

# MAXIMAL REGULARITY FOR FLEXIBLE STRUCTURAL SYSTEMS IN LEBESGUE SPACES

CLAUDIO FERNÁNDEZ, CARLOS LIZAMA, AND VERÓNICA POBLETE

ABSTRACT. We study abstract equations of the form  $\lambda u'''(t) + u''(t) = c^2 Au(t) + c^2 \mu Au'(t) + f(t)$ ,  $0 < \lambda < \mu$  which is motivated by the study of vibrations of flexible structures possessing internal material damping. We introduce the notion of  $(\alpha; \beta; \gamma)$ -regularized families, which is a particular case of  $(a; k)$ -regularized families, and characterize maximal regularity in  $L^p$ -spaces based on the technique of Fourier multipliers. Finally, an application with the Dirichlet-Laplacian in a bounded smooth domain is given.

## 1. INTRODUCTION

During the last few decades, the use of flexible structural systems has steadily increased importance. The study of a flexible aerospace structure are problems of dynamical system theory governed by partial differential equations.

We consider here the problem of characterize  $L^p$ -maximal regularity (or well-posedness) for a mathematical model of a flexible space structure like a thin uniform rectangular panel. For example, a solar cell array or a spacecraft with flexible attachments. This problem is motivated by both engineering and mathematical considerations.

The study of *vibrations* of flexible structures possessing internal material damping was first derived by Bose and Gorain [9]. The consideration of external forces leads to more general equations of the form

$$(1.1) \quad \alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad \alpha, \beta, \gamma \in \mathbb{R},$$

where  $A$  is a closed linear operator acting in a Banach space  $X$  and  $f$  is a  $X$ -valued function. We emphasize that the abstract Cauchy problem associated with (1.1) is in general ill-posed, see e.g. [29]. Also is well known that in order to analyze well-posedness, a direct approach leads to better results than those obtained by a reduction to a first-order equation.

Maximal regularity in Hölder spaces for equation (1.1) has been recently characterized in [14]. In case  $\alpha = 0$  there are more literature. For example, stability of the solution was studied by Gorain in [16]. In [10], Gorain and Bose studied exact controllability and boundary stabilization. More recently, Batkai and Piazzera [6, p.188] have obtained the exact decay rate. We note that well-posedness in Lebesgue spaces in the case of a damped wave equation has been only recently considered by Chill and Srivastava in [12], and in Hölder spaces by Poblete [25]. We note that the class studied in [25] includes equations

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with delay. In particular, well-posedness of the homogeneous abstract Cauchy problem has been observed in [23] for  $\alpha = 0$  under certain assumptions on  $A$ .

This paper is organized as follows: The preliminary Section 2 collects results essentially contained in [20] and standard literature on  $R$ -boundedness and maximal regularity (see [15] and [2]). In Section 3 we study, by a operator theoretical method, sufficient conditions for existence of solutions for (1.1). We obtain two results: A description of the solution by means of certain regularized families (Proposition 3.1), and the existence of such families in the particular case of positive selfadjoint operators (Theorem 3.2). In Section 4, we succeed in *characterize* well posedness of (1.1) in terms of  $R$ -boundedness of a resolvent set which involves  $A$  (Theorem 4.2). This will be achieved in the Lebesgue spaces  $L^p(\mathbb{R}, X)$ , where  $X$  is a  $UMD$  space (see below for the definition). The methods to obtain this goal are those incorporated in [4] where a similar problem in case of the first order abstract Cauchy problem has been studied. Our main result (Theorem 4.2) is a combination of the well-known (and deep) result due to Weis [28] stated in Theorem 2.8 and a direct calculation involving the parameters  $\alpha, \beta$  and  $\gamma$ .

## 2. PRELIMINARES

Let  $\alpha, \beta, \gamma > 0$  be given. In what follows we denote

$$k(t) = \frac{1}{\alpha} \int_0^t (t-s)e^{-s/\alpha} ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+$$

and

$$a(t) = \beta k(t) + \frac{\gamma}{\alpha} \int_0^t e^{-s/\alpha} ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+.$$

In order to give an operator theoretical approach to (1.1) we introduce the following definition.

**Definition 2.1.** *Let  $A$  be a closed and linear operator with domain  $D(A)$  defined on a Banach space  $X$ . We call  $A$  the generator of an  $(\alpha, \beta, \gamma)$ -regularized family  $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$  if the following conditions are satisfied:*

- (R1)  $R(t)$  is strongly continuous on  $\mathbb{R}_+$  and  $R(0) = 0$ ;
- (R2)  $R(t)D(A) \subset D(A)$  and  $AR(t)x = R(t)Ax$  for all  $x \in D(A), t \geq 0$ ;
- (R3) The following equation holds:

$$(2.1) \quad R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax ds$$

for all  $x \in D(A), t \geq 0$ . In this case,  $R(t)$  is called the  $(\alpha, \beta, \gamma)$ -regularized family generated by  $A$ .

*Remark 2.2.* It is proved in [20], in the more general context of  $(a, k)$ -regularized families, that an operator  $A$  is the generator of an  $(\alpha, \mu)$ -regularized family if and only if there exists  $\omega \geq 0$  and a strongly continuous function  $R : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  such that  $\{\frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} : Re\lambda > \omega\} \subset \rho(A)$  and

$$H(\lambda)x := \frac{1}{\beta + \gamma\lambda} \left( \frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} R(t)x dt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness of the Laplace transform, we note that an  $(\alpha, \beta, \gamma)$ -regularized family corresponds to an  $(a, k)$ -regularized family studied in [20]. In fact, we have

$$\hat{a}(\lambda) = \frac{\beta + \gamma\lambda}{\lambda^2 + \alpha\lambda^3} \quad \text{and} \quad \hat{k}(\lambda) = \frac{1}{\lambda^2 + \alpha\lambda^3}, \quad \text{for all } \operatorname{Re}\lambda > \omega.$$

As in the situation of  $C_0$ -semigroups we have diverse relations of an  $(\alpha, \beta, \gamma)$ -regularized family and its generator. The following result is a direct consequence of [20, Proposition 3.1 and Lemma 2.2].

**Proposition 2.3.** *Let  $R(t)$  be an  $(\alpha, \beta, \gamma)$ -regularized family on  $X$  with generator  $A$ . Then the following holds:*

(a) *For all  $x \in D(A)$  we have  $R(\cdot)x \in C^3(\mathbb{R}_+; X)$ .*

(b) *Let  $x \in X$  and  $t \geq 0$ . Then  $\int_0^t a(t-s)R(s)x ds \in D(A)$  and*

$$R(t)x = k(t)x + A \int_0^t a(t-s)R(s)x ds.$$

Results on perturbation, approximation, asymptotic behavior, representation as well as ergodic type theorems for  $(\alpha, \beta, \gamma)$ -regularized families can be also deduced from the more general context of  $(a, k)$ -regularized families (see [19, 20, 21, 22] and [27]).

We will need the following results on Laplace transform (see [3, Theorem 2.5.1 and Corollary 2.5.2] for a detailed proof).

**Lemma 2.4.** *Suppose  $q : \mathbb{C}_+ \rightarrow \mathbb{C}$  is holomorphic and satisfies  $\sup_{\operatorname{Re}\lambda > 0} |\lambda q(\lambda)| < \infty$  and let  $b > 0$ . Then there exists  $f \in C(\mathbb{R}_+)$  with  $\sup_{t > 0} |e^{-\omega t} t^{-b} f(t)| < \infty$  such that  $q(\lambda) = \lambda^b \int_0^\infty e^{-\lambda t} f(t) dt$  for all  $\operatorname{Re}\lambda > 0$ .*

**Lemma 2.5.** *Suppose  $q : \mathbb{C}_+ \rightarrow \mathbb{C}$  is holomorphic and satisfies  $|\lambda q(\lambda)| + |\lambda^2 q'(\lambda)| \leq M$  for all  $\operatorname{Re}\lambda > 0$ . Then there exists a bounded function  $f \in C(\mathbb{R}_+)$  such that  $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$  for all  $\operatorname{Re}\lambda > 0$ .*

We introduce the means

$$\|(x_1, \dots, x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \epsilon_j x_j \right\|$$

for  $x_1, \dots, x_n \in X$ .

**Definition 2.6.** *Let  $X, Y$  be Banach spaces. A subset  $\mathcal{T}$  of  $\mathcal{B}(X, Y)$  is called  $R$ -bounded if there exists a constant  $c \geq 0$  such that*

$$(2.2) \quad \|(T_1 x_1, \dots, T_n x_n)\|_R \leq c \|(x_1, \dots, x_n)\|_R$$

for all  $T_1, \dots, T_n \in \mathcal{T}, x_1, \dots, x_n \in X, n \in \mathbb{N}$ . The least  $c$  such that (2.2) is satisfied is called the  $R$ -bound of  $\mathcal{T}$  and is denoted  $R(\mathcal{T})$ .

The notion of  $R$ -boundedness was implicitly introduced and used by Bourgain [11] and later on also by Zimmermann [31]. Explicitly it is due to Berkson and Gillespie [7] and to Clément, de Pagter, Sukochev and Witvliet [13].

$R$ -boundedness clearly implies boundedness. If  $X = Y$ , the notion of  $R$ -boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [5, Proposition 1.17]. Some useful criteria for  $R$ -boundedness are provided in [5] and [15].

*Remark 2.7.* a) Let  $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$  be  $R$ -bounded sets, then  $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$  is  $R$ -bounded.

b) Let  $\mathcal{T} \subset \mathcal{B}(X, Y)$  and  $\mathcal{S} \subset \mathcal{B}(Y, Z)$  be  $R$ -bounded sets, then  $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X, Z)$  is  $R$ -bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T}).$$

c) Also, each subset  $M \subset \mathcal{B}(X)$  of the form  $M = \{\lambda I : \lambda \in \Omega\}$  is  $R$ -bounded whenever  $\Omega \subset \mathbb{C}$  is bounded.

We recall that those Banach spaces  $X$  for which the Hilbert transform is bounded on  $L^p(\mathbb{R}, X)$ , for some  $p \in (1, \infty)$ , are called  $UMD$ -spaces. For more information and details on the Hilbert transform and the  $UMD$  Banach spaces we refer to [2]. Examples of  $UMD$  spaces include Hilbert spaces, Sobolev spaces  $W_p^s(\Omega)$ ,  $1 < p < \infty$  (see [1]), Lebesgue spaces  $L^p(\Omega, \mu)$ ,  $1 < p < \infty$ ,  $L^p(\Omega, \mu; X)$ ,  $1 < p < \infty$ , when  $X$  is a  $UMD$  space and the Schatten-von Neumann classes  $C_p(H)$ ,  $1 < p < \infty$  of operators on Hilbert spaces.

After this preliminaries, we state the following operator-valued Fourier multiplier theorem. It is fundamental in our treatment. A proof can be founded in [15].

**Theorem 2.8.** *Suppose  $X$  is a  $UMD$  space and let  $1 < p < \infty$ . Let  $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$  be such the following conditions are satisfied*

- (i) *The set  $\{M(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded and*
- (ii) *the set  $\{\rho M'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded.*

*Then the operator  $T$  defined by*

$$(2.3) \quad Tf = (M(\cdot)[\hat{f}(\cdot)])^\vee \text{ where } f \in \mathcal{S}(X)$$

*extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .*

### 3. EXISTENCE OF SOLUTIONS

Let  $\alpha, \beta, \gamma \in (0, \infty)$ . Consider the equation

$$(3.1) \quad u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t),$$

with initial conditions  $u(0) = u'(0) = u''(0) = 0$ , where  $A$  is the generator of a  $(\alpha, \beta, \gamma)$ -regularized family  $R(t)$ . By a solution of (3.1) we understand a function  $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$  such that  $u' \in C(\mathbb{R}_+; D(A))$  and verify (3.1).

**Proposition 3.1.** *Let  $R(t)$  be an  $(\alpha, \beta, \gamma)$ -regularized family on  $X$  with generator  $A$ . If  $f \in L^1_{loc}(\mathbb{R}_+, D(A^2))$  then  $u(t)$  given by*

$$(3.2) \quad u(t) = \int_0^t R(t-s)f(s)ds, \quad t \geq 0.$$

*is a solution of (3.1).*

*Proof.* Given  $x \in D(A)$ , we obtain from Proposition 2.3 that  $R(\cdot)x$ , and hence  $u$ , is of class  $C^3(\mathbb{R}_+, X)$ . For all  $x \in D(A)$ , we have

$$R'(t)x = (1 - e^{-t/\alpha})x + \int_0^t [\beta + (\gamma/\alpha - \beta)e^{-(t-s)/\alpha}]R(s)Ax ds.$$

If  $x \in D(A^2)$  then  $R'(t)x \in D(A)$ . Moreover,

$$\begin{aligned} R''(t)x &= -\frac{1}{\alpha^2}e^{-t/\alpha}x + \frac{\gamma}{\alpha}R(t)Ax + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x ds \\ R'''(t)x &= -\frac{1}{\alpha^2}e^{-t/\alpha} + \frac{\gamma}{\alpha}R'(t)Ax + \frac{\beta}{\alpha}R(t)Ax - \frac{\gamma}{\alpha^2}AR(t)x \\ &\quad + \int_0^t \left(\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x ds. \end{aligned}$$

Since  $f \in L^1_{loc}(\mathbb{R}_+, D(A^2))$ , from (3.2), we have that  $u(t), u'(t) \in D(A)$  and

$$(3.3) \quad \begin{aligned} u'(t) &= \int_0^t R'(t-s)f(s)ds, \quad u''(t) = \int_0^t R''(t-s)f(s)ds, \\ u'''(t) &= R''(0)f(t) + \int_0^t R'''(t-s)f(s)ds. \end{aligned}$$

Hence,

$$(3.4) \quad \begin{aligned} u''(t) + \alpha u'''(t) - \beta Au(t) - \gamma Au'(t) &= \int_0^t R''(t-s)f(s)ds + f(t) + \alpha \int_0^t R'''(t-s)f(s)ds \\ &\quad - \beta A \int_0^t R(t-s)f(s)ds - \gamma A \int_0^t R'(t-s)f(s)ds. \end{aligned}$$

By the other side, for all  $x \in D(A^2)$ , we obtain

$$\begin{aligned} R''(t)x + \alpha R'''(t)x - \beta AR(t)x - \gamma AR'(t)x &= \\ \frac{1}{\alpha}e^{-t/\alpha}x + \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x ds - \frac{1}{\alpha}e^{-t/\alpha}x + \gamma AR'(t)x \\ + \beta AR(t)x - \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\gamma}{\alpha^2} - \frac{\beta}{\alpha}\right)e^{-(t-s)/\alpha}AR(s)x ds - \beta AR(t)x - \gamma AR'(t)x \\ &= 0. \end{aligned}$$

Since  $f(t) \in D(A^2)$  and  $A$  is closed, from (3.4) we conclude that  $u(t)$  verify (3.1).  $\square$

The following remarkable result provide a wide class of generators of  $(\alpha, \beta, \gamma)$ -regularized families. In what follows we denote

$$\varphi(\lambda) := \frac{1}{\hat{a}(\lambda)} = \frac{\lambda^2(1 + \alpha\lambda)}{\beta + \gamma\lambda}.$$

**Theorem 3.2.** *Let  $-B$  be a positive selfadjoint operator on a Hilbert space  $H$  such that*

$$\alpha\beta \leq \gamma.$$

*Then  $B$  is the generator of a bounded  $(\alpha, \beta, \gamma)$ -regularized family on  $H$ .*

*Proof.* Since  $-B$  is a positive selfadjoint operator in  $H$ , the spectrum  $\sigma(B)$  is a subset of the negative real axis and the resolvent operator  $(\mu - B)^{-1}$  is defined at least for all negative non real  $\mu$ . Let  $\lambda \in \mathbb{C}$  such that  $Re\lambda > 0$ . If  $Im\varphi(\lambda) \neq 0$  then clearly  $\varphi(\lambda) \in \rho(B)$ . If  $Im\varphi(\lambda) = 0$  then we claim that  $Re\varphi(\lambda) > 0$ . In fact, for  $\lambda = a + bi \in \mathbb{C}$ ,  $a > 0$ , with a direct computation we obtain

$$Re\varphi(\lambda) = \frac{(a^2 - b^2)(1 + \alpha a)(\beta + \gamma a) - 2ab^2\alpha(\beta + \gamma a) + \alpha\gamma b^2(a^2 - b^2) + 2ab^2(1 + \alpha a)}{(\beta + \gamma a)^2 + \gamma^2 b^2}$$

and

$$Im\varphi(\lambda) = \frac{\alpha b(a^2 - b^2)(\beta + \gamma a) + 2ab(1 + \alpha a)(\beta + \gamma a) - \gamma b(a^2 - b^2)(1 + \alpha a) + 2ab^3\alpha\gamma}{(\beta + \gamma a)^2 + \gamma^2 b^2}.$$

Note that  $Im\varphi(\lambda) = 0$  if and only if  $b = 0$  or  $\alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma = 0$ .

Since  $\alpha\beta \leq \gamma$  we have that

$$\begin{aligned} & \alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma = \\ & 2\alpha\gamma ab^2 + b^2(\gamma - \alpha\beta) + \gamma a^2 + 3\alpha\beta a^2 + 2\beta a + 2\alpha\gamma a^2 > 0. \end{aligned}$$

Hence, we have only consider the case  $b = 0$ . Since  $a > 0$ , a direct calculation gives

$$Re\varphi(\lambda) = \frac{a^2(1 + \alpha a)}{\beta + \gamma a} > 0,$$

proving the claim. We conclude that  $\varphi(\lambda) \in \rho(B)$  for all  $Re\lambda > 0$ . Hence (see Kato [17, Section V.3.5]),

$$(3.5) \quad \|(\varphi(\lambda) - B)^{-1}\| = \frac{1}{dist(\varphi(\lambda), \sigma(B))} \text{ for all } Re\lambda > 0.$$

Note that

$$(3.6) \quad \sup_{Re\lambda > 0} \left( \frac{|\lambda|^2 + 1}{dist(\varphi(\lambda), \sigma(B))} \right) < M,$$

since  $dist(\varphi(\lambda), \sigma(B))$  has order  $|\lambda|^2$ . Define  $Q(\lambda) = \frac{1}{(\beta + \gamma\lambda)} (\varphi(\lambda) - B)^{-1}$ . We have by (3.6) and (3.5) that for all  $Re\lambda > 0$

$$\|\lambda Q(\lambda)\| = \left\| \frac{\lambda}{(\beta + \gamma\lambda)} (\varphi(\lambda) - B)^{-1} \right\| \leq \frac{|\lambda|}{|\beta + \gamma\lambda|} \frac{1}{\text{dist}(\varphi(\lambda), \sigma(B))} < M.$$

On the other hand,

$$(3.7) \quad \lambda^2 Q'(\lambda) = \frac{-\gamma\lambda}{\beta + \gamma\lambda} [\lambda Q(\lambda)] + [\lambda Q(\lambda)] [\lambda^2 (\varphi(\lambda) - B)^{-1}] \left[ \lambda \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)} \right] \frac{1}{\lambda^2 \hat{a}(\lambda)},$$

where  $\frac{1}{\lambda^2 \hat{a}(\lambda)} = \frac{1 + \alpha\lambda}{\beta + \gamma\lambda}$ ,  $\lambda \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)} = -\frac{2\alpha\gamma\lambda^2 + (\gamma + 3\alpha\beta)\lambda + 2\beta}{(1 + \alpha\lambda)(\beta + \gamma\lambda)}$  and, by (3.6),

$$\|\lambda^2 (\varphi(\lambda) - B)^{-1}\| \leq \frac{|\lambda^2|}{\text{dist}(\varphi(\lambda), \sigma(B))} < M$$

for all  $Re\lambda > 0$ . We conclude that  $\sup_{Re\lambda > 0} \|\lambda^2 Q'(\lambda)\| < \infty$ .

By Lemma 2.5 there exists a strongly continuous family  $R(t)$  such that  $\|R(t)\| \leq K$  and  $Q(\lambda) = \hat{R}(\lambda)$  for  $Re\lambda > 0$ . In consequence for all  $Re\lambda > 0$  we have

$$\hat{R}(\lambda) = \frac{\varphi(\lambda)}{\lambda^2(1 + \alpha\lambda)} (\varphi(\lambda) - B)^{-1} = \frac{1}{\beta + \gamma\lambda} \left( \frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - B \right)^{-1},$$

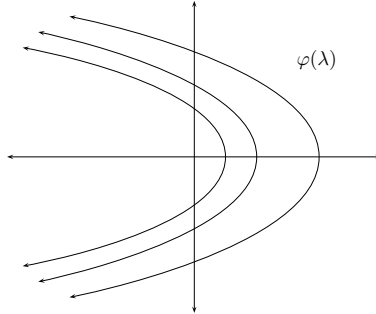
and, by Remark 2.2, it shows that  $R(t)$  is a bounded  $(\alpha, \beta, \gamma)$ -regularized family generated by  $B$ . □

Since is a know fact that the Dirichlet-Laplacian operator is a selfadjoint operator on  $L^2(\Omega)$  and  $\sigma(\Delta) \subset (-\infty, 0)$  we obtain the following corollary.

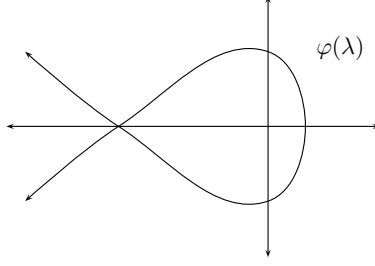
**Corollary 3.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ , and assume  $\alpha\beta \leq \gamma$ . Then the Dirichlet-Laplacian operator  $\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$  is the generator of an  $(\alpha, \beta, \gamma)$ -regularized family on  $X = L^2(\Omega)$ .*

*Remark 3.4.*

In Theorem 3.2 the condition  $\alpha\beta \leq \gamma$  is fundamental to have  $\varphi(\lambda) \in \rho(B)$  for all  $\lambda \in \mathbb{C}$  with  $Re\lambda > 0$ , which is the key in the proof. The following figure shows the typical situation, where we have mapped by  $\varphi$  the lines  $Re(\lambda) = 1, 2$  and  $3$  with  $\alpha = 3, \beta = 1$  and  $\gamma = 4$ :



Note that in case  $\alpha\beta > \gamma$  it can happens that  $\varphi(\lambda) \in \sigma(B)$ . For example, taking  $\alpha = 1, \beta = 5$  and  $\gamma = 1$  we obtain the following picture of  $\varphi(\lambda)$  for  $Im(\lambda) \in \mathbb{R}$  and  $Re(\lambda) = 1$ :



#### 4. $L^p$ - WELL POSEDNESS

Having presented preliminary material on  $R$ -boundedness and Fourier multipliers we will now show how these tools can be used to handle the equation (3.1). Our main result give concrete conditions on the operator  $A$  under which equation (3.1) have  $L^p$ -maximal regularity.

The definition of  $L^p$ -maximal regularity which we investigate in this section is as follows.

**Definition 4.1.** *We say that (3.1) has  $L^p$ - maximal regularity (or is  $L^p$ - well posed ) on  $\mathbb{R}_+$  if for each  $f \in L^p(\mathbb{R}_+, X)$  there is a unique function  $u \in W^{3,p}(\mathbb{R}_+, X) \cap W^{1,p}(\mathbb{R}_+, [D(A)]) \cap W^p(\mathbb{R}_+, [D(A)])$  such that (3.1) holds a.e.*

The following is the main abstract result of this section. It completely characterizes the maximal regularity of solutions for (3.1) in Lebesgue spaces.

**Theorem 4.2.** *Let  $X$  be a UMD space,  $1 < p < \infty$ , and let  $A$  be the generator of a bounded  $(\alpha, \beta, \gamma)$ -regularized family  $R(t)$ . The following statements are equivalent*

- (i) Equation (3.1) has  $L^p$ - maximal regularity on  $\mathbb{R}_+$ .
- (ii)  $b(\rho) := -\rho^2 \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \in \rho(A)$  for all  $\rho \in \mathbb{R} \setminus \{0\}$  and the set

$$(4.1) \quad \left\{ \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A) \right\}_{\rho \in \mathbb{R} \setminus \{0\}} \text{ is } R\text{- bounded.}$$

*Proof.* (i)  $\Rightarrow$  (ii). By (3.1) and Definition 4.1 together with Proposition 3.1, the convolution operator with kernel

$$(4.2) \quad K_4(t) := R'''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}$$

is a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ . Note that the Fourier transform  $\tilde{R}(\rho)$  exists for  $\rho \neq 0$  because  $R(t)$  is bounded and  $\hat{R}(\lambda)$  ( $Re\lambda > 0$ ) can be analytically extended from  $Re\lambda > 0$  to the imaginary axis. Then the symbol of this convolution operator is given by

$$(4.3) \quad M(\rho) = \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A), \quad \rho \in \mathbb{R} \setminus \{0\},$$

and the conclusion follows from [15, Proposition 3.17].

(ii)  $\Rightarrow$  (i). Define  $N(\rho) := \frac{1}{\beta + i\gamma\rho} R(b(\rho), A)$  and

$$(4.4) \quad N_1(\rho) := AN(\rho)$$



We check that the set  $\{N_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded.

Since  $(b(\rho) - A)R(b(\rho), A) = I$ , we have that  $A R(b(\rho), A) = b(\rho) R(b(\rho), A) - I$ . Replacing in (4.4)

$$N_1(\rho) = \frac{b(\rho)}{\beta + i\gamma\rho} R(b(\rho), A) - \frac{1}{\beta + i\gamma\rho} I = -\frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^2 N(\rho) - \frac{1}{\beta + i\gamma\rho} I.$$

Note that  $\left| \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \right|^2 = \frac{1 + \alpha^2\rho^2}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2} + \frac{\alpha^2}{\gamma^2}$  and  $\left| \frac{1}{\beta + i\gamma\rho} \right|^2 = \frac{1}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2}$ .

Since the sum of  $R$ -bounded sets is  $R$ -bounded, see [15], we obtain that  $\{N_1(\rho)\}$  is  $R$ -bounded.

We now check that the set  $\{\rho N'_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded. With a direct computation, we obtain

$$\begin{aligned} N'_1(\rho) &= b'(\rho)N(\rho) + b(\rho)N'(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I \\ &= \frac{2\alpha\gamma}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^2 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho N(\rho) \\ &\quad - \frac{i\gamma}{\beta + i\gamma\rho} b(\rho)N(\rho) - \frac{2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta}{\beta + i\gamma\rho} \rho b(\rho) N(\rho)N(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I. \end{aligned}$$

Hence

$$\begin{aligned} \rho N'_1(\rho) &= \frac{2\alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + \frac{i\gamma - \alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \\ &\quad + (2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta) \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^4 N(\rho)N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I \\ &= \frac{2\alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + \frac{i\gamma - \alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \\ &\quad + 2\alpha\gamma \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^2 N(\rho) \\ &\quad - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I \\ &= \frac{\alpha\gamma\rho - 3\alpha\beta i}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + 2\alpha\gamma \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) \\ &\quad - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^2 N(\rho) - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I. \end{aligned}$$

Since the set  $\{\rho^3 N(\rho)\}$  is  $R$ -bounded and the complex functions appearing in the above equality are bounded, we obtain the claim from the fact that the sum of  $R$ -bounded sets is again  $R$ -bounded. We employ now Theorem 2.8 to conclude that the operator  $T_1$  defined by

$$(4.5) \quad T_1 f = (N_1(\cdot)[\hat{f}(\cdot)])^\vee \text{ where } f \in \mathcal{S}(X),$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .

Define

$$(4.6) \quad N_2(\rho) := \frac{\rho}{\beta + i\gamma\rho} AR(b(\rho), A).$$

We will prove that the sets  $\{N_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  and  $\{\rho N_2'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  are  $R$ -bounded.

In fact, note that  $N_2(\rho) = \rho N_1(\rho) = -\frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) - \frac{\rho}{\beta + i\gamma\rho} I$ . Hence the set  $\{N_2(\rho)\}$  is  $R$ -bounded. Moreover, we have

$$\begin{aligned} \rho N_2'(\rho) &= \rho^2 N_1'(\rho) + \rho N_1(\rho) \\ &= \frac{\alpha\gamma\rho^2 - 3\alpha\beta\rho i}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) + 2\alpha\gamma \frac{\rho + i\alpha\rho^2}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) \\ &\quad - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^2 N(\rho) \\ &\quad + \frac{i\gamma\rho^2}{(\beta + i\gamma\rho)^2} I + N_2(\rho), \end{aligned}$$

obtaining that the set  $\{\rho N_2'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded. By Theorem 2.8 we conclude that the operator  $T_2$  defined by

$$(4.7) \quad T_2 f = (N_2(\cdot)[\hat{f}(\cdot)])^\vee \text{ where } f \in \mathcal{S}(X),$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ .

Finally, define

$$(4.8) \quad N_3(\rho) := \frac{\rho^2}{\beta + i\gamma\rho} R(b(\rho), A) = \rho^2 N(\rho).$$

The set  $\{N_3(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded from hypothesis and also note that the set  $\{\rho N_3'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$  is  $R$ -bounded, since

$$\begin{aligned} \rho N_3'(\rho) &= 2\rho^2 N(\rho) - \frac{i\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) - \frac{2\alpha\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^3 N(\rho) \\ &\quad + \frac{\gamma + 3\alpha\beta}{\beta + i\gamma\rho} i\rho^3 N(\rho) \rho^2 N(\rho) + \frac{2\beta}{\beta + i\gamma\rho} \rho N(\rho) \rho^3 N(\rho). \end{aligned}$$

Again by Theorem 2.8 we conclude that the operator  $T_3$  defined by

$$(4.9) \quad T_3 f = (N_3(\cdot)[\hat{f}(\cdot)])^\vee \text{ where } f \in \mathcal{S}(X)$$

extends to a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$ . From (4.5), (4.7) and (4.9) and since is clear that (3.1) has  $L^p$ -maximal regularity if the convolution operator with each one of the kernels

(4.10)

$$K_1(t) := AR(t)\chi_{(0,\infty)}(t), \quad K_2(t) := AR'(t)\chi_{(0,\infty)}(t), \quad K_3(t) := R''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}$$

is a bounded operator from  $L^p(\mathbb{R}, X)$  to  $L^p(\mathbb{R}, X)$  (see [15]), we conclude (i) and the proof is complete.  $\square$

Of course,  $R$ -boundedness in (4.1) can be replaced by boundedness in case  $X = H$  is a Hilbert space.

**Corollary 4.3.** *The solution  $u$  of Equation (3.1), under the conditions given by Theorem 4.2, satisfies the following maximal regularity property:  $u, u' \in L^p(\mathbb{R}_+; [D(A)])$  and  $Au, Au', u'', u''' \in L^p(\mathbb{R}_+; X)$ . Moreover, there exists a constant  $C > 0$  independent of  $f \in L^p(\mathbb{R}_+; X)$  such that*

$$(4.11) \quad \|u\|_p + \|u'\|_p + \|u''\|_p + \|u'''\|_p + \|Au\|_p + \|Au'\|_p \leq C\|f\|_p.$$

The proof follows by the closed graph theorem.

As an example, we consider for  $A = \Delta$  the vibration equation subject to the action of an external force. Explicitly, we consider,

$$(4.12) \quad \begin{cases} v_{tt}(t, x) + \lambda v_{ttt}(t, x) &= c^2(\Delta v(t, x) + \mu \Delta v_t(t, x)) + f(t, x) \text{ in } ]0, T] \times \Omega \\ v(t, x) &= 0 \text{ on } ]0, T] \times \Omega \\ v(0, x) &= 0 \text{ in } \Omega \\ v_t(0, x) &= 0 \text{ in } \Omega \\ v_{tt}(0, x) &= 0 \text{ in } \Omega \end{cases}$$

in a smooth bounded region  $\Omega \subset \mathbb{R}^n$ . Also, we assume that  $f \in L^2(\mathbb{R}; L^2(\mathbb{R}^n))$ . We have the following application in the Hilbert space setting.

**Theorem 4.4.** *Let  $\Omega$  be a bounded domain with smooth boundary  $\partial\Omega$  in  $\mathbb{R}^n$ . Suppose that  $0 < \lambda < \mu$ . Then the initial value problem (4.12) defined on  $L^2(\Omega)$  with Dirichlet boundary conditions has  $L^2$ -maximal regularity on  $\mathbb{R}_+$ .*

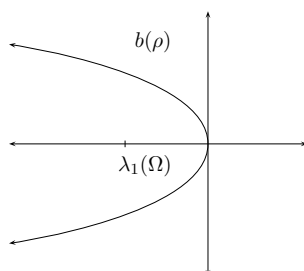
*Proof.* Let  $\alpha = \lambda, \beta = c^2$  and  $\gamma = c^2\mu$  and note that  $\alpha\beta < \gamma$  if and only if  $\lambda < \mu$ . By Corollary 3.3 we conclude that  $\Delta$  generates a bounded  $(\alpha, \beta, \gamma)$ -regularized family on  $L^2(\Omega)$ .

Note that we have  $b(\rho) = -\rho^2 \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \in \rho(\Delta)$  and there exists a constant  $C > 0$  such that

$$\begin{aligned} \left\| \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), \Delta) \right\| &= \left\| \rho \frac{b(\rho)}{1 + i\alpha\rho} (b(\rho) - \Delta)^{-1} \right\| \\ &= \frac{|\rho|}{|1 + i\alpha\rho|} \frac{|b(\rho)|}{\text{dist}(b(\rho), \lambda_1(\Omega))} \leq C, \end{aligned}$$

for all  $\rho \in \mathbb{R}$ . Here  $\lambda_1(\Omega)$  is the first eigenvalue of the Dirichlet-Laplacian. Hence, by Theorem 4.2 the assertion follows.  $\square$

*Remark 4.5.* In the following figure, we show  $b(\rho)$  in case  $\lambda = 3, \mu = 4$  and  $c^2 = 1$ .



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PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE.

*E-mail address:* cfenandez@puc.cl

UNIVERSIDAD DE SANTIAGO DE CHILE, DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS, CASILLA 307-CORREO 2, SANTIAGO-CHILE.

*E-mail address:* carlos.lizama@usach.cl

UNIVERSIDAD DE CHILE, DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS, LAS PALMERAS 3425 ÑUÑO A, SANTIAGO-CHILE.

*E-mail address:* vpoblete@uchile.cl