena, in space and time, are associated to integral or integro-differential equations, which appear in multiple contexts: Potential theory, geophysics, electricity and magnetism, hereditary phenomena in physics and biology [40], evolution of populations, optimization, communication and mathematical economy [51].

In this paper, we equip (1.1) with boundary condition of one of two types. First we study the case in which the boundary conditions are local. For example, we use the left-focal type boundary condition $u'(0) = 0$ together with an inhomogeneous condition $u(1) = T$ at the right end point. After this we show how the methodology can be adapted to the case of nonlocal boundary conditions. In this case the boundary condition is represented by a suitable Stieltjes integral, and so, includes a wide variety of conditions such as multipoint-type boundary conditions as well as integral boundary conditions.

A final achievement concerns the case in which the derivative term in (1.1) has a nonlocal coefficient. The general equation we study in this setting has the form

$$A\left(\int_{[0,1]} (g \circ u)(s)\, d\alpha(s)\right)\left(a * u''\right)(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1), \tag{1.6}$$

where g is a convex, continuous, and strictly increasing function, and α is a monotone increasing integrator. Problem (1.6) is related to one-dimensional Kirchhoff-type equations – see, for example, Cao and Dai [12], where the authors study a problem of the form

$$-\left(a + b\int_0^1 |u'|^2\, dx\right) u'' = \lambda u + h(x, u, \lambda), \quad t \in (0,1)$$

or Liu, Luo, and Dai [45], where the authors study a problem of the form

$$-\left(\int_0^1 |u'|^2\, dx\right) u'' = \lambda u^3 + h(x, u, \lambda), \quad t \in (0,1).$$

More directly, problem (1.6) is a generalization of a problem of the form

$$A\left(\int_0^1 |u|^q\, ds\right) u''(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1) \tag{1.7}$$

in which $\alpha(t) := t$ and $g(t) = |t|^q$ for $q \geq 1$, and these occur widely in the differential equations literature – see, for example, Corrêa [14], Corrêa, Menezes, and Ferreira [15], do Ó, Lorca, Sánchez, and Ubilla [18], and Stańczy [52]. Applications of (1.7) may be found, for example, amongst problems in thermodynamics and fluid mechanics – see, for example, Aly [4], Bavaud [7], Biler, Krzywicki, and Nadzieja [8], Biler and Nadzieja [9, 10], Caglioti, Lions, Marchioro, and Pulvirenti [11], Esposito, Grossi, and Pistoia [19], and Rosier and Rosier [50], and the references therein. So, in the case of (1.6) we consider a more general formulation than (1.7) since we allow for a convolution term, which, as we have already mentioned, allows us to study many different types (e.g., fractional) differential equations at once. By both adapting and extending some new methods introduced by the first author in a recent paper [31] we are able to study problem (1.6) in a very general context, allowing, for example, the coefficient function $t \mapsto A(t)$ to be nonpositive for some t – see Remark 5.2.

In each of the aforementioned settings, (1.1) and (1.6), our goal is to analyze not only the existence and uniqueness of solutions to these problems but also to deduce qualitative information regarding the solutions – e.g., whether the solutions are positive or monotone. In order to complete this latter task we use some results regarding convolutional differential inequalities – i.e., inequalities having the form

$$\frac{d^n}{dt^n}(a * u)(t) \geq 0,\ t \geq 0, \tag{1.8}$$

for some positive integer n and functions a and u specified later. Inequalities of the form given by (1.8) are strongly related to an old problem posed by Ehrenpreis [23] in 1960 – namely, given function a and v what are the regularity properties of any such function u satisfying the convolution equation $a * u = v$.

In a recent paper by the authors [33] this open question of Ehrenpreis was answered in the case where the convolution equation is replaced by a convolution inequality of the form $a * u \geq v$. And this study yielded results related to inequalities of the form (1.8). So, in this paper we, in part, investigate how these convolution inequalities can yield interesting qualitative information regarding problems formulated like (1.1) or (1.6). For example, we investigate conditions under which a solution to (1.1) equipped with some relevant boundary conditions must be monotone. The study of the connection between (1.8) and the positivity, monotonicity, or convexity of u has been ongoing for some time in the specialized context of the continuous and discrete fractional calculus – see, for example Dahal and Goodrich [20], Diethelm [22], Goodrich and Lizama [32]. But the connection with regard to solutions of convolution equations with boundary values does not seem to have been investigated.

The study of such convolution boundary value problems is not merely for the sake of abstraction. Indeed, one of the interests in studying convolution differential equations is because they capture a wide variety of important special cases. For example, fractional differential equations (see, for example Wang, Liu, and Wu [54] and Zhu [61]) are presently a rather popular area of research, due both to the mathematical difficulties encountered in their study but also their interesting applications to a wide variety of problems in physics and biology – see in the continuous case ether the textbook by Kilbas, Srivastava, and Trujillo [43] or the textbook by Podlubny [49] and in the discrete case the textbook by Goodrich and Peterson [34]. And as we have mentioned earlier in this section each of problems (1.1) and (1.6) can be specialized to the fractional differential equations setting by choosing the function a in an appropriate way.

We conclude by mentioning some of the additional literature relevant to the problems we study here. In particular, regarding the study of convolutions as they occur naturally in problems related to differential and difference equations many papers exist – e.g., one may consult Abadias and Miana [1], Abadias, Lizama, Miana, and Velasco [2, 3], Diekmann and Kaper [21], Gómez-Callado and Jordá

[24], Lipovan [44], Lizama [46], and Lizama and Murillo-Arcila [47]. At the same time the study of nonlocal boundary value problems has been extensive. In fact, the subject dates back to the work of Picone [48] and Whyburn [56], the former having occurred in the first few years of the 1900s. More recently, a seminal paper by Webb and Infante [55] demonstrated an elegant method for treating *linear* nonlocal boundary conditions. Some related, interesting work was completed by Graef and Webb [35], Jankowski [41], Karakostas [42], Infante and Pietramala [38], and Yang [58]. In the case of *nonlinear*, nonlocal boundary conditions recent works include papers by Anderson [5], Cianciaruso, Infante, and Pietramala [13], Goodrich [25, 26, 28, 29, 30], Infante [37], and Yang [59].

In spite of the vast literature separately on convolution operators and on nonlocal boundary value problems, there is very little combining these two areas and, in particular, very little in the way of using convolution operators to inform the analysis of nonlocal boundary value problems. So, in this paper we combine these two seemingly disparate topics: convolution equations and nonlocal differential equations. And, at the same time, demonstrate a useful application of existing results on convolution differential inequalities. We hope this provides some new insights into these areas and their application.

2. Preliminaries

Recall that by $*$ we denote the finite convolution – i.e.,

$$(f * g)(t) := \int_0^t f(t-s)g(s)\ ds, \quad t \geq 0,$$

for $f, g : [0, \infty) \to \mathbb{R}$. Given $\alpha > 0$, we denote by g_α the standard kernel

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0.$$

We say that a kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is of type $\mathcal{PC}_1$, if the following condition is satisfied

$(\mathcal{PC}_1)$: There exists a nonnegative kernel $b \in AC(\mathbb{R}_+)$ such that $b * a = g_1$ on $(0, \infty)$.

The condition $\mathcal{PC}_1$ was introduced by Vergara and Zacher in [53]. However, note that our definition is different because, in contrast with [53], we have omitted the condition of nonincreasing to the kernel b. Examples of kernels of type $\mathcal{PC}_1$ are the kernels

$$a(t) = g_\alpha(t), \quad a(t) = \int_0^1 g_\beta(t)\ d\beta, \quad a(t) = \int_0^\infty \frac{e^{-st}}{1+s}\ ds, \quad t > 0.$$

For other examples, see [53, Section 6]. In what follows, for $n \in \mathbb{N}$, we denote by $\mathscr{C}^n(\mathbb{R}_+)$ the space of n-times continuously differentiable functions on $\mathbb{R}_+$.

Concerning monotonicity, we have the following theorem.

Theorem 2.1. *[33, Theorem 2.2] Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$ and nonincreasing. Assume that $u \in \mathscr{C}^2(\mathbb{R}_+)$ and*

$$\frac{d^2}{dt^2}(a * u)(t) \geq 0 \quad \text{for each } t \geq 0.$$

If $u(0) \geq 0$, then u is nondecreasing.

A consequence of Theorem 2.1 is the following.

Corollary 2.2. *[33, Corollary 2] Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$, nonnegative, and nonincreasing. Assume that $u \in \mathscr{C}^2(\mathbb{R}_+)$ and*

$$(a * u'')(t) \geq 0 \quad \text{for each } t \geq 0.$$

If $u(0) = 0$ and $u'(0) \geq 0$ then u is nondecreasing.

We conclude this section with a classical fixed point result. We will use it in some of the existence proofs we provide later in this paper. One may consult, for example, Cianciaruso, Infante, and Pietramala [13, Lemma 2.3], Guo and Lakshmikantham [36], Infante, Pietramala, and Tenuta [39], or Zeidler [60] for further details on these types of results.

Lemma 2.3. *Let U be a bounded open set and, with $\mathcal{K}$ a cone in a real Banach space $\mathscr{X}$, suppose both that $U_{\mathscr{K}} := U \cap \mathcal{K} \supseteq \{0\}$ and that $\overline{U_{\mathscr{K}}} \neq \mathscr{K}$. Assume that $T : \overline{U_{\mathscr{K}}} \to \mathscr{K}$ is a compact map such that $x \neq Tx$ for each $x \in \partial U_{\mathscr{K}}$. Then the fixed point index $i_{\mathscr{K}}(T, U_{\mathscr{K}})$ has the following properties.*

1. *If there exists $e \in \mathscr{K} \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for each $x \in \partial U_{\mathscr{K}}$ and each $\lambda > 0$, then $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 0$.*
2. *If $\mu x \neq Tx$ for each $x \in \partial U_{\mathscr{K}}$ and for each $\mu \geq 1$, then $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 1$.*
3. *If $i_{\mathscr{K}}(T, U_{\mathscr{K}}) \neq 0$, then T has a fixed point in $U_{\mathscr{K}}$.*
4. *Let U^1 be open in X with $\overline{U^1_{\mathscr{K}}} \subseteq U_{\mathscr{K}}$. If $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 1$ and $i_{\mathscr{K}}\left(T, U^1_{\mathscr{K}}\right) = 0$, then T has a fixed point in $U_{\mathscr{K}} \setminus \overline{U^1_{\mathscr{K}}}$. The same result holds if $i_{\mathscr{K}}(T, U_{\mathscr{K}}) = 0$ and $i_{\mathscr{K}}\left(T, U^1_{\mathscr{K}}\right) = 1$.*

3. Boundary conditions for nonlocal convolution equations

In this section we consider problem (1.1) to show the main ideas behind our analysis of monotonicity. We will study this type of equation even more extensively in the next sections, where we will consider both nonlocal boundary conditions (instead of the local boundary conditions considered in the present section) as well as a nonlocal element directly in the convolution equation itself akin to (1.6) mentioned in Section 1.

So, let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$, nonincreasing, $f : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ and $\lambda, T > 0$. We consider the existence of positive and nonincreasing solutions for the boundary value problem

$$\begin{cases} (a * u'')(t) + \lambda f(t, u(t)) = 0, & t \in (0,1), \\ u'(0) = 0, \quad u(1) = T. \end{cases} \tag{3.1}$$

Since a is of type $\mathcal{PC}_1$ there exists a nonnegative kernel $c \in AC(\mathbb{R}_+)$ such that $c * a = g_1$. Let $b = g_1 * c$. Then b is nonnegative, increasing and $b * a = g_2$. Convolving the equation (3.1) with b, and using the prescribed boundary conditions, it is easy to see that a solution of (3.1) is a fixed point of the equation

$$u(t) = T + \lambda \int_0^1 b(1-s) f(s, u(s)) ds - \lambda \int_0^t b(t-s) f(s, u(s)) ds. \tag{3.2}$$

Note that $b \in AC^1(\mathbb{R}_+)$. Consequently, $u \in \mathscr{C}\left([0,1]\right)$ is a solution of (3.2) if and only if $u \in \mathscr{C}^2\left([0,1]\right)$ is a solution of (3.1). We begin with the following simple but illuminating result.

Theorem 3.1. *Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$, nonincreasing, and suppose that f is Lipschitz, i.e. there exists a constant $L > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|, \quad x, y \in \mathbb{R}, \quad t \in [0,1],$$

and the Lipschitz constant $L > 0$ satisfies the estimate

$$L < \frac{1}{2\lambda \|b\|_{L^1[0,1]}}.$$

Then there exists a unique positive and nonincreasing solution of problem (3.1).

Proof. We claim that assuming existence of one solution $u \in \mathscr{C}([0,1])$ for (3.2), then both the positivity and monotonicity of such a solution follow immediately from Theorem 2.1. Indeed, define momentarily $h(t) := \lambda f(t, u(t))$ and let $v(t) := (b * h)(t)$. Then $(a * v)(t) = (g_2 * h)(t)$ and hence by hypothesis

$$\frac{d^2}{dt^2}(a * v)(t) = h(t) \geq 0 \quad \text{for each } t \in [0,1],$$

and therefore we are in the hypotheses of Theorem 2.1. Observe that $a(t)$ satisfies all the hypotheses of such theorem. Moreover, $v(0) = 0$. Therefore, we conclude by Theorem 2.1 that v is nondecreasing, which implies:

$$\int_0^t b(t-s)\lambda f(s,u(s))ds = (b*h)(t) \leq (b*h)(\tau) = \int_0^\tau b(\tau - s)\lambda f(s,u(s))ds, \tag{3.3}$$

for all $0 \leq t < \tau \leq 1$. Taking $\tau = 1$ in (3.3) we obtain from (3.2) that u is positive. Moreover,

$$u(t) - u(\tau) = (b*h)(\tau) - (b*h)(t) \geq 0,$$

i.e. the solution u is nonincreasing.

To show existence, we define the mapping $F : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ by

$$F(u)(t) = T + \lambda \int_0^1 b(1-s) f(s,u(s))ds - \lambda \int_0^t b(t-s) f(s,u(s))ds. \tag{3.4}$$

where $\mathscr{C}([0,1])$ is a Banach space endowed with its natural norm $\|\cdot\|_\infty$ and the kernel b is nonnegative. In order to see that (3.2) has a solution it is enough to show that (3.4) has a fixed point. To see this, we have the estimate

$$\begin{aligned}
|F(u)(t) - F(v)(t)| &\leq \lambda \int_0^1 b(1-s)|f(s,u(s)) - f(s,v(s))|ds \\
&\quad + \lambda \int_0^t b(t-s)|f(s,u(s)) - f(s,v(s))|ds \\
&\leq \lambda L \int_0^1 b(1-s)|u(s) - v(s)|ds + \lambda L \int_0^t b(t-s)|u(s)-v(s)|ds \\
&\leq \lambda L \|u - v\|_\infty \Big[\int_0^1 b(1-s) + \int_0^t b(t-s)ds \Big], \quad t \in [0,1] \\
&\leq 2\lambda L \int_0^1 b(s)ds \, \|u-v\|_\infty,
\end{aligned}$$

valid for all $u, v \in C(\mathbb{R}_+)$. Consequently

$$\|F(u) - F(v)\|_\infty = \sup_{t\in[0,1]} |F(u)(t) - F(v)(t)| \le 2\lambda L \|b\|_{L^1[0,1]} \|u - v\|_\infty$$

Then, the hypothesis and the Banach fixed point theorem imply the existence of a unique fixed point $u \in \mathscr{C}\big([0,1]\big)$. □

Example 3.2. For given $f : [0,1] \times \mathbb{R} \to \mathbb{R}_+$ and $\lambda, T > 0$, we consider the existence of positive and nonincreasing solutions for the boundary value problem

$$\begin{cases} {}_C D_t^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \quad 1 < \alpha < 2; \\ u'(0) = 0, \quad u(1) = T. \end{cases} \tag{3.5}$$

where ${}_C D_t^\alpha$ denotes the Caputo fractional derivative of order α defined by

$${}_C D_t^\alpha u(t) = (g_{2-\alpha} * u'')(t).$$

Here $a(t) = g_{2-\alpha}(t)$ is of type $\mathcal{C}_1$ because there exists $b(t) = g_{\alpha-1}(t)$ verifying $a * b = g_1$. Clearly a is positive and nonincreasing. Moreover, since $1 < \alpha$, we have

$$\|b\|_{L^1[0,1]} = \frac{1}{\Gamma(\alpha)}.$$

From Example 3.2 we conclude the following result.

Corollary 3.3. *Suppose that f is Lipschitz with Lipschitz constant $L > 0$ satisfying the following estimate*

$$L < \frac{\Gamma(\alpha)}{2\lambda}. \tag{3.6}$$

Then, there exists a unique positive, nonincreasing solution of (3.5).

Remark 3.4. A different way to interpret (3.6) in Corollary 3.3 is to say that if f is Lipschitz, then the problem (3.5) has always a unique, positive and nonincreasing solution for all values of λ small enough.

More general hypothesis on the boundary conditions and the nonlinear term f will be given in the next section.

4. Nonlocal boundary conditions for nonlocal convolution equations

We again consider problem (3.1), but here we consider a nonlocal boundary condition. More specifically, let us consider the problem

$$\begin{aligned} \big(a * u''\big)(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1) \\ u'(0) = 0 \\ u(1) = \varphi(u). \end{aligned} \tag{4.1}$$

In (4.1) the nonlocal element $\varphi(u)$ is realized as a Stieltjes integral of the form

$$\varphi(u) := \int_{[0,1]} u(s)\ d\alpha(s),$$

where the integrator $t \mapsto \alpha(t)$ is of bounded variation on $[0, 1]$ but not necessarily monotone. In this way, we can include a variety of nonlocal boundary conditions, including both integral and multipoint nonlocal conditions even in the case of sign-changing coefficients (since α need not be monotone).

Let $F \ : \ \mathscr{C}\big([0, 1]\big) \to \mathscr{C}\big([0, 1]\big)$ be the operator defined by

$$(Fu)(t) := \varphi(u) + \lambda \int_0^1 G(t, s) f\big(s, u(s)\big) \ ds,$$

where

$$G(t, s) := \begin{cases} b(1 - s) - b(t - s), & 0 \le s \le t \le 1 \\ b(1 - s), & 0 \le t \le s \le 1 \end{cases}.$$

This is nearly the same as the operator in (3.4) except that the inhomogeneous boundary value T has been replaced by the nonlocal inhomogeneous element $\varphi(u)$.

Define the function $\mathscr{G} \ : \ [0, 1] \to [0, +\infty)$ by

$$\mathscr{G}(s) := \sup_{t \in [0,1]} \big|G(t, s)\big|.$$

Using ideas from the recent papers [27, 28, 29, 30] we consider the cone

$$\mathscr{K} := \Big\{u \in \mathscr{C}\big([0, 1]\big) \ : \ u \ge 0, \, \varphi(u) \ge C_0 \|u\|_\infty\Big\},$$

where the coercivity constant C_0 is defined as follows:

$$C_0 := \min \left\{\varphi(\mathbf{1}), \inf_{s \in (0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t, s) \ d\alpha(t)\right\}; \tag{4.2}$$

here in (4.2) and in the sequel by $\mathbf{1}$ we mean the function that is identically unity on $\mathbb{R}$. We will assume from now on that C_0 exists as a positive real number. We also note that the set $[0, 1]$ over which the infimum is taken in the definition of C_0 can be replaced by any measurable set $S_0 \subseteq [0, 1]$ such that $\big|S_0\big| = 1$ – i.e., that S_0 has full measure. In addition, since φ is a linear functional, we also note the existence of a number $C_1 \ge 0$ such that

$$\varphi(u) \le C_1 \|u\|_\infty,$$

for each $u \in \mathscr{C}\big([0, 1]\big)$. We will assume that $C_1 > C_0$.

In addition, for any $\rho > 0$ we will use the following open set.

$$\widehat{V}_\rho := \big\{u \in \mathscr{K} \ : \ \varphi(u) < \rho\big\}$$

Note that (see, for example, [30]) this set has the property that

$$\partial \widehat{V}_\rho := \big\{u \in \mathscr{K} \ : \ \varphi(u) = \rho\big\}.$$

We will use this property in what follows. In addition, we will also use the notation

$$\Omega_\rho := \big\{u \in \mathscr{K} \ : \ \|u\|_\infty < \rho\big\}.$$

In order to use the fixed point theory that we wish to employ, we need to establish that $F(\mathscr{K}) \subseteq \mathscr{K}$. This is the content of the next lemma.

Lemma 4.1. *It holds that $F(\mathscr{K}) \subseteq \mathscr{K}$.*

Proof. Let $u \in \mathscr{K}$ be fixed but arbitrary. Then we first calculate

$$(Fu)(t) \geq C_0\|u\|_\infty + \lambda \int_0^1 G(t,s) f\big(s, u(s)\big)\ ds \geq 0,$$

where we note that $G(t,s) \geq 0$ by virtue of the fact that b is nonnegative and increasing. In addition, we note that

$$\begin{aligned}
\varphi(Fu) &= \int_0^1 \varphi(u)\ d\alpha(s) + \lambda \int_0^1 \int_0^1 G(t,s) f\big(s,u(s)\big)\ ds\ d\alpha(t)\\
&= \varphi(u) \int_0^1\ d\alpha(s) + \lambda \int_0^1 \int_0^1 G(t,s) f\big(s,u(s)\big)\ d\alpha(t)\ ds\\
&= \varphi(u)\varphi(\mathbf{1}) + \lambda \int_0^1 \left[\frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ d\alpha(t)\right] \mathscr{G}(s) f\big(s,u(s)\big)\ ds\\
&\geq \varphi(u)\varphi(\mathbf{1}) + \left(\inf_{s\in(0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ d\alpha(t)\right) \lambda \int_0^1 \mathscr{G}(s) f\big(s,u(s)\big)\ ds\\
&\geq \underbrace{\min\left\{\varphi(\mathbf{1}),\ \inf_{s\in(0,1)} \frac{1}{\mathscr{G}(s)} \int_0^1 G(t,s)\ d\alpha(t)\right\}}_{=:C_0} \left[\varphi(u) + \lambda \int_0^1 \mathscr{G}(s) f\big(s,u(s)\big)\ ds\right]\\
&\geq C_0\|Fu\|_\infty.
\end{aligned} \tag{4.3}$$

Therefore, we conclude that $F(\mathscr{K}) \subseteq \mathscr{K}$, as claimed. □

We now establish that F has at least one nontrivial fixed point. This then establishes that problem (4.1) has at least one positive solution. Note that in the sequel we use the following notation for a continuous function $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ with $[a,b] \subseteq [0,1]$ and $[c,d] \subseteq [0,+\infty)$.

- $f^m_{[a,b]\times[c,d]} := \min_{(t,y)\in[a,b]\times[c,d]} f(t,y)$
- $f^M_{[a,b]\times[c,d]} := \max_{(t,y)\in[a,b]\times[c,d]} f(t,y)$

Theorem 4.2. *Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$ and nonincreasing. Suppose that for some numbers $\rho_1,\ \rho_2 \in (0,+\infty)$, where $\rho_1 \neq \rho_2$, each of the following conditions is satisfied:*

(1) $f^m_{[0,1]\times\left[0,\frac{\rho_1}{C_0}\right]} \lambda \int_0^1 \int_0^1 G(t,s)\ d\alpha(t)\ ds > \rho_1$; *and*

(2) $\rho_2\varphi(\mathbf{1}) + f^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]} \lambda \int_0^1 \int_0^1 G(t,s)\ ds\ d\alpha(t) < \rho_2$.

Then problem (4.1) has at least one positive solution, $u_0 \in \mathscr{K}$, satisfying the localization

$$u_0 \in \widehat{V}_{\max\{\rho_1,\rho_2\}} \setminus \overline{\widehat{V}_{\min\{\rho_1,\rho_2\}}}.$$

Proof. We first assume for contradiction that there exists $u \in \partial \widehat{V}_{\rho_1}$ and $\mu \geq 0$ such that $u = Fu + \mu e$, where $e(t) := 1$. Note that $e \in \mathscr{K}$ since

$$\varphi(e) = \int_{[0,1]} d\alpha(s) = \varphi(\mathbf{1}) \geq C_0 \|\mathbf{1}\|,$$

due to the definition of C_0; incidentally, this demonstrates that $\mathscr{K} \neq \varnothing$. Then applying φ to both sides of the equation $u = Tu + \mu e$ and using the fact that $u \in \partial \widehat{V}_{\rho_1}$ we deduce with the help of Fubini's theorem that

$$\begin{aligned}
\rho_1 &= \varphi(Fu) + \mu\varphi(e) \\
&\geq \lambda \int_0^1 \int_0^1 G(t,s) f\big(s, u(s)\big)\ ds\ d\alpha(t) \\
&= \lambda \int_0^1 \underbrace{\left[\int_0^1 G(t,s)\ d\alpha(t)\right]}_{>0} f\big(s,u(s)\big)\ ds \\
&\geq f^m_{[0,1]\times\left[0,\frac{\rho_1}{C_0}\right]} \lambda \int_0^1 \int_0^1 G(t,s)\ d\alpha(t)\ ds \\
&> \rho_1,
\end{aligned}$$

which is a contradiction; here we have used condition (1) in the statement of the theorem. Consequently, we deduce that

$$i_{\mathscr{K}}\left(F, \widehat{V}_{\rho_1}\right) = 0.$$

Note that in the preceding paragraph we have used the fact that if $u \in \partial \widehat{V}_{\rho_1}$, then it follows that

$$\rho_1 = \varphi(u) \leq C_1 \|u\|_\infty$$

so that

$$\|u\|_\infty \geq \frac{\rho_1}{C_1}.$$

At the same time, if $u \in \partial \widehat{V}_{\rho_1}$, then it follows that

$$\rho_1 = \varphi(u) \geq C_0 \|u\|_\infty$$

so that

$$\|u\|_\infty \leq \frac{\rho_1}{C_0}.$$

Therefore, putting these facts together we deduce that

$$f\big(s, u(s)\big) \geq f^m_{[0,1]\times\left[0,\frac{\rho_1}{C_0}\right]},$$

for each $s \in [0,1]$ and $u \in \partial \widehat{V}_{\rho_1}$.

Conversely, we suppose next for contradiction that there exists $u \in \partial \widehat{V}_{\rho_2}$ such that $\mu u = Fu$ for some $\mu \geq 1$. Then applying φ to both sides of the equation $\mu u = Fu$ we obtain

$$\begin{aligned}\rho_2 &\leq \varphi(Fu)\\ &= \int_0^1 \varphi(u)\ d\alpha(t) + \lambda \int_0^1 \int_0^1 G(t,s) f\big(s, u(s)\big)\ ds\ d\alpha(t)\\ &\leq \rho_2 \varphi(\mathbf{1}) + f^M_{[0,1]\times\left[0,\frac{\rho_2}{C_0}\right]} \lambda \int_0^1 \int_0^1 G(t,s)\ ds\ d\alpha(t)\\ &< \rho_2,\end{aligned}$$

which is a contradiction; here we have used condition (2) in the statement of the theorem. Consequently, we deduce that

$$i_{\mathscr{K}}\left(F, \widehat{V}_{\rho_2}\right) = 1.$$

Finally, since $i_{\mathscr{K}}\left(F, \widehat{V}_{\rho_1}\right) = 0$ and $i_{\mathscr{K}}\left(F, \widehat{V}_{\rho_2}\right) = 1$, it follows by the properties of the fixed point index together with Lemma 2.3 that there exists $u_0 \in \mathscr{K}$ satisfying the localization

$$u_0 \in \widehat{V}_{\max\{\rho_1,\rho_2\}} \setminus \overline{\widehat{V}_{\min\{\rho_1,\rho_2\}}}$$

such that $\big(Fu_0\big)(t) = u_0(t)$. Thus, u_0 is a positive solution of problem (4.1). And this completes the proof of the theorem. □

Finally, we establish that any nontrivial solution of problem (4.1), including the one whose existence was established in the previous theorem, must satisfy certain qualitative properties.

Theorem 4.3. *Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$ and nonincreasing. Any solution $u \in \mathscr{K}$ of problem (4.1) must be nonincreasing.*

Proof. Suppose that u is a solution of problem (4.1). Then it holds that

$$u(t) = (Fu)(t) = \varphi(u) + \lambda \int_0^1 b(1-s) f\big(s, u(s)\big)\ ds - \lambda \int_0^t b(t-s) f\big(s, u(s)\big)\ ds.$$

But one can then repeat the calculations in the first paragraph of the proof of Theorem 3.1. In particular, it again holds that

$$\frac{d^2}{dt^2}(a * v)(t) = h(t) \geq 0,$$

for each $t \in [0,1]$, since the functions h and v are defined exactly as in the proof of Theorem 3.1. So, we see that

$$\begin{aligned}u(t) - u(\tau) &= \left[\varphi(u) + \lambda \int_0^1 b(1-s) f\big(s,u(s)\big)\ ds - \lambda \int_0^t b(t-s) f\big(s,u(s)\big)\ ds\right]\\ &\quad - \left[\varphi(u) + \lambda \int_0^1 b(1-s) f\big(s,u(s)\big)\ ds - \lambda \int_0^\tau b(\tau-s) f\big(s,u(s)\big)\ ds\right]\\ &= -\lambda \int_0^t b(t-s) f\big(s,u(s)\big)\ ds + \lambda \int_0^\tau b(\tau-s) f\big(s,u(s)\big)\ ds\\ &= \lambda (b*h)(\tau) - \lambda (b*h)(t)\\ &\geq 0,\end{aligned}$$

for any $0 \leq t < \tau \leq 1$, where we have used inequality (3.3), which still holds true in spite of the different boundary condition at $t = 1$. Therefore, we conclude, as in the proof of Theorem 3.1, that a solution of problem (4.1) must be nonincreasing. And this completes the proof. $\square$

Remark 4.4. If the kernel of F, namely $(t, s) \mapsto G(t, s)$, happens to satisfy a Harnack-like inequality of the form $\min_{t\in[a,b]} G(t, s) \geq \eta_0 \mathscr{G}(s)$ for each $s \in [0, 1]$ and some $\eta_0(a, b) := \eta_0 \in (0, 1)$, where $(a, b) \Subset (0, 1)$, then condition (1) in the statement of Theorem 4.2 can be refined a bit to read

$$f^m_{[0,1]\times\left[\frac{\rho_1\eta_0}{C_1},\frac{\rho_1}{C_0}\right]} \lambda \int_0^1 \int_0^1 G(t, s)\ d\alpha(t)\ ds \geq \rho_1.$$

One may compare this, for example, with [29, Theorem 3.1].

Remark 4.5. Concerning condition (1), which recall is

$$\left[\min_{(t,y)\in[0,1]\times\left[0,\frac{\rho_1}{C_0}\right]} f(t, y)\right] \lambda \int_0^1 \int_0^1 G(t, s)\ d\alpha(t)\ ds > \rho_1,$$

we note that the function

$$\psi_1(\rho) := \min_{(t,y)\in[0,1]\times\left[0,\frac{\rho}{C_0}\right]} f(t, y)$$

is decreasing. Hence, if we assume $\psi_1(0) = \min_{t\in[0,1]} f(t, 0) > 0$ then we have that the function

$$\phi(\rho) := \frac{\rho}{\psi_1(\rho)}$$

is continuous, increasing, and satisfies $\phi(0) = 0$. Therefore, there exists a number ρ_1 such that the condition (1) is satisfied.

On the other hand, define

$$\psi_2(\rho) := \max_{(t,y)\in[0,1]\times\left[0,\frac{\rho}{C_0}\right)} f(t, y)$$

and observe that is continuous and increasing. Assuming $\psi(\infty) < +\infty$, we have that

$$\frac{\rho}{\psi(\rho)} \to +\infty$$

as $\rho \to +\infty$ and therefore under the additional hypothesis that $\varphi(\mathbf{1}) \neq 1$ we obtain that the condition (2) always holds, too. Consequently, we obtain the following, easier to check, result.

Corollary 4.6. *Let $a \in AC(\mathbb{R}_+)$ be of type $\mathcal{PC}_1$ and nonincreasing. Suppose that*

(1a) $\min_{t\in[0,1]} f(t, 0) > 0$;

(2a) $\varphi(\mathbf{1}) \neq 1$ *and* $\max_{(t,y)\in[0,1]\times[0,\infty)} f(t, y) < \infty$.

Then problem (4.1) has at least one positive and nonincreasing solution.

We next consider the following example, which applies the preceding result to the case of a fractional boundary value problem.

Example 4.7. We consider the existence of positive and nonincreasing solutions for the nonlocal boundary value problem

$$\begin{cases} {}_C D_t^{\alpha} u(t) + \lambda \frac{(t+1)^2}{(u(t)+1)^4} = 0, & t \in (0,1), \quad 1 < \alpha < 2; \\ u'(0) = 0, \quad u(1) = 2\displaystyle\int_0^1 u(s)(s+1)ds. \end{cases} \tag{4.4}$$

In this example, we have $f(t,y) = \dfrac{(t+1)^2}{(y+1)^4}$ and $a(t) = g_{2-\alpha}(t)$. It is clear that $\min\limits_{t\in[0,1]} f(t,0) = 1$ and hence condition (1a) holds. On the other hand, we note that

$$\varphi(u) = \int_0^1 u(s)\ d\alpha(s),$$

with $\alpha(s) = (s+1)^2$. Therefore, $\varphi(\mathbf{1}) = 3$ and $\max\limits_{(t,y)\in[0,1]\times[0,\infty)} f(t,y) = 4$ which proves that the condition (2a) is also verified. We conclude by Corollary 4.6 that the nonlocal problem (4.4) has at least one positive and nonincreasing solution.

5. Multiplicative perturbations for nonlocal convolution equations

We analyze in this section a modification of problem (4.1). Specifically, for a given constant $T > 0$ and a continuous function $A : \mathbb{R} \to \mathbb{R}$ we consider the following problem:

$$\begin{aligned} A\big(\varphi_g(u)\big)\big(a * u''\big)(t) + \lambda f\big(t, u(t)\big) &= 0, \quad t \in (0,1) \\ u'(0) &= 0 \\ u(1) &= T, \end{aligned} \tag{5.1}$$

where we put

$$\varphi_g(u) := \int_{[0,1]} (g \circ u)(s)\ d\alpha(s) \tag{5.2}$$

for an *increasing* integrator α of bounded variation on $[0,1]$ and a convex, continuous function $g : \mathbb{R} \to [0,+\infty)$. If g is the identity map on $\mathbb{R}$, then we will use the notation $\varphi_g(u) \equiv \varphi(u)$ since in this case $\varphi_g(u) = \displaystyle\int_{[0,1]} u(s)\ d\alpha(s)$. Note that in the definition of (5.2) and the sequel, we use the notation

$$(g \circ u)(s) := g\big(u(s)\big).$$

Remark 5.1. The fact that we require in this section the integrator $t \mapsto \alpha(t)$ to be increasing, which, recall, we did not assume in Section 4, is for technical reasons. Essentially, as will become clear in the existence theorem to follow, we need that the integral in (5.2) is monotone, which fails, in general, if the integrator is non-monotone.

Compared to (4.1) we note that in (5.1) the nonlocal element has been moved to the differential equation itself instead of the boundary condition at $t = 1$. As was mentioned in Section 1 equations of the form

$$A\left(\int_0^1 |u|^q\ ds\right) u''(t) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1) \tag{5.3}$$

in which $\alpha(t) := t$ and $g(t) = |t|^q$ for $q \geq 1$, occur widely in the differential equations literature. Moreover, as has been noted elsewhere in this paper if we select $a(t) = g_{2-\alpha}(t)$, then (5.1) includes as a special case the nonlocal fractional equation

$$A\left(\int_{[0,1]}(g\circ u)(s)\ d\alpha(s)\right){}_{C}D_t^{\alpha}u(t)+\lambda f\big(t,u(t)\big)=0,\qquad t\in(0,1).$$

Recently in [31] the use of a nonstandard cone was introduced to analyze boundary value problems for equation (5.3). Here we will modify and generalize this method in order to analyze problem (5.1).

Remark 5.2. One observation we would like to make straightaway is that the methodology introduced in [31] and which we modify here permits the function $t \mapsto A(t)$ to be possibly zero or negative on some non-degenerate interval $I \subset \mathbb{R}$ or even a collection of such intervals. This is noteworthy because as one immediately sees from the formulation of the problems (5.1) and (5.3), the positivity of A is essential if one wishes to avoid a degeneracy (i.e., killing off the derivative term if A assumes the value zero) or nonexistence of a positive solution (i.e., if A assumes negative values). Normally in the literature (e.g., see [14, (6) and (7)], [52, Theorem 2.2], [18, Condition (H1), p. 299], and [57, Theorem 4.1, p. 84]) it is simply assumed that $A(t) > 0$ for all $t \in \mathbb{R}$. And, frequently, it is further assumed that $t \mapsto A(t)$ is monotone or satisfies some sort of growth condition – e.g., sub- or super-linearity growth condition as $t \to +\infty$. Because of our use of a nonstandard cone and associated nonstandard open set (as described below) we can avoid these restrictive assumptions.

So, we begin by defining the constant

$$C_0 := \min\left\{\int_0^1\ d\alpha(t),\ \inf_{s\in(0,1)}\frac{1}{\mathscr{G}(s)}\int_0^1 G(t,s)\ d\alpha(t)\right\}$$

and then the cone $\mathscr{K}_0$ by

$$\mathscr{K}_0 := \left\{u\in\mathscr{C}\big([0,1]\big)\ :\ u\geq 0,\ \int_0^1 u(s)\ d\alpha(s)\geq C_0\|u\|_\infty\right\}.$$

Furthermore, define the associated open set $\widehat{V}_\rho^g$ by

$$\widehat{V}_\rho^g := \left\{u\in\mathscr{K}_0\ :\ \int_0^1(g\circ u)(s)\ d\alpha(s)<\rho\right\}.$$

Note that

$$\partial\widehat{V}_\rho^g = \left\{u\in\mathscr{K}_0\ :\ \int_0^1(g\circ u)(s)\ d\alpha(s)=\rho\right\}.$$

We then see that if we define the operator $S : \mathscr{C}\big([0,1]\big) \to \mathscr{C}\big([0,1]\big)$ by

$$(Su)(t) := T + \lambda\int_0^1 G(t,s)f\big(s,u(s)\big)\left(A\left(\int_{[0,1]}(g\circ u)(\xi)\ d\alpha(\xi)\right)\right)^{-1}\ ds,$$

where, once again,

$$G(t,s) := \begin{cases} b(1-s)-b(t-s), & 0\leq s\leq t\leq 1\\ b(1-s), & 0\leq t\leq s\leq 1\end{cases},$$

then a fixed point of S corresponds to a solution of the nonlocal boundary value problem (5.1). In order to apply the classical fixed point theory to S, we need to know that S is a reflexive map on $\mathscr{K}_0$. This is our next lemma. Note that Lemma 5.3 requires only that S be a reflexive map on a annular sector of $\mathscr{K}_0$ rather than its entirety.

Lemma 5.3. *Let ρ_1, $\rho_2 > 0$ be two given numbers such that $\rho_1 \neq \rho_2$. Assume that $A(t) > 0$ for each $t \in \big[\rho_1, \rho_2\big]$. Then*

$$T\left(\widehat{V}_{\rho_2}^{g} \setminus \widehat{V}_{\rho_1}^{g}\right) \subseteq \mathscr{K}_0. \tag{5.4}$$

Proof. Since $u \in \widehat{V}_{\rho_2}^{g} \setminus \widehat{V}_{\rho_1}^{g}$ it follows that

$$0 < \rho_1 \leq \int_0^1 (g \circ u)(\xi)\ d\alpha(\xi) \leq \rho_2 < +\infty.$$

Therefore, it follows that for any $u \in \widehat{V}_{\rho_2}^{g} \setminus \widehat{V}_{\rho_1}^{g}$ it holds that

$$A\left(\underbrace{\int_{[0,1]} (g \circ u)(\xi)\ d\alpha(\xi)}_{\in[\rho_1,\rho_2]\subset(0,+\infty)}\right) > 0.$$

Hence, the operator S is well defined on the set $\widehat{V}_{\rho_2}^{g} \setminus \widehat{V}_{\rho_1}^{g}$. In addition, that $(Su)(t) \geq 0$ for each $t \in [0,1]$ is obvious from the assumptions on f and A as well as the definition of G and the fact that $T > 0$.

It remains to show that

$$\int_0^1 (Su)(s)\ d\alpha(s) \geq C_0\|Su\|_\infty. \tag{5.5}$$

To demonstrate that (5.5) holds we directly calculate (using, without any loss, t instead of s as the variable of integration on the left-hand side)

$$\begin{aligned}
&\int_0^1 (Su)(t)\ d\alpha(t)\\
&\quad = T\int_0^1\ d\alpha(t) + \lambda\int_0^1\int_0^1 G(t,s)f\big(s,u(s)\big)\left(A\left(\int_{[0,1]}(g\circ u)(\xi)\ d\alpha(\xi)\right)\right)^{-1}\ ds\ d\alpha(t)\\
&\quad \geq T\int_0^1\ d\alpha(t) + \int_0^1\left[\frac{1}{\mathscr{G}(s)}G(t,s)\ d\alpha(t)\right]f\big(s,u(s)\big)\mathscr{G}(s)\left(A\left(\int_{[0,1]}(g\circ u)(\xi)\ d\alpha(\xi)\right)\right)^{-1}\ ds\\
&\quad \geq \min\left\{\int_0^1\ d\alpha(t), C_0\right\}\left[T + \int_0^1 f\big(s,u(s)\big)\left(A\left(\int_{[0,1]}(g\circ u)(\xi)\ d\alpha(\xi)\right)\right)^{-1}\ ds\right]\\
&\quad \geq C_0\|Su\|_\infty,
\end{aligned}$$

using calculations very similar to (4.3) earlier. Therefore, (5.5) holds, and so, inclusion (5.4) is established, as desired. □

We next require a technical lemma, which is the analogue of [31, Lemma 2.4]. This lemma establishes a relationship between the standard open set

$$\Omega_\rho := \{u \in \mathscr{K}_0 \;:\; \|u\|_\infty < \rho\}$$

and the nonstandard open set $\widehat{V}_\rho^g$. The lemma uses in an important way the assumption that g is a convex map. Before stating and proving our lemma, we recall a preliminary result, which provides for a Jensen's-type inequality in the case of Stieljtes integrals with a monotone increasing integrator (i.e., the type we are studying here). This Jensen's-type inequality may be found in a paper of Barnett, Cerone, and Dragomir [6, Corollary 1].

Lemma 5.4. *Assume that $F \;:\; I \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous and convex function. If $x,\, p \;:\; [a,b] \to I$ are continuous, $p(t) \geq 0$ for $t \in [a,b]$, and $u \;:\; [a,b] \to \mathbb{R}$ is monotone nondecreasing on $[a,b]$ with $\displaystyle\int_a^b p(t)\; du(t) > 0$, then*

$$\int_a^b p(t)F\big(x(t)\big)\; du(t) \geq \left(\int_a^b p(t)\; du(t)\right) F\left(\left(\int_a^b p(t)\; du(t)\right)^{-1}\int_a^b p(t)x(t)\; du(t)\right).$$

Now we can prove our technical lemma.

Lemma 5.5. *Assume that $u \in \partial\widehat{V}_\rho^g$ for some real number $\rho > 0$. Then*

$$u \in \overline{\Omega_{\frac{\varphi(\mathbf{1})}{C_0}g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right)}} \setminus \Omega_{g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right)}.$$

Proof. Suppose that $u \in \partial\widehat{V}_\rho^g$. Then it follows that

$$\int_{[0,1]} (g \circ u)(\xi)\; d\alpha(\xi) = \rho. \tag{5.6}$$

Since g is a convex map, the version of Jensen's inequality as given in Lemma 5.4 with $p(t) \equiv 1$ implies together with (5.6) that

$$\begin{aligned}\varphi(\mathbf{1})g\left(\frac{1}{\varphi(\mathbf{1})}C_0\|u\|_\infty\right) &\leq \left(\int_0^1 \; d\alpha(\xi)\right) g\left(\left(\int_0^1 \; d\alpha(\xi)\right)^{-1}\underbrace{\int_{[0,1]} u(\xi)\; d\alpha(\xi)}_{\geq C_0\|u\|_\infty}\right)\\ &\leq \int_{[0,1]} (g\circ u)(\xi)\; d\alpha(\xi)\\ &= \rho,\end{aligned} \tag{5.7}$$

where we have also used the fact that g is an increasing function. Note that due to the fact that α is assumed to be a monotone increasing integrator, it follows that

$$\int_0^1 \; d\alpha(t) > 0$$

so that inequality (5.7) is well defined. Since g is strictly increasing, it follows that g^{-1} exists and itself is strictly increasing, and so, from inequality (5.7) we deduce that

$$\|u\|_\infty \leq \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right). \tag{5.8}$$

On the other hand, using again both that g is increasing and that α is an monotone increasing integrator we deduce that

$$\rho = \int_{[0,1]} (g \circ u)(\xi)\ d\alpha(\xi) \leq \int_{[0,1]} g\big(\|u\|_\infty\big)\ d\alpha(\xi) = g\big(\|u\|_\infty\big)\varphi(\mathbf{1})$$

so that

$$\|u\|_\infty \geq g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right). \tag{5.9}$$

Then putting (5.8)–(5.9) together we conclude that

$$g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right) \leq \|u\|_\infty \leq \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right),$$

which establishes the desired inclusion.

Finally, in order for the inclusion to be sensible it must hold that

$$g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right) \leq \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho}{\varphi(\mathbf{1})}\right).$$

However, this inequality is equivalent to the inequality

$$C_0 \leq \varphi(\mathbf{1}),$$

and this inequality is an immediate consequence of the definition of C_0. Therefore, it follows that the inclusion is sensible. And this completes the proof. □

Remark 5.6. Suppose we choose $\alpha(t) = t$ so that the Stieltjes integral reduces to a Riemann integral – hence

$$\varphi(\mathbf{1}) = \int_0^1 1\ dt = 1.$$

In addition, suppose that $g(t) := t^q$ for $q > 1$. Then $g^{-1}(t) = t^{\frac{1}{q}}$, and the conclusion of Lemma 5.5 reads as

$$u \in \partial \widehat{V}_\rho^g \Longrightarrow u \in \overline{\Omega_{\frac{\rho^{\frac{1}{q}}}{C_0}}} \setminus \Omega_{\rho^{\frac{1}{q}}}.$$

And this exactly recovers the recent result [31, Lemma 2.4]. Consequently, we see that the methodology used here is a generalization of that used in [31].

With Lemma 5.5 in hand we are ready to state and prove an existence result for problem (5.1). We remark that in the statement and proof of Theorem 5.7 we use, for a continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and given intervals $[a,b] \subseteq [0,1]$ and $[c,d] \subseteq [-\infty,+\infty]$, the notation $f^m_{[a,b]\times[c,d]} := \min_{(t,u)\in[a,b]\times[c,d]} f(t,u)$ and $f^M_{[a,b]\times[c,d]} := \max_{(t,u)\in[a,b]\times[c,d]} f(t,u)$ – just as we did in Section 4.

Theorem 5.7. *Suppose that each of the following conditions holds.*

1. *There exists a number $\rho_1 > 0$ such that*

$$\left(\frac{\varphi(\boldsymbol{T})}{\varphi(\mathbf{1})} + \frac{\lambda f^m_{[0,1]\times\left[0,\frac{\varphi(\mathbf{1})}{C_0}g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right)\right]}}{\varphi(\mathbf{1})A\left(\rho_1\right)}\int_0^1\int_0^1 G(t,s)\ ds\ d\alpha(t)\right)^{-1} g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right) < 1.$$

2. *There exists a number $\rho_2 > 0$, with $\rho_1 \neq \rho_2$, such that*

$$\varphi(\mathbf{1})\int_0^1 g\left(T + \frac{\lambda f^M_{[0,1]\times\left[0,\frac{\varphi(\mathbf{1})}{C_0}g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A\left(\rho_2\right)}\mathscr{G}(s)\right)\ ds < 1.$$

3. *It holds that $A(t) > 0$ for each $t \in \Big[\min\{\rho_1,\rho_2\},\max\{\rho_1,\rho_2\}\Big]$.*

Then problem (5.1) has at least one positive solution, u_0, satisfying the localization

$$u_0 \in \widehat{V}^g_{\rho_2} \setminus \overline{\widehat{V}^g_{\rho_1}}.$$

Proof. Set $e(t) \equiv 1$. Observe that $e \in \mathscr{K}_0$ because $e(t) \geq 0$, trivially, and

$$\int_0^1 e(s)\ d\alpha(s) = \varphi(\mathbf{1}) = \varphi(\mathbf{1})\|\mathbf{1}\|_\infty \geq C_0\|\mathbf{1}\|_\infty.$$

Incidentally, this demonstrates that $\mathscr{K}_0 \neq \varnothing$.

We first endeavor to show that for each $\mu \geq 0$ and $u \in \partial\widehat{V}^g_{\rho_1}$ that $u \neq Su + \mu e$. To this end suppose instead that $u(t) = (Su)(t) + \mu e(t)$ for some $u \in \partial\widehat{V}^g_{\rho_1}$ and $\mu \geq 0$ and each $t \in [0,1]$. Then for each $t \in [0,1]$ it holds that

$$g\big(u(t)\big) = g\big((Su)(t) + \mu e(t)\big) \geq g\big((Su)(t)\big), \tag{5.10}$$

where to obtain the inequality we use the fact that g is an increasing function. Since $u \in \partial\widehat{V}^g_{\rho_1}$ it follows that

$$\int_0^1 (g\circ u)(s)\ d\alpha(s) = \rho_1. \tag{5.11}$$

In addition, recalling Lemma 5.5, since $u \in \widehat{V}^g_{\rho_1}$ it follows that

$$g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right) \leq \|u\|_\infty \leq \frac{\varphi(\mathbf{1})}{C_0}g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right). \tag{5.12}$$

So, integrating both sides of inequality (5.10) from 0 to 1 against $d\alpha(t)$ (here we use that α is monotone increasing) and combining with equality (5.11) as well as inequality (5.12) we deduce that

$$
\begin{aligned}
\rho_1 &= \int_0^1 (g \circ u)(s)\ d\alpha(t) \\
&\geq \int_0^1 (g \circ Su)(t)\ d\alpha(t) \\
&= \int_0^1 g\left(T + \lambda \int_0^1 G(t,s) f\big(s,u(s)\big) \left(A\left(\underbrace{\int_{[0,1]} (g \circ u)(\xi)\ d\alpha(\xi)}_{=\rho_1}\right)\right)^{-1} ds\right)\ d\alpha(t) \\
&= \int_0^1 g\left(T + \frac{\lambda}{A(\rho_1)} \int_0^1 G(t,s) f\big(s,u(s)\big)\ ds\right)\ d\alpha(t) \\
&\geq \varphi(\mathbf{1}) g\left(\frac{\varphi(\boldsymbol{T})}{\varphi(\mathbf{1})} + \frac{\lambda}{\varphi(\mathbf{1}) A(\rho_1)} \int_0^1 \int_0^1 G(t,s) f\big(s,u(s)\big)\ ds\ d\alpha(t)\right) \\
&\geq \varphi(\mathbf{1}) g\left(\frac{\varphi(\boldsymbol{T})}{\varphi(\mathbf{1})} + \frac{\lambda f^m_{[0,1]\times\left[0,\frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right)\right]}}{\varphi(\mathbf{1}) A(\rho_1)} \int_0^1 \int_0^1 G(t,s)\ ds\ d\alpha(t)\right),
\end{aligned}
\tag{5.13}
$$

where we have used the version of Jensen's inequality given by Lemma 5.4 with $p(t) \equiv 1$; we have also used that g is increasing. Note that in (5.13) we have used the notation

$$\varphi(\boldsymbol{T}) := \int_0^1 T\ d\alpha(t).$$

Then using that g is invertible we obtain from condition (1) in the statement of this theorem that

$$\varphi(\mathbf{1}) g\left(\frac{\varphi(\boldsymbol{T})}{\varphi(\mathbf{1})} + \frac{\lambda f^m_{[0,1]\times\left[0,\frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right)\right]}}{\varphi(\mathbf{1}) A(\rho_1)} \int_0^1 \int_0^1 G(t,s)\ ds\ d\alpha(t)\right) > \rho_1, \tag{5.14}$$

which is a contradiction to inequality (5.13). Therefore, from (5.14) we conclude that

$$i_{\mathscr{K}_0}\left(S, \widehat{V}^g_{\rho_1}\right) = 0. \tag{5.15}$$

We next show that $\mu u \neq Su$ for each $u \in \partial\widehat{V}^g_{\rho_2}$ and $\mu \geq 1$. Again suppose for contradiction that this is not so. Then there exist $u \in \partial\widehat{V}^g_{\rho_2}$ and $\mu \geq 1$ such that $\mu u(t) = (Su)(t)$ for each $t \in [0,1]$. Applying g to both sides of this equality yields

$$(g \circ u)(t) \leq (g \circ \mu u)(t) = (g \circ Su)(t),$$

using that $\mu \geq 1$, $u(t) \geq 0$, and g is increasing, and so, we deduce that

$$\begin{aligned}
\rho_2 &= \int_0^1 (g \circ u)(t)\ d\alpha(t) \\
&\leq \int_0^1 (g \circ Su)(t)\ d\alpha(t) \\
&= \int_0^1 g\left(T + \lambda \int_0^1 G(t,s) f\big(s, u(s)\big) \left(A\left(\underbrace{\int_{[0,1]} (g \circ u)(\xi)\ d\alpha(\xi)}_{=\rho_2}\right)\right)^{-1} ds\right) d\alpha(t) \\
&= \int_0^1 g\left(T + \frac{\lambda}{A(\rho_2)} \int_0^1 G(t,s) f\big(s, u(s)\big)\ ds\right) d\alpha(t) \\
&\leq \int_0^1 g\left(T + \frac{\lambda f^M_{[0,1]\times\left[0, \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A(\rho_2)} \int_0^1 G(t,s)\ ds\right) d\alpha(t) \\
&\leq \int_0^1 g\left(\int_0^1 \left(\frac{T}{\mathscr{G}(s)} + \frac{\lambda f^M_{[0,1]\times\left[0, \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A(\rho_2)}\right) \mathscr{G}(s)\ ds\right) d\alpha(t) \\
&\leq \int_0^1 \int_0^1 g\left(\left(\frac{T}{\mathscr{G}(s)} + \frac{\lambda f^M_{[0,1]\times\left[0, \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A(\rho_2)}\right) \mathscr{G}(s)\right) ds\ d\alpha(t) \\
&= \varphi(\mathbf{1}) \int_0^1 g\left(T + \frac{\lambda f^M_{[0,1]\times\left[0, \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A(\rho_2)} \mathscr{G}(s)\right) ds
\end{aligned} \tag{5.16}$$

where we have again used inequality (5.12) (but with ρ_2 replacing ρ_1) as well as the version of Jensen's inequality given by Lemma 5.4 with $p(t) \equiv 1$ and $u(s) = s$ – i.e., so that the integrals are just Riemann integrals, and so, in the notation of Lemma 5.4 we have that $\displaystyle\int_a^b p(s)\ du(s) = \int_0^1\ ds = 1$. Since by condition (2) in the statement of this theorem it holds that

$$\varphi(\mathbf{1}) \int_0^1 g\left(T + \frac{\lambda f^M_{[0,1]\times\left[0, \frac{\varphi(\mathbf{1})}{C_0} g^{-1}\left(\frac{\rho_2}{\varphi(\mathbf{1})}\right)\right]}}{A(\rho_2)} \mathscr{G}(s)\right) ds < 1, \tag{5.17}$$

it follows that inequalities (5.16)–(5.17) jointly imply a contradiction. Therefore, we conclude that

$$i_{\mathscr{K}_0}\left(S, \widehat{V}^g_{\rho_2}\right) = 1. \tag{5.18}$$

Finally, putting (5.15) and (5.18) together we deduce by means of Lemma 2.3 the existence of a fixed point of S, say u_0, such that

$$u_0 \in \widehat{V}^g_{\rho_2} \setminus \overline{\widehat{V}^g_{\rho_1}}.$$

Since such a fixed point is a solution of problem (5.1), we obtain the existence of a positive solution to (5.1). Note that u_0 is not identically zero since

$$u_0 \in \mathscr{K}_0 \setminus \overline{\widehat{V}^g_{\rho_1}} \Longrightarrow u_0 \in \mathscr{K}_0 \setminus \overline{\Omega_{g^{-1}\left(\frac{\rho_1}{\varphi(\mathbf{1})}\right)}}.$$

And this completes the proof. □

We make some brief remarks regarding Theorem 5.7.

Remark 5.8. Theorem 5.7 is a generalization of [31, Theorem 2.6] since in the latter case only the case in which $g(t) = t^q$, for $q \geq 1$, and $\alpha(t) \equiv t$ is considered.

Remark 5.9. Notice that due the use of the nonstandard cone and associated open set we are able to achieve condition (3) in the statement of Theorem 5.7. This, in particular, means that $A(t) < 0$ might hold on non-degenerate subintervals of $[0, +\infty)$. Moreover, $A(t) = 0$ may also hold at one or more $t \in [0, +\infty)$. As was mentioned earlier these allowances are not generally found in the existing literature due to the use of more standard cones and open sets.

Remark 5.10. Similar to Remark 5.9 the use of the nonstandard cone and associated open set also facilitates the "pointwise" conditions in conditions (1)–(2) in the statement of Theorem 5.7. By this we mean that a condition is placed on A only at the numbers ρ_1 and ρ_2. As also mentioned earlier this is very unusual since in nearly all other works either asymptotic or interval-type conditions are imposed on the equivalent of A.

Remark 5.11. Note that if one allow the more restrictive condition $A(t) > 0$ for all $t \in \mathbb{R}$ then arguing as in the previous sections we can additionally obtain that the solution must be nonincreasing.

We conclude with an example of the application of Theorem 5.7.

Example 5.12. Letting $\alpha(t) = t$ consider the convolution nonlocal boundary value problem given below.

$$\begin{aligned} \left(\sin\left(\int_0^1 \left(u(s)\right)^2\, ds\right)\right)\left(a * u''\right)(t) &= -\lambda f\big(t, u(t)\big),\ t \in (0,1) \\ u'(0) &= 0 \\ u(1) &= T \end{aligned} \tag{5.19}$$

Here we have selected $g(t) := t^2$, which is continuous, nonnegative, convex, and strictly increasing on $[0, +\infty)$, and $A(t) := \sin t$. Note that

$$\varphi(\boldsymbol{T}) = \int_0^1 T\, dt = T \text{ and } \varphi(\mathbf{1}) = \int_0^1\, dt = 1.$$

Then since $g^{-1}(t) = \sqrt{t}$, for $t \geq 0$, it follows that condition (1) in Theorem 5.7 is

$$\left(T + \frac{\lambda f^m_{[0,1]\times\left[0,\frac{1}{C_0}\sqrt{\rho_1}\right]}}{\sin \rho_1}\int_0^1\int_0^1 G(t,s)\, ds\, d\alpha(t)\right)^{-1}\sqrt{\rho_1} < 1. \tag{5.20}$$

whereas condition (2) is

$$\int_0^1 \left(T + \frac{\lambda f^M_{[0,1]\times\left[0,\frac{1}{C_0}\sqrt{\rho_2}\right]}}{\sin \rho_2}\mathscr{G}(s)\right)^2\, ds < 1. \tag{5.21}$$

Thus, so long as the interval $\Big[\min\{\rho_1,\rho_2\},\max\{\rho_1,\rho_2\}\Big]\subseteq[0,+\infty)$ can be chosen such that

1. each of (5.20) and (5.21) is satisfied; and
2. $\sin t>0$ for $t\in\Big[\min\{\rho_1,\rho_2\},\max\{\rho_1,\rho_2\}\Big]$;

then problem (5.19) will have at least one positive solution, u_0, satisfying the localization

$$u_0\in\widehat{V}_{\rho_2}^g\setminus\overline{\widehat{V}_{\rho_1}^g}.$$

Notice, in particular, as Remark 5.9 mentioned the function A does not satisfy any global positivity condition. Indeed, $A(t)\le 0$ on countably many non-degenerate subintervals of $[0,+\infty)$.

Finally, we note that if, as in Example 4.7, we select $a(t)=g_{2-\alpha}(t)$, where $1<\alpha<2$, then problem (5.19) becomes the following.

$$\begin{aligned}\left(\sin\left(\int_0^1\big(u(s)\big)^2\,ds\right)\right)\left({}_CD_t^{\alpha}u\right)(t)&=-\lambda f\big(t,u(t)\big),\ t\in(0,1)\\ u'(0)&=0\\ u(1)&=T\end{aligned}$$

Thus, we recover the existence of at least one positive solution to the fractional-order boundary value problem above. Consequently, we obtain, in this instance, a fractional-order analogue of some of the results presented in [14, 15, 18, 31, 52].

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