EXISTENCE OF ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS FOR DAMPED WAVE EQUATIONS

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ABSTRACT. In this paper, a class of nonlinear damped wave equations of the form $\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t))$, $t \geq 0$, satisfying $\alpha\beta < \gamma$ with prescribed initial conditions are studied. Some sufficient conditions are established for the existence and uniqueness of an asymptotically almost periodic solution. These results have significance in the study of vibrations of flexible structures possessing internal material damping. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

1. INTRODUCTION

It is well known, that the dynamics of linear vibrations of elastic structures are mathematically governed by the wave equation. However, the dynamics of elastic vibrations of flexible structures are actually nonlinear in practice. In 1998, Bose and Gorain [2] studied a more realistic model of vibrations of elastic structure in which the stress is not simply proportional to the strain. As a result, they shown that the dynamics of vibrations of elastic structures are governed by the following third order differential equation

$$\alpha u'''(t) + u''(t) - \beta\Delta u(t) - \gamma\Delta u'(t) = 0, \qquad t \geq 0 \tag{1.1}$$

with suitable boundary and initial conditions, and where α, β, γ are positive constants satisfying $\alpha\beta < \gamma$. Concerning qualitative properties, Bose and Gorain studied boundary stabilization and obtained the explicit exponential energy decay rate for the solution of (1.1) subject to mixed boundary conditions (see [3, 9, 10, 11] and references therein). Motivated by these works, abstract linear equations of the form

$$\alpha u'''(t) + u''(t) - \beta Au(t) - \gamma Au'(t) = f(t), \quad \alpha, \beta, \gamma \in \mathbb{R}_+, \quad t \geq 0, \tag{1.2}$$

where A is a closed linear operator acting in a Banach space X and f is a X-valued function has been treated in recent papers [7, 8]. We emphasize that the abstract Cauchy problem associated with (1.2) is in general ill-posed, see e.g. [20]. Also is well known that in order to analyze well-posedness, a direct approach leads to better results than those obtained by a reduction to a first-order equation, see e.g. [4].

The study of the asymptotic behavior of solutions of a differential equation is one of the most interesting themes of the qualitative theory of differential equations and for this reason has attracted interest of many researchers over the years. In the early 40s, this theory was boosted by the introduction of the concept of asymptotically almost periodic

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functions by the mathematician M. Fréchet. Nevertheless, the existing literature for the study of existence of asymptotically almost periodic solutions to differential equations is limited, in general, to equations of first and second order. Our purpose in this paper is analyze and to prove, by the first time, the existence of asymptotically almost periodic mild solutions for an abstract semilinear equation of the form

$$\alpha u'''(t) + u''(t) - \beta Au(t) - \gamma Au'(t) = f(t, u(t)), \quad \alpha, \beta, \gamma \in \mathbb{R}_+, \tag{1.3}$$

with appropriate initial conditions. The motivation for incorporating f as an input disturbance in the governing differential equation arises from the fact that very small amount of these, are always present in real materials (see e.g. [5]) as long as the system vibrates. Hence, is also reasonable the study of existence of asymptotically almost periodic solutions when $f(t, x)$ is asymptotically almost periodic in t; that is, asymptotically almost periodic stability of the system.

A surprising fact is that in order to get asymptotic behavior, some initial conditions should be forced to be zero. This leads to an unexpected property that is not present in the study of the same qualitative property for the Cauchy problem of order less than 3, see [1].

To achieve our goal we use a mixed method, combining tools of certain strongly continuous families in operator theory, that we introduce in this paper, and fixed point theory.

This paper is organized as follows: The preliminary Section 2 collects results essentially contained in [15] and standard literature of almost periodic and asymptotically almost periodic functions (see [21, 22]). In particular we establish a result of composition for asymptotically almost periodic functions (see Lemma 2.8) which is very important in our investigations. In Section 3 we study sufficient conditions for existence of solutions for equation (1.3). In fact, Proposition 3.1 gives a complete description of the solutions in terms of (α, β, γ)-regularized families. It corresponds to an extension of the standard variation of parameters formula. In Section 4, we study conditions for existence and uniqueness of asymptotically almost periodic vibrations. We have two situations: In the linear case, we can ensure conditions for existence of asymptotically almost periodic solution (see Theorem 4.2). For the semilinear case, we establish sufficient conditions for existence of asymptotically almost periodic mild solutions (see Theorem 4.4, Theorem 4.5 and Theorem 4.7). Finally, we show that our abstract results apply to equation (1.3) in case of $A = \Delta$, the Laplacian.

2. Preliminares

Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$ be given. In what follows we denote

$$k(t) = \frac{1}{\alpha}\int_0^t (t-s)e^{-s/\alpha}ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+$$

and

$$a(t) = \beta k(t) + \frac{\gamma}{\alpha}\int_0^t e^{-s/\alpha}ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+ .$$

In order to give a consistent definition of mild solution for equation (1.3) based on an operator theoretical approach, we introduce the following definition (see [13] for a recent discussion about the concept of mild solutions for nonlinear equations).

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an (α,β,γ)-regularized family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied:*

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0)=0$;

(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A), t \geq 0$;

(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Axds \tag{2.1}$$

for all $x \in D(A), t \geq 0$. In this case, $R(t)$ is called the (α,β,γ)-regularized family generated by A.

Remark 2.2. It is proved in [15], in the more general context of (a,k)- regularized families, that an operator A is the generator of an (α,β,γ)-regularized family if and only if there exists $\omega \geq 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda} : Re\lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \frac{1}{\beta+\gamma\lambda}\left(\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t}R(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness of the Laplace transform, we note that an (α,β,γ)-regularized family corresponds to an (a,k)-regularized family studied in [15]. In fact, we have

$$\hat{a}(\lambda) = \frac{\beta+\gamma\lambda}{\lambda^2+\alpha\lambda^3} \text{ and } \hat{k}(\lambda) = \frac{1}{\lambda^2+\alpha\lambda^3}, \text{ for all } Re\lambda > \omega.$$

As in the situation of semigroup theory, we have diverse relations of an (α,β,γ)-regularized family and its generator. The following result is a direct consequence of [15, Proposition 3.1 and Lemma 2.2].

Proposition 2.3. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A. Then the following holds:*

(a)For all $x \in D(A)$ we have $R(\cdot)x \in C^2(\mathbb{R}_+;X)$.

(b) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)R(s)xds \in D(A)$ and

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds.$$

Results on perturbation, approximation, asymptotic behavior, representation as well as ergodic type theorems for (α,β,γ)-regularized families can be also deduced from the more general context of $(a,k)-$regularized families (see [14, 15, 16, 17] and [19]).

Let us recall the notion of almost periodicity and asymptotically almost periodicity which shall come into play later on.

Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be Banach spaces.

Definition 2.4 ([21])**.** *A function $f : \mathbb{R} \to X$ is called almost periodic if f is continuous, and for each $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that for every interval of length $l(\epsilon)$ contains a number τ with the property that $\|f(t+\tau) - f(t)\| \leq \epsilon$ for each $t \in \mathbb{R}$. The number τ*

above is called an ϵ-translation number for f, and the colection of such functions will be denoted $AP(X)$.

We remark that the set $AP(X)$ endowed with the sup norm is a Banach space.

Definition 2.5. *A function $f : \mathbb{R} \times Y \to X$ is called almost periodic if f is continuous, and for each $\epsilon > 0$ and any compact $K \subset Y$ there exists $l(\epsilon) > 0$ such that every interval I of length $l(\epsilon)$ contains a number τ with the property that $\|f(t+\tau, x) - f(t, x)\| \leq \epsilon$ for all $t \in \mathbb{R}$, $x \in K$. The collection of such functions will be denoted by $AP(\mathbb{R} \times Y, X)$.*

We recall the following standard result in the theory of almost periodic functions.

Lemma 2.6 ([22])**.** *Let $f \in AP(\mathbb{R} \times Y, X)$ and $h \in AP(Y)$. Then the function $f(\cdot, h(\cdot)) \in AP(X)$.*

Let $C_0(\mathbb{R}_+, X)$ be the subspace of $BC(\mathbb{R}_+, X)$ such that $\lim_{t\to\infty} \|x(t)\| = 0$ and $C_0(\mathbb{R}_+ \times Y, X)$ denotes the space of all continuous functions $h : \mathbb{R}_+ \times Y \to X$ such that $\lim_{t\to\infty} h(t, x) = 0$ uniformly for x in any compact subset of Y.

Definition 2.7. *A continuous function $f : \mathbb{R}_+ \to X$ (resp., $\mathbb{R}_+ \times Y \to X$) is called asymptotically almost periodic if it admits a decomposition $f = g + \phi$, where $g \in AP(X)$ (resp., $g \in AP(\mathbb{R} \times Y, X)$) and $\phi \in C_0(\mathbb{R}_+, X)$ (resp., $\phi \in C_0(\mathbb{R}_+ \times Y, X)$). Denote by $AAP(X)$ (resp., $AAP(\mathbb{R}_+ \times Y, X)$) the set all such functions.*

We observe that $AAP(X)$ is a Banach space with the sup norm. The next lemma will be very useful for our results. We note that it improves the conditions on a more general result recently established in [18, Theorem 4.1].

Lemma 2.8. *Let X and Y two Banach spaces. Suppose that $f \in AAP(\mathbb{R}_+ \times Y; X)$ is uniformly continuous on any bounded subset $K \subset Y$, uniformly for $t \geq 0$. Then, $u \in AAP(Y)$ implies $f(\cdot, u(\cdot)) \in AAP(X)$.*

Proof. Let $f = g + h$ be the decomposition of f with $g \in AP(\mathbb{R}_+ \times Y; X)$ and $h \in C_0(\mathbb{R}_+ \times Y, X)$. Writing $u(t) = v(t) + \theta(t)$, where $v \in AP(Y)$ and $\theta \in C_0(\mathbb{R}_+; Y)$, we observe that

$$f(t, u(t)) = g(t, v(t)) + f(t, u(t)) - g(t, v(t)) = g(t, v(t)) + g(t, u(t)) - g(t, v(t)) + \phi(t, u(t)).$$

By Lemma 2.6 we have that $g(\cdot, v(\cdot)) \in AP(X)$. Since u is a bounded function and $\phi \in C_0(\mathbb{R}_+ \times Y; X)$ we have $\lim_{t\to\infty} \|\phi(t, u(t))\| = 0$. It remains show that

$$g(\cdot, u(\cdot)) - g(\cdot, v(\cdot)) \in C_0(\mathbb{R}_+; X).$$

We can see that there is a bounded subset $K \subset X$ such that $u(t), v(t) \in K$, for all $t \geq 0$. Furthermore, for any fixed $\delta > 0$ there is $T > 0$ for which $\|u(t) - v(t)\| = \|\theta(t)\| \leq \delta$, for all $|t| > T$. Then, given $\epsilon > 0$ we have

$$\|g(t, u(t)) - g(t, v(t))\| \leq \epsilon,$$

for $|t| > T$. □

Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}_+$, and $h(t) \to \infty$ as $t \to \infty$. We consider the space

$$C_h(Z) = \{u \in C(\mathbb{R}_+, Z) : \lim_{t\to\infty} \frac{u(t)}{h(t)} = 0\}$$

endowed with the norm $\|u\|_h = \sup_{t\geq 0} \frac{\|u(t)\|}{h(t)}$.

We use the following result (cf. [6], Lemma 2.2).

Lemma 2.9. *A subset $K \subseteq C_h(X)$ is a relatively compact set if it verifies the following conditions:*
(c-1) The set $K_b = \{u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0,b];X)$ for all $b \geq 0$.
(c-2) $\lim_{t\to\infty} \frac{\|u(t)\|}{h(t)} = 0$ uniformly for all $u \in K$.

3. Existence of Solutions

Let $\alpha, \beta, \gamma \in (0,\infty)$. Consider the linear equation

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \tag{3.1}$$

with initial conditions $u(0) = x, u'(0) = y, u''(0) = z$, where A is the generator of a (α,β,γ)-regularized family $R(t)$. By a solution of (3.1) we understand a function $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$ such that $u' \in C(\mathbb{R}_+; D(A))$ and verify (3.1).

The following result gives a complete description of the solutions for equation (3.1) in terms of (α,β,γ)-regularized families. It corresponds to an extension of the standard variation of parameters formula for the second order Cauchy problem.

Proposition 3.1. *Let $R(t)$ be an $(\alpha,\beta,\gamma)-$regularized family on X with generator A. If $f \in L^1_{loc}(\mathbb{R}_+, D(A^2))$, $x \in D(A^3)$, $y \in D(A^2)$ and $z \in D(A^2)$ then $u(t)$ given by*

$$\begin{aligned} u(t) &= \alpha R''(t)x + R'(t)x - \gamma AR(t)x + \alpha R'(t)y + R(t)y + \alpha R(t)z \\ &\quad + \int_0^t R(t-s)f(s)ds, \quad t \geq 0, \end{aligned} \tag{3.2}$$

is a solution of (3.1).

Proof. For all $i = 1, ..., 5$, we can write $R^{(i)}(t)w$ as follows:

$$R'(t)w = (1 - e^{-t/\alpha})w + \int_0^t \left[\beta + \left(\frac{\gamma}{\alpha} - \beta\right) e^{-\frac{1}{\alpha}(t-s)}\right] R(s)Awds, \quad w \in D(A),$$

and we conclude from Proposition 2.3(b) that $R'(t)w \in D(A)$ for $w \in D(A)$.

$$R''(t)w = \frac{1}{\alpha}e^{-t/\alpha}w + \frac{\gamma}{\alpha}R(t)Aw + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right) e^{-\frac{1}{\alpha}(t-s)}R(s)Awds, \quad w \in D(A),$$

and hence by (R2), we have $R''(t)w \in D(A)$ for $w \in D(A^2)$.

$$\begin{aligned} R'''(t)w &= -\frac{1}{\alpha^2}e^{-t/\alpha}w + \frac{\gamma}{\alpha}R'(t)Aw + \frac{\beta}{\alpha}R(t)Aw - \frac{\gamma}{\alpha^2}R(t)Aw \\ &+ \int_0^t \left(\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2}\right) e^{-\frac{1}{\alpha}(t-s)}R(s)Awds, \quad w \in D(A^2), \end{aligned}$$

and $R'''(t)w \in D(A)$ for $w \in D(A^2)$.

$$\begin{aligned} R^{(iv)}(t)w &= \frac{1}{\alpha^3}e^{-t/\alpha}w + \frac{\gamma}{\alpha}R''(t)Aw + \frac{\beta}{\alpha}R'(t)Aw - \frac{\gamma}{\alpha^2}R'(t)Aw + \frac{\gamma}{\alpha^3}R(t)Aw \\ &- \frac{\beta}{\alpha^2}R(t)Aw + \int_0^t \left(\frac{\beta}{\alpha^3} - \frac{\gamma}{\alpha^4}\right) e^{-\frac{1}{\alpha}(t-s)}R(s)Awds, \quad w \in D(A^2), \end{aligned}$$

and $R^{(iv)}(t)w \in D(A)$ for $w \in D(A^3)$.

$$\begin{aligned} R^{(v)}(t)w &= -\frac{1}{\alpha^4}e^{-t/\alpha}w + \frac{\gamma}{\alpha}R'''(t)Aw + \frac{\beta}{\alpha}R''(t)Aw - \frac{\gamma}{\alpha^2}R''(t)Aw + \frac{\gamma}{\alpha^3}R'(t)Aw \\ &- \frac{\beta}{\alpha^2}R'(t)Aw + \frac{\beta}{\alpha^3}R(t)Aw - \frac{\gamma}{\alpha^4}R(t)Aw \\ &+ \int_0^t \left(\frac{\gamma}{\alpha^5} - \frac{\beta}{\alpha^4}\right) e^{-\frac{1}{\alpha}(t-s)}R(s)Awds, \quad w \in D(A^3), \end{aligned}$$

and $R^{(v)}(t)w \in D(A)$ for $w \in D(A^4)$. From the above, we deduce that if $x \in D(A^3)$, $y \in D(A^2)$ and $z \in D(A^2)$ then $R(\cdot)x \in C^5(\mathbb{R}_+, X)$, $R(\cdot)y \in C^4(\mathbb{R}_+, X)$ and $R(\cdot)z \in C^3(\mathbb{R}_+, X)$. Since $f \in L^1_{loc}(\mathbb{R}_+, D(A^2))$ we have

$$u'(t) = \alpha R'''(t)x + R''(t)x - \gamma R'(t)Ax + \alpha R''(t)y + R'(t)y + \alpha R'(t)z$$
$$+\int_0^t R'(t-s)f(s)ds,$$

and hence $u'(t) \in D(A)$ for $x, y \in D(A^2)$ and $z \in D(A)$.

$$u''(t) = \alpha R^{(iv)}(t)x + R'''(t)x - \gamma R''(t)Ax + \alpha R'''(t)y + R''(t)y + \alpha R''(t)z$$
$$+\int_0^t R''(t-s)f(s)ds,$$

and hence $u''(t) \in D(A)$ for $x \in D(A^3)$ and $y, z \in D(A^2)$.

$$u''(t) = \alpha R^{(v)}(t)x + R^{(iv)}(t)x - \gamma R'''(t)Ax + \alpha R^{(iv)}(t)y + R'''(t)y + \alpha R'''(t)z$$
$$+\frac{1}{\alpha}f(t) + \int_0^t R'''(t-s)f(s)ds,$$

and hence $u \in C^3(\mathbb{R}_+, X)$. Using the fact that A is closed and the expressions for $R^{(i)}(t)$, $i = 1, \cdots, 5$, we can conclude that $u(t)$ verify (3.1). Finally, a straightforward computation shows that $u(0) = x$, $u'(0) = y$ and $u''(0) = z$. □

Remark 3.2. *Observe that in the border case $\alpha = 0$, with $\gamma = 0$, the above theorem recover the variation of parameters formula for the second order Cauchy problem $u''(t) = Au(t) + f(t)$, so that the $(0, \beta, 0)$-regularized family $R(t)$ corresponds in this case to the sine family generated by A and $R'(t)$ is the respective cosine family.*

We recall the following result which provide a wide class of generators of (α, β, γ)-regularized families.

Theorem 3.3 ([8]). *Let $-B$ be a positive selfadjoint operator on a Hilbert space H such that*

$$\alpha\beta \leq \gamma\,.$$

Then B is the generator of a bounded (α, β, γ)-regularized family on H.

4. Asymptotically almost periodic solutions

In this section we study existence of asymptotically almost periodic solutions for the equation

$$u''(t) + \alpha u'''(t) = \beta A u(t) + \gamma A u'(t) + f(t), \tag{4.1}$$

with initial conditions $u(0) = x, u'(0) = y, u''(0) = z$, where A is the generator of a (α, β, γ)-regularized family $R(t)$.

Assume that $R(t)$ is differentiable. We introduce the following assumption.

(ED) There are constants $M > 0$ and $\omega > 0$ such that

$$||R'(t)|| + ||R(t)|| \leq M e^{-\omega t}, \quad t \geq 0.$$

We say in short that $R(t)$ and $R'(t)$ are exponentially stable. The following result on regularity of the convolution under asymptotically almost periodic functions is one of the keys to obtain our results.

Lemma 4.1. *Let $R(t)$ be an exponentially stable (α, β, γ)-regularized family on X with generator A. If $f \in AAP(X)$ then the function*

$$F(t) = \int_0^t R(t-s) f(s) ds$$

belongs to $AAP(X)$.

Proof. If $f = g + h$ with $g \in AP(X)$ and $h \in C_0(\mathbb{R}_+, X)$ then we have that $F(t) = G(t) + H(t)$, where

$$G(t) := \int_{-\infty}^t R(t-s) g(s) ds$$

and

$$H(t) := \int_0^t R(t-s) h(s) ds - \int_{-\infty}^0 R(t-s) g(s) ds.$$

For $\epsilon > 0$, we take $l(\epsilon)$ involved in Definition 2.4, then for every interval of length $l(\epsilon)$ contains a number τ such that $\|g(t+\tau) - g(t)\| < \epsilon$ for each $t \in \mathbb{R}$. The estimate

$$\begin{aligned} \|G(t+\tau) - G(t)\| &\leq \int_0^\infty \|R(s)\| \|g(t-s+\tau) - g(t-s)\| ds \\ &\leq \left(M \int_0^\infty e^{-\omega s} ds \right) \epsilon. \end{aligned}$$

is responsible for the fact that $G \in AP(X)$. We claim that $\|H(t)\| \to 0$ as $t \to \infty$. In fact, for each $\epsilon > 0$ there exists a $T > 0$ such that $\|h(s)\| \leq \epsilon$ for all $s > T$. Then for all $t > 2T$ we deduce

$$\begin{aligned} \|H(t)\| &\leq \int_0^{t/2} M e^{-\omega(t-s)} \|h(s)\| ds + \int_{t/2}^t M e^{-\omega(t-s)} \|h(s)\| ds + \int_t^\infty M e^{-\omega s} \|g(t-s)\| ds \\ &\leq M(\|h\|_\infty + \|g\|_\infty) \int_t^\infty e^{-\omega s} ds + \epsilon M \int_0^\infty e^{-\omega s} ds. \end{aligned}$$

Therefore, $\lim_{t \to \infty} H(t) = 0$, that is, $H \in C_0(\mathbb{R}_+, X)$. This completes the proof. □

We begin our results on existence of asymptotically almost periodic functions for the linear equation, with the following theorem.

Theorem 4.2. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *If $f \in AAP(X)$ is such that $f(t) \in D(A^2)$ for all $t \geq 0$, then Eq. (4.1) with initial conditions $u(0) = 0$, $u'(0) = y \in D(A^2)$ and $u''(0) = z \in D(A^2)$ has a unique solution $u \in AAP(X)$.*

Proof. Let $f \in AAP(X)$ such that $f(t) \in D(A^2)$ and $y, z \in D(A^2)$. From Proposition 3.1 we have that the solution for Eq. (4.1) is given by

$$u(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s)ds.$$

From Lemma 4.1 we have that $g(t) = \int_0^t R(t-s)f(s)ds$ belongs to $AAP(X)$. On the other hand, if $t \to \infty$ we have that $\|\alpha R'(t)y\| \leq \alpha\|y\|Me^{-\omega t} \to 0$, $\|R(t)y\| \leq \|y\|Me^{-\omega t} \to 0$ and

$$\|\alpha R(t)z\| \leq \alpha\|z\|Me^{-\omega t} \to 0.$$

Therefore, $u \in AAP(X)$. □

From now we study the semilinear version of Eq. (4.1), that is, we consider the initial value problem

$$\begin{cases} u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t)), & t \geq 0; \\ u(0) = 0, u'(0) = y, u''(0) = z, & . \end{cases} \tag{4.2}$$

where $\alpha, \beta, \gamma \in (0, \infty)$, A is the generator of a (α,β,γ)-regularized family $R(t)$ and $f : \mathbb{R}_+ \times X \to X$ is a suitable function.

Motivated by the results of the previous section we introduce the following concept.

Definition 4.3. *Let $R(t)$ be an (α,β,γ)-generalized family on X with generator A. A continuous function $u : \mathbb{R}_+ \to X$ satisfying the integral equation*

$$u(t) \;=\; \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s,u(s))ds, \;\; \forall\; t \geq 0,$$

where $y, z \in X$ is called a mild solution to the equation (4.2).

Initially we study conditions to existence and uniqueness of a mild solution for equation (4.2) when the function f is Lipschitz continuous.

Theorem 4.4. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *Let $f \in AAP(\mathbb{R}_+ \times X, X)$ and suppose that there exists an integrable bounded function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \;\forall x, y \in X, \;\; t \geq 0. \tag{4.3}$$

Then equation (4.2) has a unique asymptotically almost periodic mild solution.

Proof. We define the operator Λ on the space $AAP(X)$ by

$$\Lambda u(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s,u(s))ds. \tag{4.4}$$

We show that $\Lambda u \in AAP(X)$. Initially we observe that since $R(t)y \to 0$, $\alpha R(t)z \to 0$ and $\alpha R'(t)y \to 0$ as $t \to \infty$, then $R(\cdot)y \in AAP(X)$, $\alpha R(\cdot)z \in AAP(X)$ and $\alpha R'(\cdot)y \in AAP(X)$. It follows from Lemma 2.8 that the function $s \to f(s, u(s))$ is asymptotically almost periodic; then by Lemma 4.1

$$\int_0^t R(t-s)f(s,u(s))ds \in AAP(X).$$

Furthermore, for $u_1, u_2 \in AAP(X)$ we have that

$$\begin{aligned}\|\Lambda u_1(t) - \Lambda u_2(t)\| &\leq M\int_0^t e^{-\omega(t-s)}L(s)ds\|u_1-u_2\|_\infty \\ &\leq M\int_0^t L(s)ds\|u_1-u_2\|_\infty \\ &\leq M\|L\|_1\|u_1-u_2\|_\infty,\end{aligned}$$

hence,

$$\begin{aligned}\|(\Lambda^2 u_1)(t) - (\Lambda^2 u_2)(t)\| &\leq M^2\left(\int_0^t L(s)\left(\int_0^s L(\tau)d\tau\right)ds\right)\|u_1-u_2\|_\infty \\ &\leq \frac{M^2}{2}\left(\int_0^t L(\tau)d\tau\right)^2\|u_1-u_2\|_\infty \\ &\leq \frac{(M\|L\|_1)^2}{2}\|u_1-u_2\|_\infty.\end{aligned}$$

In general, we get the following estimate

$$\|(\Lambda^n u_1)(t) - (\Lambda^n u_2)(t)\| \leq \frac{(C\|L\|_1)^n}{n!}\|u_1-u_2\|_\infty.$$

Since $\frac{(M\|L\|_1)^n}{n!} < 1$ for n sufficiently large, by the fixed point iteration method Λ has a unique fixed point $u \in AAP(X)$. This completes the proof. □

A new Lipschitz condition is considered in the next result.

Theorem 4.5. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)***. Let $f \in AAP(\mathbb{R}_+ \times X, X)$ and suppose that f satisfies the Lipschitz condition (4.3) with L a bounded continuous function. Let $\beta(t) = \int_0^t e^{-\omega(t-s)}L(s)ds$. If there is a constant $k > 1$ such that $M\beta(t) \leq k < 1$, where $M > 0$ is the constant given in the assumption* **(ED)***, then equation (4.2) has a unique mild solution $u \in AAP(X)$.*

Proof. Define the map $\Lambda : AAP(X) \to AAP(X)$ by expression (4.4). A similar argument as the proof of Theorem 4.4 shows that Λ is well defined. We next proof that Λ is a k-contraction. In fact, given $u, v \in AAP(X)$ we have that

$$\begin{aligned}\|\Lambda u(t) - \Lambda v(t)\| &\leq M\int_0^t e^{-\omega(t-s)}L(s)\|u(s)-v(s)\|ds \\ &\leq M\int_0^t e^{-\omega(t-s)}L(s)ds\|u-v\|_\infty \\ &\leq M\theta(t)\|u-v\|_\infty \\ &\leq k\|u-v\|_\infty.\end{aligned}$$

Hence, by the Banach's fixed point theorem we conclude the proof. □

The next result is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *Let $f \in AAP(\mathbb{R}_+ \times X, X)$ and suppose that f satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq k\|x - y\|$$

for all $x, y \in X$ and $t \geq 0$. If $\frac{Mk}{-\omega} < 1$, where M and ω are the constants given in the assumption **(ED)**, *then equation (4.2) has a unique mild solution $u \in AAP(X)$.*

We next study the existence of asymptotically almost periodic mild solutions of the problem (4.2) when the function f is not Lipschitz continuous. To establish our result, we consider functions $f : \mathbb{R}_+ \times X \to X$ that satisfies the following boundedness condition.

(B) There exists a continuous nondecreasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x)\| \leq W(\|x\|),$$

for all $t \in \mathbb{R}_+$ and $x \in X$. We have the following result.

Theorem 4.7. *Let $f \in AAP(\mathbb{R}_+ \times X; X)$ be a function that satisfies assumption (B), and the following conditions:*

(a) For each $\nu \geq 0$,

$$\lim_{t\to\infty} \frac{1}{h(t)} \int_0^t e^{-\omega(t-s)} W(\nu h(s))ds = 0.$$

(b) For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|v - u\|_h \leq \delta$ implies that

$$M \int_0^t e^{-\omega(t-s)} \|f(s, v(s)) - f(s, u(s))\| ds \leq \epsilon,$$

for all $t \in \mathbb{R}_+$.

(c) For all $a, b \in \mathbb{R}_+$, $a \leq b$, and $r > 0$, the set

$$\{f(s, h(s)x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$$

is relatively compact in X.

(d) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$, where

$$\beta(\nu) := \left\| \|\alpha R'(\cdot)y\| + \|R(\cdot)y\| + \|\alpha R(\cdot)z\| + M \int_0^{\cdot} e^{-\omega(\cdot - s)} W(\nu h(s))ds \right\|_h.$$

Then equation (4.2) has an asymptotically almost periodic mild solution.

Proof. We define the operator Λ on $C_h(X)$ by

$$\Lambda u(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s, u(s))ds.$$

We show that Λ has a fixed point in $AAP(X)$. We divide the proof in several steps.
(i) For $u \in C_h(X)$, we have that

$$\|\Lambda u(t)\| \leq (\alpha + 1)M\|y\| + \alpha M\|z\| + M \int_0^t e^{-\omega(t-s)} W(\|u\|_h h(s))ds. \tag{4.5}$$

It follows from condition (a) that $\Lambda : C_h(X) \to C_h(X)$.
(ii) The map Λ is continuous. In fact, for $\varepsilon > 0$, we take δ involved in condition (b). If $u, v \in C_h(X)$ and $\|u - v\|_h \leq \delta$, then

$$\|\Lambda u(t) - \Lambda v(t)\| \leq M \int_0^t e^{-\omega(t-s)} \|f(s, u(s)) - f(s, v(s))\| ds \leq \varepsilon,$$

which shows the assertion.
(iii) We next show that Λ is completely continuous. We set $B_r(Z)$ for the closed ball with center at 0 and radius r in a space Z. Let $V = \Lambda(B_r(C_h(X)))$ and $v = \Lambda(u)$ for $u \in B_r(C_h(X))$. Initially, we prove that $V_b(t)$ is a relatively compact subset of X for each $t \in [0, b]$. We get

$$\begin{aligned} v(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z \quad &+ \int_0^t R(s) f(t-s, u(t-s)) ds \\ &\in \alpha R'(t)y + R(t)y + \alpha R(t)z + t\overline{c(K)}, \end{aligned}$$

where $c(K)$ denotes the convex hull of K and $K = \{R(s) f(\xi, h(\xi)x) :\ 0 \leq s \leq t,\ 0 \leq \xi \leq t,\ \|x\| \leq r\}$. Using the fact that $R(\cdot)$ is strongly continuous and the property (c), we infer that K is a relatively compact set, and $V_b(t) \subseteq \alpha R'(t)y + R(t)y + \alpha R(t)z + t\overline{c(K)}$, which establishes our assertion. We next show that the set V_b is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) &= \alpha\big(R'(t+s) - R'(t)\big)y + \big(R(t+s) - R(t)\big)y + \alpha\big(R(t+s) - R(t)\big)z \\ &+ \int_t^{t+s} R(t+s-\xi) f(\xi, u(\xi)) d\xi \\ &+ \int_0^t \big(R(\xi+s) - R(\xi)\big) f(t-\xi, u(t-\xi)) d\xi. \end{aligned}$$

For each $\epsilon > 0$, we can choose $\delta_1 > 0$ such that

$$\Big\| \int_t^{t+s} R(t+s-\xi) f(\xi, u(\xi)) d\xi \Big\| \leq M \int_t^{t+s} e^{-\omega(t+s-\xi)} W(rh(\xi)) d\xi \leq \epsilon/5,$$

for $s \leq \delta_1$. Moreover, since $\{f(t-\xi, u(t-\xi)) : 0 \leq \xi \leq t, u \in B_r(C_h(X))\}$ is relatively compact set, $R(\cdot)$ and $R'(\cdot)$ are strongly continuous, we can choose $\delta_i > 0$, $i = 2, ..., 5$ such that

$$\|\alpha\big(R'(t+s) - R'(t)\big)y\| \leq \epsilon/5,\ s \leq \delta_2,$$
$$\|\big(R(t+s) - R(t)\big)y\| \leq \epsilon/5,\ s \leq \delta_3,$$
$$\|\alpha\big(R(t+s) - R(t)\big)z\| \leq \epsilon/5,\ s \leq \delta_4$$

and

$$\|\big(R(\xi+s) - R(\xi)\big) f(t-\xi, u(t-\xi))\| \leq \frac{\epsilon}{5t},\ s \leq \delta_5.$$

Combining these estimate, we get $\|v(t+s) - v(t)\| \leq \varepsilon$ for s small enough and independent of $u \in B_r(C_h(X))$.

Finally, applying condition (a), we can show that

$$\frac{\|v(t)\|}{h(t)} \leq \frac{(\alpha+1)M\|y\|}{h(t)} + \frac{\alpha M \|z\|}{h(t)} + \frac{M}{h(t)} \int_0^t e^{-\omega(t-s)} W(rh(s)) ds \to 0,\ t \to \infty,$$

and this convergence is independent of $u \in B_r(C_h(X))$. Hence V satisfies conditions (c-1) and (c-2) of Lemma 2.9, which completes the proof that V is a relatively compact set in $C_h(X)$.
(iv) If $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda\Lambda(u^\lambda)$ for some $0 < \lambda < 1$, we have the estimate $\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \leq 1$ and, combining with condition (d), we conclude that the set $\tilde{K} := \{u^\lambda : u^\lambda = \lambda\Lambda(u^\lambda), \lambda \in (0,1)\}$ is bounded.
(v) It follows, from Lemma 2.8 and Lemma 4.1, that $\Lambda(\overline{AAP(X)}) \subseteq \overline{AAP(X)}$ and, consequently, we can consider $\Lambda : \overline{AAP(X)} \to \overline{AAP(X)}$. Using properties (i)-(iii), we have that this map is completely continuous. Taking into account that $\tilde{K}$ is bounded and using Leray-Schauder alternative theorem ([12, Theorem 6.5.4]), we infer that Λ has a fixed point $u \in \overline{AAP(X)}$. Let $(u_n)_n$ be a sequence in $AAP(X)$ that converges to u. We see that $(\Lambda u_n)_n$ converges to $\Lambda u = u$ uniformly in $\mathbb{R}_+$. This implies that $u \in AAP(X)$, and completes the proof. □

We finish this paper with the following application, which gives sufficient conditions for the existence of asymptotically almost periodic solutions of the problem stated in the introduction.

Example 4.8. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $0 < \lambda < \mu$. Consider the equation* [2]*:*

$$u''(t) + \lambda u'''(t) = c^2\Delta u(t) + c^2\mu\Delta u'(t) + f(t, u(t)). \tag{4.6}$$

Define $\alpha = \lambda, \beta = c^2$ and $\gamma = c^2\mu$. Then $\alpha\beta < \gamma$. It is well known that the Dirichlet-Laplacian operator Δ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is a selfadjoint operator on $L^2(\Omega)$ and $\sigma(\Delta) \subset (-\infty, 0)$. It follows from Theorem 3.3 that $B = \Delta$ is the generator of an (α, β, γ)-regularized family $R(t)$ on $X = L^2(\Omega)$. By Proposition 3.1 it follows that $u(t) = \alpha R(t)w$ is the unique solution of (4.6) with initial conditions $u(0) = u'(0) = 0$ and $u''(0) = w \in D(\Delta^2)$.

Define $v = u + \lambda u'$. Then the equation (4.6) becomes the system

$$v''(t) = c^2\Delta v(t) + c^2(\mu - \lambda)\Delta u'(t) \tag{4.7}$$

with initial condition $v(0) = 0$. The energy functional of this system is given by

$$E(t) = \frac{1}{2}\int_\Omega v'^2 + c^2|\nabla v|^2 + c^2\lambda(\nu - \lambda)|\nabla u'|^2 dx. \tag{4.8}$$

It was proved by Bose and Gorain [2] *that the energy of the system tends to zero exponentially as $t \to \infty$, that is, there exists constants $M > 0$ and $\nu > 0$ such that*

$$E(t) \leq Me^{-\nu t}, \quad t \geq 0. \tag{4.9}$$

In particular, from the definition of $E(t)$ follows that $||\nabla v(t)||_{L^2} \leq Me^{-\nu t}$ and $||\nabla u'(t)||_{L^2} \leq Me^{-\nu t}$. Hence, Poincare's inequality and the definition of v implies that there exists a constant $C_w > 0$ such that

$$||\alpha R(t)w|| = ||u(t)|| \leq ||v(t)|| + \lambda||u'(t)|| \leq C_w e^{-\nu t}$$

In particular, $||R'(t)w|| = ||u'(t)|| \leq C_w e^{-\nu t}$. Finally, the uniform boundedness principle implies that there exists $C > 0$ such that

$$||R'(t)|| + ||R(t)|| \leq Ce^{-\nu t}, \quad t \geq 0.$$

Now, from Theorem 4.2 we conclude that for each $f \in AAP(L^2(\Omega))$ *such that* $f(t) \in D(\Delta^2)$ *for all* $t \geq 0$*, the linear equation*

$$u''(t) + \lambda u'''(t) = c^2 \Delta u(t) + c^2 \mu \Delta u'(t) + f(t)$$

with initial conditions $u(0) = 0$, $u'(0) = y \in D(\Delta^2)$ *and* $u''(0) = z \in D(\Delta^2)$ *has a unique solution* $u \in AAP(L^2(\Omega))$*. On the other hand, from Theorem 4.4, given* $f \in AAP(\mathbb{R}_+ \times X, X)$ *and assuming that there exists an integrable bounded function* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \ \forall x, y \in X, \ t \geq 0, \tag{4.10}$$

we conclude that the nonlinear equation (4.6) with initial conditions $u(0) = 0$ *and* $u'(0) = y \in L^2(\Omega)$ *and* $u''(0) = z \in L^2(\Omega)$ *has a unique asymptotically almost periodic mild solution (AAP).*

This result shows that the output solution u *is AAP for every AAP input disturbance* f*. Consequently, amplitude of the vibrations remains bounded and decay to zero when amplitude of the disturbances is small and decay to zero. Hence, the equation (4.6) is AAP-input AAP-output stable.*

References

[1] W. Arendt, C.J.K. Batty, *Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half-line.* Bull. London Math. Soc. 31 (1999), no. 3, 291–304.

[2] S.K. Bose, G.C. Gorain, *Stability of the boundary stabilised damped wave equation* $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ *in a bounded domain in* $\mathbb{R}^n$. Indian J. Math. **40** (1) (1998), 1-15.

[3] S.K. Bose, G.C. Gorain, *Exact controllability and boundary stabilization of torsional vibrations of an internally damped flexible space structure.* J. Optim. Theory Appl. **99** (2) (1998), 423-442.

[4] R. Chill, S. Srivastava. *L^p-maximal regularity for second order Cauchy problems.* Math. Z. **251** (2005), 751-781.

[5] R.M. Christensen, Theory of Viscoelasticity, Academic Press, New York, 1971.

[6] C. Cuevas, H. Henríquez, *Solutions of second order abstract retarded functional differential equations on the line*, Journal of Nonlinear and Convex Analysis, to appear.

[7] C. Cuevas, C. Lizama. *Well posedness for a class of flexible structure in Hölder spaces.* Mathematical Problems in Engineering Volume 2009, Article ID 358329, 13 pages, doi:10.1155/2009/358329.

[8] C. Fernández, C. Lizama, V. Poblete, *Maximal regularity for flexible structural systems in Lebesgue spaces,* Mathematical Problems in Engineering, Volume 2010, Article ID 196956, 15 pages, doi:10.1155/2010/196956.

[9] G. Gorain, *Exponential energy decay estimate for the solutions of internally damped wave equation in a bounded domain.* J. Math. Anal. Appl. **216** (1997), 510-520.

[10] G. Gorain, *Boundary stabilization of nonlinear vibrations of a flexible structure in a bounded domain in* $\mathbb{R}^N$. J. Math. Anal. Appl. **319** (2006), 635-650.

[11] G. Gorain, *Stabilization for the vibrations modeled by the "standard linear model" of viscoelasticity.* Proc. Indian Acad. Sci. (Math. Sci.) **120** (4) (2010), 495–506.

[12] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[13] E. Hernández, D. O' Regan, K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, Nonlinear Analysis, to appear.

[14] Y.C. Li, S.Y. Shaw.*Mean ergodicity and mean stability of regularized solution families.* Mediterr. J. Math. 1(2004), 175–193.

[15] C. Lizama. *Regularized solutions for abstract Volterra equations.* J. Math. Anal. Appl. 243 (2000), 278-292.

[16] C. Lizama. *On approximation and representation of k-regularized resolvent families.* Integral Equations Operator Theory 41 (2), (2001), 223-229.

[17] C. Lizama, J. Sanchez. *On perturbation of K-regularized resolvent families.* Taiwanese J. Math. 7(2) (2003), 217–227.

[18] C. Lizama, G. M. N'Guérékata. *Bounded mild solutions for semilinear integro-differential equations in Banach spaces.* Integral Equations Operator Theory, 68 (2) (2010), 207-227.

[19] S.Y.Shaw, J.C. Chen. *Asymptotic behavior of (a,k)-regularized families at zero.* Taiwanese J. Math. 10(2) (2006), 531-542.

[20] T.J. Xiao, J. Liang. *The Cauchy problem for higher-order abstract differential equations.* Lecture Notes in Mathematics, 1701. Springer-Verlag, Berlin, 1998.

[21] S. Zaidman, Almost Periodic Functions in Abstract Spaces, Research Notes in Mathematics, Pitman, London, 1985.

[22] C. Zhang, Almost Periodic Type and Ergodicity, Kluwer Academic Publishers and Science Press, 2003.

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