REGULARITY OF SOLUTIONS FOR A THIRD ORDER DIFFERENTIAL EQUATION IN HILBERT SPACES

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Abstract. We study regularity of mild and strong solutions for an abstract mathematical model of a flexible space structure under appropriate initial conditions. We apply our results showing qualitative properties of the trajectories in case of the negative Laplacian operator.

1. Introduction

Let H be a Hilbert space and A an unbounded, selfadjoint, elliptic operator in H. In this paper we consider the problem

$$
\begin{cases} \alpha u'''(t) + u''(t) + \beta A u(t) + \gamma A u'(t) = f(t), \quad t \geqslant 0; \\ u(0) = u_0, \\ u'(0) = u_1, \\ u''(0) = u_2. \end{cases} \tag{1.1}
$$

Here $f : \mathbb{R}_+ \to H$, $\alpha, \beta, \gamma \in \mathbb{R}_+$ as well as $u_0, u_1, u_2 \in H$ are given data. In case $A = -\Delta$ is the negative Laplacian in $H = L^2(\Omega)$ equipped with suitable boundary conditions, equation (1.1) was first derived by Bose and Gorain [4] to model flexible structural systems possessing internal damping. See also [2] and [3] for further information and a complete description of the model. Maximal regularity properties of equation (1.1) in Lebesgue spaces has been recently studied in [7].

We consider in this paper existence, uniqueness and regularity properties for strong and mild solutions of (1.1). One popular approach should be to reduce the third order problem to a first order system in a suitable phase space and use operator semigroup theory. The disadvantage of this approach is that, finding an ideal space is generally difficult, and the structure of the phase space (if any) may be complicated so that inconvenient to computational application; also is well known that some inherent properties of problems of order greater or equal than two can not always be reflected precisely from the corresponding first order systems (see [6] or [9]). Our idea is to establish a more inclusive theory about third order problems and therefore we give a direct treatment of (1.1).

Is also important to remark that abstract differential equations of order greater than two, are in general ill-posed [9]. As a consequence, in our case we necessarily have to assume $\alpha\beta < \gamma$.

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It is shown in Section 3 that under these assumption, for any $u_0 \in D(A)$, $u_1 \in D(\sqrt{A})$, $u_2 \in H$ and $f \in L^1(0,+\infty;H)$, the solution u satisfies

$$u \in C^1([0,\infty);D(\sqrt{A})) \cap C([0,\infty);H).$$

For smoother data, $u_0 \in D(A^{3/2})\,, u_1 \in D(A), u_2 \in D(\sqrt{A})$ and $f \in L^1([0,+\infty);D(\sqrt{A}))$ we prove that the solution u given is more regular, that is

$$u \in C^3([0,\infty);H) \cap C^2([0,\infty);D(\sqrt{A})) \cap C^1([0,\infty);D(A)) \cap C([0,\infty);D(A)).$$

This paper is organized in the following way. Section 2 is mainly devoted to the construction of a resolvent operator family for (1.1) with $f \equiv 0$. In passing, we define an energy functional naturally associated to third order problems, by means of the expression

$$E(t) = \frac{1}{2}\left[||\alpha S''(t)x + S'(t)x||^2 + \beta||\alpha\sqrt{A}S'(t)x + \sqrt{A}S(t)x||^2 \ + \alpha(\gamma-\alpha\beta)||\sqrt{A}S'(t)x||^2\right].$$

In Section 3 of this paper we state our main results on regularity of mild and strong solutions for (1.1), as well as important a priori-estimates, necessary for applications in nonlinear problems.

2. The homogeneous problem

Let H be a real Hilbert space with scalar product $\langle , \rangle$. We consider in this section the homogeneous equation

$$\alpha u'''(t) + u''(t) + \beta A u(t) + \gamma A u'(t) = 0, \quad t \geqslant 0\,, \tag{2.1}$$

where A is unbounded, selfadjoint linear operator, with domain $D(A)$ dense in H. The domain $D(A)$ will be regarded as a Hilbert space with the graph-norm. We suppose moreover that A is elliptic, that is, there exists a constant $M > 0$ such that

$$\langle Ax, x\rangle \geqslant M||x||^2 \quad \text{for all} \;\; x \in D(A). \tag{2.2}$$

Since $D(A)$ is dense in $D(\sqrt{A})$, we easily obtain the abstract version of the Poincare's inequality with Dirichlet boundary conditions

$$||\sqrt{A}x|| \geqslant M||x|| \quad \text{for all} \;\; x \in D(\sqrt{A}). \tag{2.3}$$

In this section, we represent the solution of (2.1) by means of a strongly continuous family of bounded and linear operators, that we denote by $S(t)$. We define also the abstract energy functional associated to equation (2.1) and get a priori estimates.

Recall that a family $\{C(t)\}_{t\in\mathbb{R}}$ of continuous linear operators on H is called a strongly continuous operator cosine family if $C(0) = I$, $2C(t)C(s) = C(t+s) + C(t-s)$ and $\lim_{t\to 0} C(t)x = x$ for all $x \in H$. For further literature on cosine families we refer to the monographs of Fattorini [6] and Arendt-Batty-Hieber-Neubrander [1] and the references given there. The following observation, stated as a Lemma, will be the key to associate to equation (2.1) a family of operators which will then provide the solution of the non homogeneous problem (1.1) by means of a kind of variation of parameters formula.

Lemma 2.1. *If A is selfadjoint and strictly positive operator then $-A$ generates a strongly continuous operator cosine family.*

Proof. Applying Stone's Theorem to $\sqrt{A}$, we obtain the existence of a strongly continuous group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ with generator $i\sqrt{A}$. Then defining

$$C(t)x = \frac{U(t)x + U(-t)x}{2}, \quad t \in \mathbb{R},\, x \in H,$$

we obtain a strongly continuous operator cosine family with generator $C''(0)x = -Ax$, $x \in D(A)$. □

We now define the appropriate concept of resolvent family for the homogeneous problem.

Definition 2.2. *A family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators in H is called a regular resolvent for equation (2.1) if the following conditions are satisfied:*

(S1) *$S(t)$ is strongly continuous on $[0,\infty)$ and $S(0) = I$;*
(S2) *for any $x \in D(A)$ and $t \geqslant 0$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;*
(S3) *for any $x \in D(A)$, $S(\cdot)x$ is twice continuously differentiable in H on $[0,\infty)$,*
(S4) *for any $x \in D(A^2)$ and $t \geqslant 0$, $S'(t)x \in D(A)$, $AS'(\cdot)x$ is continuous and $S(\cdot)x$ is three times continuously differentiable in H on $[0,\infty)$; Moreover, the equation*

$$\alpha S'''(t)x + S''(t)x + \gamma AS'(t)x + \beta AS(t)x = 0 \quad \textit{hold.} \tag{2.4}$$

In order to establish existence of a regular resolvent for (2.1) we are going to use the theory of integral evolutionary equations [8]. To do that, we note that integrating three times equation (2.1) we obtain that it is equivalent to the perturbed integral Volterra equation

$$u(t) = g(t) + (a * (-A)u)(t) + (b * u)(t), \tag{2.5}$$

where

$$a(t) = \frac{\beta}{2\alpha}t^2 + \frac{\gamma}{\alpha}t, \qquad b(t) = -\frac{1}{\alpha}, \tag{2.6}$$

and

$$g(t) = (1 + \frac{t}{\alpha} + \frac{\gamma}{2\alpha}t^2 A)\, u(0) + (t + \frac{t^2}{2\alpha})u'(0) + \frac{t^2}{2}u''(0). \tag{2.7}$$

The following is the main result on this section.

Theorem 2.3. *Equation (2.1) admits a regular resolvent $\{S(t)\}_{t\geqslant 0}$ which moreover satisfies*

$$\alpha S'(t)x + S(t)x + \gamma A(1 * S)(t)x + \beta A(t * S)(t)x = 0 \tag{2.8}$$

and

$$\alpha S''(t)x + S'(t)x + \gamma AS(t)x + \beta A(1 * S)(t)x = 0 \tag{2.9}$$

for all $x \in D(A)$.

Proof. From Lemma 2.1 we have that $-A$ is the generator of cosine family, say $C(t)$. Hence the equation

$$v(t) = g(t) + \frac{\gamma}{\alpha}\int_0^t (t-s)(-A)v(s)ds \tag{2.10}$$

is well posed and $(t * C)(t) \in D(A)$ for all $t \geqslant 0$ (see [8, Proposition 1.1]). Consider now the equation (2.5) where $a(t)$ is given in (2.6). Define $c(t) = \frac{\gamma}{\alpha} t$, $k(t) = \frac{\beta}{\gamma}$ and observe that equation (2.5) can be rewritten as

$$\begin{aligned} u(t) = & \; g(t) + \int_0^t a(t-s)(-A)u(s)ds + \int_0^t b(t-s)u(s)ds \\ = & \; g(t) + \frac{\beta}{2\alpha}\int_0^t (t-s)^2(-A)u(s)ds \; + \; \frac{\gamma}{\alpha}\int_0^t (t-s)(-A)u(s)ds + (b*u)(t) \\ = & \; g(t) + (c*k*(-A)u)(t) + (c*(-A)u)(t) + (b*u)(t). \end{aligned}$$

Hence

$$u(t) = g(t) + [(c + c*k)*(-A)]u(t) + (b*u)(t)\,. \tag{2.11}$$

Since it is a perturbed form of (2.10) it follows from [8, Theorem 1.2] that equation (2.5) admits a resolvent family. Furthermore, $c * u \in C([0,+\infty); D(A))$ if and only if $(c + c*k)*u \in C([0,+\infty); D(A))$ (see the remark after formula (2.16) in [8, p.39]). Then there exists a family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators in H and

(i) $S(0) = I$ and $S(t)$ is strongly continuous on $[0,\infty)$;
(ii) $S(t)$ commutes with A, which means that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geqslant 0$;
(iii) the resolvent equation

$$S(t)x = x + (a*(-A)S)(t)x \, + (b*S)(t)x \quad t \geqslant 0\,, \;\; x \in D(A), \tag{2.12}$$

holds.

By [8, Proposition 1.1] for any $x \in H$ the function $(a*S)(\cdot)x$ belongs to $C([0,\infty); D(A))$, $A(a*S)$ is strongly continuous in H and

$$S(t)x = x - A(a*S)(t)x + (b*S)(t)x, \quad x \in H. \tag{2.13}$$

Observe that for all $x \in D(A)$, from (2.12) and the fact that $S(t)$ commutes with A, we have that $(b*S)(t)x = S(t)x - x + (a*S)(t)Ax$ and hence $(1*S)(t)$ is leaving $D(A)$ invariant and $A(1*S)(\cdot)$ is strongly continuous.
Moreover, from the proof of [8, Theorem 1.2], we deduce that $(t*S)(t)$ maps H into $D(A)$ and $A(t*S)(\cdot)$ is strongly continuous.

By (i) and (ii), it is clear that **(S1)** and **(S2)** are satisfied. Differentiating the equations in (iii), for $x \in D(A)$ we have (2.8). Now, differentiating again the equations in (iii), for $x \in D(A)$ we obtain (2.9). We conclude that **(S3)** holds.

Now, for $x \in D(A^2)$, from (2.8) it follows that $S'(t)x \in D(A)$ and $AS'(\cdot)$ is strongly continuous. Finally, differentiating (2.9) we obtain that the resolvent equation in **(S4)** holds. □

The following theorem states a number of bounds and inequalities that are the key pieces for the main results obtained later in this paper.

Theorem 2.4. *Suppose $\alpha\beta < \gamma$. Then the regular resolvent $S(t)$ for equation (2.1) verifies the following properties:*

(i) For any $x \in H$ the functions

$$t \to (1 * S)(t)x = \int_0^t S(s)xds \quad \text{and} \quad t \to (t * S)(t)x = \int_0^t (t-s)S(s)xds$$

belong to $C([0,\infty); D(\sqrt{A}))$ and

$$||\sqrt{A}(1 * S)(t)x|| \leqslant C||x||, \tag{2.14}$$

$$||\sqrt{A}(t * S)(t)x|| \leqslant C||x||, \quad t \geqslant 0. \tag{2.15}$$

(ii) For any $x \in D(\sqrt{A})$, $S(\cdot)x$ belongs to $C([0,\infty); D(\sqrt{A}))$ and

$$||\sqrt{A}\, S(t)x|| \leqslant C||\sqrt{A}x||, \quad t \geqslant 0. \tag{2.16}$$

(iii) For any $x \in D(\sqrt{A})$ the functions
$t \to (1*S)(t)x = \int_0^t S(s)xds$ *and* $t \to (t*S)(t)x = \int_0^t (t-s)S(s)xds$ *belong to* $C([0,\infty); D(A))$ *and*

$$||A(1 * S)(t)x|| \leqslant C||\sqrt{A}x||, \tag{2.17}$$

$$||A(t * S)(t)x|| \leqslant C||\sqrt{A}x||, \quad t \geqslant 0. \tag{2.18}$$

(iv) For any $x \in D(\sqrt{A})$, $S(\cdot)x$ is continuously differentiable in H on $[0,\infty)$ and

$$||S'(t)x|| \leqslant C(||x|| + ||\sqrt{A}x||), \quad t \geqslant 0.$$

(v) For any $x \in D(A)$, $S(\cdot)x$ belongs to $C([0,\infty); D(A))$ and

$$||A\, S(t)x|| \leqslant C\, ||Ax||, \quad t \geqslant 0.$$

(vi) For any $x \in D(A)$, $S'(\cdot)x$ belongs to $C([0,\infty); D(\sqrt{A}))$ and

$$||\sqrt{A}\, S'(t)x|| \leqslant C\, ||Ax||, \quad t \geqslant 0.$$

Here C denote suitable positive constants depending of α, β and γ.

Proof. (i) Let $x \in D(A)$ and $(1*S)(t)x = \int_0^t S(s)xds$. Multiplying (2.8) by $\alpha S(t)x + (1*S)(t)x$, we obtain

$$\alpha^2\langle S'(t)x, S(t)x\rangle + \alpha||S(t)x||^2 + \alpha\beta\langle A(t*S)(t)x, S(t)x\rangle + \alpha\gamma\langle A(1*S)(t)x, S(t)x\rangle$$

$$+\alpha\langle S'(t)x, (1*S)(t)x\rangle + \langle S(t)x, (1*S)(t)x\rangle + \beta\langle A(t*S)(t)x, (1*S)(t)x\rangle$$

$$+\gamma\langle A(1*S)(t)x, (1*S)(t)x\rangle = 0$$

or equivalently,

$$\frac{\alpha^2}{2}\frac{d}{dt}||S(t)x||^2+\alpha||S(t)x||^2+\alpha\beta\langle A(t*S)(t)x,S(t)x\rangle+\frac{\alpha\gamma}{2}\frac{d}{dt}||\sqrt{A}(1*S)(t)x||^2$$

$$+\alpha\langle S'(t)x,(1*S)(t)x\rangle+\frac{1}{2}\frac{d}{dt}||(1*S)(t)x||^2+\frac{\beta}{2}\frac{d}{dt}||\sqrt{A}(t*S)(t)x||^2$$

$$+\gamma\langle A(1*S)(t)x,(1*S)(t)x\rangle=0\,.$$

Integrating the last identity on $[0,t]$, we have

$$\frac{\alpha^2}{2}||S(t)x||^2-\frac{\alpha^2}{2}||S(0)x||^2+\alpha\int_0^t||S(\tau)x||^2d\tau+\alpha\beta\int_0^t\langle A(t*S)(\tau)x,S(\tau)x\rangle d\tau$$

$$+\frac{\alpha\gamma}{2}||\sqrt{A}(1*S)(t)x||^2+\alpha\int_0^t\langle S'(\tau)x,(1*S)(\tau)x\rangle d\tau+\frac{1}{2}||(1*S)(t)x||^2$$

$$+\frac{\beta}{2}||\sqrt{A}(t*S)(t)x||^2+\gamma\int_0^t\langle A(1*S)(\tau)x,(1*S)(\tau)x\rangle d\tau=0\,.$$

Integrating by parts the fourth and sixth terms on the left-hand side of the above equality, we obtain

$$\frac{\alpha^2}{2}||S(t)x||^2-\frac{\alpha^2}{2}||S(0)x||^2+\alpha\int_0^t||S(\tau)x||^2d\tau+\alpha\beta\langle A(t*S)(t)x,(1*S)(t)x\rangle$$

$$-\alpha\beta\int_0^t||\sqrt{A}(1*S)(\tau)x||^2d\tau+\frac{\alpha\gamma}{2}||\sqrt{A}(1*S)(t)x||^2+\alpha\langle S(t)x,(1*S)(t)x\rangle$$

$$-\alpha\int_0^t||S(\tau)x||^2d\tau+\frac{1}{2}||(1*S)(t)x||^2+\frac{\beta}{2}||\sqrt{A}(t*S)(t)x||^2+\gamma\int_0^t||\sqrt{A}(1*S)(\tau)x||^2d\tau=0\,.$$

Taking in account the Hilbert space structure, we can reorganize the above equality as follows

$$\frac{1}{2}||\alpha S(t)x+(1*S)(t)x||^2+\alpha\beta\langle A(t*S)(t)x,(1*S)(t)x\rangle+(\gamma-\alpha\beta)\int_0^t||\sqrt{A}(1*S)(\tau)x||^2d\tau$$

$$+\frac{\alpha\gamma}{2}||\sqrt{A}(1*S)(t)x||^2+\frac{\beta}{2}||\sqrt{A}(t*S)(t)x||^2=\frac{\alpha^2}{2}||S(0)x||^2\,.$$

or equivalently,

$$\frac{1}{2}||\alpha S(t)x+(1*S)(t)x||^2+\frac{\beta}{2}||\sqrt{A}(t*S)(t)x+\alpha\sqrt{A}(1*S)(t)x||^2$$

$$+(\gamma-\alpha\beta)\int_0^t||\sqrt{A}(1*S)(\tau)x||^2d\tau+\frac{\alpha\gamma}{2}||\sqrt{A}(1*S)(t)x||^2-\frac{\alpha^2\beta}{2}||\sqrt{A}(1*S)(t)x||^2$$

$$=\frac{\alpha^2}{2}||S(0)x||^2\,.$$

Hence, we obtain the following identity

$$||\alpha S(t)x+(1*S)(t)x||^2+\beta||\alpha\sqrt{A}(1*S)(t)x+\sqrt{A}(t*S)(t)x||^2$$

$$+2(\gamma-\alpha\beta)\int_0^t||\sqrt{A}(1*S)(s)x||^2ds+\alpha(\gamma-\alpha\beta)||\sqrt{A}(1*S)(t)x||^2=\alpha^2||x||^2. \tag{2.19}$$

Since $\alpha\beta<\gamma$ we have that the sum in left-hand side of equality above is positive, for all x nonzero. We then obtain that

$$||\sqrt{A}(1*S)(t)x||\leqslant C||x||\,,\quad \text{for any}\ \ t\geqslant 0\,,\,x\in D(A)\,.$$

Moreover, again from (2.19) we obtain

$$||\sqrt{A}(t*S)(t)x||=||\sqrt{A}(1*S)(t)x+\sqrt{A}(t*S)(t)x-\sqrt{A}(1*S)(t)x||$$

$$\leqslant||\sqrt{A}(1*S)(t)x+\sqrt{A}(t*S)(t)x||+||\sqrt{A}(1*S)(t)x||\leqslant C||x||\,,\quad t\geqslant 0,$$

where C is a generic constant which depends on α,β,γ. The proof of (i) follows using the density of $D(A)$ in H.

(ii) Let $x\in D(A)$ be fixed. Multiplying (2.8) by $\alpha S'(t)x+S(t)x$, and integrating on $[0,t]$, $t\geqslant 0$, we obtain analogously as in the proof of (i), the following identity

$$||\alpha S'(t)x+S(t)x||^2+\beta||\alpha\sqrt{A}S(t)x+\sqrt{A}(1*S)(t)x||^2+\alpha(\gamma-\alpha\beta)||\sqrt{A}S(t)x||^2=\alpha\gamma||\sqrt{A}x||^2. \tag{2.20}$$

Since $\alpha\beta<\gamma$, we deduce that $||\sqrt{A}S(t)x||\leqslant C||\sqrt{A}x||$, for any $t\geqslant 0$ and $x\in D(A)$. Finally, the proof of (ii) is completed using the density of $D(A)$ in $D(\sqrt{A})$.

(iii) Let $x\in D(A)$ be fixed. From (2.8), we have

$$-\gamma A(1*S)(t)x=\alpha S'(t)x+S(t)x+\beta A(t*S)(t)x.$$

Then,

$$||A(1*S)(t)x||\leqslant\frac{1}{\gamma}||\alpha S'(t)x+S(t)x||+\frac{\beta}{\gamma}||A(t*S)(t)x||\,.$$

From (2.20) we have $||\alpha S'(t)x+S(t)x||\leqslant C||\sqrt{A}x||$ and then by (2.15) we obtain that

$$||A(t*S)(t)x||=||\sqrt{A}(t*S)(t)\sqrt{A}x||\leqslant C||\sqrt{A}x||\,.$$

We conclude that

$$||A(1*S)(t)x||\leqslant C||\sqrt{A}x||.$$

Finally, observe that the previous inequality hold for any $x \in D(\sqrt{A})$ thanks to the density of $D(A)$ in $D(\sqrt{A})$. This proves (2.17) and (2.18).

(iv) Let $x \in D(A)$ be fixed. From (2.8) we have

$$||S'(t)x|| \leqslant \frac{1}{\alpha}||S(t)x|| + \frac{\beta}{\alpha}||A(t*S)(t)x|| + \frac{\gamma}{\alpha}||A(1*S)(t)x|| \leqslant C(||x|| + ||\sqrt{A}x||),$$

for $t \geqslant 0$, and then (iv) holds using again the density of $D(A)$ in $D(\sqrt{A})$.

(v) Let $x \in D(A)$. From (ii) it follows that

$$||AS(t)x|| = ||\sqrt{A}S(t)\sqrt{A}x|| \leqslant C||Ax||.$$

(vi) Let $x \in D(A^2)$ be fixed. Multiplying (2.4) by $\alpha S''(t)x + S'(t)x$ we obtain

$$\alpha||S''(t)x||^2 + \frac{\alpha^2}{2}\frac{d}{dt}||S''(t)x||^2 + \alpha\beta\langle AS(t)x, S''(t)x\rangle + \frac{\alpha\gamma}{2}\frac{d}{dt}||\sqrt{A}S'(t)x||^2$$

$$+\frac{1}{2}\frac{d}{dt}||S'(t)x||^2 + \alpha\langle S'''(t)x, S'(t)x\rangle + \frac{\beta}{2}\frac{d}{dt}||\sqrt{A}S(t)x||^2 + \gamma\langle AS'(t)x, S'(t)x\rangle = 0.$$

Integrating on $[0,t]$, we obtain

$$\alpha\int_0^t ||S''(s)x||^2 ds + \frac{\alpha^2}{2}||S''(t)x||^2 + \alpha\beta\int_0^t \langle AS(s)x, S''(s)x\rangle ds + \frac{\alpha\gamma}{2}||\sqrt{A}S'(t)x||^2$$

$$+\frac{1}{2}||S'(t)x||^2 + \alpha\int_0^t \langle S'''(s)x, S'(s)x\rangle ds + \frac{\beta}{2}||\sqrt{A}S(t)x||^2 + \gamma\int_0^t \langle AS'(s)x, S'(s)x\rangle ds$$

$$= \frac{\alpha^2}{2}||S''(0)x||^2 + \frac{1}{2}||S'(0)x||^2 + \frac{\alpha\gamma}{2}||\sqrt{A}S'(0)x||^2 + \frac{\beta}{2}||\sqrt{A}S(0)x||^2.$$

Integrating by parts the third and sixth terms on the left-hand side of the above equality, we obtain

$$\frac{1}{2}\left[||\alpha S''(t)x + S'(t)x||^2 + \beta||\alpha\sqrt{A}S'(t)x + \sqrt{A}S(t)x||^2 + \alpha(\gamma - \alpha\beta)||\sqrt{A}S'(t)x||^2\right]$$

$$+(\gamma - \alpha\beta)\int_0^t ||\sqrt{A}S'(s)x||^2 ds = \frac{\alpha^2}{2}||S''(0)x||^2 + \frac{1}{2}||S'(0)x||^2 + \frac{\alpha\gamma}{2}||\sqrt{A}S'(0)x||^2$$

$$+\tfrac{\beta}{2}||\sqrt{A}S(0)x||^2 + \alpha\langle S'(0)x, S''(0)x\rangle + \alpha\beta\langle\sqrt{A}S'(0)x, \sqrt{A}S(0)x\rangle.$$

Note from (2.8) and (2.9) that $S'(0)x = -\frac{1}{\alpha}x$, $S''(0)x = \frac{1}{\alpha^2}x - \frac{\gamma}{\alpha}Ax$. Replacing in above identity, we get

(2.21)
$$||\alpha S''(t)x + S'(t)x||^2 + \beta||\alpha\sqrt{A}S'(t)x + \sqrt{A}S(t)x||^2 + \alpha(\gamma-\alpha\beta)||\sqrt{A}S'(t)x||^2$$
$$+2(\gamma-\alpha\beta)\int_0^t ||\sqrt{A}S'(s)x||^2 ds = \frac{\gamma-\alpha\beta}{\alpha}||\sqrt{A}x||^2 + \gamma^2||Ax||^2 .$$

Since $\alpha\beta < \gamma$, and A is elliptic we deduce that $||\sqrt{A}S'(t)x|| \leqslant C||Ax||$, for any $t \geqslant 0$ and $x \in D(A^2)$. Finally, the proof of (vi) is completed using the density of $D(A^2)$ in $D(A)$.

□

Corollary 2.5. *Under the assumptions of Theorem 2.4,*

(a) *For $x \in H$, there exists $C > 0$ such that*
$$||(t * S)(t)x|| \leqslant C||x|| \quad \textit{and} \quad ||(1 * S)(t)x|| \leqslant C||x||, \ t \geqslant 0.$$

(b) *For $x \in H$, there exists $C > 0$ such that*
$$||S(t)x|| \leqslant C||x||, \ t \geqslant 0.$$

(c) *For $x \in D(\sqrt{A})$, there exists $K > 0$ such that*
$$||S(t)x|| \leqslant K||\sqrt{A}x||, \ t \geqslant 0.$$

(d) *The equality (2.8) holds by $x \in D(\sqrt{A})$.*

Proof. (a) For $x \in H$, by Theorem 2.4 (i), we have that $(t * S)(t)x$ and $(1 * S)(t)x$ belong to $D(\sqrt{A})$. From (2.3), (2.14) and (2.15) it follows (a).

(b) For $x \in H$, note that
$$\alpha||S(t)x|| \leqslant ||\alpha S(t)x + (1 * S)(t)x|| + ||(1 * S)(t)x||, \quad t \geqslant 0.$$
From (2.19) and (a), we obtain (b).

(c) For $x \in D(\sqrt{A})$, by Theorem 2.4 (ii), we have that $S(t)x$ belongs to $D(\sqrt{A})$. From (2.3) and (2.16) we obtain (b).

(d) Follows from Theorem 2.4 (iii), (2.8) and the density of $D(A)$ in $D(\sqrt{A})$. □

Based on the proof of the previous theorem, we define the following energy functional associated to our problem.

Definition 2.6. *For $x \in D(A)$, $\alpha\beta \leqslant \gamma$, we define the abstract energy functional associated to equation (2.1) by*

(2.22)
$$E(t) = \frac{1}{2}\Big[||\alpha S''(t)x + S'(t)x||^2 + \beta||\alpha\sqrt{A}S'(t)x + \sqrt{A}S(t)x||^2$$
$$+ \quad \alpha(\gamma-\alpha\beta)||\sqrt{A}S'(t)x||^2\Big], \qquad t \geqslant 0 .$$

Note that by the previous Theorem $E : [0,\infty) \to \mathbb{R}$ is a well defined, positive and continuous function. Moreover, we observe that in case $\alpha = 0, \gamma = 0$ and $\beta = 1$ we obtain

$$E(t) = \frac{1}{2}||S'(t)x||^2 + \frac{1}{2}||\sqrt{A}S(t)x||^2,$$

which is the usual expression of the energy functional for the second order equation

$$u''(t) + Au(t) = f(t).$$

Compare also with Sforza [5, p. 160]. Moreover, note that (2.22) is the abstract version of the energy functional defined by Bose-Gorain [3] in case of $-A = \Delta$, the Laplacian.

We finish this section with the following result on an important property of the energy functional $E(t)$.

Theorem 2.7. *Suppose $\alpha\beta < \gamma$. For any $x \in D(A)$, $E(t)$ is decreasing function, in particular $E(t) \leqslant E(0)$.*

Proof. Note that

$$\begin{aligned} E(0) &= \frac{1}{2}\left[||\alpha S''(0)x + S'(0)x||^2 + \beta||\alpha\sqrt{A}S'(0)x + \sqrt{A}S(0)x||^2 + \alpha(\gamma - \alpha\beta)||\sqrt{A}S'(0)x||^2\right] \\ &= \frac{\gamma - \alpha\beta}{2\alpha}||\sqrt{A}x||^2 + \frac{\gamma^2}{2}||Ax||^2 . \end{aligned}$$

From (2.21) we have that

$$E(t) + (\gamma - \alpha\beta)\int_0^t ||\sqrt{A}S'(s)x||^2 ds = E(0) .$$

Since $\alpha\beta < \gamma$ we obtain that $h(t) = (\gamma - \alpha\beta)\int_0^t ||\sqrt{A}S'(s)x||^2 ds$ is a positive and increasing function, we conclude that $E(t)$ is a decreasing function and $E(t) \leqslant E(0)$.

□

3. Regularity of strong and mild solutions

In this section, we assume that A is self-adjoint linear operator which satisfies the ellipticity condition (2.2). The existence of the resolvent $S(t)$ proved in the previous section allows us to solve the nonhomogeneous equation

$$\text{(3.23)} \qquad \begin{cases} \alpha u'''(t) + u''(t) + \beta Au(t) + \gamma Au'(t) = f(t), \quad t \geqslant 0, \\ u(0) = u_0, \\ u'(0) = u_1, \\ u''(0) = u_2, \end{cases}$$

for which we will now state the notions of mild and strong solutions.

Definition 3.8. *Let $f \in C([0,\infty); H)$. We say that u is a strong solution of (3.23) in $[0,\infty)$ if $u \in C^3([0,\infty); H) \cap C^1([0,\infty); D(A))$ and u verifies (3.23) in $[0,\infty)$.*

Let $f \in L^1(0,\infty; H)$ and $u_0 \in D(A)$ u_1, $u_2 \in H$. The mild solution of (3.23) in $[0,\infty)$ with initial conditions

$$u(0) = u_0 , \quad u'(0) = u_1 , \quad u''(0) = u_2 ,$$

is the function $u \in C([0,\infty); H)$ *defined by*

(3.24)
$$u(t) = S(t)u_0 + \frac{1}{\alpha}(1*S)(t)u_0 + \frac{\gamma}{\alpha}(t*S)(t)Au_0 + (1*S)(t)u_1 + \frac{1}{\alpha}(t*S)(t)u_1 + (t*S)(t)u_2 + \frac{1}{\alpha}(t*S*f)(t).$$

We begin with the following result.

Theorem 3.9. *If* u *is a strong solution of equation (3.23) then* u *is a mild solution of (3.23).*

Proof. A strong solution u of the equation (3.23) is also a strong solution of the integral equation

$$u(t) = (1 + \frac{t}{\alpha} + \frac{\gamma}{2\alpha}t^2 A)\, u_0 + (t + \frac{t^2}{2\alpha})u_1 + \frac{t^2}{2}u_2 + \frac{1}{\alpha}(1 * t * f)(t) + \int_0^t \mathcal{A}(t-s)u(s)ds\,,$$

where $\mathcal{A}(t) = -(\frac{\beta}{2\alpha}t^2 + \frac{\gamma}{\alpha}t)A - \frac{1}{\alpha}I$. Therefore, by [8, Proposition 6.3] we have

$$\begin{aligned} u(t) &= \frac{d}{dt}\int_0^t S(s)\, \Big[(1 + \frac{1}{\alpha}(t-s) + \frac{\gamma}{2\alpha}(t-s)^2 A)u_0 + (t-s)u_1 + \frac{(t-s)^2}{2\alpha}u_1 \\ &+ \frac{(t-s)^2}{2}u_2 + \frac{1}{\alpha}(1*t*f)(t-s)\Big]\, ds \\ &= \int_0^t S(s)\, \Big[(\frac{1}{\alpha} + \frac{\gamma}{\alpha}(t-s)A)u_0 + u_1 + \frac{1}{\alpha}(t-s)u_1 + (t-s)u_2 + \frac{1}{\alpha}(t*f)(t-s)\Big]\, ds \\ &+ S(t)u_0 \end{aligned}$$

$$\begin{aligned} &= S(t)u_0 + \frac{1}{\alpha}(1*S)(t)u_0 + \frac{\gamma}{\alpha}(t*S)(t)Au_0 + (1*S)(t)u_1 + \frac{1}{\alpha}(t*S)(t)u_1 \\ &+ (t*S)(t)u_2 + \frac{1}{\alpha}(t*S*f)(t). \end{aligned}$$

□

In what follows we will assume the condition $\alpha\beta < \gamma$. Our next main regularity result, establish conditions on the initial data in order to have mild solutions. We also state a priori bounds that will be useful later.

Theorem 3.10. *If* $f \in L^1([0,\infty); H)$, $u_0 \in D(A)$, $u_1, \in D(\sqrt{A})$ *and* $u_2 \in H$, *then the mild solution of equation (3.23) belongs to* $C^1([0,\infty); D(\sqrt{A})) \cap C^1([0,\infty); H)$. *Moreover the following estimate holds for any* $t \in [0,\infty)$

(3.25)
$$||u''(t)|| + ||u'(t)|| + ||\sqrt{A}u'(t)|| + ||u(t)|| + ||\sqrt{A}u(t)||$$
$$\leqslant C(||u_0|| + ||u_1|| + ||u_2|| + ||Au_0|| + ||\sqrt{A}u_1|| + ||f||_1)\,.$$

Proof. From (i) and (ii) of Theorem 2.4 we have that $u \in C([0,\infty); D(\sqrt{A}) \cap C([0,\infty); H)$. By Theorem 2.4 and Remark 2.5 we obtain

$$\begin{aligned} ||u(t)|| &\leqslant ||S(t)u_0|| + \frac{1}{\alpha}||(1*S)(t)u_0|| + \frac{\gamma}{\alpha}||(t*S)(t)Au_0|| + ||(1*S)(t)u_1|| \\ &+\frac{1}{\alpha}||(t*S)(t)u_1|| + ||(t*S)(t)u_2|| + \frac{1}{\alpha}||(t*S*f)(t)|| \\ &\leqslant C(||\sqrt{A}u_0|| + ||u_0|| + ||Au_0|| + ||u_1|| + ||u_2|| + ||f||_1) \\ &\leqslant C(||u_0|| + ||Au_0|| + ||u_1|| + ||u_2|| + ||f||_1), \end{aligned}$$

where in the last inequality we made use of (2.2). On the other hand, by Theorem 2.4 we also have

$$\begin{aligned} ||\sqrt{A}u(t)|| &\leqslant ||\sqrt{A}S(t)u_0|| + \frac{1}{\alpha}||\sqrt{A}(1*S)(t)u_0|| + \frac{\gamma}{\alpha}||\sqrt{A}(t*S)(t)Au_0|| \\ &+||\sqrt{A}(1*S)(t)u_1|| + \frac{1}{\alpha}||\sqrt{A}(t*S)(t)u_1|| + ||\sqrt{A}(t*S)(t)u_2|| + \frac{1}{\alpha}||\sqrt{A}(t*S*f)(t)|| \\ &\leqslant C(||u_0|| + ||Au_0|| + ||u_1|| + ||u_2|| + ||f||_1), \end{aligned}$$

Furthermore, note that

$$u'(t) = S'(t)u_0 + \frac{1}{\alpha}S(t)u_0 + \frac{\gamma}{\alpha}(1*S)(t)Au_0 + S(t)u_1 + \frac{1}{\alpha}(1*S)(t)u_1 + (1*S)(t)u_2 + \frac{1}{\alpha}(1*S*f)(t).$$

Since $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$ by Theorem 2.4, we have that $u \in C^1([0,\infty); D(\sqrt{A}) \cap C([0,\infty); H)$.

The second derivative of u is

$$u''(t) = S''(t)u_0 + \frac{1}{\alpha}S'(t)u_0 + \frac{\gamma}{\alpha}S(t)Au_0 + S'(t)u_1 + \frac{1}{\alpha}S(t)u_1 + S(t)u_2 + \frac{1}{\alpha}(S*f)(t).$$

From (2.21), (2.20), Corollary 2.5 (c) and Theorem 2.4 we have that $u \in C^2([0,\infty); H)$.

Using repeatedly Theorem 2.4, we obtain the estimative (3.25). □

The following is the main result of this section, on regularity of strong solutions for the nonhomogeneous problem (3.23).

Theorem 3.11. *Let $f \in L^1([0,\infty); D(\sqrt{A}))$, $u_0 \in D(A^{3/2})$, $u_1 \in D(A)$ and $u_2 \in D(\sqrt{A})$. Then, the mild solution of problem (3.23) is a strong one, it belongs to $C^2([0,\infty); D(\sqrt{A}))$ and the following estimate holds for any $t \geqslant 0$*

$$\begin{aligned} &||u'''(t)|| + ||u''(t)|| + ||u'(t)|| + ||u(t)|| + ||\sqrt{A}u''(t)|| + ||Au'(t)|| + ||Au(t)|| \\ &\leqslant C(||u_0|| + ||u_1|| + ||u_2|| + ||A^{3/2}u_0|| + ||Au_1|| + ||\sqrt{A}u_2|| + ||f||_1), \end{aligned}$$

where C is a positive constant depending on α, β and γ.

Proof. We define the auxiliary functions $v(t) = \frac{1}{\alpha}(t*S*f)(t)$, $w(t) = S(t)u_0$, $r(t) = \frac{1}{\alpha}(1*S)(t)u_0 + \frac{\gamma}{\alpha}(t*S)(t)Au_0$, $p(t) = (1*S)(t)u_1 + \frac{1}{\alpha}(t*S)(t)u_1$ and $q(t) = (t*S)(t)u_2$.

We are going to show that each one of the functions v, w, r, p and q are strong solutions of the equation (3.23) and that corresponding estimates hold. Then the claim follows by the superposition principle.

Let $u_0 = u_1 = u_2 = 0$. By the closedness of A and from Theorem 2.4 (iii), we have that v and v' belong to $C([0,+\infty);D(A))$. By differentiation we obtain

$$\alpha v'''(t) + v''(t) + \beta Av(t) + \gamma Av'(t) = f(t) + (S' * f)(t) + \frac{1}{\alpha}(S * f)(t) + \frac{\beta}{\alpha}A(t * S * f)(t)$$

$$+\frac{\gamma}{\alpha}A(1 * S * f)(t)$$

$$= f(t) + \int_0^t S'(t-s)f(s)ds + \frac{1}{\alpha}\int_0^t S(t-s)f(s)ds + \frac{1}{\alpha}\int_0^t \beta At * S(t-s)f(s)ds$$

$$+\frac{1}{\alpha}\int_0^t \gamma A(1 * S)(t-s)f(s)ds$$

$$= f(t) + \frac{1}{\alpha}\int_0^t [\alpha S'(t-s) + S(t-s) + \beta A(t * S)(t-s) + \gamma A(1 * S)(t-s)]f(s)\; ds$$

Since $f(t) \in D(\sqrt{A})$, by Corollary 2.5 (c), we have that $[\alpha S'(t-s) + S(t-s) + \beta A(t*S)(t-s) + \gamma A(1*S)(t-s)]f(s) = 0$, hence (3.23) holds.

Now, we show that the estimate holds for v. By Corollary 2.5 (a), (b), we obtain that $||v(t)|| \leqslant C||f||_{L^1}$, $||v'(t)|| \leqslant C||f||_{L^1}$ and $||v''(t)|| \leqslant C||f||_{L^1}$. From Theorem 2.4 (iv) it follows $||v'''(t)|| \leqslant C||f||_{L^1}$. Applying the Theorem 2.4 (iii) we have also $||Av(t)|| + ||Av'(t)|| \leqslant C||f||_{L^1}$. Again by Theorem 2.4 (ii) $v''(t) \in D(\sqrt{A})$ and $||\sqrt{A}v''(t)|| \leqslant C||f||_{L^1}$. Hence

$$||v'''(t)|| + ||v''(t)|| + ||v'(t)|| + ||v(t)|| + ||Av(t)|| + ||Av'(t)|| + ||\sqrt{A}v''(t)|| \leqslant C||f||_{L^1}.$$

and hence the claim is proved for v.

Suppose now $u_1 = u_2 = 0$ and $f = 0$. By hypothesis $u_0 \in D(A^{3/2})$, hence $w(t) \in D(A)$. From (2.8) we have that $w'(t) = -\frac{1}{\alpha}S(t)u_0 - \frac{\beta}{\alpha}(t * S)(t)Au_0 - \frac{\gamma}{\alpha}(1 * S)(t)Au_0$.

Since $Au_0 \in D(\sqrt{A})$, by Theorem 2.4 (iii), (v) we obtain that $w'(\cdot) \in C([0,+\infty);D(A))$. Now, using (2.9) we have

$$w''(t) = -\frac{1}{\alpha}w'(t) - \frac{\beta}{\alpha}(1 * S)(t)Au_0 - \frac{\gamma}{\alpha}S(t)Au_0 .$$

Due to Theorem 2.4 (i), (ii) and (vi), it follows that $w''(\cdot) \in C([0,+\infty);D(\sqrt{A}))$. Note that

$$w'''(t) = -\frac{1}{\alpha}w''(t) - \frac{\beta}{\alpha}S(t)Au_0 - \frac{\gamma}{\alpha}S'(t)Au_0 .$$

We obtain from Theorem 2.4 (iv) and Corollary 2.5 (b), that $w'''(\cdot) \in C([0,+\infty);H)$. Finally, is easy to verify that $w(t)$ satisfy (3.23).

Now, we show that the estimate holds for w. From Corollary 2.5 (b) and Theorem 2.4 (v), it follows that

$$||w(t)|| \leqslant C||\sqrt{A}u_0|| \quad \text{and} \quad ||Aw(t)|| \leqslant C||Au_0||.$$

By Corollary 2.5 (a), (b) we obtain $||w'(t)|| \leqslant C(||Au_0|| + ||\sqrt{A}u_0||)$. Using Theorem 2.4 (iii), (v) we have $||Aw'(t)|| \leqslant C(||A^{3/2}u_0|| + ||Au_0||)$. Considered again Theorem 2.4 and Corollary

2.5 we obtain

$$||w''(t)|| + ||\sqrt{A}w''(t)|| \leqslant C(||u_0|| + ||\sqrt{A}u_0|| + ||A^{3/2}u_0|| + ||Au_0||)$$

and

$$||w'''(t)|| \leqslant C(||u_0|| + ||\sqrt{A}u_0|| + ||A^{3/2}u_0|| + ||Au_0||).$$

Let now $u_1 = u_2 = 0$ and $f = 0$. Thanks to Theorem 2.4 (iii), (v), we obtain that $r(\cdot)$ and their first derivative belong to $C([0,+\infty); D(A))$. Moreover, by (vi), their second derivative belongs to $C([0,+\infty); D(\sqrt{A}))$. The third derivative of r is given by

$$r'''(t) = \frac{1}{\alpha}w''(t) + \frac{\gamma}{\alpha}S'(t)Au_0,$$

and belongs to $C([0,+\infty); H)$ because $w'' \in C([0,+\infty); D(\sqrt{A}))$ and $S'(\cdot)Au_0 \in C([0,+\infty); H)$, by Theorem 2.4 (iv).

From Corollary 2.5 (a), (b), it follows that

$$||r(t)|| + ||r'(t)|| \leqslant C(||u_0|| + ||\sqrt{A}u_0|| + ||Au_0||).$$

By Theorem 2.4 and Corollary 2.5, we have that

$$||r''(t)|| + ||r'''(t)|| \leqslant C(||A^{3/2}u_0|| + ||\sqrt{A}u_0|| + ||Au_0|| + ||u_0||)$$

and

$$||Ar'(t)|| + ||Ar(t)|| \leqslant C(||A^{3/2}u_0|| + ||\sqrt{A}u_0|| + ||Au_0||).$$

Since $Au_0 \in D(A^{1/2})$, by Theorem 2.4 (ii), the second derivative of $r(t)$ belongs to $D(\sqrt{A})$ and

$$||\sqrt{A}r''(t)|| \leqslant C(||A^{3/2}u_0|| + ||Au_0||).$$

Using (2.9) and Corollary 2.5 (c), we finally obtain that $r(t)$ verify (3.23).

Let $u_0 = u_2 = 0$ and $f = 0$. Similarly as in the above cases, applying Theorem 2.4 and Corollary 2.5, we obtain that $p(t)$ is the strong solution of (3.23) and

$$||p'''(t)||+||p''(t)||+||p'(t)||+||p(t)||+||\sqrt{A}p''(t)||+||Ap'(t)||+||Ap(t)|| \leqslant C(||u_1||+||\sqrt{A}u_1||+||Au_1||).$$

Analogously, for $q(t)$ and since $u_2 \in D(\sqrt{A})$, $u_0 = u_1 = 0$, $f = 0$, using repeatedly Theorem 2.4 and Corollary 2.5 we obtain that $q(t)$ is strong solution of (3.23) and

$$||q'''(t)|| + ||q''(t)|| + ||q'(t)|| + ||q(t)|| + ||\sqrt{A}q''(t)|| + ||Aq'(t)|| + ||Aq(t)|| \leqslant C(||u_2|| + ||\sqrt{A}u_2||).$$

Finally, concerning the regularity of u'', note that since

$$u''(t) = S''(t)u_0 + \frac{1}{\alpha}S'(t)u_0 + \frac{\gamma}{\alpha}S(t)Au_0 + S'(t)u_1 + \frac{1}{\alpha}S(t)u_1 + S(t)u_2 + \frac{1}{\alpha}(S*f)(t), \quad t \in [0,\infty)$$

we have from Theorem 2.4 and Corollary 2.5 that $\sqrt{A}u''(t)$ is continuous. □

As an example, we now consider the problem

$$(3.26)\qquad \begin{cases} u_{tt}(t,x)+\lambda u_{ttt}(t,x) &=& c^2(\Delta u(t,x)+\mu\Delta u_t(t,x))+f(t,x) \text{ in }]0,T]\times\Omega\\ u(t,x) &=& 0 \text{ on }]0,T]\times\partial\Omega\\ u(0,x) &=& u_0(x) \text{ in } \Omega\\ u_t(0,x) &=& u_1(x) \text{ in } \Omega\\ u_{tt}(0,x) &=& u_2(x) \text{ in } \Omega \end{cases}$$

in a smooth bounded region $\Omega\subset\mathbb{R}^n$, where $0<\lambda<\mu$; see [3]. Also, we assume that $f\in L^1([0,\infty);L^2(\Omega))$.

In what follows, we apply the previous results to the operator $A\equiv-\Delta$, with Dirichlet boundary conditions on $\partial\Omega$. The properties of the corresponding resolvent $S(t)$ and the fact that the solutions have bounded energy will allow us to obtain compactness of the trajectories for the problem (3.26).

Notice that the solution of the Cauchy problem (3.26) is giving by the formula (3.24), that is $u=v+w$, where v is the solution of the non homogeneous problem with zero data, and w is the solution of the homogeneous problem,

$$(3.27)\qquad \begin{cases} w_{tt}(t,x)+\lambda w_{ttt}(t,x) &=& c^2(\Delta w(t,x)+\mu\Delta w_t(t,x)) \text{ in }]0,T]\times\Omega\\ w(t,x) &=& 0 \text{ on }]0,T]\times\partial\Omega\\ w(0,x) &=& u_0(x) \text{ in } \Omega\\ w_t(0,x) &=& u_1(x) \text{ in } \Omega\\ w_{tt}(0,x) &=& u_2(x) \text{ in } \Omega. \end{cases}$$

Moreover, with the notation of the previous sections, we have that,

$$w(t)=S(t)u_0+\frac{1}{\alpha}(1*S)(t)u_0+\frac{\gamma}{\alpha}(t*S)(t)Au_0+(1*S)(t)u_1+\frac{1}{\alpha}(t*S)(t)u_1+(t*S)(t)u_2$$

and

$$v(t)=\frac{1}{\alpha}(t*S*f)(t)\ .$$

The following is our main result of L^2-well posedness of (3.26).

Theorem 3.12. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ and $0<\lambda<\mu$. Suppose that $(u_0,u_1,u_2)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L^2(\Omega)$. Then, for every $f\in L^1([0,\infty),L^2(\Omega))$ the initial value problem (3.26) has a unique mild solution solution u which satisfies*

$$(3.28)\qquad ||u''(t)||+||u'(t)||+||\nabla u'(t)||+||u(t)||+||\nabla u(t)||$$

$$\leqslant C(||u_0||+||u_1||+||u_2||+||\Delta u_0||+||\nabla u_1||+||f||_1)\,.$$

Proof. Recall that $A=-\Delta$ is positive and elliptic, by Poincare's inequality. Since $0<\lambda<\mu$, the numbers $\alpha=\lambda,\beta=c^2$ and $\gamma=c^2\mu$ satisfy the condition $\alpha\beta<\gamma$. Moreover, with Dirichlet boundary conditions, we have $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$ and $D(\sqrt{A})=H_0^1(\Omega)$. Then the result follows directly from Theorem 3.10. ☐

Hence, the homogeneous equation (3.27) is well posed if the initial data (u_0, u_1, u_2) are in the space $\mathcal{X} := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$. In this case, the energy functional take the explicit form

$$E(t) = \frac{1}{2}\int_\Omega (\varphi_t^2 + c^2|\nabla\varphi|^2 + \lambda c^2(\mu-\lambda)|\nabla\omega_t|^2)dx.$$

By Theorem 2.7, we have that,

$$E(t) \leqslant E(0).$$

Moreover,

$$E'(t) = -c^2(\mu-\lambda)\int_\Omega |\nabla\omega_t|^2 dx,$$

where the integral shows that the energy of the system is dissipating throughout the domain due to the presence of the internal damping of the system. We consider next the dynamic of the system defined by $T_0(t)(u_0, u_1, u_2) = (w, w_t, w_{tt})$.

Corollary 3.13. *Given $\rho > 0$ there exists $R > 0$ such that if the initial data (u_0, u_1, u_2) satisfies $||(u_0, u_1, u_2)||_{\mathcal{X}} \leqslant \rho$, then the following a-priori estimate hold*

$$||w|| + ||w_t|| + ||w_{tt}|| + ||\nabla w_t|| \leqslant \rho.$$

Let $P : \mathcal{X} \to \mathcal{X}$ the projection given by $P(u_0, u_1, u_2) = (0, u_1, u_2)$. The following result shows that the projected dynamic $T_0(t)P$ is absorbing in $H_0^1(\Omega) \times L^2(\Omega)$.

Corollary 3.14. *Given $\rho > 0$ there exists $R > 0$ such that if the initial data (u_0, u_1, u_2) satisfies $||(u_0, u_1, u_2)||_{\mathcal{X}} \leqslant \rho$, then $||T_0(t)P(u_0, u_1, u_2)||_{\mathcal{X}} \leqslant R$, for all $t \in \mathbb{R}$.*

A third consequence of Theorem 3.12 is the compactness of the trajectories of the equation 3.26 on the Hilbert space.

Corollary 3.15. *Under the hypothesis of Theorem 3.12, the trajectory,*

$$\{u(t,\cdot) : \; t \in \mathbb{R}\}$$

is a compact set in the Hilbert space $L^2(\Omega)$.

Proof. By Poincare's inequality,

$$||u(t)|| \;\leqslant c(\Omega)\; ||\nabla u(t)||,$$

we have that the above trajectory is a bounded set in the Solobev space $H^1(\Omega)$, hence, it is compact in $L^2(\Omega)$. □

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