WEIGHTED BOUNDED SOLUTIONS FOR A CLASS OF NONLINEAR FRACTIONAL EQUATIONS

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Abstract

Let T be a bounded linear operator defined on a Banach space X. We investigate the existence of solutions for a class of nonlinear fractional equation in the form

$$(*)\begin{cases} \Delta^\alpha u(n) &= Tu(n) + f(n, u(n)), \quad n \in \mathbb{N}_0, \quad 0 < \alpha \leq 1; \\ u(0) &= x, \end{cases}$$

on the vector-valued weighted sequence space

$$l_f^\infty(\mathbb{N}; X) = \left\{ x : \mathbb{N} \to X \ / \ \sup_{n \in \mathbb{N}} \frac{\|x(n)\|}{nn!} < \infty \right\}.$$

Our analysis relies on the fixed point theory and operator-theoretical methods.

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1. **Introduction**

The study of existence and qualitative properties of time-discrete solutions for fractional equations is a matter of great deal of interest in the last decade ([1, 2, 5, 6, 16, 17, 18, 19, 30] and [33]). In spite of the significant increase of research in this area, there are still many open questions regarding fractional difference equations. In particular, the study of well posedness for the semidiscretisation in time of evolution equations, involving bounded linear operators defined on Banach spaces remains largely untreated. These abstract fractional models are closely connected with numerical methods

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for partial differential equations, integro-differential equations [14, 32], evolution equations with memory [25] and lattices models [27, 28]. The theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears [8, 9].

In this paper, we study the existence and uniqueness of solutions for the non-linear abstract problem

$$\Delta^{\alpha} u(n) = Tu(n) + f(n, u(n)), \quad 0 < \alpha < 1, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $T : X \to X$ is a bounded linear operator defined on a Banach space X; $f : \mathbb{N}_0 \times X \to X$ a suitable function and

$$\Delta^{\alpha} u(n) = \int_0^{\infty} \frac{t^n}{n!} e^{-t} D_t^{\alpha} u(t)dt, \quad n \in \mathbb{N}_0, \tag{1.2}$$

corresponds to the Poisson transformation of the fractional differential operator D_t^{α} in the sense of Riemann-Liouville. See the paper [23] where this important relation has been recently highlighted.

Note that (1.1) is an abstract way to write the modeling of some classes of fractional integro-differential equations and PDE discretized only in time. For instance, the fractional nonconvolution equation

$$D_t^{\alpha} u(t,x) = \int k(x,s)u(t,s)ds + f(t, u(t,x)), \quad t \geq 0 \quad x \in \Omega \subset \mathbb{R}^N,$$

can be discretized in time, using (1.2), as the equation

$$\Delta^{\alpha} u(n,x) = \int k(x,s)u(n,s)ds + f(n, u(n,x)), \quad n \in \mathbb{N}_0, \quad x \in \Omega \subset \mathbb{R}^N,$$

where the kernel k is a complex-valued measurable function and f is a suitable forcing term. It admits the form (1.1) where

$$Tf(x) = \int k(x,s)f(s)ds,$$

is a bounded operator. In the case $\alpha = 1$, such time-discrete equations arise in a variety of contexts [4], [13]. In the continuous case, it is well known that in order to have existence of solutions, one of the main ingredients is the compactness of the operator T. However, in the discrete case, there are important practical situations where this property does not occur. For example, if we consider a boundary value problem as a second-kind boundary integral equation, the resulting integral operator T will, in general, be non-compact if the boundary of the domain of interest is only piecewise smooth. See [4, Section 7].

As a consequence of the results in this paper, we will show that when T is non-compact, we can still have existence of solutions for the fractional model (1.1). See Corollary 4.1 and Example 5.1. In this way, new insights into the analysis of integral equations with non-compact operators can be gained by considering a "tuning" in the way of a fractional order in the equation or, from another point of view, by consideration of some amount of discrete-time memory in the system.

Our method of proof uses a new technique which consists in the appropriate handling of a special sequence of bounded operators introduced in [22], called α-resolvent families. They play a central role in the representation of the solution by means of a kind of discrete variation of parameters formula. A second main ingredient is the introduction of a special vector-valued Banach space of weighted sequences $l^{\infty}_{f}(\mathbb{N};X)$, whose exceptional properties allows to prove the existence of solutions for (1.1) under certain conditions on the nonlinear term.

The outline of this paper is as follows: Section 2 is devoted to preliminaries, recalling the definition of the fractional difference operator that we will use and that seems to be more convenient for our purposes. In this line of ideas, we note the recent paper [10], where it was proved that many concepts of fractional differences currently used in the literature are simply related by translation. We remark that our definition is at the basis of this equivalence.

In Section 3 we recall the concept of α-resolvent sequences, denoted by $S_{\alpha}(n)$, that was introduced in [22]. This α-resolvent families have an interesting characterization [22, Theorem 3.4] connected to the concept of Mittag-Leffler operator sequence. This tool allows to obtain an explicit representation of the solution for the fractional linear difference equation associated to (1.1) with initial value $u(0) = u_0$, namely (Theorem 3.1)

$$u(n) = S_{\alpha}(n)u_0 + (S_{\alpha} * f)(n-1), \quad n \in \mathbb{N}.$$

In Section 4 we study the fully nonlinear problem (1.1). First, an equivalent formulation of the solution is motivated by the representation of the solution in the linear case (Theorem 4.1). Next, we introduce the vector-valued Banach space of weighted sequences

$$l^{\infty}_{f}(\mathbb{N};X) = \left\{ \xi : \mathbb{N} \to X / \sup_{n\in\mathbb{N}} \frac{\|\xi(n)\|}{nn!} < \infty \right\},$$

that we called the factorial number system space (fns-space). This space will play a central role in the development of this section. The main ingredient for the success of our analysis is the important observation that the special sequence $nn!$ - that represents the factorial expression of a positive integer - provides a very suitable weight in order to find existence of solutions for (1.1) in the vector-valued space $l^{\infty}_{f}(\mathbb{N};X)$. See the exceptional properties of this sequence in the identities (4.2) below. In this way, our analysis gives mild restrictions on the ingredients of the problem. In passing, we obtain information about the growth rate of $u(n)$ as $n \to \infty$. We give two positive results in this direction. See Theorem 4.1 and Theorem 4.3. They are based on the Banach fixed point theorem and the Leray-Schauder alternative theorem, respectively. Of particular interest is the following result: Suppose $\|T\| < \alpha^\alpha(1-\alpha)^{1-\alpha}$ and $f : \mathbb{N}_0 \times X \to X$ verify the following hypothesis:

(F) $f(n,0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists a positive sequence $a \in \ell^1(\mathbb{N}_0)$ and constants $c \geq 0, b > 0$ such that $\|f(k,x)\| \leq a(k)(c\|x\| + b)$ for all $k \in \mathbb{N}_0$ and $x \in X$.

(L) The function f satisfies a Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, with Lipschitz constant $L_f < \frac{18}{5}(\alpha^\alpha(1-\alpha)^{1-\alpha} - \|T\|)$.

Then the problem (1.1) with initial condition $u(0) = 0$ has an unique solution in $l^\infty_f(\mathbb{N}; X)$. Finally, Section 5 provides concrete examples and applications of our general results. We remark our Example 5.1, where the dependence of the nonlinearity f, the operator T and the fractional order is highlighted.

2. Preliminaries

In this section, we provide the necessary preliminaries on fractional differences, needed in the forthcoming sections. Additional information on this topics can be found in the monograph [3].

Let $\mathbb{N}_0$ be the set of non-negative integer numbers and let X be a complex Banach space. We denote by $s(\mathbb{N}_0; X)$ the vectorial space consisting of all vector-valued sequences $u : \mathbb{N}_0 \to X$. In this context, the forward Euler operator $\Delta : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ is defined by

$$\Delta u(n) := u(n+1) - u(n), \quad n \in \mathbb{N}_0.$$

and for $m \in \mathbb{N}$, we define recursively the m-th order forward difference operator $\Delta^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta^m u(n) := \Delta^{m-1} \circ \Delta = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} u(n+j), \quad n \in \mathbb{N}_0.$$

We also denote $\Delta^0 \equiv I$, where $I : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ is the identity operator.

Recall that the finite convolution $*$ of two sequences $u(n), v(n) \in s(\mathbb{N}_0; X)$ is defined by

$$(u * v)(n) := \sum_{j=0}^{n} u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

Definition 2.1. ([22, Definition 2.5]) Let $\alpha > 0$ be given and $u : \mathbb{N}_0 \to X$. We define the fractional sum of order α as follows

$$\Delta^{-\alpha} f(n) = \sum_{k=0}^{n} k^{\alpha}(n-k)u(k) = (k^{\alpha} * u)(n), \quad n \in \mathbb{N}_0, \tag{2.1}$$

where

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0.$$

This definition of fractional sum corresponds to a particular case of the definition proposed by Atici and Eloe [7] in 2009. The importance of this kernel began only recently to be highlighted [23], [29]. For an explicit development, see [22] and references therein. One of the reasons to choose this operator is because their flexibility to be handled by means of z-transform methods. We recall that the *z-transform* of a vector-valued sequence $u \in s(\mathbb{N}_0; X)$, is defined by

$$\widetilde{u}(z) := \sum_{j=0}^{\infty} z^{-j} u(j),$$

where z is a complex number. Note that convergence of the series is given for $|z| > R$ with R sufficiently large. The fractional sum has a better behavior for mathematical analysis when we ask, for example, for definitions of fractional sums and differences on subspaces of $s(\mathbb{N}_0; X)$ like e.g. l_p spaces. We notice that, recently, this approach by means of the z-transform has been followed by other authors, see [11, 12]. It is also interesting to note that it was recently proved that different notions of fractional sum, existing in the current literature, are equivalent with Definition 2.1 modulo translation, see [10].

The next concept is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [26] and [5]. In other words, to a given vector-valued sequence, first fractional summation and then integer difference are applied.

Definition 2.2. ([22, Definition 2.7], [23]) The fractional difference operator of order $\alpha > 0$ (in the sense of Riemann-Liouville) is defined by

$$\Delta^{\alpha} u(n) := \Delta^{m} \circ \Delta^{-(m-\alpha)} u(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

We observe that the above definition corresponds to sampling by means of the Poisson distribution for the well-known fractional differential operator in the sense of Riemann-Liouville. We remark that such strong and important relation has been only freshly discovered [23].

3. α-resolvent sequences

In this section, we recall the operator theoretical method introduced in [22] to study the linear fractional difference equation

$$\Delta^{\alpha}u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}, \tag{3.1}$$

with initial condition $u(0) = x \in X$ and for $T \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the set of all bounded linear operators on a Banach space X.

Definition 3.1. ([22, Definition 3.1]) Let T be bounded operator defined on a Banach space X and $\alpha > 0$. We call T the generator of an α-resolvent sequence if there exists a sequence of bounded and linear operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ that satisfies the following properties

(i) $S_\alpha(0) = I$
(ii) $S_\alpha(n+1) = k^\alpha(n+1)I + T(k^\alpha * S_\alpha)(n)$ for all $n \in \mathbb{N}_0$.

In this case, $S_\alpha(n)$ is called the α-resolvent sequence generated by T.

Notice that if T generates an α-resolvent family, then it is unique ([22, Lemma 3.2]).

Example 3.1. In case $\alpha = 1$ we have the recurrence relation

$$S_1(0) = I \quad \text{and} \quad S_1(n) = I + T\sum_{j=0}^{n-1} S_1(j) = (I+T)^n, \quad n \in \mathbb{N}_0.$$

In the general case, an explicit representation of α-resolvent families is given in [22, Theorem 3.4] and reads as follows:

$$S_\alpha(n) = \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma((j+1)\alpha)} T^j, \quad n \in \mathbb{N}_0. \tag{3.2}$$

Moreover, the sequence of operators $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ allows to obtain an explicit representation for the solution of equation (3.1) by means of a kind of variation of parameters formula. More precisely, we have the following theorem.

Theorem 3.1. ([22, Theorem 3.7]) *Let $0 < \alpha \leq 1$, $T \in \mathcal{B}(X)$ and $f : \mathbb{N} \to X$ be given. The unique solution of* (3.1) *with initial condition $u(0) = u_0$ can be represented by*

$$u(n) = S_\alpha(n)u_0 + (S_\alpha * f)(n-1), \quad n \in \mathbb{N}.$$

REMARK 3.1. In the border case $\alpha = 1$ we obtain the representation proved in [3, Proposition 1.3.1].

Using the property of the z-transform on the convolution, we obtain formally from the definition

$$\widetilde{S}_\alpha(z) = z\left(\frac{z}{\widetilde{k^\alpha}(z)} - T\right)^{-1}$$

whenever $\dfrac{z}{\widetilde{k^\alpha}(z)} \in \rho(T)$, where $\rho(T)$ denotes the resolvent set of T. Since

$$\widetilde{k^\alpha}(z) = \frac{z^\alpha}{(z-1)^\alpha},$$

for $|z| > 1$, we have

$$\widetilde{S}_\alpha(z) = z((z-1)^\alpha z^{1-\alpha} - T)^{-1},$$

whenever the right hand side exists. Then, applying the inverse z-transform, we obtain

$$S_\alpha(n) = \frac{1}{2\pi i}\int_C z^n((z-1)^\alpha z^{1-\alpha} - T)^{-1} dz, \tag{3.3}$$

where C is a circle, centered at the origin of the complex plane, that encloses all spectral values of $(z-1)^\alpha z^{1-\alpha} - T$.

The main result in this section is the following theorem that gives important information concerning qualitative properties of α-resolvent sequences.

THEOREM 3.2. *Let $0 < \alpha \le 1$, $T \in \mathcal{B}(X)$ and $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ the α-resolvent sequence generated by T. Then the following properties hold:*

(i) *If X is an ordered Banach space and $T \ge 0$ then $S_\alpha(n) \ge 0$ for all $n \in \mathbb{N}_0$.*
(ii) *Let X be an ordered Banach space and $T \ge 0$. If $\alpha < \beta$, then $S_\alpha(n) < S_\beta(n)$ for all $n \in \mathbb{N}_0$.*
(iii) *If $\|T\| < \alpha^\alpha(1-\alpha)^{1-\alpha}$ then $\|S_\alpha(n)\| \to 0$ as $n \to \infty$ and the estimate*

$$\sup_{n\in\mathbb{N}_0} \|S_\alpha(n)\| \le \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha} - \|T\|}$$

holds.

P r o o f. (i) follows from (3.2) directly. To prove (ii), we note that for $k \in \mathbb{N}$ and $\lambda_1 < \lambda_2 \in \mathbb{R}$,

$$\begin{aligned}\frac{\Gamma(k+\lambda_1)}{\Gamma(\lambda_1)} &= (k-1+\lambda_1)(k-2+\lambda_1)\cdots\lambda_1 \\ &< (k-1+\lambda_2)(k-2+\lambda_2)\cdots\lambda_2 = \frac{\Gamma(k+\lambda_2)}{\Gamma(\lambda_2)}.\end{aligned}$$

Then,

$$\begin{aligned}S_\alpha(n) &= \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma((j+1)\alpha)} T^j \\ &< \sum_{j=0}^{n} \frac{\Gamma(n-j+(j+1)\beta)}{\Gamma(n-j+1)\Gamma((j+1)\beta)} T^j = S_\beta(n).\end{aligned}$$

(iii) Let $0 < R < 1$. From (3.3) we have

$$\begin{aligned}S_\alpha(n) &= \frac{1}{2\pi i}\int_{|z|=R} z^n((z-1)^\alpha z^{1-\alpha} - T)^{-1} dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} R^n e^{int}((Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t} - T)^{-1} Rie^{it} dt\end{aligned}$$

and then

$$\|S_\alpha(n)\| \leq \frac{1}{2\pi}\int_0^{2\pi} R^n \|((Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t} - T)^{-1}\| R dt.$$

Define $f(R) = (1-R)^\alpha R^{1-\alpha}$, $R \in (0,1)$. It is not difficult to see that the minimum value of f is attained at $R = 1-\alpha$ and that $f(1-\alpha) = \alpha^\alpha(1-\alpha)^{1-\alpha}$. Therefore

$$\begin{aligned}\|T\| &< \alpha^\alpha(1-\alpha)^{1-\alpha} \leq (1-R)^\alpha R^{1-\alpha} \\ &\leq |(Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t}|, \quad t \in (0,2\pi),\ R \in (0,1),\end{aligned}$$

and then by [21, Theorem 7.3-4, p.377] we have that $(Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t} - T$ is invertible for each $t \in (0,2\pi), R \in (0,1)$ and

$$((Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t} - T)^{-1} = \sum_{j=0}^{\infty} \frac{T^j}{((Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t})^{j+1}}.$$

We deduce that

$$\begin{aligned}\|((Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t} - T)^{-1}\| &\leq \frac{1}{|(Re^{it}-1)^\alpha R^{1-\alpha} e^{i(1-\alpha)t}| - \|T\|} \\ &\leq \frac{1}{(1-R)^\alpha R^{1-\alpha} - \|T\|}.\end{aligned}$$

Consequently,

$$\|S_\alpha(n)\| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{R^{n+1}}{(1-R)^\alpha R^{1-\alpha} - \|T\|} dt$$
$$\leq \frac{R^{n+1}}{\alpha^\alpha (1-\alpha)^{1-\alpha} - \|T\|}, \quad R \in (0,1).$$

It proves that $\|S_\alpha(n)\| \to 0$ as $n \to \infty$ and the inequality

$$\sup_{n\in\mathbb{N}_0} \|S_\alpha(n)\| \leq \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha} - \|T\|},$$

is valid for all $0 < \alpha < 1$. The proof is complete. □

The following picture illustrates the function $\alpha \to \alpha^\alpha (1-\alpha)^{1-\alpha}$

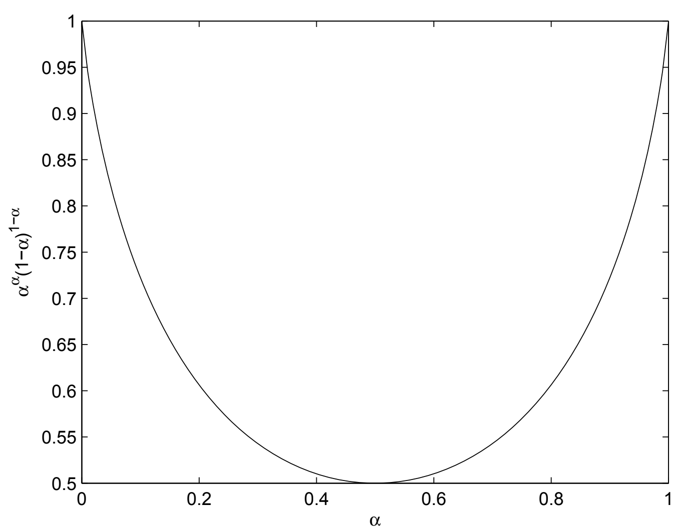

FIGURE 1. $\alpha \to \alpha^\alpha (1-\alpha)^{1-\alpha}$

4. **Non-linear fractional difference equations on Banach spaces**

Let T be a bounded linear operator defined on a Banach space X. In this section we study the problem

$$\begin{cases} \Delta^\alpha u(n) &= Tu(n) + f(n, u(n)), \quad n \in \mathbb{N}_0,\ 0 < \alpha \leq 1; \\ u(0) &= 0. \end{cases} \tag{4.1}$$

REMARK 4.1. Note that $u(0) = 0$ implies $u(1) = f(n, 0)$, for all $n \in \mathbb{N}_0$. Indeed, by definition we have $\Delta^\alpha u(n) = (k^{1-\alpha} * u)(n+1) - (k^{1-\alpha} * u)(n)$

and hence from (4.1) we obtain

$$\begin{aligned} f(n,0) = \Delta^\alpha u(0) &= (k^{1-\alpha} * u)(1) - (k^{1-\alpha} * u)(0) \\ &= k^{1-\alpha}(1)u(0) + k^{1-\alpha}(0)u(1) - k^{1-\alpha}(0)u(0) = u(1). \end{aligned}$$

DEFINITION 4.1. Let $T \in \mathcal{B}(X)$, $f : \mathbb{N}_0 \times X \to X$ and $0 < \alpha \leq 1$ be given. By a solution of (4.1) we understand a sequence $u : \mathbb{N}_0 \to X$ that satisfy (4.1) for all $n \in \mathbb{N}_0$.

The following result is a consequence of Theorem 3.1. It gives an equivalent representation of the solution of (4.1) in terms of the family of bounded operators $S_\alpha(n)$ generated by the operator T. It is the key tool in order to analyze existence of solutions for nonlinear equations.

THEOREM 4.1. *Let $T \in \mathcal{B}(X)$, $f : \mathbb{N}_0 \times X \to X$ and $0 < \alpha \leq 1$ be given. Let $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ be the α-resolvent sequence generated by T. The following assertions are equivalent:*

(i) *u is a solution of (4.1);*

(ii) *$u(0) = 0$ and $u(n) = \sum_{k=0}^{n-1} S_\alpha(n-1-k)f(k,u(k)),\ n \in \mathbb{N}$.*

P r o o f. $(ii) \implies (i)$. It follows from (the proof of) Theorem 3.1. $(i) \implies (ii)$. By hypothesis, we obtain

$$(S_\alpha * f)(n-1) = (S_\alpha * \Delta^\alpha u)(n-1) - T(S_\alpha * u)(n-1).$$

By [22, Lemma 3.6] we have the identities

$$(S_\alpha * \Delta^\alpha u)(n-1) = \Delta^\alpha(S_\alpha * u)(n-1) - S_\alpha(n)u(0),$$

and

$$\Delta^\alpha(S_\alpha * u)(n-1) = (\Delta^\alpha S_\alpha * u)(n-1) + S_\alpha(0)u(n).$$

Taking into account that $S_\alpha(0) = I$ and $u(0) = 0$ we conclude that

$$(S_\alpha * f)(n-1) = (\Delta^\alpha S_\alpha * u)(n-1) + u(n) - T(S_\alpha * u)(n-1).$$

By Theorem 3.1 we know that $\Delta^\alpha S_\alpha(n) = TS_\alpha(n)$. Then, we arrive from the above identity to (ii). The proof is complete. □

We notice that there is a rich literature on qualitative properties of difference equations of Volterra type, as the given in (ii) above. However, most of them are referred to the scalar case or finite dimensional case, i.e. when $S_\alpha(n)$ is scalar-valued or matrix-valued, instead of operator-valued as given here. For instance, in [15] the author give a survey of some of the fundamental results on stability and asymptotic properties in the scalar-valued and matrix-valued case, and states some open problems. See also [20] for another survey containing related results on stability and the use of resolvents.

REMARK 4.2. Observe that Theorem 4.1 is valid only in the range $0 < \alpha \leq 1$. This is due to the fact that the representation given in Theorem 3.1 is only true for such values of α. In the case of the range $1 < \alpha \leq 2$ a representation of the solution, solely in terms of certain family of operators, is given in [24]. Note that in such case two initial conditions for the problem (4.1) are needed.

DEFINITION 4.2. We call the factorial number system space *(fns-space)* the vector-valued weighted space defined by

$$l_f^\infty(\mathbb{N};X) = \left\{ \xi : \mathbb{N} \to X / \sup_{n\in\mathbb{N}} \frac{\|\xi(n)\|}{nn!} < \infty \right\},$$

and endowed with the norm $\|\xi\|_f = \sup_{n\in\mathbb{N}} \dfrac{\|\xi(n)\|}{nn!}$.

Note that the sequence $\dfrac{1}{nn!}\displaystyle\sum_{k=0}^{n-1} kk!$ is decreasing for $n \geq 3$ and by [31, formula 33 p.598] it is verified that

$$\sup_{n\in\mathbb{N}} \frac{1}{nn!}\sum_{k=0}^{n-1} kk! = \frac{5}{18} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{nn!}\sum_{k=0}^{n-1} kk! = \lim_{n\to\infty} \frac{n!-1}{nn!} = 0. \tag{4.2}$$

The following is our first positive result on existence of solutions for the problem (4.1). It uses a Lipschitz type condition.

THEOREM 4.2. *Let $T \in \mathcal{B}(X)$ be the generator of a bounded α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$, for $0 < \alpha \leq 1$. Let $f : \mathbb{N}_0 \times X \to X$ be given and verifying the following hypotheses:*

- (F) *$f(n,0) \neq 0$ for all $n \in \mathbb{N}_0$ and there exists a positive sequence $a \in \ell^1(\mathbb{N}_0)$ and constants $c \geq 0, b > 0$ such that $\|f(k,x)\| \leq a(k)(c\|x\| + b)$ for all $k \in \mathbb{N}_0$ and $x \in X$.*
- (L) *The function f satisfies a Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, that is, there exists a constant $L > 0$ such that $\|f(k,x) - f(k,y)\| \leq L\|x-y\|$, for all $x, y \in X$, $k \in \mathbb{N}_0$, with $L < \dfrac{18}{5\|S_\alpha\|_\infty}$.*

Then the problem (4.1) has an unique solution in $l_f^\infty(\mathbb{N};X)$.

P r o o f. Let us define the operator $G : l^\infty_f(\mathbb{N};X) \to l^\infty_f(\mathbb{N};X)$ given by

$$Gu(n) = \sum_{k=0}^{n-1} S_\alpha(n-1-k) f(k,u(k)), \quad n \geq 1.$$

First, we show that G is well defined: Let $u \in l^\infty_f(\mathbb{N};X)$ be given. By using the assumption (F) and the boundedness of $S_\alpha(n)$ we have

$$\begin{aligned}
\|Gu(n)\| &\leq \sum_{k=0}^{n-1} \|S_\alpha(n-1-k)\| \|f(k,u(k))\| \\
&\leq \sum_{k=0}^{n-1} \|S_\alpha(n-1-k)\| a(k)[c\|u(k)\| + b] \\
&\leq c\|S_\alpha\|_\infty \|a\|_\infty \sum_{k=0}^{n-1} \|u(k)\| + b\|S_\alpha\|_\infty \sum_{k=0}^{n-1} a(k) \\
&\leq c\|S_\alpha\|_\infty \|a\|_\infty \|u\|_f \sum_{k=0}^{n-1} kk! + b\|S_\alpha\|_\infty \|a\|_1,
\end{aligned}$$

for each $n \in \mathbb{N}$. Hence,

$$\frac{\|Gu(n)\|}{nn!} \leq c\|S_\alpha\|_\infty \|a\|_\infty \|u\|_f \frac{1}{nn!} \sum_{k=0}^{n-1} kk! + b\|S_\alpha\|_\infty \|a\|_1.$$

It proves, in view of (4.2), that $Gu \in l^\infty_f(\mathbb{N};X)$. We next prove that G is a contraction on $l^\infty_f(\mathbb{N};X)$. Indeed, let $u, v \in l^\infty_f(\mathbb{N};X)$ be given. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|Gu(n) - Gv(n)\| &\leq \sum_{k=0}^{n-1} \|S_\alpha(n-1-k)\| \|f(k,u(k)) - f(k,v(k))\| \\
&\leq \|S_\alpha\|_\infty \sum_{k=0}^{n-1} \|f(k,u(k)) - f(k,v(k))\| \\
&\leq \|S_\alpha\|_\infty \sum_{k=0}^{n-1} L\|u(k) - v(k)\| \\
&\leq \|S_\alpha\|_\infty L \|u-v\|_f \sum_{k=0}^{n-1} kk!,
\end{aligned}$$

where we have used the assumption (L). Therefore

$$\frac{\|Gu(n) - Gv(n)\|}{nn!} \leq \|S_\alpha\|_\infty L \|u-v\|_f \frac{1}{nn!} \sum_{k=0}^{n-1} kk!,$$

and consequently,

$$\|Gu - Gv\|_f \leq \|S_\alpha\|_\infty \frac{5}{18} L \|u - v\|_f,$$

where we have used (4.2). Then, G has a unique fixed point in $l^\infty_f(\mathbb{N}; X)$, by the Banach fixed point theorem. $\square$

The next corollary gives an explicit bound on the Lipschitz constant. Note that such Lipschitz constant vary with α.

COROLLARY 4.1. *Let $T \in \mathcal{B}(X)$ and $0 < \alpha < 1$. Suppose $\|T\| < \alpha^\alpha(1-\alpha)^{1-\alpha}$ and let $f : \mathbb{N}_0 \times X \to X$ be given satisfying condition (F) and the following hypothesis:*

> *The function f satisfies a Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, with Lipschitz constant $L < \frac{18}{5}(\alpha^\alpha(1-\alpha)^{1-\alpha} - \|T\|)$.*

Then the problem (4.1) has an unique solution in $l^\infty_f(\mathbb{N}; X)$.

The following lemma provides a necessary tool for the use of the Schauder's fixed point theorem, needed in the second main result on existence of solutions.

LEMMA 4.1. *Let $U \subset l^\infty_f(\mathbb{N}; X)$ such that:*

(a) *The set $H_n(U) = \left\{ \frac{u(n)}{nn!} : u \in U \right\}$ is relatively compact in X, for all $n \in \mathbb{N}$.*
(b) $\displaystyle\lim_{n\to\infty} \frac{1}{nn!} \sup_{u \in U} \|u(n)\| = 0$, *that is, for each $\varepsilon > 0$, there are $N > 0$ such that $\dfrac{\|u(n)\|}{nn!} < \varepsilon$, for each $n \geq N$ and for all $u \in U$.*

Then U is relatively compact in $l^\infty_f(\mathbb{N}; X)$.

P r o o f. Let $\{u_m\}_m$ be a sequence in U, then by (a) for $n \in \mathbb{N}$ there is a convergent subsequence $\{u_{m_j}\}_j \subset \{u_m\}_m$ such that $\displaystyle\lim_{j\to\infty} \frac{u_{m_j}(n)}{nn!} = a(n)$, that is, for each $\varepsilon > 0$ there exists $N(n, \varepsilon) > 0$ such that $\|\frac{u_{m_j}(n)}{nn!} - a(n)\| < \varepsilon$ for all $j \geq N(n, \varepsilon)$. Let $\varepsilon > 0$ and N the value of the assumption (b). If we consider $N^* := \min_{0\le n<N} N(n,\varepsilon)$, then for $j,k \ge N^*$ we have

$$\sup_{0\le n<N} \frac{\|u_{m_j}(n)-u_{m_k}(n)\|}{nn!} \le \sup_{0\le n<N} \|\frac{u_{m_j}(n)}{nn!}-a(n)\| + \sup_{0\le n<N} \|\frac{u_{m_k}(n)}{nn!}-a(n)\| < \varepsilon/2+\varepsilon/2=\varepsilon,$$

and also

$$\sup_{n\ge N} \frac{\|u_{m_j}(n)-u_{m_k}(n)\|}{nn!} \le \sup_{n\ge N} \frac{\|u_{m_j}(n)\|}{nn!} + \sup_{n\ge N} \frac{\|u_{m_k}(n)\|}{nn!} < \varepsilon/2+\varepsilon/2=\varepsilon.$$

Consequently,

$$\|u_{m_j}-u_{m_k}\|_! = \sup_{n\in\mathbb{N}} \frac{\|u_{m_j}(n)-u_{m_k}(n)\|}{nn!} < \varepsilon,$$

therefore $\{u_{m_j}\}_j$ is a Cauchy subsequence in $l^\infty_f(\mathbb{N};X)$ which finish the proof of the lemma. $\square$

For $f : \mathbb{N}\times X \to X$ we recall that the Nemytskii operator $\mathcal{N}_f : l^\infty_f(\mathbb{N};X) \to l^\infty_f(\mathbb{N};X)$ is defined by

$$\mathcal{N}_f(u)(n) := f(n,u(n)), \quad n \in \mathbb{N}.$$

The next theorem is the second main result for this section. It gives one useful criteria for the existence of solutions without use of Lipschitz type conditions.

THEOREM 4.3. *Let $T \in \mathcal{B}(X)$ be a compact operator, generator of a bounded α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ for $0<\alpha\le 1$ and $f : \mathbb{N}_0 \times X \to X$ be a function. Suppose that the condition (F) is satisfied and the Nemytskii operator is continuous in $l^\infty_f(\mathbb{N};X)$. Then the problem* (4.1) *has a solution in $l^\infty_f(\mathbb{N};X)$.*

P r o o f. By the hypothesis, T is the generator of an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ where $S_\alpha(n)$ can be described by formula (3.2). Let us define the operator $G : l^\infty_f(\mathbb{N};X) \to l^\infty_f(\mathbb{N};X)$ by

$$Gu(n) = \sum_{k=0}^{n-1} S_\alpha(n-1-k) f(k,u(k)), \quad n\in\mathbb{N}.$$

To prove that G has a fixed point in $l^\infty_f(\mathbb{N},X)$, we will use Leray-Schauder alternative theorem. We verify that the conditions of the theorem are satisfied:

- G is well defined: It follows from condition (F) and was proved in the first part of the proof of Theorem 4.2.

- G is continuous: Let $\varepsilon > 0$ and $u, v \in l_f^\infty(\mathbb{N}; X)$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\|Gu(n) - Gv(n)\| &\leq \sum_{k=0}^{n-1} \|S_\alpha(n-1-k)\| \|f(k, u(k)) - f(k, v(k))\| \\
&\leq \|S_\alpha\|_\infty \sum_{k=0}^{n-1} \|f(k, u(k)) - f(k, v(k))\| \\
&\leq \|S_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_f \sum_{k=0}^{n-1} kk!.
\end{aligned}$$

  Therefore

$$\frac{\|Gu(n) - Gv(n)\|}{nn!} \leq \|S_\alpha\|_\infty \|\mathcal{N}_f(u) - \mathcal{N}_f(v)\|_f \frac{1}{nn!} \sum_{k=0}^{n-1} kk!.$$

  Hence, by the assumption (ii) we obtain $\|Gu - Gv\|_f < \varepsilon$.
- G is compact: For $R > 0$ given, let $B_R(l_f^\infty(\mathbb{N}; X) := \{w \in l_f^\infty(\mathbb{N}; X) : \|w\|_f < R\}$. To prove that $V := G(B_R(l_f^\infty(\mathbb{N}; X)))$ is relatively compact, we will use Lemma 4.1. We check that the conditions in this lemma are satisfied:
  (a) Let $u \in B_R(l_f^\infty(\mathbb{N}; X))$ and $v = Gu$. We have

$$\begin{aligned}
v(n) = Gu(n) &= \sum_{k=0}^{n-1} S_\alpha(n-1-k) f(k, u(k)) \\
&= \sum_{k=0}^{n-1} S_\alpha(k) f(n-1-k, u(n-1-k)),
\end{aligned}$$

  and then,

$$\frac{v(n)}{nn!} = \frac{1}{n!} \left(\frac{1}{n} \sum_{k=0}^{n-1} S_\alpha(k) f(n-1-k, u(n-1-k)) \right).$$

  Therefore $\dfrac{v(n)}{nn!} \in \dfrac{1}{n!} co(K_n)$, where $co(K_n)$ denotes the convex hull of K_n for the set

$$K_n = \bigcup_{k=0}^{n-1} \{S_\alpha(k) f(\xi, x) : \xi \in \{0, 1, 2, \ldots, n-1\}, \|x\| \leq R\}, \quad n \in \mathbb{N}.$$

  Since T is compact, for all $0 < \alpha \leq 1$ the sequence of operators $\{S_\alpha(n)\}_{n \in \mathbb{N}_0}$ generated by T is compact. See formula (3.2). Also, for all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set $\{f(k, x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is bounded because from condition (F) we have

$\|f(k,x)\| \leq a(k)[c\|x\|+b] \leq \|a\|_\infty[c\sigma+b]$ for all $0 \leq k \leq a$ and $\|x\| \leq \sigma$. Consequently, the set $\{S_\alpha(n)f(k,x) : 0 \leq k \leq a, \|x\| \leq \sigma\}$ is relatively compact in X for all $n \in \mathbb{N}_0$. Then it follows that each set K_n is relatively compact. From the inclusions $H_n(V) = \left\{\dfrac{v(n)}{nn!} : v \in V\right\} \subseteq \dfrac{1}{n!}co(K_n) \subseteq \dfrac{1}{n!}co(\overline{K_n})$, we conclude that the set $H_n(V)$ is relatively compact in X, for all $n \in \mathbb{N}$.

(b) Let $u \in B_R(l_f^\infty(\mathbb{N};X))$ and $v = Gu$. Using condition (F), for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{\|v(n)\|}{nn!} &\leq \frac{1}{nn!}\sum_{k=0}^{n-1}\|S_\alpha(n-1-k)\|\|f(k,u(k))\| \\
&\leq \frac{1}{nn!}\sum_{k=0}^{n-1}\|S_\alpha(n-1-k)\|a(k)[c\|u(k)\|+b] \\
&\leq c\|S_\alpha\|_\infty\|a\|_\infty\|u\|_f\frac{1}{nn!}\sum_{k=0}^{n-1}kk! + \frac{1}{nn!}b\|S_\alpha\|_\infty\|a\|_1 \\
&\leq c\|S_\alpha\|_\infty\|a\|_\infty R\frac{1}{nn!}\sum_{k=0}^{n-1}kk! + \frac{1}{nn!}b\|S_\alpha\|_\infty\|a\|_1,
\end{aligned}$$

then $\displaystyle\lim_{n\to\infty}\frac{\|v(n)\|}{nn!} = 0$ independently of $u \in B_R(l_f^\infty(\mathbb{N};X))$.

Therefore, $V = G(B_R(l_f^\infty(\mathbb{N};X)))$ is relatively compact in $l_f^\infty(\mathbb{N};X)$ by Lemma 4.1 and we conclude that G is a compact operator.

- The set $U := \{u \in l_f^\infty(\mathbb{N};X) : u = \gamma Gu, \gamma \in (0,1)\}$ is bounded: In fact, let us consider $u \in l_f^\infty(\mathbb{N};X)$ such that $u = \gamma Gu$, $\gamma \in (0,1)$. Again by condition (F),

$$\begin{aligned}
\|u(n)\| = \|\gamma Gu(n)\| &\leq \sum_{k=0}^{n-1}\|S_\alpha(n-1-k)\|\|f(k,u(k))\| \\
&\leq c\|S_\alpha\|_\infty\|a\|_\infty\|u\|_f\sum_{k=0}^{n-1}kk! + b\|S_\alpha\|_\infty\|a\|_1.
\end{aligned}$$

Then for each $n \in \mathbb{N}$ we have

$$\frac{\|u(n)\|}{nn!} \leq c\|S_\alpha\|_\infty\|a\|_\infty\|u\|_f\frac{1}{nn!}\sum_{k=0}^{n-1}kk! + b\|S_\alpha\|\|a\|_1.$$

In view of (4.2), we deduce that U is a bounded set in $l_f^\infty(\mathbb{N};X)$.

Finally, by using the Leray-Schauder alternative theorem, we conclude that G has a fixed point $u \in l^\infty_f(\mathbb{N}; X)$. $\square$

5. **Examples and applications**

In this section, we provide several concrete examples and applications of the abstract results developed in the previous sections.

Example 5.1. Let $P : C[0,1] \to C[0,1]$ be the integral operator given by

$$Pf(x) = \int_0^1 k(x,s)f(s)ds.$$

Suppose that

(H1) P is bounded, non compact and $\|P\| = 1$.

For a concrete example of a kernel $k(x,s)$ such that T is non compact, see [4, Section 7].

We study the existence of solutions for the problem

$$\begin{cases} \Delta^\alpha u(n,x) = \dfrac{1}{10}\displaystyle\int_0^1 k(x,s)u(n,s)ds + \dfrac{1+u(n,x)}{1+\sup\limits_{x\in[0,1]}|u(n,x)|}, \\ u(0,x) = 0, \end{cases} \tag{5.1}$$

for $n \in \mathbb{N}_0$, $x \in [0,1]$ and $0 < \alpha < 1$. Define

$$Tf(x) = \frac{1}{10}\int_0^1 k(x,s)f(s)ds.$$

We will apply Corollary 4.1. We have $\|T\| = \frac{1}{10} < \alpha^\alpha(1-\alpha)^{1-\alpha}$. Then, by Theorem 3.2, the operator T generates a bounded α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ on $C[0,1]$ given by

$$\begin{aligned} S_\alpha(n)f(x) &= \sum_{j=0}^n \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma((j+1)\alpha)} T^j f(x) \\ &= \sum_{j=0}^n \frac{\Gamma(n-j+(j+1)\alpha)}{\Gamma(n-j+1)\Gamma((j+1)\alpha)} \int_0^1 k^j(x,s)f(s)ds, \quad n \in \mathbb{N}_0. \end{aligned}$$

where $k^0(t,s) := \delta_t(s)$, the Dirac delta concentrated at t, $k^1(t,s) := k(t,s)$ and

$$k^j(t,s) := \int_0^1 k^{j-1}(t,\tau)k(\tau,s)d\tau, \quad j = 2,3,...$$

Define $v(n)(x) := u(n,x)$ and $f : \mathbb{N}_0 \times C[0,1] \to C[0,1]$ by $f(n,v) := \dfrac{\mathbf{1}+v}{1+\|v\|_\infty}$, where $\mathbf{1}(t) \equiv 1$. Then the problem (5.1) can be rewritten as

$$\Delta^\alpha v(n) = Tv(n) + f(n, v(n)), \quad n \in \mathbb{N}_0.$$

with initial condition $u(0) = 0$. On the one hand, we observe that condition (F) is satisfied by f defined above with $a(k) = \dfrac{1}{2^k}$, $c = 0$ and $b = 1$. On the other hand, note that for $u_1, u_2 \in l_f^\infty(\mathbb{N}; C[0,1])$ and each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
\|f(n, u_1(n)) - f(n, u_2(n))\| &\leq \left\| \frac{1 + u_1(n)}{1 + \|u_1(n)\|} - \frac{1 + u_2(n)}{1 + \|u_2(n)\|} \right\| \\
&= \left\| \frac{(u_1(n) - u_2(n))(1 + \|u_2(n)\|) + (1 + u_2(n))(\|u_2(n)\| - \|u_1(n)\|)\|}{(1 + \|u_1(n)\|)(1 + \|u_2(n)\|)} \right\| \\
&\leq \frac{\|u_1(n) - u_2(n)\|}{1 + \|u_1(n)\|} + \frac{(1 + \|u_2(n)\|)\|u_1(n) - u_2(n)\|}{(1 + \|u_1(n)\|)(1 + \|u_2(n)\|)} \\
&\leq 2\|u_1(n) - u_2(n)\|.
\end{aligned}$$

Therefore f is a Lipschitz function with constant $L = 2$. Since $\|T\| = \frac{1}{10}$, we notice that the condition (L) in Corollary 4.1 is satisfied if and only if

$$2 < \frac{18}{5}\Big[\alpha^\alpha (1-\alpha)^{1-\alpha} - \frac{1}{10}\Big]$$

if and only if

$$0 < \alpha < \frac{3}{2} \quad \text{or} \quad \frac{17}{20} < \alpha < 1. \tag{5.2}$$

For this and Corollary 4.1 we conclude that for all fractional order $\alpha > 0$ in the range of values indicated in (5.2) we have that the fractional integro-difference equation of Fredholm type

$$\Delta^\alpha u(n,x) = \frac{1}{10}\int_0^1 k(x,s)u(n,s)ds + f(n, u(n,x)), \quad n \in \mathbb{N}_0, \ x \in [0,1]$$

with initial condition $u(0,x) = 0$ admits an unique solution in the space

$$l_f^\infty(\mathbb{N}; C[0,1]) = \left\{ \xi : \mathbb{N} \to C[0,1] \,/\, \sup_{n\in\mathbb{N}} \frac{\|\xi(n)\|_\infty}{nn!} < \infty \right\}.$$

Alternatively, suppose that

(H2) P is bounded, compact and $\|P\| = 1$.

Then the conditions of Theorem 4.3 applied to the problem (5.1) are satisfied, without restriction on the fractional values $\alpha > 0$. Consequently, the problem (5.1) has an unique solution $u \in l_f^\infty(\mathbb{N}, C[0,1])$.

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References

[1] T. Abdeljawad, On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **62** (2011), 1602–1611.

[2] T. Abdeljawad and F.M. Atici, On the definitions of nabla fractional operators. *Abstr. Appl. Anal.* **2012** (2012), 1–13.

[3] R.P. Agarwal, C. Cuevas and C. Lizama, *Regularity of Difference Equations on Banach Spaces*. Springer, Cham (2014).

[4] S. Amini and I.H. Sloan, Collocation method for second kind integral equations with non-compact operators. *J. Int. Eq. Appl.* **2**, No 1 (1989), 1–30.

[5] F.M. Atici and P.W. Eloe, A transform method in discrete fractional calculus. *Int. J. Difference Equ.* **2**, No 2 (2007), 165–176.

[6] F.M. Atici and P.W. Eloe, Initial value problems in discrete fractional calculus. *Proc. Amer. Math. Soc.* **137**, No 3 (2009), 981–989.

[7] F.M. Atici and P.W. Eloe, Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ.* **3** (2009), 1–12.

[8] F.M. Atici and S. Sengül, Modeling with fractional difference equations. *J. Math. Anal. Appl.* **369** (2010), 1–9.

[9] D. Baleanu, K. Diethelm and E. Scalas, *Fractional Calculus: Models and Numerical Methods*. World Scientific, Singapore (2012).

[10] S. Calzadillas, C. Lizama and J.G. Mesquita, A unified approach to discrete fractional calculus and applications. *Submitted*.

[11] J. Cermák, T. Kisela and L. Nechvátal, Stability and asymptotic properties of a linear fractional difference equation. *Adv. Difference Equ.* **122** (2012), 1–14.

[12] J. Cermák, T. Kisela and L. Nechvátal, Stability regions for linear fractional differential systems and their discretizations. *Appl. Math. Comput.* **219**, No 12 (2013), 7012–7022.

[13] G.A. Chandler and I.G. Graham, Product integration-collocation methods for noncompact integral operator equations, *Math. Comp.* **50**, No 181 (1988), 125–138.

[14] E. Cuesta, C. Lubich and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations. *Math. Comp.* **75**, No 254 (2006), 673–696.

[15] S. Elaydi, Stability and asymptoticity of Volterra difference equations: A progress report. *J. Comput. Appl. Math.* **228**, No 2 (2009), 504–513.

[16] C.S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. *Comput. Math. Appl.* **61** (2011), 191–202.

[17] C.S. Goodrich, On a discrete fractional three-point boundary value problem. *J. Differ. Equ. Appl.* **18** (2012), 397–415.

[18] C.S. Goodrich, On a first-order semipositone discrete fractional boundary value problem. *Arch. Math.* **99** (2012), 509–518.

[19] F. Jarad, B. Kaymakcalan and K. Tas, A new transform method in nabla discrete fractional calculus. *Adv. Diff. Eq.* **190** (2012), 1–17.

[20] V.B. Kolmanovskii, E. Castellanos-Velasco and J.A. Torres-Munoz, A survey: Stability and boundedness of Volterra difference equations. *Nonlinear Anal.* **53**, No 7-8 (2003), 861–928.

[21] E. Kreyszig, *Introductory Functional Analysis with Applications.* John Wiley & Sons, New York (1989).

[22] C. Lizama, ℓ_p-maximal regularity for fractional difference equations on UMD spaces. *Math. Nach.* **288**, No 17-18 (2015), 2079–2092.

[23] C. Lizama, The Poisson distribution, abstract fractional difference equations, and stability. *Proc. Amer. Math. Soc.*, To appear.

[24] C. Lizama and M. Murillo-Arcila, ℓ_p-maximal regularity for a class of fractional difference equations on UMD spaces: The case $1 < \alpha \leq 2$. *Banach J. Math. Anal.*, To appear.

[25] Ch. Lubich, I.H. Sloan and V. Thomée, Nonsmooth data error estimates for approximations of an evolution equation with a positive type memory term. *Math. Comp.* **65** (1996), 1–17.

[26] K.S. Miller and B. Ross, Fractional difference calculus. In: *Proc. Internat. Symp. "Univalent Functions, Fractional Calculus and Their Applications"*, Koriyama (1988) and Horwood, Chichester (1989), 139–152.

[27] J. Mallet-Paret, Traveling waves in spatially discrete dynamical systems of diffusive type. In: *Lecture Notes in Math.* **1822**, Springer, Berlin (2003), 231–298.

[28] J. Mallet-Paret and G.R. Sell, Differential systems with feedback: time discretizations and Lyapunov functions. Special issue dedicated to Victor A. Pliss on the occasion of his 70th birthday. *J. Dynam. Differential Equations* **15**, No 2-3 (2003), 659–698.

[29] M.D. Ortigueira, F.J. V. Coito and J.J. Trujillo, Discrete-time differential systems. *Signal Processing* **107**, (2015), 198–217.

[30] Y. Pan, Z. Han, S. Sun and Z. Huang, The existence and uniqueness of solutions to boundary value problems of fractional difference equations. *Math. Sci.* **6** (2012), 1–7.

[31] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*, Vol. 1. Gordon and Breach Science Publishers, Amsterdam (1986).
[32] J. M. Sanz-Serna, A numerical method for a partial integro-differential equation. *SIAM J. Numer. Anal.* **25** (1988), 319–327.
[33] L. Xiao-Yan and J. Wei, Solving fractional difference equations using the Laplace transform method. *Abstr. Appl. Anal.* **2014** (2014), 1–6.

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