FUNDAMENTAL SOLUTIONS FOR DISCRETE DYNAMICAL SYSTEMS INVOLVING THE FRACTIONAL LAPLACIAN

JORGE GONZÁLEZ-CAMUS, VALENTIN KEYANTUO, CARLOS LIZAMA, AND MAHAMADI WARMA

Abstract. We prove representation results for solutions of a time-fractional differential equation involving the discrete fractional Laplace operator in terms of generalized Wright functions. Such equations arise in the modeling of many physical systems, for example chain processes in chemistry and radioactivity. Our focus is in the problem $\mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t)$, where $0 < \beta \le 2$, $0 < \alpha \le 1$, $n \in \mathbb{Z}$, $(-\Delta_d)^\alpha$ is the discrete fractional Laplacian and $\mathbb{D}_t^\beta$ is the Caputo fractional derivative of order β. We discuss important special cases as consequences of the representations obtained.

1. Introduction

In the present article, we shall mainly focus on the following discrete dynamical system:

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n),\ u_t(n,0) = \psi(n), & n \in \mathbb{Z}. \end{cases} \tag{1.1}$$

Here, g, φ and ψ are given data, $0 < \alpha \le 1$, $(-\Delta_d)^\alpha$ is the discrete fractional Laplacian defined by

$$(-\Delta_d)^\alpha f(n) = \sum_{k\in\mathbb{Z}} K^\alpha(n-k) f(k),\ n \in \mathbb{Z},\ f \in l^2(\mathbb{Z}), \tag{1.2}$$

where the coefficients K^α are given by

$$K^\alpha(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (4\sin^2(\theta/2))^\alpha e^{-in\theta} d\theta,\ n \in \mathbb{Z}, \tag{1.3}$$

and for a real number $\gamma > 0$, $\mathbb{D}_t^\gamma$ is the Caputo time fractional derivative of order γ defined by

$$\mathbb{D}_t^\gamma u(t) := \int_0^t g_{m-\gamma}(t-s) u^{(m)}(s) ds,$$

2010 *Mathematics Subject Classification.* 35R11, 35A08, 39A12.

Key words and phrases. Caputo fractional derivative; discrete fractional Laplacian; fundamental solutions; Wright and Mittag-Leffler functions.

The first author was supported by CONICYT under grant Doctorado Nacional Chile 2016 NO:21160120. The work of the second and fourth authors is partially supported by the Air Force Office of Scientific Research (AFOSR) under Award NO: FA9550-18-1-0242. The third author is partially supported by FONDECYT under grant NO: 1180041.

where $g_\beta(t) := \dfrac{t^{\beta-1}}{\Gamma(\beta)}$, $\beta > 0$, $t > 0$, and $m := \lceil \gamma \rceil$ is the smallest integer greatest than or equal to γ, $u^{(m)}$ is the m^{th}-order distributional derivative of $u(\cdot)$, under appropriate assumptions. Then, when $\gamma = n$ is a natural number, we get the ordinary derivative of order n, that is, $\mathbb{D}_t^n u := u^{(n)}$.

In the present paper we shall firstly investigate explicit representation of solutions of the semi discrete model (1.1) in the case $0 < \alpha \le 1$ and $1 < \beta \le 2$. In the second part the case $0 < \beta \le 1$ will be studied. Our analysis relies heavily on the following generalized Wright functions

$$F(z) \equiv {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| \, z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \quad z \in \mathbb{C}. \tag{1.4}$$

We shall describe the range of the parameters and give rigorous results on the convergence of the above series in Section 2.

It is known that, the representation of solutions of differential equations in an explicit form is important but, not always easily obtained. In this regard, the fundamental solutions (or Green functions) represent an important tool, since knowledge of these functions allows us to obtain analytical representations of solutions. Fundamental solutions for many problems governed by fractional derivatives can be found in the literature. For instance, Ferreira, Rodrigues and Vieira in [17] obtained the fundamental solution of the time-fractional telegraph Dirac operator, and the multidimensional time-fractional telegraph equation [16] and Mirza and Vieru [34] found the fundamental solutions to the advection-diffusion equation with time-fractional Caputo-Fabrizio derivative. See also the references therein. However, to the best of our knowledge, there is no explicit representation of solutions of the semi discrete model (1.1) in the full range $0 < \alpha \le 1$ and $0 < \beta \le 2$.

Our main result in the present paper shows that the unique solution of the system (1.1) has the following explicit representation:

$$\begin{aligned} u(n,t) = & \sum_{m \in \mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m \in \mathbb{Z}} H_{\alpha,\beta}(n-m,t)\phi(m) \\ & + \sum_{m \in \mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s) g(m,s)\, ds, \quad n \in \mathbb{Z},\ t \ge 0, \end{aligned} \tag{1.5}$$

where the functions $G_{\alpha,\beta}(n,t), H_{\alpha,\beta}(n,t), L_{\alpha,\beta}(n,t)$ are given for $n \in \mathbb{Z}$ and $t \ge 0$ by

$$\begin{aligned} G_{\alpha,\beta}(n,t) :=& (-1)^n \, {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| \, -t^\beta \right] \\ =& \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+1}(t), \end{aligned} \tag{1.6}$$

and

$$H_{\alpha,\beta}(n,t)) := (-1)^n t \, {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (2,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right]$$
$$= \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+2}(t), \tag{1.7}$$

and

$$L_{\alpha,\beta}(n,t)) := (-1)^n t^{\beta-1} \, {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right]$$
$$= \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+\beta}(t). \tag{1.8}$$

We call $G_{\alpha,\beta}(\cdot,t)$ the fundamental solution of the system (1.1). Using the classical Stirling formula and the Fubini-Tonelli Theorem, we shall show that $G_{\alpha,\beta}(\cdot,t) \in l^1(\mathbb{Z})$. This result will allow us to solve the system (1.1) for arbitrary initial values $\varphi, \psi \in l^p(\mathbb{Z})$ for every $1 \leq p \leq \infty$.

We observe that only very recently, Lizama and Roncal [28] obtained the fundamental solutions in the cases $\beta = 1$ and $\beta = 2$. They combined operator theory techniques with the properties of the Bessel functions to develop a theory of analytic semigroups and cosine operators generated by $-(-\Delta_d)^\alpha$. Such theory was then applied to prove existence and uniqueness of almost periodic solutions to nonlinear versions of the equation (1.1). Moreover, they proved a characterization of well-posedness on periodic Hölder spaces and applied their results to the Nagumo and Fisher-KPP models with a discrete Laplacian. However, the fundamental solutions that we have obtained in the case $1 < \beta < 2$, that is, the fractional wave equation, have not yet been considered elsewhere before and therefore are completely new.

The interest in the study of these lattice differential models comes partially from an equation studied by Mainardi, Luchko and Pagnini [32]. In the years 1940, H. Bateman [6] made an extensive study of differential-difference equations with applications to a wide array of physical and engineering systems. Other specific areas of applications include weather forecast by numerical processes and the study of crystal lattices in vibration initiated by Born and von Kármán (more details on this and further topics can be found in [6]). Several authors have studied equations involving the discrete fractional Laplacian, for example, [12, 13, 21, 22, 28].

As pointed out in [28], lattice differential equations with a discrete fractional Laplacian arise to understand the behavior of processes related to small objects where the continuum limit cannot describe events on length scales comparable to nanometers. The lattice approach gives a possible microstructural basis for anomalous diffusion in media that are characterized by the non-locality of power law type. Examples of such behavior can be found on the deformations and diffusion processes in solid objects like nanocrystalline and ultrafine grain polycrystals, both with and without external forces, nanomechanics, and N-dimensional physical lattices with long-range particle interactions. In all of these cases, the main advantage of the semidiscrete analysis over the continuum is the better accuracy in the description of the phenomena. The references [2, 11, 39, 40, 41, 42] cover several of these phenomena and demonstrate the importance of fractional discrete equations.

On the other hand, the class of functions given in (1.4) was studied by Wright (see e.g. [45, 44, 46] and the references therein), who carried out an extensive analysis of their asymptotic behavior. His primary motivation was the study of the asymptotic theory of partitions in number theory. These functions have nowadays found important applications notably in fractional calculus. Hyper-geometric functions belong to this class and thus, most of the special functions of mathematical physics can be represented using them. We note, however, that the class is much wider than that of hyper-geometric functions. A precise reference on the subject is [18]. For the particular case of the Mittag-Leffler functions (which are of older origin), we refer for example to [18] and the references therein. It is remarkable that Wright [44] introduced the now called Wright functions in his study of the asymptotic theory of partitions.

Evolution equations of fractional order have become an active area of study due to the fact that many phenomena have been found to be more accurately modeled using such equations. The idea of fractional derivative is not new, but the realization that they provide better models for a large array of problems in science and engineering is rather recent. Phenomena with memory effects, anomalous diffusion, polymer science, rheology, material science are but some of the areas where modeling with fractional derivatives has proven successful in uncovering properties that were not detectable with the usual differential equations. The references [18, 29, 31, 30, 33, 37] cover several of these phenomena and demonstrate the importance of the fractional model. The basic functions relevant to these studies are the Mittag-Leffler functions, the Wright functions and their generalizations. The thesis [8] is an early reference on the topic with a functional analytic perspective. Several monographs have appeared on fractional calculus and its applications, e.g. [30, 33, 36, 37].

The structure of the paper is the following. In Section 2, we present preliminary facts on the discrete fractional Laplacian, the Mittag-Leffler and Wright functions and their generalizations. Also, we give some results related to the well-known Bessel functions which will be useful in the sequel. Some results on discrete Fourier transform will be also given. In Section 3, we consider the system (1.1) with $1 < \beta \leq 2$ and $0 < \alpha \leq 1$ where we give a detailed proof of the representation (1.5). In Section 4, we make a deeper study of some special cases. These special cases are the discrete wave equation, that is, when $\beta = 2$, $\alpha = 1$ and also the fractional wave equation, that is, $\alpha = 1$ and $1 < \beta < 2$. In Section 5 we revisit the case $0 < \beta \leq 1$ and relate our results to those of [25] that are obtained through the subordination principle. Finally in Section 6 we study the translated operator $\Delta_{d,h} := \Delta_d - hI$ which is a perturbation of the operator Δ_d.

2. Preliminaries

In this section we introduce some notations and give some known results as they are needed throughout the paper.

2.1. **The discrete fractional Laplace operator.** Let $(-\Delta)^\alpha$ be the fractional Laplacian of order $0 < \alpha < 1$. We recall from [12, Section 3] and [13], that its discrete counterpart is defined as the discrete convolution operator in the following way (see e.g. [9] and references therein):

$$(-\Delta_d)^\alpha f(n) = \sum_{k\in\mathbb{Z}} K^\alpha(n-k)f(k),\ n\in\mathbb{Z},\ f\in l^2(\mathbb{Z}), \tag{2.1}$$

where the coefficients K^α are given by (see [13]):

$$K^\alpha(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(4\sin^2(\theta/2))^\alpha e^{-in\theta}d\theta = \frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)},\ n\in\mathbb{Z}. \tag{2.2}$$

This sequence encapsulates all the information about the discrete fractional Laplacian, and will be very important in what follows. A graphical representation is given in [13]. Some properties are: $K^\alpha(n) > 0$ only when $n = 0$, $\sum_{n\in\mathbb{Z}} K^\sigma(n) = 0$ ([28, Proposition 1]), $\|K^\sigma\|_1 = 2\frac{\Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}$([28, Lemma 3.2]) and $|K^\alpha(n)| \sim \frac{\Gamma(2\alpha+1)}{\pi}|n|^{-2\alpha-1}$.

In the borderline case $\alpha = 1$, we have that

$$\Delta_d f(n) = f(n+1) - 2f(n) + f(n-1), \quad n\in\mathbb{Z},$$

which is in agreement with the definition of the discrete version of the well-know Laplace operator Δ. This is of course related to the approximation of the second derivative. In fact, discrete fractional models have been used to approximate the continuous models (see e.g. [14]). We notice that the above representation (2.1) of the discrete fractional Laplace operator may be also obtained from the semigroup generated by Δ_d using the general theory of fractional powers of generators of bounded C_0-semigroups or by using the so called Caffarelli-Silvestre extension (see e.g. [5, 10, 38] and their references).

We observe that the discrete fractional Laplace operator $(-\Delta_d)^\alpha$ is bounded on $\ell^p(\mathbb{Z}, X)$ for any Banach space X and $1 \le p \le \infty$. Indeed, by [13, p.121], we have that $K^\alpha(n) \sim \frac{C}{n^{2\alpha+1}}$ and hence, the result follows from a simple application of the Young inequality. That is,

$$\|(-\Delta_d)^\alpha f\|_p = \|K^\alpha * f\|_p \le \|K^\alpha\|_1\|f\|_p.$$

For more information on the discrete fractional Laplacian, we refer to [12, 13, 28] and the references therein.

2.2. **The discrete Fourier transform.** For a given sequence $f = (f(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, the discrete Fourier transform is defined by

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{in\theta} = \sum_{n\in\mathbb{Z}} f(n)z^n \equiv \hat{f}(z), \quad \theta\in\mathbb{T}, \quad z = e^{i\theta},$$

where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ is the one-dimensional torus, that we identify with the interval $[-\pi, \pi]$.

Given $\varphi \in L^1[-\pi,\pi]$, its inverse discrete Fourier transform is given by the formula

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta) e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Therefore

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\mathbb{Z}}(f)(\theta) e^{-in\theta}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} z^{n-1}\hat{f}(z)dz, \quad n \in \mathbb{Z}.$$

The convolution theorem

$$\mathcal{F}_{\mathbb{Z}}(f * g) = \mathcal{F}_{\mathbb{Z}}(f)\mathcal{F}_{\mathbb{Z}}(g),$$

holds, where $*$ denotes the usual convolution in $l^1(\mathbb{Z})$.

In what follows, we denote $\mathcal{F} := \mathcal{F}_{\mathbb{Z}}$ and $\mathcal{F}(f)(z) := f(z)$. Also we let

$$J(z) := z + \frac{1}{z} - 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.3}$$

We observe that the function $J(z)$ is a translation of the well known Joukowsky transform. It is obvious that for $|z| = 1$ the values of $J(z)$ lie in the interval $[-4, 0]$.

We conclude this subsection with the following observation.

Remark 2.1. *The following properties hold.*

(i) $-J(z) \geq 0$ *for all* $z \in \mathbb{C},\ |z| = 1$.
(ii) $J(z) = -4\sin^2(\theta/2)$, *if* $z = e^{i\theta}$.
(iii) $\mathcal{F}(K^\alpha(n))(z) = (-J(z))^\alpha$ *for all* $z \in \mathbb{C},\ z = e^{i\theta},\ \theta \in \mathbb{R}$, *where* K^α *is given in* (2.2).

2.3. **Useful properties of a class of special functions.** The well-known Mittag-Leffler function (see e.g. [19, 31, 36, 37]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha > 0,\ \beta > 0, z \in \mathbb{C}. \tag{2.4}$$

A useful identity about the derivative of the Mittag-Leffler function is the following. Let $\alpha, \gamma > 0$ and $m \in \mathbb{N} \cup \{0\}$. Then by [36, p. 22, Formula (1.83)],

$$\left(\frac{d}{dz}\right)^m [z^{\gamma-1}E_{\alpha,\gamma}(z^\alpha)] = z^{\gamma-m-1}E_{\alpha,\gamma-m}(z^\alpha). \tag{2.5}$$

Recall the definition of the Wright type function with one parameter from [19, Formula (28)] (see also [36, 37, 46]):

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{2\pi i}\int_\gamma \mu^{\alpha-1} e^{\mu - z\mu^\alpha} d\mu, \quad 0 < \alpha < 1, \tag{2.6}$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counter-clockwise. This special case has sometimes been called the Mainardi function.

An interesting fact is the following relationship between the Wright and the Mittag-Leffler functions.

Remark 2.2. Let $z \in \mathbb{C}$, $t > 0$ and $0 < \alpha, \gamma < 1$. Then the following properties hold:

(i) $E_{\gamma,1}(z) = \displaystyle\int_0^\infty \Phi_\gamma(t) e^{zt} dt$.

(ii) $\Phi_\alpha(t) \geq 0$.

(iii) $\displaystyle\int_0^\infty \Phi_\alpha(t) dt = 1$.

It follows from (ii) and (iii) that Φ_α is a probability density function on $\mathbb{R}_+$. Actually, the Wright function has been used for models in stochastic processes [19, 31].

The following formula on the moments of the Wright function will be useful:

$$\int_0^\infty x^p \Phi_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)}, \quad p > -1, \quad 0 < \alpha < 1. \tag{2.7}$$

The formula (2.7) is derived from the representation (2.6) and can be found in [19, formula after (38)]. Note that in this reference the notation $M(x, \alpha) := \Phi_\alpha(x)$ is used. For more details on the Wright type functions, we refer to [8, 19, 29, 46] and the references therein. These functions were studied by Wright initially in connection with the asymptotic theory of partitions.

We will need the following function, called stable Lévy process, defined for $0 < \alpha < 1$ by

$$f_{t,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^\alpha} dz, & \sigma > 0, \quad t > 0, \quad \lambda \geq 0, \\ 0 & \lambda < 0, \end{cases} \tag{2.8}$$

where the branch of z^α is taken so that $\mathrm{Re}(z^\alpha) > 0$ for $\mathrm{Re}(z) > 0$. This branch is single-valued in the z-plane cut along the negative real axis.

Remark 2.3. The following properties hold:

(i) $\displaystyle\int_0^\infty e^{-\lambda a} f_{t,\alpha}(\lambda) d\lambda = e^{-ta^\alpha}, \quad t > 0, \quad a > 0, \quad 0 < \alpha < 1.$

(ii) $f_{t,\alpha}(\lambda) \geq 0, \quad \lambda > 0, \; t > 0, \quad 0 < \alpha < 1.$

(iii) $\displaystyle\int_0^\infty f_{t,\alpha}(\lambda) d\lambda = 1, \quad t > 0, \quad 0 < \alpha < 1.$

(iv) $f_{t+s,\alpha}(\lambda) = \displaystyle\int_0^\lambda f_{t,\alpha}(\lambda - \mu) f_{s,\alpha}(\mu) d\mu, \quad \lambda > 0, \quad t, s > 0, \quad 0 < \alpha < 1.$

(v) $\displaystyle\int_0^\infty e^{\lambda z} f_{\lambda,\alpha}(t) d\lambda = t^{\alpha-1} E_{\alpha,\alpha}(zt^\alpha), \quad z \in \mathbb{C}, \quad t > 0, \quad 0 < \alpha < 1.$

For a proof of (i)-(iv), see [47, p.260-262]. Concerning (iv) we have to observe in [47, Proposition 1, p.260] that by definition $f_{t,\alpha}(\lambda) = 0$ for $\lambda < 0$. For the interesting property (v) we refer to the recent paper [1, Theorem 3.2 (iii)].

Now, we present the generalized Wright function (see [26, 27, 45, 44]) that we have briefly mentioned in the introduction. It is defined by

$$F(z) \equiv {}_p\Psi_q \left[\begin{array}{c|} (a_i, \alpha_i)_{1,p} \\ \\ (b_j, \beta_j)_{1,q} \end{array} \; z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \tag{2.9}$$

where $z, a_i, b_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, ..., p$ and $j = 1, ..., q$. Results on the radius of convergence of the series (2.9) are presented in the following theorem. We refer to [27, Theorem 1.5 p.56.] for the proof which relies on the well-known Stirling's formula.

Theorem 2.4. *Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}, (i = 1, ..., p; j = 1, ..., q)$, and let:*

$$\kappa = \textstyle\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l,$$

$$\delta = \prod_{l=1}^{p} |\alpha_l|^{-\alpha_l} \prod_{j=1}^{q} |\beta_j|^{\beta_j},$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{l=1}^{p} a_l - \frac{p-q}{2}.$$

(i) *If $\kappa > -1$, then the series in (2.9) is absolutely convergent for all $z \in \mathbb{C}$.*
(ii) *If $\kappa = -1$, then the series in (2.9) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\mathrm{Re}(\mu) > \frac{1}{2}$.*

Furthermore, if $c_k = \dfrac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \dfrac{1}{k!}$, then (using the well-known) Stirling's formula, for k sufficiently large, we have that

$$|c_k| \sim A \left(\frac{k}{e}\right)^{-(\kappa+1)k} \delta^{-k} k^{-(\mathrm{Re}(\mu) + \frac{1}{2})}, \tag{2.10}$$

where

$$A = (2\pi)^{\frac{p-q-1}{2}} \frac{\prod_{l=1}^{p} |\alpha_l|^{\mathrm{Re}(a_l) - \frac{1}{2}}}{\prod_{j=1}^{q} |\beta_j|^{\mathrm{Re}(b_j) - \frac{1}{2}}}.$$

Remark 2.5. The Mittag-Leffler and Wright functions can be recovered as special cases of the generalized Wright function (with an obvious extension of notation). Indeed,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = {}_0\Psi_1 \left[\begin{array}{c|} -- \\ \\ (\beta, \alpha) \end{array} \; z \right],$$

and

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = {}_0\Psi_1 \left[\begin{array}{c|} -- \\ \\ (1-\alpha, -\alpha) \end{array} \; -z \right].$$

The Wright function with two-parameters $W_{\lambda,\mu}(z)$ has the representation:

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)} = {}_0\Psi_1 \left[\begin{array}{c|} -- \\ \\ (\mu, \lambda) \end{array} \; z \right].$$

Note that the Lévy functions can be defined using the Wright functions of one and two parameters. In the case of one parameter, we have the representation

$$f_{t,\alpha}(\lambda) = \frac{\alpha t}{\lambda^{\alpha+1}} \Phi_\alpha(t\lambda^{-\alpha}),$$

which follows from [24, (2.11)] and [47, Prop. 1, p.260]. For the case of the Wright function with two parameters, it follows from [1, Formulas (17), (18) and (32)] that

$$f_{t,\alpha}(\lambda) = \frac{1}{\lambda} \Phi_{-\alpha,0}(-t\lambda^{-\alpha}).$$

For $\nu \in \mathbb{R}$, let J_ν denote the *Bessel function* defined by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \geq 0. \tag{2.11}$$

We recall some properties of the Bessel functions. More precisely, the generating function, symmetry and the Gegenbauer addition formula (see [20, Formula 8.511], [3]). We have

$$\sum_{n\in\mathbb{Z}} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0, \tag{2.12}$$

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x), \tag{2.13}$$

$$J_n(x+y) = \sum_{k\in\mathbb{Z}} J_{n-k}(x) J_k(y). \tag{2.14}$$

The integral representation of J_ν (see [3, Chapter 4]) is given by

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{ixt}(1-t^2)^{\nu-\frac{1}{2}} dt, \ \mathrm{Re}(\nu) > -\frac{1}{2}. \tag{2.15}$$

From (2.15), one easily deduces the Poisson integral representation

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_0^\pi \cos(x\cos\theta) \sin^{2\alpha}(\theta) \, d\theta. \tag{2.16}$$

We will need this in the estimate of the fundamental solution. For $\nu \in \mathbb{R}$, I_ν denotes the *Modified Bessel functions of the first kind*, defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.17}$$

This function satisfies, among others, the following properties:

$$I_n(x) \geq 0, \quad n \in \mathbb{Z}, \ x \geq 0, \tag{2.18}$$

$$\sum_{n\in\mathbb{Z}} I_n(x) z^n = e^{\frac{x}{2}(z+\frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0, \tag{2.19}$$

$$I_{-n}(x) = I_n(x) = (-1)^n I_n(-x). \tag{2.20}$$

From the definition of $J_\nu(x)$ and $I_\nu(x)$ it is clear that

$$|J_\nu(x)| \leq I_\nu(x), \quad \nu \in \mathbb{R}_+. \tag{2.21}$$

The next identity will be crucial for obtaining the explicit representation of our fundamental solution (see e.g. [35]).

Lemma 2.6. *Assume that $Re(a) > 0$ and $b \in \mathbb{R}$. Then*

$$\int_0^\pi (\sin(t))^{a-1} e^{ibt} dt = \frac{\pi}{2^{a-1}} \frac{e^{i\pi b/2}}{a\mathbb{B}((a+b+1)/2, (a-b+1)/2)}, \tag{2.22}$$

where $\mathbb{B}(\cdot,\cdot)$ is the usual Beta function.

3. Main results in the case $1 < \beta \leq 2$

In this section, we give an explicit representation of solutions for the following time/space fractional evolution equation:

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) + (-\Delta_d)^\alpha u(n,t) = g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}. \end{cases} \tag{3.1}$$

Here, $\alpha \in (0,1]$ and $\beta \in (1,2]$ are real numbers, $(-\Delta_d)^\alpha$ is the discrete fractional Laplace operator defined in (2.1) and we recall that $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative given by

$$\mathbb{D}_t^\beta v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t).$$

The main result in this section is the following Theorem.

Theorem 3.1. *Let $\varphi, \phi \in l^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}_+$, $g(\cdot,t) \in l^\infty(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot,s)||_\infty < \infty$. Then the function*

$$\begin{aligned} u(n,t) = &\sum_{m\in\mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} H_{\alpha,\beta}(n-m,t)\phi(m) \\ &+ \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s) g(m,s) ds, \end{aligned} \tag{3.2}$$

is the unique solution of the initial value problem (3.1), where $G_{\alpha,\beta}(n,t)$, $H_{\alpha,\beta}(n,t))$ and $L_{\alpha,\beta}(n,t))$ are given by (1.6), (1.7) and (1.8), respectively. Furthermore, $G_{\alpha,\beta}(\cdot,t), H_{\alpha,\beta}(\cdot,t)$, $L_{\alpha,\beta}(\cdot,t) \in l^1(\mathbb{Z})$, for every $t > 0$.

Proof. We prove the result in several steps.

Step 1. First we show the explicit solution. Taking the discrete Fourier trasformation of (3.1), we obtain that

$$\begin{cases} \mathbb{D}_t^\beta u(z,t) = -(-J(z))^\alpha u(z,t) + g(z,t) \\ u(z,0) = \varphi(z), \ u_t(z,0 = \phi(z). \end{cases} \tag{3.3}$$

It is known that a solution of the ODE (3.3) is given by

$$\begin{aligned} u(z,t) =& E_{\beta,1}\left(-(-J(z))^{\alpha}t^{\beta}\right)\varphi(z) + tE_{\beta,2}\left(-(-J(z))^{\alpha}t^{\beta}\right)\phi(z) \\ &+ t^{\beta-1}E_{\beta,\beta}\left(-(-J(z))^{\alpha}t^{\beta}\right) * g(z,t). \end{aligned} \tag{3.4}$$

Calculating the inverse discrete Fourier transform of (3.4) and using (2.1) we get that

$$E_{\beta,1}(-(-J(z))^{\alpha}t^{\beta}) = E_{\beta,1}\left(-4^{\alpha}\sin^{2\alpha}(\theta/2)\,t^{\beta}\right) = \sum_{k=0}^{\infty}\frac{(-4^{\alpha}\sin^{2\alpha}\left(\frac{\theta}{2}\right)t^{\beta})^{k}}{\Gamma(\beta k+1)}. \tag{3.5}$$

Thus

$$\begin{aligned} G_{\alpha,\beta}(n,t) :=& \mathcal{F}^{-1}\left(\left(E_{\beta,1}(-(-J(z))^{\alpha}t^{\beta})\right)(n)\right. \\ =& \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta n}\sum_{k=0}^{\infty}\frac{(-4^{\alpha}\sin^{2\alpha}\left(\frac{\theta}{2}\right)t^{\beta})^{k}}{\Gamma(\beta k+1)}d\theta \\ =& \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k}4^{\alpha k}t^{\beta k}}{\Gamma(\beta k+1)}\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)\,d\theta. \end{aligned} \tag{3.6}$$

Using the change of variable $\frac{\theta}{2} = \rho$, we get that

$$\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)\,d\theta = 2\int_0^{\pi} e^{ib\rho}(\sin(\rho))^{a-1}d\rho, \quad b=-2n, \quad a = 2\alpha k+1.$$

It follows from the integral representation (2.22) that

$$\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)d\theta = \frac{2\pi}{2^{2\alpha k}}\frac{(-1)^{n}}{(2\alpha k+1)B(\alpha k-n+1,\alpha k+n+1)}, \tag{3.7}$$

Now, using the relation $\Gamma(z+1) = z\Gamma(z)$ together with (3.6) and (3.7) we get that

$$\begin{aligned} G_{\alpha,\beta}(n,t) =& \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k}4^{\alpha k}t^{\beta k}}{\Gamma(\beta k+1)}\frac{2\pi}{2^{2\alpha k}}\frac{(-1)^{n}}{(2\alpha k+1)\mathbb{B}(\alpha k-n+1,\alpha k+n+1)} \\ =& (-1)^{n}\sum_{k=0}^{\infty}\frac{(-1)^{k}t^{\beta k}}{\Gamma(\beta k+1)}\frac{\Gamma(2\alpha k+2)}{(2\alpha k+1)\Gamma(\alpha k-n+1)\Gamma(\alpha k+n+1)}. \end{aligned}$$

We have shown that

$$G_{\alpha,\beta}(n,t) = (-1)^{n}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)}\frac{(-t^{\beta})^{k}}{k!}. \tag{3.8}$$

From the definition of the Wright function (2.9), we can deduce that

$$G_{\alpha,\beta}(n,t) = (-1)^{n}\,{}_2\Psi_3\left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix}\,\middle|\, -t^{\beta}\right].$$

Furthermore, notice that by (2.5), we have that

$$tE_{\beta,2}(-(-J(z))^{\alpha}t^{\beta}) = \int_0^t E_{\beta,1}(-(-J(z))^{\alpha}x^{\beta})dx. \tag{3.9}$$

We shall use this fact to derive the expression $H_{\alpha,\beta}$. In fact, by (3.9), we have that

$$\begin{aligned} H_{\alpha,\beta}(n,t)) :=& \mathcal{F}^{-1}(tE_{\beta,2}(-(-J(z))^{\alpha}t^{\beta})) \\ =& (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)(\beta k+1)} \frac{(-1)^k t^{\beta k+1}}{k!} \\ =& (-1)^n t \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^{\beta})^k}{k!} \\ =& (-1)^n t \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (2,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^{\beta} \right]. \end{aligned} \tag{3.10}$$

Analogously to the computations of $G_{\alpha,\beta}(n,t)$, we have that

$$\begin{aligned} L_{\alpha,\beta}(n,t)) :=& \mathcal{F}^{-1}(t^{\beta-1}E_{\beta,\beta}(-(-J(z))^{\alpha}t^{\beta}))(n) \\ =& (-1)^n t^{\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^{\beta})^k}{k!} \\ =& (-1)^n t^{\beta-1} \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ & & \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^{k} \right]. \end{aligned} \tag{3.11}$$

We have shown that the solution $u(n,t)$ of (3.1) has the representation (3.2). In addition since $\kappa = \beta - 1 > -1$ in each of the series (3.8), (3.10) and (3.11), it follows from Theorem 2.4(i) that the series converge uniformly and that $G_{\alpha,\beta}(n,z), H_{\alpha,\beta}(n,z)$, and $L_{\alpha,\beta}(n,z)$ are entire functions of z.

Step 2. We show uniqueness. Assume that the system (3.1) has two solutions u_1 and u_2 with the same initial data φ, ϕ, and set $v := u_1 - u_2$. Then v is a solution of the following ODE

$$\mathbb{D}_t^{\beta} v(z,t) = -(-J(z))^{\alpha} v(z,t), \;\; v(z,0) = 0, \;\; v_t(z,0) = 0.$$

Since the above fractional differential equation has zero as its unique solution (see e.g. [8]), we have that $v(z,t) = 0$. The uniqueness of the inverse discrete Fourier transform implies that $v(n,t) = 0$ for every $n \in \mathbb{Z}$ and $t \geq 0$. Hence, $u_1 = u_2$.

Step 3. Next, we show that $G_{\alpha,\beta}(\cdot,t) \in l^1(\mathbb{Z})$, for every $t > 0$. Indeed, it follows from the representation (3.8) that

$$G_{\alpha,\beta}(n,t) = G_{\alpha,\beta}(-n,t) \;\; \text{for every} \;\; n \in \mathbb{Z} \;\; \text{and} \;\; t \geq 0.$$

Therefore, using the Fubini-Tonelli Theorem we get that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|G_{\alpha,\beta}(n,t)| &\le \sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\ &\le 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\ &\le 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}.\end{aligned}$$

Let $c_k=\dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$. Using (2.10), we obtain that, for k sufficiently large,

$$|c_k|\sim\frac{(\alpha k)^n}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}. \tag{3.12}$$

Therefore, for k_0 sufficiently large, it follows from (3.12) that

$$\begin{aligned}&2\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\ &\sim 2\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}\sum_{n=0}^{\infty}\frac{(\alpha k)^n}{n!}\\ &=2\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}e^{\alpha k}\\ &=\frac{2}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\frac{1}{2}}}(et)^{\beta k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}.\end{aligned}$$

Since $\beta>\alpha$, then applying the root test, we can deduce that $G_{\alpha,\beta}(\cdot,t)\in l^1(\mathbb{Z})$ for every $t>0$.

Step 4. Next we show $H_{\alpha,\beta}(\cdot,t)\in l^1(\mathbb{Z})$ for every $t>0$. For $t>0$, we have that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|H(n,t)| &\le t\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\ &\le 2t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\ &\le 2t\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}.\end{aligned}$$

Letting $d_k = \dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$ and using (2.10), we obtain that, for k sufficiently large,

$$|d_k| \sim \frac{(\alpha k)^n}{k(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}. \tag{3.13}$$

Therefore, for k_0 sufficiently large, it follows from (3.13) that

$$\begin{aligned}
&2t\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\
&\sim 2t\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{k(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!}\\
&=\frac{2t}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\frac{3}{2}}}(et)^{\beta k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}.
\end{aligned}$$

Since $\beta>\alpha$, then using the root test again, we get that $H_{\alpha,\beta}(\cdot,t)\in l^1(\mathbb{Z})$ for every $t>0$.

Step 5. Finally we show that $L_{\alpha,\beta}(\cdot,t)\in l^1(\mathbb{Z})$ for every $t>0$. Proceeding as above, for each $t>0$, we have that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}|L(n,t)| \le& t^{\beta-1}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\
&\le 2t^{\beta-1}\sum_{k=0}^{\infty}\sum_{n\in\mathbb{N}_0}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}.
\end{aligned}$$

Letting $r_k = \dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$ and using (2.10) again, we obtain that, for k sufficiently large,

$$|r_k| \sim \frac{(\alpha k)^n}{k^{\beta-\frac{1}{2}}(\pi\beta)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}. \tag{3.14}$$

Therefore, for k_0 sufficiently large (by using (3.14)) we have that

$$\begin{aligned}
&2t^{\beta-1}\sum_{k=k_0}^{\infty}\sum_{n\in\mathbb{N}_0}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}\\
&\sim 2t^{\beta-1}\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{k^{\beta-\frac{1}{2}}(\pi\beta)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!}\\
&=\frac{2t^{\beta-1}}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\beta-\frac{1}{2}}}(et)^{\beta k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}.
\end{aligned}$$

Since $\beta>\alpha$, we can deduce that $L_{\alpha,\beta}(\cdot,t)\in l^1(\mathbb{Z})$ for every $t>0$. The proof of the theorem is finished. $\square$

Remark 3.2. We mention that the fundamental solution for the continuous models similar to the semi-discrete problem that is under study in the present paper was investigated in [32] by Mainardi, Luchko and Pagnini. Here we have defined the fundamental solution for the equation of order $\beta \in (1,2]$ by requiring that the initial value φ be the sequence $\varphi(n)$ such that $\varphi(0)=1$ and $\varphi(n)=0$ for all $n \neq 0$ and the initial velocity is $\phi \equiv 0$. In the study of the wave equation, it is sometimes the choice $\varphi \equiv 0$ and $\phi(n)=\delta_0^n$ (the Kronecker symbol) that is used. It is easy to see that when $\beta = 2$ then $H_{\alpha,\beta}(n,t)$ and $L_{\alpha,\beta}(n,t)$ coincide. In fact, in the study of the second order nonhomogeneous evolution equation

$$\begin{cases} w''(t) = Aw(t) + g(t), \quad t>0, \\ w(0) = x, \\ w'(0) = y, \end{cases} \tag{3.15}$$

in a Banach space X, if we assume that the problem is well posed, thus A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$, denoting the associated sine function by $(S(t))_{t\geq 0}$, the mild solution is given by:

$$w(t) = C(t)x + S(t)y + \int_0^t S(t-s)g(s)ds, \; t \geq 0.$$

Therefore, two operator families suffice to describe the solutions of the problem. This subject is treated in [4, Section 3.14]. In the fractional case with $1<\beta<2$ three operator families are needed in the representation of the solutions (see e.e. [23, 24]). However, we have the following relations between $G_{\alpha,\beta}(n,t)$, $H_{\alpha,\beta}(n,t)$ and $L_{\alpha,\beta}(n,t)$.

Remark 3.3. Let $t \geq 0$ and $n \in \mathbb{Z}$. Then

$$H_{\alpha,\beta}(n,t) = \int_0^t G_{\alpha,\beta}(n,s)\, ds$$

and

$$L_{\alpha,\beta}(n,t) = \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} G_{\alpha,\beta}(n,s)ds = g_{\beta-1} * G_{\alpha,\beta}(n,t),$$

where for any $\gamma>0$ and $t>0$, $g_\gamma(t) := \frac{t^{\gamma-1}}{\Gamma(\gamma)}$.

Remark 3.4. From the series representation (3.8) and using (2.2) we obtain the following identity

$$G_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n) \frac{(-t^\beta)^j}{\Gamma(1+\beta j)} = \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+1}(t).$$

Interpreting the above series as an inner product, we observe that we can separate the variables n and t in two sequences indexed by j. One depending on α and n and the other depending on β and t. These sequences tell us the different role played by α and β in the equation. Since α is linked to $K^{\alpha j}(n)$, it means that this sequence is the responsible for the control the lattice-behavior of the equation i.e. the discrete fractional Laplacian operator and, on the other hand, β is linked to $\frac{(-t^\beta)^j}{\Gamma(1+\beta j)}$ which means that this term expresses the behavior of the time-fractional derivative. For example, $\beta = 2$ produces $\frac{(-1)^j t^{2j}}{(2j)!}$ which corresponds to the coefficient in the series of the cosine function, i.e. the wave equation, and for $\beta = 1$ we have $\frac{(-1)^j t^j}{j!}$ that corresponds to the coefficient in the series of the exponential function, i.e. the heat equation. For instance, $G_{\alpha,2}(n,t) = \displaystyle\sum_{j=0}^{\infty} K^{\alpha j}(n) \frac{(-1)^j t^{2j}}{(2j)!}$ represents, when reading the coefficients in such way, the wave equation combined with the discrete fractional Laplacian of order α. We also note that

$$H_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n) \frac{(-1)^j t^{\beta j+1}}{\Gamma(2+\beta j)},$$

and

$$L_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n) \frac{(-1)^j t^{\beta j+\beta-1}}{\Gamma(\beta+\beta j)},$$

thanks to the semigroup property $g_\gamma * g_\delta = g_{\gamma+\delta}$ valid for all $\gamma, \delta > 0$.

The following subordination principle is a direct consequence of Theorem 3.1.

Corollary 3.5. *Let $0 < \alpha \leq 1$ and $1 < \beta < 2$. Then the fundamental solution $G_{\alpha,\beta}(n,t)$ of* (3.1) *has an integral representation given by*

$$G_{\alpha,\beta}(n,t) = \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) G_{\alpha,2}(n, \tau t^{\frac{\beta}{2}}) d\tau, \quad t > 0,\ n \in \mathbb{Z}, \tag{3.16}$$

where $\Phi_{\frac{\beta}{2}}$ is the Wright function defined by (2.6).

Proof. Indeed, using (3.8) we obtain that for every $t > 0$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
G_{\alpha,\beta}(n,t) &= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^\beta)^k}{k!} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k t^{\beta k}}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k t^{\beta k}}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau)\tau^{2k}\, d\tau \\
&= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau)(-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} (\tau^2 t^\beta)^k\, d\tau \\
&= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) G_{\alpha,2}(n, \tau t^{\frac{\beta}{2}})\, d\tau.
\end{aligned}$$

We have shown (3.16) and the proof is finished. $\square$

4. Special cases

In this section we will discuss several special cases of interest which are a direct consequence of Theorem 3.1.

4.1. **The wave equation.**

This corresponds to the case $(\alpha, \beta) = (1, 2)$. The corresponding equation is:

$$\begin{cases} u_{tt}(n,t) = \Delta_d u(n,t) + g(n,t), & t > 0, \quad n \in \mathbb{Z} \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}, \\ u_t(n,0) = \phi(n) & n \in \mathbb{Z}. \end{cases} \tag{4.1}$$

We derive the following complementary result.

Corollary 4.1. *Let $\varphi, \phi \in l^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ be such that, for each $t > 0$, $g(\cdot, t) \in l^\infty(\mathbb{Z})$ and $\sup\limits_{s \in [0,t]} ||g(\cdot, t)||_\infty < \infty$. Then the wave equation (4.1) admits a unique solution in $l^1(\mathbb{Z})$ given by*

$$\begin{aligned} u(n,t) = &\sum_{m \in \mathbb{Z}} J_{2(n-m)}(2t)\varphi(m) + \sum_{m \in \mathbb{Z}} \int_0^t J_{2(n-m)}(2x)\phi(m)\, dx \\ &+ \sum_{m \in \mathbb{Z}} \int_0^t \left(\int_0^{t-s} J_{2(n-m)}(2x)\, dx \right) g(m,s)\, ds. \end{aligned} \tag{4.2}$$

Proof. Taking $\alpha = 1$ and $\beta = 2$ in (3.8), we get that

$$\begin{aligned} G_{1,2}(n,t) &= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2k+1)(-1)^k}{\Gamma(2k+1)\Gamma(k+n+1)\Gamma(k-n+1)} t^{2k} \\ &= (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+n+1)\Gamma(k-n+1)} t^{2k} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{\Gamma(k+2n+1)k!} t^{2(k+n)} \\ &= J_{2n}(2t). \end{aligned} \tag{4.3}$$

Since $H_{1,2}(n,t) = L_{1,2}(n,t)$, it follows from (3.9) that $H_{1,2}(n,t) = \int_0^t G_{1,2}(n,x)dx$. The proof is finished. □

We define the following operator family, called semidiscrete cosine function (associated with the wave equation) [28, Formula (11)]

$$C_t f(n) := \sum_{m \in \mathbb{Z}} G_{1,2}(n-m,t) f(m) = \sum_{m \in \mathbb{Z}} J_{2(n-m)}(2t) f(m). \tag{4.4}$$

As observed in [28, Theorem 1.2 (i)], we note that $C_t \in \mathcal{L}(l^2(\mathbb{Z}))$. In fact

$$\sum_{n \in Z} |G_{1,2}(n,t)| = \sum_{n \in Z} |J_{2n}(2t)| = \sum_{n \in \mathbb{Z}} I_{2n}(2t) \le \sum_{n \in \mathbb{Z}} I_n(2t) = e^t,$$

and the claim follows from Young's inequality.

The corresponding representation of the solution can be found in [6, Formula (4.2)] in the case where $g \equiv 0$. The author in [6] provides a wealth of concrete situations, e.g. weather prediction by numerical processes, the theory of elasticity among others where the equations under consideration here (heat and wave type equations) play an important role.

From the relation $J_k(-t) = (-1)^k J_k(t)$, we see that we can extend $(C_t)_{t\geq 0}$ to $(C_t)_{t\in\mathbb{R}}$ by setting $C_t = C_{-t}$. We shall prove the following result.

Theorem 4.2. *Let $f \in l^p(\mathbb{Z})$, $1 \leq p \leq \infty$. Then the family $\{C_t\}_{t\geq 0}$ satisfies the following properties.*

(i) $C_0 f = f$.
(ii) $C_{t+s}f + C_{t-s}f = 2C_tC_sf$.
(iii) $\lim_{t\to 0} C_t f = f$ *in* $l^1(\mathbb{Z})$.

Proof. (i) Using (2.11) and the definition of C_t we get that

$$
\begin{aligned}
C_0 f(n) &= \sum_{m\in\mathbb{Z}} J_{2m}(0) f(n-m) \\
&= \sum_{m\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2m+1)} (0)^{2(k+m)} f(n-m) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-k+1)} f(n+k).
\end{aligned}
$$

Since if $m+k \neq 0$, all those terms in the sum are 0, whence only appear the term $k=-m$, obtaining the last expression. Now we note that for $k \geq 1$, the terms in the last sum are 0. Therefore, the only nonzero term appears when $k=0$, yielding $C_t f(n) = f(n)$.

(ii) We prove the cosine property of C_t. Using (2.13), (2.14) and the Fubini-Tonelli Theorem, we obtain

$$
\begin{aligned}
C_{t+s}f(n) + C_{t-s}f(n) &= \sum_{m\in\mathbb{Z}} J_{2(n-m)}(t+s)f(m) + \sum_{m\in\mathbb{Z}} J_{2(n-m)}(t-s)f(m) \\
&= \sum_{m\in\mathbb{Z}} \left(J_{2(n-m)}(t+s) + J_{2(n-m)}(t-s)\right) f(m) \\
&= \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-m)-l}(2t)J_k(2s) + \sum_{l\in\mathbb{Z}} (-1)^l J_{2(n-m)-l}(2t)J_l(2s) \right) f(m) \\
&= 2\sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-m-l)}(2t)J_{2l}(2s) \right) f(m) \\
&= 2\sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t)J_{2(l-m)}(2s) \right) f(m) \\
&= 2\sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} J_{2(n-l)}(2t)J_{2(l-m)}(2s)f(m) \\
&= 2\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t) \sum_{m\in\mathbb{Z}} J_{2(l-m)}(2s)f(m) \\
&= 2\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t)C_sf(l) = 2C_t(C_sf(n))
\end{aligned}
$$

(iii) Let $\mathcal{M}_t(n) = J_{2n}(2t)$ and $\delta_0(n) = 1$ if $n = 0$ and 0 in other case. We notice that

$$
(C_tf - f)(n) = \sum_{k\in\mathbb{Z}} (\mathcal{M}_t(n-k) - \delta_0(n-k))f(k) = ((\mathcal{M}_t - \delta_0) * f)(n). \tag{4.5}
$$

From Young's inequality, it is enough to estimate $||\mathcal{M}_t - \delta_0||_{l^1(\mathbb{Z})}$. It follows from (2.15) that

$$
|J_{2n}(2t)| = \frac{2t^{2n}}{\sqrt{\pi}\Gamma(2n+\frac{1}{2})} \le \frac{2t^{2n}}{\sqrt{\pi}\Gamma(2n+1)} \frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})}.
$$

Since

$$
\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \tag{4.6}
$$

for $|z|$ large enough, we have that

$$
\frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})} = \frac{\Gamma(\frac{4n+1}{2})}{\Gamma(2n+1)} \sim \frac{\left(\frac{4n+1}{2}\right)^{\frac{4n+1}{2e}}}{\left(\frac{2n+1}{e}\right)^{2n+1}} \frac{\sqrt{(4n+1)\pi}}{\sqrt{(2n+1)2\pi}} \le \left(\frac{4n+1}{2n+1}\right)^{\frac{4n+1}{2}} \frac{\sqrt{e}}{(2n+1)^{\frac{1}{2}}} 2^{-\frac{4n+1}{2}}.
$$

Since $\frac{4n+1}{2n+1} < 2$, it follows that

$$
\left(\frac{4n+1}{2n+1}\right)^{\frac{4n+1}{2}} \frac{\sqrt{e}}{(2n+1)^{\frac{1}{2}}} 2^{-\frac{4n+1}{2}} < 2^{\frac{4n+1}{2}}\sqrt{e}2^{-\frac{4n+1}{2}} = \sqrt{e}.
$$

Therefore, for some $M > 0$,

$$|J_{2n}(2t)| \leq M \frac{t^{2n}}{\Gamma(2n+1)}, \ t > 0. \tag{4.7}$$

Using (2.13) and (4.7), we get that

$$\begin{aligned}
||\mathcal{M}_t - \delta_0||_{l^1(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} |(\mathcal{M}_t - \delta_0)(n)| = \sum_{n\neq 0} |J_{2n}(2t)| + |J_0(2t) - 1| \\
&= 2\sum_{n=1}^{\infty} |J_{2n}(2t)| + |J_0(2t) - 1| \\
&\leq 2\sum_{n=1}^{\infty} M \frac{t^{2n}}{\Gamma(2n+1)} + |J_0(2t) - 1| \\
&= 2M(\cosh(t) - 1) + |J_0(2t) - 1|.
\end{aligned}$$

Thus $||\mathcal{M}_t - \delta_0||_{l^1(\mathbb{Z})} \to 0$ when $t \to 0$ and we have shown that $\lim_{t\to 0} C_t f = f$, in $l^1(\mathbb{Z})$. The proof is finished. □

Remark 4.3. We point out the following facts.

(a) From Theorem 4.2(iii), we note that $||C_t|| \leq Me^t, \ t > 0$, since $|J_0(2t)| \leq 1, t \in \mathbb{R}$.

(b) Theorem 4.2 also appears in the recent reference [28, Theorem 1.2 (i)] and is the counterpart of [12, Proposition 1] where the case of the heat semigroup W_t is proved. We stress that the point is to have a concrete verification using the properties of the special functions of mathematical physics, for the corresponding properties follow from the general theory of differential equations. The case of the heat equation is covered by semigroup theory whereas the case of the wave equation is governed by cosine function theory (see e.g. [4, Section 3.14] and [15]). The latter reference also coveres some semigroup theory and the relation to Hadamard's well-posedness principle is discussed.

4.2. **The super diffusive case with the discrete Laplacian.** In this subsection, we analyze the case $\alpha = 1$ and $1 < \beta \leq 2$. This is given by the system

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = \Delta_d u(n,t) + g(n,t), & t > 0, \quad n \in \mathbb{Z} \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \\ u_t(n,0) = \phi(n), & n \in \mathbb{Z}. \end{cases} \tag{4.8}$$

The following result shows that the fundamental solution can be represented in terms of the Wright function.

Corollary 4.4. *Let $1 < \beta \leq 2$. Then the fundamental solution $G_{1,\beta}(n,t)$ has an integral representation given by*

$$G_{1,\beta}(n,t) = \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\rho) J_{2n}(2\rho t^{\beta}) d\rho, \quad t > 0, \ n \in \mathbb{Z}, \tag{4.9}$$

where for $0 < \nu < 1$, Φ_ν is the Wright function defined by (2.6).

Proof. The result follows by applying Corollaries 3.5 and 4.1. □

Example 4.5. (Transverse vibrations of an infinite light string) The partial differential/difference equation

$$u_{tt}(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t) + g(n,t), \quad n \in \mathbb{Z}, \quad t > 0, \tag{4.10}$$

was studied by Bateman [7, Section 4.9] with forcing term $g(n,t) = -\delta_0(n) f'(t)$, where $f \in W^{1,q}(\mathbb{R}_+),\ 1 < q < \infty$, and

$$\delta_0(n) = \begin{cases} 1, & n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Physically, this means that the motion of the particle labeled 0 is forced while the motion of the other particles is free. We consider their super diffusive version, i.e. equation (4.8) with g as just described. Assuming $\sup_{s\in[0,t]} |f'(s)| < \infty$ for all $t > 0$ and zero initial conditions, we obtain the following explicit representation of the solution

$$u(n,t) = -\int_0^t (g_{\beta-1} * G_{1,\beta})(n, t-s) f'(s) ds,$$

where $1 < \beta \le 2$. In the border case $\beta = 2$ we get after integration by parts

$$u(n,t) = \int_0^t J_{2n}(2(t-s)) f(s) ds,$$

so that we recover the explicit solution of (4.10) proposed by Bateman in [7, p.644] and give new insight on the representation in the super diffusive case.

5. The case $0 < \beta \le 1$

In this section, we give an explicit representation of solutions for the following time/space fractional diffusion equation

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0 \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}, \end{cases} \tag{5.1}$$

where $0 < \alpha, \beta \le 1$. Note that this problem was studied in [23].

Theorem 5.1. *Let $\varphi \in l^\infty(\mathbb{Z})$ and $g : \mathbb{Z}\times\mathbb{R}_+ \to \mathbb{C}$ be such that for each $t > 0$, $g(\cdot,t) \in l^\infty(\mathbb{Z})$ and $\sup\limits_{s\in[0,t]} ||g(\cdot,t)||_\infty < \infty$. Then the function*

$$u(n,t) = \sum_{m\in\mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s) g(m,s) ds. \tag{5.2}$$

is the unique solution of (5.1)*, where here*

$$G_{\alpha,\beta}(n,t) := (-1)^n\ {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right] \tag{5.3}$$

and

$$L_{\alpha,\beta}(n,t)) := (-1)^n t^{\beta-1} \,{}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right]. \tag{5.4}$$

Furthermore, for each $t \in \mathbb{R}_+$, $G_{\alpha,\beta}(\cdot,t)$ and $L_{\alpha,\beta}(\cdot,t)$ belong to $l^1(\mathbb{Z})$

Proof. As in the proof of Theorem 3.1, taking the discrete Fourier transform of (5.1) we get that

$$\begin{cases} \mathbb{D}_t^\beta u(z,t) = -(-J(z))^\alpha u(z,t) + g(z,t) \\ u(z,0) = \varphi(z). \end{cases} \tag{5.5}$$

Nowadays it is well-known that the solution of the ODE (5.5) is given by

$$u(z,t) = E_{\beta,1}\left(-(-J(z))^\alpha t^\beta\right) u(z,0) + t^{\beta-1} E_{\beta,\beta}\left(-(-J(z))^\alpha t^\beta\right) * g(z,t). \tag{5.6}$$

Now taking the inverse discrete Fourier transform we get that for $t > 0$ and $n \in \mathbb{Z}$,

$$u(n,t) = \sum_{m\in\mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s) g(m,s) ds.$$

Next, we show that $G_{\alpha,1}(\cdot,t) \in l^1(\mathbb{Z})$ for every $t > 0$. Since

$$G_{\alpha,1}(n,t) = (-1)^n \sum_{k=0}^\infty \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t)^k}{k!},$$

then using that $G_{\alpha,1}(n,t) = G_{\alpha,1}(-n,t)$ and Fubini-Tonelli Theorem, we can deduce that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |G_{\alpha,1}(n,t)| &= \sum_{k=0}^\infty \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|} \frac{t^k}{k!} \\ &\le 2 \sum_{n\in\mathbb{N}_0} \sum_{k=0}^\infty \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)| n!} \frac{t^k}{k!} = 2 \sum_{k=0}^\infty \sum_{n\in\mathbb{N}_0} \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)| n!} \frac{t^k}{k!}. \end{aligned}$$

Let $c_k := \dfrac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|} \dfrac{1}{k!}$. Then, using (2.10) with $\kappa = -\alpha$, $\delta = 2^{-2\alpha}\alpha^\alpha$, $\mu = -n$, $A = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\alpha}}{\alpha^{-n+\frac{1}{2}}}$), we get that for k sufficiently large

$$|c_k| \sim \frac{(\alpha k)^n}{\sqrt{\pi}} \left(\frac{k}{e}\right)^{-(1-\alpha)k} \frac{2^{2\alpha k}\alpha^{\alpha k}}{k^{\frac{1}{2}}}.$$

Thus for k_0 large enough, we have that

$$
\begin{aligned}
2\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|n!}\frac{t^k}{k!} &\sim \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}\left(\frac{k}{e}\right)^{-(1-\alpha)k}\frac{(4\alpha)^{\alpha k}}{k^{\frac{1}{2}}}t^k\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!}\\
&= \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}\left(\frac{k}{e}\right)^{-(1-\alpha)k}\frac{(4\alpha)^{\alpha k}}{k^{\frac{1}{2}}}t^k e^{\alpha k}\\
&= \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}k^{-(1-\alpha)k-\frac{1}{2}}e^k(4\alpha)^{\alpha k}t^k. \qquad (5.7)
\end{aligned}
$$

Hence $G_{\alpha,1}(.,t)\in l_1(\mathbb{Z})$ for each $t>0$.

Now, using the subordination principle we get that

$$G_{\alpha,\beta}(n,t)=\int_0^{\infty}\Phi_\beta(s)G_{\alpha,1}(n,st^\beta)ds. \qquad (5.8)$$

This implies that

$$\sum_{n\in\mathbb{Z}}|G_{\alpha,\beta}(n,t)|=\sum_{n\in\mathbb{Z}}\left|\int_0^{\infty}\Phi_\beta(s)G_{\alpha,1}(n,st^\beta)\,ds\right|\le\int_0^{\infty}\Phi_\beta(s)\sum_{n\in\mathbb{Z}}\left|G_{\alpha,1}(n,st^\beta)\right|ds.$$

Let k_0 be sufficiently large. Let $K_t=\sum_{k=0}^{k_0-1}c_kt^k<\infty$ and $b_k=\frac{2}{\sqrt{\pi}}k^{-(1-\alpha)k-\frac{1}{2}}e^k(4\alpha)^{\alpha k}t^k$ in the above estimate for $G_{\alpha,1}(n,t)$. Then

$$
\begin{aligned}
\int_0^{\infty}\Phi_\beta(s)\sum_{n\in\mathbb{Z}}|G_{\alpha,1}(n,st^\beta)|ds &\le \int_0^{\infty}\Phi_\beta(s)K_t\,ds+\int_0^{\infty}\sum_{k=k_0}^{\infty}b_k(st^\beta)^k\Phi_\beta(s)\,ds\\
&= K_t+\sum_{k=k_0}^{\infty}b_kt^{\beta k}\int_0^{\infty}s^k\Phi_\beta(s)\,ds.
\end{aligned}
$$

Using (2.7) we obtain that

$$G_{\alpha,\beta}(n,t)\le C_t+\sum_{k=k_0}^{\infty}b_kt^{\beta k}\frac{\Gamma(k+1)}{\Gamma(\beta k+1)}.$$

Let $d_k=b_k\frac{\Gamma(k+1)}{\Gamma(\beta k+1)}$. Using (4.6), we get that

$$d_k=b_k\frac{\Gamma(k+1)}{\Gamma(\beta k+1)}\sim\frac{2}{\sqrt{\pi}}k^{-(1-\alpha)k-\frac{1}{2}}e^k(4\alpha)^{\alpha k}\frac{\left(\frac{k}{e}\right)^k\sqrt{2\pi k}}{\left(\frac{\beta k}{e}\right)^k\sqrt{2\pi\beta k}}=\frac{k^{-(1-\alpha)k-\frac{1}{2}}}{\beta^{k+\frac{1}{2}}}(4\alpha)^{\alpha k}e^k.$$

Therefore, applying the root test again, we can conclude that $G_{\alpha,\beta}(\cdot,t)\in l_1(\mathbb{Z})$, for each $t>0$. The proof is finished. $\square$

In the case $\alpha = \beta = 1$, the corresponding equation is the discrete heat equation

$$\begin{cases} u_t(n,t) = \Delta_d u(n,t) + g(n,t) & n \in \mathbb{Z},\ t > 0 \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}. \end{cases} \tag{5.9}$$

From the general results on ordinary differential equations in Banach spaces, the unique solution of (5.9) is given by $u(n,t) = e^{t\Delta_d}\varphi(n)$, where $e^{\Delta_d t} =: W_t$ is the semigroup generated by Δ_d. This semigroup, which is called the discrete heat semigroup in analogy with the classical continuous case, has the following representation:

$$W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m), \quad f \in l^2(\mathbb{Z}). \tag{5.10}$$

We refer to [12] for more information, and the derivation of the semigroup property, as well as other standard properties. We note that the representation of the discrete heat semigroup was already given by Bateman in [6, p. 496]. This reference contains much information on the discrete heat equation and several related others, along with historical background and concrete applications.

Proposition 5.2. *Let $\alpha \in (0,1]$. Then, for all $n \in \mathbb{Z}$, the following identity*

$$\int_0^\infty I_n(2\lambda) e^{-2\lambda} f_{t,\alpha}(\lambda)\, d\lambda = \sum_{j=0}^\infty K^{\alpha j}(n) \frac{(-t)^j}{j!}, \tag{5.11}$$

holds, where $f_{t,\alpha}(\lambda)$ is the stable Lévy process defined in (2.8).

Proof. Indeed, using the discrete Fourier transform, Remark 2.3(i), (2.19) and Fubini-Tonelli Theorem, we get that

$$
\begin{aligned}
\mathcal{F}\left(\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda\right)(z) &= \sum_{n\in\mathbb{Z}}\left[\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda\right] z^n \\
&= \int_0^\infty \sum_{n\in\mathbb{Z}} I_n(2\lambda) z^n e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{\left(z+\frac{1}{z}\right)\lambda} e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{\left(z+\frac{1}{z}-2\right)\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{J(z)\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= e^{-(-J(z))^\alpha t} \\
&= E_{1,1}(-(-J(z))^\alpha t) \\
&= \mathcal{F}(G_{\alpha,1}(n,t))(z) \\
&= \mathcal{F}\left((-1)^n \sum_{k=0}^\infty \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t)^k}{k!}\right)(z).
\end{aligned}
$$

By the uniqueness of the discrete Fourier transform we can deduce that

$$
\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda = (-1)^n \sum_{k=0}^\infty \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t)^k}{k!},
$$

and the claim follows from Remark 3.4. □

It is clear from (5.9) that the case $\alpha = \beta = 1$ is the discretized heat equation. Therefore, the expression in (5.10) represents the heat semigroup or Gauss-Weierstrauss semigroup W_t in the present context. The semigroup property is studied in [12, Proposition 1]. Similarly to the continuous case, when $\alpha = \frac{1}{2}$ and $\beta = 1$, the corresponding semigroup is the Poisson semigroup P_t. The Poisson semigroup was considerated by Ciaurri-Gillespie in [12, Remark 2] through the subordination principle with the Lévy function. That is,

$$
(P_t f)(n) = \sum_{m\in\mathbb{Z}} p_t(n-m) f(m), \;\; f \in l^p(\mathbb{Z}).
$$

Corollary 5.3. *We have the following representations for the heat and Poisson semigroups.*

(i) *For the discrete heat kernel, we have that for all $n \in \mathbb{Z}$,*

$$
e^{-2t} I_n(2t) = \sum_{j=0}^\infty K^{2j}(n) \frac{(-t)^j}{j!}
$$

(ii) *For the Poisson kernel, we have, for all $n \in \mathbb{Z}$,*

$$p_t(n) = \int_0^\infty I_n(2\lambda) e^{-2\lambda} \frac{1}{2\sqrt{\pi}} \lambda^{-\frac{3}{2}} t e^{-\frac{t^2}{4\lambda}} d\lambda = \sum_{j=0}^\infty K^{j/2}(n) \frac{(-t)^j}{j!}.$$

Proof. (i) From the above proof, taking $\alpha = 1$, we obtain that

$$\mathcal{F}(G_{1,1}(n,t))(z) = E_{1,1}(J(z)t) = e^{J(z)t} = \mathcal{F}(e^{-2t} I_n(2t))(z),$$

and we have shown that

$$e^{-2t} I_n(2t) = (-1)^n \sum_{k=0}^\infty \frac{\Gamma(2k+1)}{\Gamma(k+n+1)\Gamma(k-n+1)} \frac{(-t)^k}{k!}.$$

(ii) For $\alpha = \frac{1}{2}$, we have that $f_{t,\frac{1}{2}}(\lambda) = \frac{1}{2\sqrt{\pi}} \lambda^{-\frac{3}{2}} t e^{-\frac{t^2}{4\lambda}}$ (seee.g. [1]). The result is a direct consequence of the subordination principle. $\square$

Now, we consider the equation

$$\begin{cases} \mathbb{D}_t^\beta(n,t) = \Delta_d u(n,t) + g(n,t) & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}. \end{cases} \tag{5.12}$$

We have the following result.

Proposition 5.4. *Let $\beta \in (0,1]$. Then, for all $n \in \mathbb{Z}$, the identity*

$$\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda = \sum_{j=0}^\infty K^{2j}(n) \frac{(-t)^j}{\Gamma(\beta j + 1)}, \tag{5.13}$$

holds, where $\Phi_\beta(\lambda)$ is the Wright function given in (2.6).

Proof. Indeed, using the discrete Fourier transform, (2.2)(i), (2.19) and Fubini-Tonelli Theorem, we obtain that

$$\begin{aligned}
&\mathcal{F}\left(\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda\right)(z) \\
=& \sum_{n \in \mathbb{Z}} \left[\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda\right] z^n = \int_0^\infty \sum_{n \in \mathbb{Z}} I_n(2\lambda t^\beta) z^n e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda \\
=& \int_0^\infty e^{\left(z + \frac{1}{z}\right)\lambda t^\beta} e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda \\
=& \int_0^\infty e^{\left(z + \frac{1}{z} - 2\right)\lambda t^\beta} \Phi_\beta(\lambda) d\lambda = \int_0^\infty e^{J(z)\lambda t^\beta} \Phi_\beta(\lambda) d\lambda \\
=& E_{\beta,1}(J(z)t^\beta) = \mathcal{F}(G_{1,\beta}(n,t))(z) \\
=& \mathcal{F}\left((-1)^n \sum_{k=0}^\infty \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)\Gamma(k+n+1)\Gamma(k-n+1)} (-t)^k\right)(z).
\end{aligned}$$

Thus

$$\int_0^\infty I_n(2\lambda t^\beta)e^{-2\lambda t^\beta}\Phi_\beta(\lambda)d\lambda = (-1)^n\sum_{k=0}^\infty \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)\Gamma(k+n+1)\Gamma(k-n+1)}(-t)^k,$$

by the uniqueness of the discrete Fourier transform. The result follows from Remark 3.4. □

An interesting case is when $\beta = \frac{1}{3}$, where the explicit representation of the Wright function is known. Namely,

$$\Phi_{\frac{1}{3}}(z) = 3^{\frac{2}{3}} Ai\left(-\frac{z}{3^{\frac{1}{3}}}\right),$$

where $Ai(z)$ is the *Airy function* (see e.g. [19]), whose integral representation is given by

$$Ai(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)dt, \;\; x \in \mathbb{R}. \tag{5.14}$$

This function appears in several applied problems, specially in optics (study of caustics) and the Schrödinger's equation of quantum physics. The reference [43] contains a great deal of information on this topic along with numerous applications.

The following is a direct consequence of Proposition 5.4.

Corollary 5.5. *The following identity*

$$\int_0^\infty I_n(2\lambda t^{\frac{1}{3}})e^{-2\lambda t^{\frac{1}{3}}} 3^{\frac{2}{3}} Ai\left(-\frac{\lambda}{3^{\frac{1}{3}}}\right)d\lambda = \sum_{j=0}^\infty K^{2j}(n)\frac{(-t)^j}{\Gamma(\frac{j}{3}+1)},$$

holds.

6. Perturbation of the discrete Laplace operator

H. Bateman considered in [6, Formula 5.1] a problem related to the operator Δ_d, which describes the dynamics of surges in springs of spirals or helical type. He singled out the case of masses concentrated on a light string and obtained the following equation:

$$f_{tt}(n,t) = f(n+1,t) - (2+h)f(n,t) + f(n-1,t), \quad h \geq 0, \; t \geq 0. \tag{6.1}$$

It appears that the operator Δ_d is now replaced by $\Delta_{d,h} := \Delta_d - 2hI$ (I being the identity operator). More recently, such additive perturbations of the discrete Laplace operator have appeared in connection with well posedness of the heat equation in Hölder spaces [28, Theorem 1.7].

In the general theory of cosine families, if A is an operator that generates a cosine family $C(t)$, and $a \in \mathbb{C}$, then $A + aI$ generates a cosine family $C_a(t)$ (see [15, Chapter VI, p.167, formula 2.8]) represented by

$$C_a(t) = C(t) + \sqrt{a}t\int_0^t \frac{I_1(\sqrt{a}(t^2-s^2)^{\frac{1}{2}})}{(t^2-s^2)^{\frac{1}{2}}}C(s)ds. \tag{6.2}$$

In the case of equation (6.1) we will obtain a representation of the solution through the methods of the previous sections.

We start by giving the representation of the fractional powers of $\Delta_{d,h}$,similar to the one given for Δ_d in the introduction. The main tool we shall use is the formula

$$(-A)^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (T(t)x - x) \frac{dt}{t^{1+\alpha}}, \ x \in D(A), \tag{6.3}$$

representing the fractional powers of $(-A)$, where A is the generator of a bounded semigroup $T(t)$ (see e.g. [47, p. 260, Formula 5 and Theorem 2, p. 264]).

6.1. **Fractional Powers of $\Delta_{d,h}$.** In this subsection, we study the operator

$$\Delta_{d,h} f(n) := f(n+1) - (2+h)f(n) + f(n-1), \quad h \geq 0, \ f \in l^p(\mathbb{Z}), \ p \in [1, \infty]. \tag{6.4}$$

Obviously, when $h = 0$, $\Delta_{d,0} f(n) = \Delta_d f(n)$. A simple computation yields

$$\mathcal{F}(\Delta_{d,h} f(n))(z) = \left[z + \frac{1}{z} - (2+h)\right] \mathcal{F}(f)(z). \tag{6.5}$$

Therefore, we have:

$$\mathcal{F}(\Delta_{d,h})(z) = z + \frac{1}{z} - (2+h) = J(z) - h. \tag{6.6}$$

Note that $z + \frac{1}{z} - (2+h) = J(z) - h \leq 0$ for all $z \in \mathbb{C}, |z| = 1, h \geq 0$.

Proposition 6.1. *Let $\Delta_{d,h}$ be defined in* (6.4)*. Then, for $0 < \alpha < 1$, the fractional power $(-\Delta_{d,h})^\alpha$ is given by*

$$(-\Delta_{d,h})^\alpha f(n) = \sum_{l \in \mathbb{Z}} K_h^\alpha(n-l) f(l), \quad n \in \mathbb{Z}, \tag{6.7}$$

where

$$K_h^\alpha(n) = \frac{1}{2\pi} \int_0^{2\pi} \left(4 \sin^2(\theta/2) + h\right)^\alpha e^{-in\theta} d\theta, \quad n \in \mathbb{Z}. \tag{6.8}$$

Proof. First, note that the semigroup generated by $\Delta_{d,h}$ is: $e^{t\Delta_{d,h}} = e^{-ht} e^{t\Delta_d}$. Then, taking the discrete Fourier transform in (6.3), with $A = \Delta_{d,h}$, and noting that

$$\mathcal{F}(e^{t\Delta_{d,h}})(z) = e^{\left[z + \frac{1}{z} - (2+h)\right]t} \mathcal{F}(f)(z) = e^{-(4\sin^2\frac{\theta}{2} + h)t} \mathcal{F}(f)(z), \tag{6.9}$$

we obtain that

$$\mathcal{F}((-\Delta_{d,h})^\alpha f)(\theta) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-(4\sin^2\frac{\theta}{2} + h)t} - 1\right) \frac{dt}{t^{1+\alpha}} \mathcal{F}(f)(\theta) = \left(4\sin^2\theta/2 + h\right)^\alpha \mathcal{F}(f)(\theta).$$

The second equality follows from an integration by parts and the fact that the Laplace transform of the function $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ is given by $\hat{g}_\beta(\lambda) = \lambda^{-\beta}$. Therefore, taking the inverse discrete Fourier transform, we obtain (6.7). $\square$

Remark 6.2. We notice that the kernel $K_h^\alpha \in l^1(\mathbb{Z})$. In fact, let $f(\theta) = (4\sin^2\theta/2 + h)^\alpha$ and $h > 0$. It is easily to check that f, f' and f'' belong to $L^1([0, 2\pi])$ and are periodic with period 2π. This, together with an integration by parts implies that

$$||K_h^\alpha||_{l^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |K_h^\alpha(n)| \leq ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^{\infty} \frac{1}{n^2}||f''||_{L^1([0,2\pi])} < \infty. \tag{6.10}$$

From Young's inequality, we can deduce that $(-\Delta_{d,h})^\alpha$ is a bounded operator on $l^p(\mathbb{Z}, X)$, for any Banach space X and $1 \leq p \leq \infty$, since $||(-\Delta_{d,h})^\alpha f|| = ||K_h^\alpha * f||_p \leq ||K_h^\alpha||_1||f||_p$.

6.2. **Explicit representation of solutions.** The following result gives an explicit representation of the fundamental solution.

Theorem 6.3. *Let $\varphi, \phi \in l^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{C}$ be such that for each $t > 0$, $g(\cdot, t) \in l^\infty(\mathbb{Z})$ and $\sup\limits_{s\in[0,t]} ||g(\cdot, s)||_\infty < \infty$. Then the unique solution of*

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_{d,h})^\alpha u(n,t) + g(n,t), \\ u(n,0) = \varphi(n), \\ u_t(n,0) = \phi(n), \end{cases} \tag{6.11}$$

is given by

$$\begin{aligned} u(n,t) =& \sum_{m\in\mathbb{Z}} G_{\alpha,\beta}^h(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} H_{\alpha,\beta}^h(n-m,t)\phi(m) \\ &+ \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}^h(n-m,t-s)g(m,s)ds, \end{aligned} \tag{6.12}$$

where

$$G_{\alpha,\beta}^h(n,t) = (-1)^n \sum_{k=0}^{\infty} \frac{(-t^\beta)^k}{\Gamma(\beta k+1)} \sum_{r=0}^{\infty} \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}, \tag{6.13}$$

and

$$H_{\alpha,\beta}^h(n,t) = (-1)^n t \sum_{k=0}^{\infty} \frac{(-t^\beta)^k}{\Gamma(\beta k+2)} \sum_{r=0}^{\infty} \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}, \tag{6.14}$$

and

$$L_{\alpha,\beta}^h(n,t) = (-1)^n t^{\beta-1} \sum_{k=0}^{\infty} \frac{(-t^\beta)^k}{\Gamma(\beta k+\beta)} \sum_{r=0}^{\infty} \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}. \tag{6.15}$$

Furthermore, $G_{\alpha,\beta}^h(\cdot,t)$, $H_{\alpha,\beta}^h(\cdot,t)$, $L_{\alpha,\beta}^h(\cdot,t) \in l^1(\mathbb{Z})$ for each $t > 0$,.

Proof. Proceeding as in the previous sections (by taking the discrete Fourier transform of (6.11)) we obtain that

$$G^h_{\alpha,\beta}(n,t) = \mathcal{F}^{-1}\left(E_{\beta,1}(-(-J(z)-h)^\alpha t^\beta)\right)(n) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{(-t^\beta)^k\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}}{\Gamma(\beta k+1)}d\theta.$$

Proceeding in the same way as we have obtained the expresion (3.8) in Theorem 3.1 and using the generalized Binomial Theorem, we obtain (6.13). Moreover,

$$||G^h_{\alpha,\beta}(\cdot,t)||_{l^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}}|G^h_{\alpha,\beta}(n,t)| = \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\left|\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{(-t^\beta)^k\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}}{\Gamma(\beta k+1)}e^{-in\theta}d\theta\right|.$$

Using Fubini-Tonelli Theorem, we get that

$$\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{(-t^\beta)^k\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}}{\Gamma(\beta k+1)}e^{-in\theta}d\theta = \sum_{k=0}^{\infty}\frac{(-t^\beta)^k}{\Gamma(\beta k+1)}\int_0^{2\pi}\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}e^{-in\theta}d\theta.$$

Let $f(\theta) = (4\sin^2\frac{\theta}{2}+h)^{\alpha k}$ and $h>0$. Notice that f, f', $f'' \in L^1([0,2\pi])$ and are periodic with period 2π. Furthermore $\mathcal{O}(f'') \le Mk^2(4+k)^{\alpha k}$ for some $M>0$ sufficiently large. Then, from integration by parts we can deduce that

$$||G^h_{\alpha,\beta}(\cdot,t)||_{l^1(\mathbb{Z})} \le ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{t^{\beta k}}{\Gamma(\beta k+1)}Mk^2(4+h)^{\alpha k}.$$

Using the root test and Stirling's formula we obtain

$$\sum_{k=0}^{\infty}\frac{t^{\beta k}}{\Gamma(\beta k+1)}Mk^2(4+h)^{\alpha k} = C < \infty.$$

Thus

$$||G^h_{\alpha,\beta}(\cdot,t)||_{l^1(\mathbb{Z})} \le ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^{\infty}\frac{1}{n^2}C = ||f||_{L^1([0,2\pi])} + \frac{C\pi^2}{3}.$$

The proof for the explicit formula and the convergence in $l^1(\mathbb{Z})$ of $H^h_{\alpha,\beta}(\cdot,t)$ and $L^h_{\alpha,\beta}(\cdot,t)$ are similar to the proof for $G^h_{\alpha,\beta}$. □

When $h=0$, we know that C_t given in (4.4) defines a cosine operator function. Then, using (6.2) we obtain the following new result.

Corollary 6.4. *The cosine operator generated by $\Delta_{d,h}$ for $(\alpha,\beta)=(1,2)$ is given by*

$$C_{-h}(t)f(n) = C_tf(n) + \sqrt{-h}t\int_0^t\frac{I_1(\sqrt{-h}(t^2-s^2)^{\frac{1}{2}})}{(t^2-s^2)^{\frac{1}{2}}}C_sf(n)\,ds,$$

where

$$C_tf(n) = \sum_{m\in\mathbb{Z}}J_{2(n-m)}(2t)f(m).$$

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Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile.
E-mail address: jorge.gonzalezcam@usach.cl

University of Puerto Rico, Rio Piedras Campus, Department of Mathematics, Faculty of Natural Sciences, 17 University AVE. STE 1701 San Juan PR 00925-2537
E-mail address: valentin.keyantuo1@upr.edu

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile.
E-mail address: carlos.lizama@usach.cl

University of Puerto Rico, Rio Piedras Campus, Department of Mathematics, Faculty of Natural Sciences, 17 University AVE. STE 1701 San Juan PR 00925-2537
E-mail address: mahamadi.warma1@upr.edu, mjwarma@gmail.com