# Asymptotic periodicity for strongly damped wave equations

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**Abstract**

This work deals with existence and uniqueness of asymptotically almost-periodic mild solutions for a class of strongly damped semilinear wave equations.

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# 1 Introduction

Let $X$ be a reflexive Banach space and let $A : D(A) \subseteq X \to X$ be a closed densely defined operator and $\eta > 0$. Consider the Cauchy problem

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + Au = f(t, u, u_t), \ t > 0, \\ u(0) = u_0 \in X^{\frac{1}{2}}, \ \ u_t(0) = v_0 \in X, \end{cases} \tag{1.1}$$

where $X^{\frac{1}{2}}$ is the fractional power space associated with $A$ as in [25]. Equations like (1.1) appear in the literature under the name of strongly damped wave equations. A example of mathematical model represented in the form (1.1) is the wave equation with structural damping (see [9, 13, 14, 15]). The strongly damped wave equations has been investigated in several contexts by many authors in the last years, for example, existence [5, 21], global classical solution [5, 20], long time asymptotic behavior [22, 30, 36], attractor [7, 8, 9, 10, 28, 33], well-posedness [12], decay estimates [26], blow up [4, 20, 31], controllability [4], bootstrapping and regularity [11]. Another important aspect of the qualitative study of the solutions of strongly damped wave equations is their asymptotic periodicity. In recent years the study of periodicity and its various extensions for evolution equations have attracted a great deal of attention of many mathematicians (see [2, 3, 6, 16, 17, 18, 24, 27, 29, 32, 34] and references therein).

To the best of our knowledge, the study of the existence of asymptotically almost-periodic solutions for strongly damped wave equations of type (1.1) is a topic not yet considered in the literature.

Problem (1.1) can be written as a first order in time Cauchy problem in $Y^0 := X^{\frac{1}{2}} \times X$

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \mathcal{A}_{(\frac{1}{2})} \begin{bmatrix} u \\ v \end{bmatrix} = F\Big(t, \begin{bmatrix} u \\ v \end{bmatrix}\Big), \ t > 0, \tag{1.2}$$

$$\begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{1.3}$$

where

$$\mathcal{A}_{(\frac{1}{2})} = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} \end{bmatrix} : D(\mathcal{A}_{(\frac{1}{2})}) \subseteq X^{\frac{1}{2}} \times X \to X^{\frac{1}{2}} \times X$$

is defined by

$$\mathcal{A}_{(\frac{1}{2})} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A\varphi + 2\eta A^{\frac{1}{2}} \psi \end{bmatrix} \text{ for } \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\mathcal{A}_{(\frac{1}{2})}) = X^1 \times X^{\frac{1}{2}} \tag{1.4}$$

and

$$F\Big(t, \begin{bmatrix} u \\ v \end{bmatrix}\Big) = \begin{bmatrix} 0 \\ f(t, u, v) \end{bmatrix}. \tag{1.5}$$

**Definition 1.1** The pair $(\eta, A)$ is said to be an *admissible pair* if there exist $\psi \in (0, \frac{\pi}{2})$ and $M > 0$ such that

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}, \tag{1.6}$$

for all $\lambda$ in the sector $\Sigma_\psi = \{\lambda \in \mathbb{C} : \psi \leq |arg\lambda| \leq \pi\} \cup \{0\}$ and $\frac{\pi}{2} > \frac{\psi}{2} + arg(\eta + \sqrt{\eta^2 - 1})$.

If $(\eta, A)$ is an admissible pair, by [11, Proposition 2.1] $\mathcal{A}_{(\frac{1}{2})}$ is a closed operator with $0 \in \rho(\mathcal{A}_{(\frac{1}{2})})$. Indeed, $\mathcal{A}_{(\frac{1}{2})}$ has the inverse expressed in the matrix form as

$$\mathcal{A}^{-1}_{(\frac{1}{2})} = \begin{bmatrix} 2\eta A^{-\frac{1}{2}} & A^{-1} \\ -I & 0 \end{bmatrix} \in \mathcal{L}(X^{\frac{1}{2}} \times X). \tag{1.7}$$

Moreover, the operator $\mathcal{A}_{(\frac{1}{2})}$ has compact resolvent whenever $A$ has compact resolvent. By [11, Theorem 2.3] the operator $\mathcal{A}_{(\frac{1}{2})}$ is sectorial in $X^{\frac{1}{2}} \times X$. The semigroup $\{e^{-\mathcal{A}_{(\frac{1}{2})}t} : t \geq 0\}$ generated by $-\mathcal{A}_{(\frac{1}{2})}$ in $X^{\frac{1}{2}} \times X$ is exponentially decaying analytic. That is, there are constants $K \geq 1$ and $C > 0$ such that

$$\|e^{-\mathcal{A}_{(\frac{1}{2})}t}\|_{\mathcal{L}(X^{\frac{1}{2}} \times X)} \leq Ke^{-Ct}, \ t \geq 0. \tag{1.8}$$

Throughout this paper we always assume that $(\eta, A)$ is an admissible pair.

This paper has four sections. In the next section, we consider some definitions, technical aspects and basic properties related with asymptotically almost-periodic functions. In the third section, we obtain general results on the existence of asymptotically almost-periodic (mild) solutions to the problem (1.1). The main abstract results are Theorems 3.1, 3.2, 3.3 and 3.4. Finally, in the fourth section we consider several applications. In particular, we consider the following class of partial differential equations

$$u_{tt} + \nu b(t)u_t + \triangle^2 u - \delta\triangle u_t = \mu a(t)(h(u)\nabla \cdot u + g(u)\triangle u), \ x \in \Omega, \ t \geq 0, \tag{1.9}$$

$$u = \triangle u = 0, \ x \in \partial\Omega, \ t > 0, \tag{1.10}$$

$$u(0, x) = u_0(x), \ u_t(0, x) = v_0(x), \ x \in \Omega, \tag{1.11}$$

in a bounded smooth domain $\Omega \subseteq \mathbb{R}^N$ and where $h$ and $g$ satisfy certain growth conditions. We prove that if $a \in AP(\mathbb{R})$, $b \in C_0(\mathbb{R}^+, \mathbb{R})$ and $|\mu| + |\nu|$ is small enough, then the above problem has an asymptotically almost-periodic mild solution. The same type of conclusion is derived for the wide class

$$u_{tt} + \triangle^2 u - \delta\triangle u_t = a(t)\left|\int_\Omega \nabla \cdot u(t, x)dx\right|^\beta \Phi_0, \ x \in \Omega, \ t \geq 0, \tag{1.12}$$

$$u = \triangle u = 0, \ x \in \partial\Omega, \ t > 0, \tag{1.13}$$

$$u(0, x) = u_0(x), \ u_t(0, x) = v_0(x), \ x \in \Omega, \tag{1.14}$$

where $\Phi_0 \in L^p(\Omega)$ and $p > \frac{N}{2}$.

## 2 Preliminaries

In this section we present some concepts and properties needed to develop the following sections. Let $X$ be a reflexive Banach space. For an interval $I \subseteq \mathbb{R}$, $C_b(I, X)$ denotes the space formed by the bounded continuous functions from $I$ into $X$, endowed with the norm of uniform convergence. When $X = \mathbb{R}$ we denote $C_b(I)$ instead of $C_b(I, \mathbb{R})$. The notation $C_0(\mathbb{R}^+, X)$ stands for the subspace of $C_b(\mathbb{R}^+, X)$ consisting of functions that vanish at infinity. We denote by $\mathcal{L}(X)$ the Banach algebra of bounded linear operators defined on $X$. For $r > 0$, the notation $B_r(X)$ stands for the closed ball $\{x \in X : \|x\| \leq r\}$. For a linear operator $A$ with domain $D(A)$ and range $\mathcal{R}(A)$ in $X$, we represent by $\sigma(A)$ (resp. $\rho(A)$) the spectrum (resp. the resolvent set) of $A$. For $\lambda \in \rho(A)$, we denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent operator of $A$. When $A$ is closed, we denote by $[D(A)]$ the domain of $A$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

Next, we present a brief summary of the main properties of asymptotically almost-periodic functions.

**Definition 2.2** ([38]) A continuous function $f : \mathbb{R} \to X$ is called almost-periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for every interval of length $l(\varepsilon)$ it contains a number $\tau$ with the property that $\|f(t + \tau) - f(t)\| \leq \varepsilon$ for each $t \in \mathbb{R}$.

The number $\tau$ above is called an $\varepsilon$-translation number for $f$. We denote by $AP(X)$ the space formed by the almost-periodic functions $f : \mathbb{R}^+ \to X$. We note that each almost-periodic function is bounded and uniformly continuous. It is well known that the range $\mathcal{R}(f) = \{f(t) : t \in \mathbb{R}\}$ of an almost periodic function $f$ is relatively compact. The space $AP(X)$ is a Banach space endowed with the norm of uniform convergence.

Let $Y$ be a Banach space. We have the following concept of parameter-dependent almost-periodic function.

**Definition 2.3** *A continuous function $f : \mathbb{R} \times Y \to X$ is called almost-periodic in $t$ uniformly for $y$ in compact subsets of $Y$ if for every compact subset $K$ of $Y$ and each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval $I$ of length $l(\varepsilon)$ it contains a number $\tau$ with the property that $\|f(t + \tau, y) - f(t, y)\| \leq \varepsilon$ for all $t \in \mathbb{R}$, $y \in K$.*

Henceforth we abbreviate the terminology by calling almost-periodic from $\mathbb{R} \times Y$ into $X$ to those functions that are almost-periodic in $t$ uniformly for $y$ in compact subsets of $Y$, and we denote by $AP(\mathbb{R} \times Y, X)$ the set formed by the almost-periodic functions from $\mathbb{R} \times Y$ into $X$.

The proof of the following result is similar to the proof of [2, Lemma 2.12] and therefore omitted.

**Lemma 2.1** *If $h : \mathbb{R} \to X^{\frac{1}{2}} \times X$ is an almost-periodic function and*

$$u(t) = \int_{-\infty}^{t} e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)} h(s) ds, \; t \in \mathbb{R},$$

*then $u$ is almost-periodic.*

It is well known that the study of composition of two functions with special properties is important and basic for deep investigations. The following result has been established in [19].

**Lemma 2.2** *Let* $f : \mathbb{R} \times Y \to X$ *be an almost-periodic function and let* $u : \mathbb{R} \to Y$ *be an almost-periodic function. Then the function* $\mathbb{R} \to X,\ t \to f(t, u(t))$ *is almost-periodic.*

We will need the following definition.

**Definition 2.4** A continuous function $f : \mathbb{R}^+ \to X$ is called asymptotically almost-periodic if there exist two functions $f_{ap} \in AP(X)$ and $\varphi_f \in C_0(\mathbb{R}^+, X)$ such that

$$f(t) = f_{ap}(t) + \varphi_f(t),\ t \in \mathbb{R}^+.$$

The function $f_{ap}$ is called the almost-periodic part of $f$. We denote by $AAP(X)$ the space formed by the asymptotically almost-periodic functions $f : \mathbb{R}^+ \to X$. The space $AAP(X)$ is a Banach space endowed with the norm of uniform convergence. Furthermore, $AAP(X) = AP(X) \oplus C_0(\mathbb{R}^+, X)$.

In what follows $C_0(\mathbb{R}^+ \times Y, X)$ denotes the space consisting of continuous functions $f : \mathbb{R}^+ \times Y \to X$ such that $\lim_{t\to\infty} f(t, y) = 0$ uniformly for $y$ in compact subsets of $Y$.

**Definition 2.5** A continuous function $f : \mathbb{R}^+ \times Y \to X$ is called asymptotically almost-periodic if there are two functions $g \in AP(\mathbb{R} \times Y, X)$ and $\varphi \in C_0(\mathbb{R}^+ \times Y, X)$ such that

$$f(t, y) = g(t, y) + \varphi(t, y),\ y \in Y,\ t \in \mathbb{R}^+.$$

We denote by $AAP(\mathbb{R}^+ \times Y, X)$ the set consisting of all asymptotically almost-periodic functions from $\mathbb{R}^+ \times Y$ into $X$.

Let $I \subseteq \mathbb{R}$ be an interval. We have the following concept of function uniformly continuous on compacts sets.

**Definition 2.6** A continuous function $f : I \times Y \to X$ is called uniformly continuous on compact sets if for all compact set $K \subseteq Y$ and all $\varepsilon > 0$ there is $\delta_{\varepsilon,K} > 0$ such that $\|f(t, y_1) - f(t, y_2)\| \le \varepsilon$ for all $t \in I$ and $y_1,\ y_2 \in K$ with $\|y_1 - y_2\| \le \delta_{\varepsilon,K}$.

**Lemma 2.3** ([2]) *Let* $f : \mathbb{R}^+ \times Y \to X$ *be an asymptotically almost-periodic and uniformly continuous on compact sets function. Let* $u : \mathbb{R}^+ \to Y$ *be an asymptotically almost-periodic function. Then the function* $\mathbb{R}^+ \to X,\ t \to f(t, u(t))$ *is asymptotically almost-periodic.*

The proof of the following result is similar to the proof of [2, Lemma 2.13]. Therefore we will omit it.

**Lemma 2.4** *If $h : \mathbb{R}^+ \to X^{\frac{1}{2}} \times X$ is an asymptotically almost-periodic function and*

$$u(t) = \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)} h(s)ds, \ t \geq 0.$$

*Then $u$ is asymptotically almost-periodic.*

# 3 Asymptotically almost-periodic mild solutions

We recall the following definition that will be essential for us.

**Definition 3.7** Let $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ be in $X^{\frac{1}{2}} \times X$. We say that $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} : \mathbb{R}^+ \to X^{\frac{1}{2}} \times X$ is a mild solution to (1.2)-(1.3) (or to (1.1)) if satisfies the Cauchy integral formula:

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{-\mathcal{A}_{(\frac{1}{2})}t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)} F\Big(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\Big) ds, \ t \geq 0. \tag{3.1}$$

**Theorem 3.1** *Let $f : \mathbb{R}^+ \times X^{\frac{1}{2}} \times X \to X$ be an asymptotically almost-periodic function and assume that there exists a locally integrable function $L_f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_X \leq L_f(t)\big[\|u_1 - u_2\|_{X^{\frac{1}{2}}} + \|v_1 - v_2\|_X\big], \tag{3.2}$$

*for all $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in X^{\frac{1}{2}} \times X, \ i = 1, 2$ and each $t \geq 0$. If*

$$K \sup_{t \geq 0} \int_0^t e^{-C(t-s)} L_f(s) ds < 1,$$

*where $K$ and $C$ are given in (1.8). Then (1.2)-(1.3) has a unique asymptotically almost-periodic mild solution.*

**Proof.** We define the map $\mathfrak{F}$ on the space $AAP(X^{\frac{1}{2}} \times X)$ by the expression

$$\mathfrak{F}\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t) = e^{-\mathcal{A}_{(\frac{1}{2})}t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)} F\Big(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\Big) ds, \ t \in \mathbb{R}^+, \tag{3.3}$$

where

$$\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix} = \begin{bmatrix} u_{ap}(\cdot) \\ v_{ap}(\cdot) \end{bmatrix} + \begin{bmatrix} \varphi_u(\cdot) \\ \varphi_v(\cdot) \end{bmatrix}$$

is an asymptotically almost-periodic function, that is

$$\begin{bmatrix} u_{ap}(\cdot) \\ v_{ap}(\cdot) \end{bmatrix} \in AP(X^{\frac{1}{2}} \times X) \text{ and } \begin{bmatrix} \varphi_u(\cdot) \\ \varphi_v(\cdot) \end{bmatrix} \in C_0(\mathbb{R}^+, X^{\frac{1}{2}} \times X).$$

Since $f \in AAP(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$, then there are two functions $\Phi \in AP(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$ and $\Psi \in C_0(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$ so that $f = \Phi + \Psi$.

We set for $\begin{bmatrix} a \\ b \end{bmatrix} \in X^{\frac{1}{2}} \times X$

$$\tilde{\Phi}\Big(s, \begin{bmatrix} a \\ b \end{bmatrix}\Big) = \begin{bmatrix} 0 \\ \Phi(s,a,b) \end{bmatrix}, \ \tilde{\Psi}\Big(s, \begin{bmatrix} a \\ b \end{bmatrix}\Big) = \begin{bmatrix} 0 \\ \Psi(s,a,b) \end{bmatrix}.$$

For $t \in \mathbb{R}^+$, we have the following decomposition

$$\begin{aligned}
\mathfrak{F}\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t) &= \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\Big(\tilde{\Phi}\Big(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\Big) - \tilde{\Phi}\Big(s, \begin{bmatrix} u_{ap}(s) \\ v_{ap}(s) \end{bmatrix}\Big)\Big)ds \\
&+ \int_{-\infty}^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Phi}\Big(s, \begin{bmatrix} u_{ap}(s) \\ v_{ap}(s) \end{bmatrix}\Big)ds \\
&- \int_{-\infty}^0 e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Phi}\Big(s, \begin{bmatrix} u_{ap}(s) \\ v_{ap}(s) \end{bmatrix}\Big)ds \\
&+ e^{-\mathcal{A}_{(\frac{1}{2})}t}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Psi}\Big(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}\Big)ds \\
&:= \mathcal{G}(t) + \mathcal{H}(t), \qquad (3.4)
\end{aligned}$$

where

$$\mathcal{G}(t) = \int_{-\infty}^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Phi}\Big(s, \begin{bmatrix} u_{ap}(s) \\ v_{ap}(s) \end{bmatrix}\Big)ds$$

and $\mathcal{H}(t)$ denotes the remained terms of the above decomposition.

Next, let us show that $\mathcal{H} \in C_0(\mathbb{R}^+, X^{\frac{1}{2}} \times X)$. By (1.8) we have that

$$t \to e^{-\mathcal{A}_{(\frac{1}{2})}t}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in C_0(\mathbb{R}^+, X^{\frac{1}{2}} \times X). \qquad (3.5)$$

We observe that

$$\mathcal{B} := \Bigg\{\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} : \ t \geq 0\Bigg\} \ \text{and} \ \mathcal{B}_{ap} := \Bigg\{\begin{bmatrix} u_{ap}(t) \\ v_{ap}(t) \end{bmatrix} : \ t \in \mathbb{R}\Bigg\}$$

are relatively compact in $X^{\frac{1}{2}} \times X$. Since $\Psi \in C_0(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$, for each $\varepsilon > 0$ there exists a constant $T > 0$ such that

$$\|\tilde{\Psi}(s,z)\|_{X^{\frac{1}{2}} \times X} \leq \varepsilon$$

for all $t \geq 2T, \ z \in \mathcal{B}$.

We deduce

$$\left\|\int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Psi}\Big(s,\begin{bmatrix} u(s) \\ v(s)\end{bmatrix}\Big)ds\right\|_{X^{\frac{1}{2}}\times X} \leq \frac{K}{C}e^{-C\frac{t}{2}}\sup_{t\in\mathbb{R}^+,\, z\in\mathcal{B}}\|\tilde{\Psi}(s,z)\|_{X^{\frac{1}{2}}\times X}+\frac{K}{C}\varepsilon. \tag{3.6}$$

On the other hand since $\Phi\in AP(\mathbb{R}^+\times X^{\frac{1}{2}}\times X,X)$ by [37, Appendix] we get

$$\left\|\int_{-\infty}^0 e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\tilde{\Phi}\Big(s,\begin{bmatrix} u_{ap}(s) \\ v_{ap}(s)\end{bmatrix}\Big)ds\right\|_{X^{\frac{1}{2}}\times X} \leq \frac{K}{C}e^{-Ct}\sup_{t\in\mathbb{R},\, z\in\mathcal{B}_{ap}}\|\Phi(s,z)\|_X,\ \text{ for } t\in\mathbb{R}^+. \tag{3.7}$$

Next we estimate the first term of (3.4). For $\varepsilon>0$, we choose $T>0$ big enough so that

$$\|\varphi_u(s)\|_{X^{\frac{1}{2}}}+\|\varphi_v(s)\|_X\leq\varepsilon,$$

for all $s\geq T$, and $\|\tilde{\Psi}(s,z)\|_{X^{\frac{1}{2}}\times X}\leq\varepsilon$ for all $s\geq T$ and all $z\in\mathcal{B}\cup\mathcal{B}_{ap}$. We have the following estimates:

$$\begin{aligned}
&\left\|\int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}\left(\tilde{\Phi}\Big(s,\begin{bmatrix} u(s) \\ v(s)\end{bmatrix}\Big)-\tilde{\Phi}\Big(s,\begin{bmatrix} u_{ap}(s) \\ v_{ap}(s)\end{bmatrix}\Big)\right)ds\right\|_{X^{\frac{1}{2}}\times X}\\
&\leq K\int_0^t e^{-C(t-s)}L_f(s)\Big(\|u(s)-u_{ap}(s)\|_{X^{\frac{1}{2}}}+\|v(s)-v_{ap}(s)\|_X\Big)ds\\
&+K\int_0^t e^{-C(t-s)}\Big\|\tilde{\Psi}\Big(s,\begin{bmatrix} u(s) \\ v(s)\end{bmatrix}\Big)-\tilde{\Psi}\Big(s,\begin{bmatrix} u_{ap}(s) \\ v_{ap}(s)\end{bmatrix}\Big)\Big\|_{X^{\frac{1}{2}}\times X}ds\\
&\leq K\int_0^T e^{-C(t-s)}L_f(s)\Big(\|\varphi_u(s)\|_{X^{\frac{1}{2}}}+\|\varphi_v(s)\|_X\Big)ds\\
&+K\int_T^t e^{-C(t-s)}L_f(s)\Big(\|\varphi_u(s)\|_{X^{\frac{1}{2}}}+\|\varphi_v(s)\|_X\Big)ds\\
&+\Big(K\int_0^T e^{-C(t-s)}ds\Big)\Big(\sup_{\xi\in\mathbb{R}^+,\, z\in\mathcal{B}}\|\tilde{\Psi}(\xi,z)\|_{X^{\frac{1}{2}}\times X}+\sup_{\xi\in\mathbb{R}^+,\, z\in\mathcal{B}_{ap}}\|\tilde{\Psi}(\xi,z)\|_{X^{\frac{1}{2}}\times X}\Big)\\
&+K\int_T^t e^{-C(t-s)}\Big(\Big\|\tilde{\Psi}\Big(s,\begin{bmatrix} u(s) \\ v(s)\end{bmatrix}\Big)\Big\|_{X^{\frac{1}{2}}\times X}+\Big\|\tilde{\Psi}\Big(s,\begin{bmatrix} u_{ap}(s) \\ v_{ap}(s)\end{bmatrix}\Big)\Big\|_{X^{\frac{1}{2}}\times X}\Big)ds\\
&\leq Ke^{-Ct}\Big(\int_0^T e^{Cs}L_f(s)ds\Big)\sup_{\xi\in\mathbb{R}^+}\Big\|\begin{bmatrix} \varphi_u(\xi) \\ \varphi_v(\xi)\end{bmatrix}\Big\|_{X^{\frac{1}{2}}\times X}+\varepsilon K\\
&+\frac{K}{C}e^{CT}e^{-Ct}\Big(\sup_{\xi\in\mathbb{R}^+,\, z\in\mathcal{B}}\|\tilde{\Psi}(\xi,z)\|_{X^{\frac{1}{2}}\times X}+\sup_{\xi\in\mathbb{R}^+,\, z\in\mathcal{B}_{ap}}\|\tilde{\Psi}(\xi,z)\|_{X^{\frac{1}{2}}\times X}\Big)\\
&+2\varepsilon\frac{K}{C}.
\end{aligned} \tag{3.8}$$

From (3.5)-(3.8) we deduce that $\lim_{t\to\infty}\mathcal{H}(t)=0$, that is $\mathcal{H}\in C_0\Big(\mathbb{R}^+,X^{\frac{1}{2}}\times X\Big)$.

Since $\tilde{\Phi}\in AP\Big(\mathbb{R}^+\times X^{\frac{1}{2}}\times X,X^{\frac{1}{2}}\times X\Big)$ and $\begin{bmatrix} u_{ap} \\ v_{ap}\end{bmatrix}\in AP\Big(X^{\frac{1}{2}}\times X\Big)$, we get from Lemma 2.2 that

$$\tilde{\Phi}\Big(\cdot,\begin{bmatrix} u_{ap}(\cdot) \\ v_{ap}(\cdot)\end{bmatrix}\Big)\in AP\Big(X^{\frac{1}{2}}\times X\Big).$$

Now, by Lemma 2.1, we obtain that $\mathcal{G} \in AP\Big(X^{\frac{1}{2}} \times X\Big)$, and hence $\mathfrak{F}$ is well defined. It suffices to show that the operator $\mathfrak{F}$ has a unique fixed point in $AAP\Big(X^{\frac{1}{2}} \times X\Big)$.

For this, we consider $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in AAP\Big(X^{\frac{1}{2}} \times X\Big)$. We can deduce that

$$\begin{aligned}
\left\| \mathfrak{F}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \mathfrak{F}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty &= \sup_{t\geq 0} \left\| \mathfrak{F}\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \mathfrak{F}\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t) \right\|_{X^{\frac{1}{2}}\times X} \\
&\leq K \sup_{t\geq 0} \int_0^t e^{-C(t-s)} \Big\| f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s)) \Big\|_X ds \\
&\leq K \sup_{t\geq 0} \int_0^t e^{-C(t-s)} L_f(s) \Big[ \|u_1(s)-u_2(s)\|_{X^{\frac{1}{2}}} + \|v_1(s)-v_2(s)\|_X \Big] ds \\
&\leq \left( K \sup_{t\geq 0} \int_0^t e^{-C(t-s)} L_f(s) ds \right) \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty ,
\end{aligned}$$

by the contraction principle, $\mathfrak{F}$ has a unique fixed point in $AAP\Big(X^{\frac{1}{2}} \times X\Big)$. This completes the proof. ∎

**Remark 3.1** We wish to emphasize that condition (3.2) is optimal in the sense that the function $L_f(\cdot)$ is locally integrable. This is the largest possible class of Lipschitz constant $L_f(\cdot)$ for which the conclusion of Theorem 3.1 holds true. However, this condition in $L_f(\cdot)$ makes our analysis much more harder, because to prove Theorem 3.1 we can not use the standard composition lemma for asymptotically almost-periodic functions (see Lemma 2.3). To overcome this difficulty we need to use a suitable decomposition for the natural operator associated to the mild solution (see (3.1) and (3.4) ). In contrast we note that in the more restrictive case of $L_f(\cdot)$ to be an integrable bounded function we can use Lemma 2.3 directly.

**Corollary 3.1** *Let $f : \mathbb{R}^+ \times X^{\frac{1}{2}} \times X \to X$ be an asymptotically almost-periodic function that satisfies the Lipschitz condition (3.2) with $L_f(\cdot) \equiv L$. If*

$$\frac{KL}{C} < 1, \tag{3.9}$$

*then the problem (1.2)-(1.3) has a unique asymptotically almost-periodic mild solution.*

**Remark 3.2** Let $f : \mathbb{R}^+ \times X^{\frac{1}{2}} \times X \to X$ be an asymptotically almost-periodic function that satisfies the Lipschitz condition (3.2). We can avoid the condition (3.9) by using the fixed point iteration method. Indeed, we consider two cases.

**Case 1:** $L_f(\cdot) \equiv L$. We consider the following space

$$AAP^0(X^{\frac{1}{2}} \times X) := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in AAP(X^{\frac{1}{2}} \times X) : u(0) = 0,\ v(0) = 0 \right\}$$

endowed with the norm of the uniform convergence. We define the map $\Gamma$ on the space $AAP^0(X^{\frac{1}{2}} \times X)$ by

$$\Gamma\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t) = \int_0^t e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)} F\Big(s, \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} + e^{-\mathcal{A}_{(\frac{1}{2})}s} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\Big) ds. \tag{3.10}$$

Combining Lemma 2.3 and Lemma 2.4 we know that $\Gamma$ is a continuous function from $AAP^0(X^{\frac{1}{2}} \times X)$ into $AAP^0(X^{\frac{1}{2}} \times X)$. Moreover, for $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in AAP^0(X^{\frac{1}{2}} \times X),\ i = 1, 2$ we have

$$\begin{aligned} &\Big\| \Gamma\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \Gamma\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t) \Big\|_{X^{\frac{1}{2}} \times X} \\ &\leq KL \int_0^t e^{-C(t-s)} \sup_{0 \leq \xi \leq s} \Big[ \|u_1(\xi) - u_2(\xi)\|_{X^{\frac{1}{2}}} + \|v_1(\xi) - v_2(\xi)\|_X \Big] ds. \end{aligned} \tag{3.11}$$

With the notation $\Phi(s) = KLe^{-Cs}$ and

$$\alpha(s) = \sup_{0 \leq \xi \leq s} \Big[ \|u_1(\xi) - u_2(\xi)\|_{X^{\frac{1}{2}}} + \|v_1(\xi) - v_2(\xi)\|_X \Big],$$

the above estimate yields

$$\Big\| \Gamma\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \Gamma\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t) \Big\|_{X^{\frac{1}{2}} \times X} \leq (\Phi * \alpha)(t),\ t \geq 0. \tag{3.12}$$

Since $\Phi * \alpha$ is a nondecreasing function, proceeding inductively we can show that

$$\Big\| \Gamma^n\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \Gamma^n\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t) \Big\|_{X^{\frac{1}{2}} \times X} \leq (\Phi^{*n} * \alpha)(t),\ t \geq 0, \tag{3.13}$$

where $\Phi^{*n}$ denotes the $n$-fold convolution of $\Phi$ with itself.

On the other hand the map $\mathcal{S} : C_b(\mathbb{R}^+) \to C_b(\mathbb{R}^+)$ given by

$$(\mathcal{S}\alpha)(t) = \int_0^t \Phi(t-s)\alpha(s) ds,\ t \geq 0, \tag{3.14}$$

is a bounded linear map. Moreover, it follows from [23, Theorem 2.3.5] that $\sigma(\mathcal{S}) = \{0\}$, which implies that $\|\mathcal{S}^n\| = \|\Phi^{*n}\|_1 \to 0$ as $n \to \infty$. This show that $\Gamma^n$ is a contraction for $n$ sufficiently large. As a consequence $\Gamma$ has a fixed point $\begin{bmatrix} u \\ v \end{bmatrix}$ in $AAP^0(X^{\frac{1}{2}} \times X)$.

We note that the function

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + e^{-\mathcal{A}_{(\frac{1}{2})}t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{3.15}$$

is an asymptotically almost-periodic mild solution to problem (1.2)-(1.3) .

**Case 2:** $L_f(\cdot)$ in (3.2) is an integrable bounded function. By Lemma 2.3 and Lemma 2.4 the space $AAP(X^{\frac{1}{2}} \times X)$ is invariant under $\mathfrak{F}$ (see (3.3)). The fixed point iteration method and the following estimate

$$\left\| \mathfrak{F}^n \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \mathfrak{F}^n \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty \leq \frac{(K\|L_f\|_1)^n}{n!} \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_\infty \tag{3.16}$$

are responsible for the fact that $\mathfrak{F}$ has a unique fixed point in $AAP(X^{\frac{1}{2}} \times X)$. This concludes the discussion of Remark 3.2.

We have the following results of existence of local type.

**Theorem 3.2** *Let $f : \mathbb{R}^+ \times X^{\frac{1}{2}} \times X \to X$ be an asymptotically almost-periodic function that satisfies the Lipschitz condition*

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\|_X \leq L_f(r)\Big[\|u_1 - u_2\|_{X^{\frac{1}{2}}} + \|v_1 - v_2\|_X\Big], \tag{3.17}$$

*for each $t \geq 0$ and $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in X^{\frac{1}{2}} \times X$ such that $\|u_i\|_{X^{\frac{1}{2}}} + \|v_i\|_X \leq r,\ i = 1, 2$ where $L_f : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $L_f(0) = 0$ and $f(t, 0, 0) = 0$ for all $t \geq 0$. Then there exists $\varepsilon > 0$ such that for each $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in X^{\frac{1}{2}} \times X$ satisfying $\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X \leq \varepsilon$, there is a unique asymptotically almost-periodic mild solution of problem (1.2)-(1.3).*

**Proof.** We choose $\lambda,\ r > 0$ small enough such that $\lambda < 1$ and $K(\lambda + \frac{1}{C}L_f(r)) < 1$. Assume that $\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X \leq \lambda r$. We consider the space

$$AAP_r := \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in AAP(X^{\frac{1}{2}} \times X) : u(0) = u_0,\ v(0) = v_0,\ \|u(t)\|_{X^{\frac{1}{2}}} + \|v(t)\|_X \leq r,\ t \geq 0 \right\},$$

endowed with the norm of the uniform convergence. We consider the map $\mathfrak{F}$ given by (3.3) on $AAP_r$. By Lemma 2.3

$$\mathfrak{F}(AAP_r) \subseteq AAP(X^{\frac{1}{2}} \times X).$$

We next prove that $\mathfrak{F}(AAP_r) \subseteq AAP_r$.

In fact, if $\begin{bmatrix} u \\ v \end{bmatrix} \in AAP_r$, we have

$$\begin{aligned}\left\|\mathfrak{F}\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} &\leq Ke^{-Ct}\Big[\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X\Big] \\ &+ K\int_0^t e^{-C(t-s)}L_f(r)\Big[\|u(s)\|_{X^{\frac{1}{2}}} + \|v(s)\|_X\Big]ds \\ &\leq K\Big(\lambda + \frac{1}{c}L_f(r)\Big)r \\ &\leq r, \end{aligned} \tag{3.18}$$

which permit us to infer that $\mathfrak{F}(AAP_r) \subseteq AAP_r$.

On the other hand, for $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in X^{\frac{1}{2}} \times X,\ i = 1, 2$ we have that

$$\begin{aligned}&\left\|\mathfrak{F}\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \mathfrak{F}\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} \\ &\leq KL_f(r)\int_0^t e^{-C(t-s)}\Big[\|u_1(s) - u_2(s)\|_{X^{\frac{1}{2}}} + \|v_1(s) - v_2(s)\|_X\Big]ds \\ &\leq \frac{K}{C}L_f(r)\left\|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right\|_\infty, \end{aligned} \tag{3.19}$$

which shows that $\mathfrak{F}$ is a contraction from $AAP_r$ into itself. Therefore, the assertion holds for $\epsilon = \lambda r$. ■

**Theorem 3.3** *Let $f \in AAP(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$ be a function that satisfies the local Lipschitz condition (3.17) with $L_f(\cdot)$ a nondecreasing function. Assume that there is a constant $r > 0$ such that*

$$\frac{K}{C}\Big(L_f(r + C_\#) + \frac{1}{r}\Big(L_f(C_\#)C_\# + \sup_{t\geq 0}\|f(t, 0, 0)\|\Big)\Big) < 1, \tag{3.20}$$

*where $C_\# = K\Big[\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X\Big]$, $K$ and $C$ are the constant given in (1.8). Then there is an asymptotically almost-periodic mild solution of problem (1.2)-(1.3).*

**Proof.** We define the map $\Gamma$ on the closed ball

$$\mathcal{B}_r = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in AAP^0(X^{\frac{1}{2}} \times X) : \|u(t)\|_{X^{\frac{1}{2}}} + \|v(t)\|_X \leq r,\ t \geq 0\right\}$$

by means of the expression (3.10).

If $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{B}_r$, we have the estimate

$$\begin{aligned}
\left\|\Gamma\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} &\leq \left\|\Gamma\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t) - \Gamma\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} + \left\|\Gamma\Big(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} \\
&\leq KL_f(r+C_{\#})\int_0^t e^{-C(t-s)}\Big[\|u(s)\|_{X^{\frac{1}{2}}} + \|v(s)\|_X\Big]ds \\
&+ K\int_0^t e^{-C(t-s)}\Big\|F\Big(s, e^{-\mathcal{A}_{(\frac{1}{2})}s}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\Big)\Big\|_{X^{\frac{1}{2}}\times X} ds \\
&\leq \frac{K}{C}L_f(r+C_{\#})r + \frac{K}{C}\Big(L_f(C_{\#})C_{\#} + \sup_{t\geq 0}\Big\|f(t,0,0)\Big\|\Big) \\
&\leq \frac{K}{C}\Big(L_f(r+C_{\#}) + \frac{1}{r}\Big(L_f(C_{\#})C_{\#} + \sup_{t\geq 0}\Big\|f(t,0,0)\Big\|\Big)\Big)r \\
&\leq r. 
\end{aligned} \tag{3.21}$$

Moreover, for $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in \mathcal{B}_r,\ i=1,2$ we have

$$\begin{aligned}
&\left\|\Gamma\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t) - \Gamma\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} \\
&\leq KL_f(r+C_{\#})\int_0^t e^{-C(t-s)}\Big[\|u_1(s)-u_2(s)\|_{X^{\frac{1}{2}}} + \|v_1(s)-v_2(s)\|_X\Big]ds \\
&\leq KL_f(r+C_{\#})\left\|\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\right\|_{\infty}.
\end{aligned} \tag{3.22}$$

Using (3.22) we get that $\Gamma$ is a contraction on $\mathcal{B}_r$. ■

In many concrete situations the operator $A$ has compact resolvent, which in turn implies that the semigroup $e^{-\mathcal{A}_{(\frac{1}{2})}t}$ generated by the operator $-\mathcal{A}_{(\frac{1}{2})}$ is compact for $t>0$. To exploit this property of compactness, we need to introduce some preliminaries.

Let $Y$ be an arbitrary Banach space and let $h:[0,\infty)\to[1,\infty)$ be a nondecreasing continuous function such that $h(t)\to\infty$ when $t\to\infty$. In next, we denote by $C_h(Y)$ the space

$$C_h(Y) = \Big\{u \in C(\mathbb{R}^+, Y) : \lim_{t\to 0}\frac{u(t)}{h(t)} = 0\Big\}$$

endowed with the norm

$$\|u\|_h = \sup_{t\geq 0}\frac{\|u(t)\|_Y}{h(t)}.$$

For reference purposes, we state the following property.

**Lemma 3.5** ([17]) *A set $\mathcal{K}\subseteq C_h(Y)$ is relatively compact in $C_h(Y)$ if the following conditions are fulfilled:*

(C1) *For all $b>0$, the set $\mathcal{K}_b = \{u|_{[0,b]} : u\in\mathcal{K}\}$ is relatively compact in $C([0,b],Y)$.*

(C2) $\lim_{t\to\infty} \dfrac{\|u(t)\|_Y}{h(t)} = 0$ *uniformly for* $u \in \mathcal{K}$.

To establish our next result we introduce the following condition.

$(W)$ There is a continuous nondecreasing function $W_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,u,v)\|_X \leq W_f(\|u\|_{X^{\frac{1}{2}}} + \|v\|_X),$$

for all $t \geq 0$ and all $\begin{bmatrix} u \\ v \end{bmatrix} \in X^{\frac{1}{2}} \times X$.

We next denote

$$\beta(L) = \sup_{t\geq 0} \frac{K}{h(t)} \int_0^t e^{-C(t-s)} W_f(Lh(s))ds.$$

We have the following result.

**Theorem 3.4** *Assume that the operator $A$ in (1.1) has compact resolvent. Suppose, in addition, that the following conditions are fulfilled:*

(a) *The function* $f \in AAP(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$ *is uniformly continuous on compact sets and satisfies the condition* $(W)$.

(b) *For each* $a \geq 0,\ r > 0$ *and* $0 \leq \xi \leq a$,

$$\Big(e^{-\mathcal{A}_{(\frac{1}{2})}(a+s-\xi)} - e^{-\mathcal{A}_{(\frac{1}{2})}(a-\xi)}\Big) F\Big(\xi, \begin{bmatrix} u \\ v \end{bmatrix}\Big) \to 0$$

*as* $s \to \infty$ *uniformly for all* $\begin{bmatrix} u \\ v \end{bmatrix}$ *so that* $\|u\|_{X^{\frac{1}{2}}} + \|v\|_X \leq r$.

(c) *For each* $L > 0$,

$$\lim_{t\to\infty} \frac{1}{h(t)} \int_0^t e^{-C(t-s)} W_f(Lh(s))ds = 0.$$

(d) *For each* $\varepsilon > 0$, *there is* $\delta > 0$ *such that for every* $\begin{bmatrix} u_i \\ v_i \end{bmatrix} \in C_h(X^{\frac{1}{2}} \times X),\ i = 1,2$ *with*

$$\frac{1}{h(t)} \Big[\|u_1(t) - u_2(t)\|_{X^{\frac{1}{2}}} + \|v_1(t) - v_2(t)\|_X\Big] \leq \delta,\ t \geq 0$$

*implies that*

$$\int_0^t e^{-C(t-s)} \|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\|_X ds \leq \varepsilon,$$

*for each* $t \geq 0$.

(e) $\liminf_{\xi\to\infty} \dfrac{\beta(\xi)}{\xi} < 1$.

*Then problem (1.2)-(1.3) has an asymptotically almost-periodic mild solution.*

**Proof.** Let $C_h^0(X^{\frac{1}{2}}\times X)$ be the space consisting of the functions $\begin{bmatrix} u(\cdot) \\ v(\cdot) \end{bmatrix}$ in $C_h(X^{\frac{1}{2}}\times X)$ such that $u(0)=0,\ v(0)=0$. It is clear that $C_h^0(X^{\frac{1}{2}}\times X)$ is a closed subspace of $C_h(X^{\frac{1}{2}}\times X)$. We define the operator $\Gamma$ on $C_h^0(X^{\frac{1}{2}}\times X)$ by (3.10). It follows from conditions $(a)$ and $(c)$ that

$$\frac{1}{h(t)}\left\|\Gamma\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} \le \frac{K}{h(t)}\int_0^t e^{-C(t-s)}W_f(Lh(s))ds \to 0,\ t\to\infty,$$

where

$$L = \left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_h + K(\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X).$$

Thus we have that $\Gamma : C_h^0(X^{\frac{1}{2}}\times X)\to C_h^0(X^{\frac{1}{2}}\times X)$.

We divide the rest of the proof into several steps.

(i) The map $\Gamma$ is continuous. For each $\varepsilon>0$ there is $\delta>0$ such that for $\begin{bmatrix} u_i \\ v_i \end{bmatrix}\in C_h^0(X^{\frac{1}{2}}\times X),\ i=1,2$ with

$$\left\|\begin{bmatrix} u_1-u_2 \\ v_1-v_2 \end{bmatrix}\right\|_h \le \delta$$

implies that

$$\frac{1}{h(t)}\left\|\Gamma\Big(\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}\Big)(t)-\Gamma\Big(\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} \le \frac{1}{h(t)}\varepsilon \le \varepsilon,$$

which shows the assertion.

(ii) The map $\Gamma$ is completely continuous. We take $r>0$ and we set $V=\Gamma(B_r(C_h^0(X^{\frac{1}{2}}\times X)))$.

For $t\ge 0$, we set

$$V(t) := \left\{\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} : \begin{bmatrix} u \\ v \end{bmatrix}\in V\right\}.$$

We first show that $V(t)$ is a relatively compact set in $X^{\frac{1}{2}}\times X$ for each $t\ge 0$. It follows from the mean value theorem that $V(t)\in\overline{tc(\mathcal{K}_0)}$, where $c(\mathcal{K}_0)$ denotes the convex hull of

$$\mathcal{K}_0=\left\{e^{-\mathcal{A}_{(\frac{1}{2})}(t-s)}F\Big(s,\begin{bmatrix} u \\ v \end{bmatrix}\Big) :\ 0\le s\le t,\ \|u\|_{X^{\frac{1}{2}}}+\|v\|_X\le\rho\right\},$$

where $\rho=h(t)(r+K(\|u_0\|_{X^{\frac{1}{2}}}+\|v_0\|_X))$, and $K$ is a constant given in (1.8).

Since

$$\sup\left\{\left\|F\Big(s,\begin{bmatrix} u \\ v \end{bmatrix}\Big)\right\|_{X^{\frac{1}{2}}\times X} : 0\le s\le t,\ \|u\|_{X^{\frac{1}{2}}}+\|v\|_X\le\rho\right\}\le W_f(\rho),$$

and taking into account that $e^{-\mathcal{A}_{(\frac{1}{2})}t}$ is compact for $t>0$, we infer that $\mathcal{K}_0$ is a relatively compact set in $X^{\frac{1}{2}}\times X$ and consequently $V(t)$ is also relatively compact. Let now $b>0$ fixed and $V_b$ the set formed by the functions $\begin{bmatrix} u \\ v \end{bmatrix} \in V$ restricted to the interval $[0,b]$. We affirm that the set $V_b$ is equi-continuous. In fact, if

$$\begin{bmatrix} u \\ v \end{bmatrix} = \Gamma\Big(\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}\Big), \text{ with } \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in B_r\Big(C_h^0(X^{\frac{1}{2}}\times X)\Big).$$

For $s\geq 0$, we obtain the following estimate

$$\begin{aligned}
&\left\|\begin{bmatrix} u(t+s)-u(t) \\ v(t+s)-v(t) \end{bmatrix}\right\|_{X^{\frac{1}{2}}\times X} \\
&\leq K\int_t^{t+s} e^{-C(t+s-\xi)}W_f(Lh(\xi))d\xi \\
&+\int_0^t \left\|\Big(e^{-\mathcal{A}_{(\frac{1}{2})}(t+s-\xi)}-e^{-\mathcal{A}_{(\frac{1}{2})}(t-\xi)}\Big)F\Big(\xi,\begin{bmatrix} \tilde{u}(\xi) \\ \tilde{v}(\xi) \end{bmatrix}+e^{-\mathcal{A}_{(\frac{1}{2})}\xi}\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\Big)\right\|_{X^{\frac{1}{2}}\times X} d\xi,
\end{aligned}$$

where $L=r+K(\|u_0\|_{X^{\frac{1}{2}}}+\|v_0\|_X)$. It is immediate that the first term on the right hand side converges to zero when $s\to 0$ and, using condition $(b)$ we obtain the second term on the right hand side also converges to zero when $s\to 0$ and the convergence is independent of the function $\begin{bmatrix} \tilde{u}(\cdot) \\ \tilde{v}(\cdot) \end{bmatrix}$.

We now show that

$$\lim_{t\to\infty}\frac{1}{h(t)}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}=0,$$

independent of $\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in B_r\Big(C_h^0(X^{\frac{1}{2}}\times X)\Big)$. This assertion is a direct consequence of the following estimate and the condition $(c)$

$$\frac{1}{h(t)}\left\|\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}\right\|_{X^{\frac{1}{2}}\times X} \leq \frac{K}{h(t)}\int_0^t e^{-C(t-s)}W_f(Lh(s))ds.$$

Combining these assertions with Lemma 3.5, we get that $V$ is a relatively compact set in $C_h^0(X^{\frac{1}{2}}\times X)$. Since $r$ was chosen arbitrary, this proves that $\Gamma$ is completely continuous.

(iii) There is $r_0>0$ such that $\Gamma\Big(B_{r_0}(C_h^0(X^{\frac{1}{2}}\times X))\Big)\subseteq B_{r_0}(C_h^0(X^{\frac{1}{2}}\times X))$. In fact, if we assume that the assertion is false, then for all $r>0$ we can choose $\begin{bmatrix} u^r \\ v^r \end{bmatrix} \in B_{r_0}(C_h^0(X^{\frac{1}{2}}\times X))$ such that

$$\frac{1}{h(t)}\left\|\Gamma\Big(\begin{bmatrix} u^r \\ v^r \end{bmatrix}\Big)(t)\right\|_{X^{\frac{1}{2}}\times X} > r$$

for all $t \in \mathbb{R}^+$. Then

$$1 \leq \left(1 + \frac{K(\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X)}{r}\right) \frac{\beta\Big(r + K(\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X)\Big)}{r + K(\|u_0\|_{X^{\frac{1}{2}}} + \|v_0\|_X)}$$

and

$$1 \leq \liminf_{\xi \to \infty} \frac{\beta(\xi)}{\xi},$$

which contradicts condition $(e)$ and establishes the assertion.

(iv) If $\begin{bmatrix} u \\ v \end{bmatrix} \in AAP^0(X^{\frac{1}{2}} \times X)$, then the function $\mathbb{R}^+ \to X^{\frac{1}{2}} \times X$ given by

$$t \to \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + e^{-\mathcal{A}_{(\frac{1}{2})} t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

is in $AAP(X^{\frac{1}{2}} \times X)$. Since $f \in AAP(\mathbb{R}^+ \times X^{\frac{1}{2}} \times X, X)$ is uniformly continuous on compact sets, we have from Lemma 2.3 that

$$t \to F\Big(t, \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + e^{-\mathcal{A}_{(\frac{1}{2})} t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\Big) \in AAP(X^{\frac{1}{2}} \times X).$$

Applying Lemma 2.4 we obtain that $\Gamma(AAP^0(X^{\frac{1}{2}} \times X)) \subseteq AAP^0(X^{\frac{1}{2}} \times X)$. Consequently, combining with $(iii)$ we infer that

$$\Gamma\Big(\overline{B_{r_0}(C_h^0(X^{\frac{1}{2}} \times X)) \cap AAP^0(X^{\frac{1}{2}} \times X)}^h\Big) \subseteq \overline{B_{r_0}(C_h^0(X^{\frac{1}{2}} \times X)) \cap AAP^0(X^{\frac{1}{2}} \times X)}^h,$$

where $\overline{\mathcal{S}}^h$ denotes the closure of $\mathcal{S}$ in $C_h(X^{\frac{1}{2}} \times X)$. Using Schauder's fixed point theorem, we deduce that $\Gamma$ has a fixed point $\begin{bmatrix} u \\ v \end{bmatrix} \in \overline{B_{r_0}(C_h^0(X^{\frac{1}{2}} \times X)) \cap AAP^0(X^{\frac{1}{2}} \times X)}^h$.

(v) Finally, we show that $\begin{bmatrix} u \\ v \end{bmatrix} \in AAP^0(X^{\frac{1}{2}} \times X)$. Let $\Big(\begin{bmatrix} u^n \\ v^n \end{bmatrix}\Big)_n$ be a sequence in $B_{r_0}(C_h^0(X^{\frac{1}{2}} \times X)) \cap AAP^0(X^{\frac{1}{2}} \times X)$ that converges to $\begin{bmatrix} u \\ v \end{bmatrix}$ for the topology in $C_h(X^{\frac{1}{2}} \times X)$. For $\varepsilon > 0$, let $\delta > 0$ be the constant in condition $(d)$, there is $n_0 \in \mathbb{N}$ so that

$$\frac{1}{h(t)} \left\| \begin{bmatrix} u^n(t) - u(t) \\ v^n(t) - v(t) \end{bmatrix} \right\|_{X^{\frac{1}{2}} \times X} \leq \delta,$$

for all $t \geq 0$ and all $n \geq n_0$. Therefore, for $n \geq n_0$

$$\sup_{t \geq 0} \left\| \Gamma\Big(\begin{bmatrix} u^n \\ v^n \end{bmatrix}\Big)(t) - \Gamma\Big(\begin{bmatrix} u \\ v \end{bmatrix}\Big)(t) \right\|_{X^{\frac{1}{2}} \times X} \leq \varepsilon.$$

Hence

$$\lim_{n\to\infty}\left\|\Gamma\begin{bmatrix} u^n \\ v^n \end{bmatrix}-\begin{bmatrix} u \\ v \end{bmatrix}\right\|_\infty=0.$$

Since $\Gamma\begin{bmatrix} u^n \\ v^n \end{bmatrix}\in AAP^0(X^{\frac{1}{2}}\times X)$ we get that $\begin{bmatrix} u \\ v \end{bmatrix}\in AAP^0(X^{\frac{1}{2}}\times X)$ and completes the proof. ■

**Remark 3.3** Note that in Theorem 3.4 we do not need to assume that the operator $A$ in (1.1) has compact resolvent if the following condition holds.

(f) For all $a\geq 0$ and $r>0$ the set $\{f(t,\varphi_1,\varphi_2): 0\leq t\leq a, ||\varphi_1||_{X^{\frac{1}{2}}}+||\varphi_2||_X\leq r\}$ is relatively compact in $X$.

## 4 Applications

Suppose that $h,\ g\in C(\mathbb{R},\mathbb{R})$, $a:\mathbb{R}\to\mathbb{R}$ ($b:\mathbb{R}^+\to\mathbb{R}$) is a bounded continuous function, $\nu,\mu\in\mathbb{R}$, $\delta>0$ and $\rho_1,\ \rho_2\in(1,+\infty)$. In a bounded smooth domain $\Omega\subseteq\mathbb{R}^N$ we consider the following partial differential equation.

$$u_{tt}+\nu b(t)u_t+\triangle^2 u-\delta\triangle u_t=\mu a(t)(h(u)\nabla\cdot u+g(u)\triangle u),\ x\in\Omega,\ t\geq 0, \tag{4.1}$$

$$u=\triangle u=0,\ x\in\partial\Omega,\ t>0, \tag{4.2}$$

$$u(0,x)=u_0(x),\ u_t(0,x)=v_0(x),\ x\in\Omega, \tag{4.3}$$

where $h$ and $g$ satisfy the following growth conditions

$$|h(s_1)-h(s_2)|\leq C_h|s_1-s_2|(1+|s_1|^{\rho_1-1}+|s_2|^{\rho_1-1}),\ s_1,\ s_2\in\mathbb{R}, \tag{4.4}$$

$$|g(s_1)-g(s_2)|\leq C_g|s_1-s_2|(1+|s_2|^{\rho_2-1}+|s_1|^{\rho_2-1}),\ s_1,\ s_2\in\mathbb{R}, \tag{4.5}$$

where $C_h$ and $C_g$ are positive constants. Here we describe the asymptotically almost-periodic behavior of solutions of problem (4.1)-(4.3) in the $L^p$-setting. To model this problem in the abstract form (1.1) we set $\eta=\frac{\delta}{2},\ p>\frac{N}{2}$, the operator $A$ is defined in $L^p(\Omega)$ by $Au=\triangle_D^2 u$ ($\triangle_D$ is the Dirichlet Laplacian in $\Omega$) on the domain

$$D(\triangle_D^2)=\{\phi\in H_p^4(\Omega):\phi=\triangle\phi=0 \text{ on } \partial\Omega\}, \tag{4.6}$$

where $H_p^4(\Omega)=W^{4,p}(\Omega)$ is the standard Sobolev space (see [35]). With this specification problem (4.1)-(4.3) will fall into the abstract formulation (1.1). Since $A^{\frac{1}{2}}=-\triangle_D$, we can choose the angle $\psi$ for the sector

$$\Sigma_{\frac{\psi}{2}}=\left\{\lambda\in\mathbb{C}:\frac{\psi}{2}\leq|arg\lambda|\leq\pi \text{ with } \psi\in(0,\frac{\pi}{2})\right\} \tag{4.7}$$

as small as needed and therefore (see [11, Example 4.3]) $(\eta,A)$ will be an admissible pair for any $\eta>0$. From [11, Section 3] we get that

$$\left[L^p(\Omega)\right]^{\frac{1}{2}}=\left\{\phi\in H_p^2(\Omega):\phi=\triangle\phi=0\,\text{on}\,\partial\Omega\right\}.$$

We define $f : \mathbb{R}^+ \times \Big[L^p(\Omega)\Big]^{\frac{1}{2}} \times L^p(\Omega) \to L^p(\Omega)$ by

$$f(t,\varphi_1,\varphi_2) = \mu a(t)\Big(h^e(\varphi_1)\nabla\cdot\varphi_1 + g^e(\varphi_1)\triangle\varphi_1\Big) - \nu b(t)\varphi_2,\ t\in\mathbb{R}^+,\ \varphi_1 \in \Big[L^p(\Omega)\Big]^{\frac{1}{2}},\ \varphi_2\in L^p(\Omega), \tag{4.8}$$

where $\theta^e$ is the Nemytskiĭ operator associated to $\theta$, and $\nabla\cdot\varphi$ represents the divergence of $\varphi$. For $\varphi_1 \in \Big[L^p(\Omega)\Big]^{\frac{1}{2}}$ and $\varphi_2 \in L^p(\Omega)$. Using Minkowski's inequality and Sobolev embedding, we have the estimate

$$\|f(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)} \quad\leq\quad |\mu||a(t)|\Big(\|h^e(\varphi_1)\|_{L^\infty(\Omega)} + \|g^e(\varphi_1)\|_{L^\infty(\Omega)}\Big)\|\varphi_1\|_{H^2_p(\Omega)} + |\nu||b(t)|\|\varphi_2\|_{L^p(\Omega)}. \tag{4.9}$$

Whence $f$ is well defined. We claim that $f$ satisfies (3.17) with

$$L_f(r) = \widetilde{C}(|\mu|+|\nu|)(\|a\|_\infty + \|b\|_\infty)(1+2r+3(r^{\rho_1}+r^{\rho_2})).$$

Indeed, it is an easy consequence of the following estimates (here $\widetilde{C}$ will stand for some positive constant independent of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ with $\tilde{\varphi}_1,\tilde{\varphi}_2 \in \Big[L^p(\Omega)\Big]^{\frac{1}{2}}$ ).

$$\|(h^e(\tilde{\varphi}_1) - h^e(\tilde{\varphi}_2))\nabla\cdot\tilde{\varphi}_1\|_{L^p(\Omega)} \leq \widetilde{C}\|\tilde{\varphi}_1-\tilde{\varphi}_2\|_{H^2_p(\Omega)}(1+\|\tilde{\varphi}_1\|^{\rho_1-1}_{H^2_p(\Omega)} + \|\tilde{\varphi}_2\|^{\rho_1-1}_{H^2_p(\Omega)})\|\tilde{\varphi}_1\|_{H^2_p(\Omega)}, \tag{4.10}$$

$$\|h^e(\tilde{\varphi}_2)\nabla\cdot(\tilde{\varphi}_1-\tilde{\varphi}_2)\|_{L^p(\Omega)} \leq \widetilde{C}(1+\|\tilde{\varphi}_2\|_{H^2_p(\Omega)} + \|\tilde{\varphi}_2\|^{\rho_1}_{H^2_p(\Omega)})\|\tilde{\varphi}_1-\tilde{\varphi}_2\|_{H^2_p(\Omega)}, \tag{4.11}$$

$$\|g^e(\tilde{\varphi}_1)\triangle(\tilde{\varphi}_1-\tilde{\varphi}_2)\|_{L^p(\Omega)} \leq \widetilde{C}(1+\|\tilde{\varphi}_1\|_{H^2_p(\Omega)} + \|\tilde{\varphi}_1\|^{\rho_2}_{H^2_p(\Omega)})\|\tilde{\varphi}_2\|_{H^2_p(\Omega)}, \tag{4.12}$$

$$\|(g^e(\tilde{\varphi}_1) - g^e(\tilde{\varphi}_2))\triangle\tilde{\varphi}_2\|_{L^p(\Omega)} \leq \widetilde{C}\|\tilde{\varphi}_1-\tilde{\varphi}_2\|_{H^2_p(\Omega)}(1+\|\tilde{\varphi}_1\|^{\rho_2-1}_{H^2_p(\Omega)} + \|\tilde{\varphi}_2\|^{\rho_2-1}_{H^2_p(\Omega)})\|\tilde{\varphi}_2\|_{H^2_p(\Omega)}. \tag{4.13}$$

Let $K$ be a compact subset of $\Big[L^p(\Omega)\Big]^{\frac{1}{2}} \times L^p(\Omega)$. For $\varphi_1 \in \Big[L^p(\Omega)\Big]^{\frac{1}{2}}$ and $\varphi_2 \in L^p(\Omega)$, we set

$$f_{ap}(t,\varphi_1,\varphi_2) = \mu a(t)(h^e(\varphi_1)\nabla\cdot\varphi_1 + g^e(\varphi_1)\triangle\varphi_1),$$

$$\Phi_f(t,\varphi_1,\varphi_2) = \nu b(t)\varphi_2,$$

$$M_K = \sup\left\{\left\|\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}\right\|_{[L^p(\Omega)]^{\frac{1}{2}}\times L^p(\Omega)} : \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}\in K\right\}. \tag{4.14}$$

We have the following estimates:

$$\|\Phi_f(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)} \leq |\nu||b(t)|M_K, \tag{4.15}$$

and

$$\begin{aligned}
&\|f_{ap}(t+\tau,\varphi_1,\varphi_2) - f_{ap}(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)}\\
&\leq |\mu||a(t+\tau)-a(t)|\Big(\|h^e(\varphi_1)\|_{L^\infty(\Omega)} + \|g^e(\varphi_1)\|_{L^\infty(\Omega)}\Big)\|\varphi_1\|_{H^2_p(\Omega)}\\
&\leq \widetilde{C}|\mu||a(t+\tau)-a(t)|(1+\|\varphi_1\|_{H^2_p(\Omega)} + \|\varphi_1\|^{\rho_1}_{H^2_p(\Omega)} + \|\varphi_1\|^{\rho_2}_{H^2_p(\Omega)})\|\varphi_1\|_{H^2_p(\Omega)}\\
&\leq \widetilde{C}(K)|a(t+\tau)-a(t)|,
\end{aligned} \tag{4.16}$$

where $\widetilde{C}(K)$ is a constant depending on $K$.

If $b \in C_0(\mathbb{R}^+, \mathbb{R})$ from (4.15) we get

$$\Phi_f \in C_0(\mathbb{R}^+ \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega)),$$

and (4.16) implies that

$$f_{ap} \in AP(\mathbb{R} \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega)).$$

Hence

$$f \in AAP(\mathbb{R}^+ \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega)).$$

Applying Theorem 3.3 we have the following result.

**Proposition 4.1** *Under the above conditions, if $a \in AP(\mathbb{R})$, $b \in C_0(\mathbb{R}^+, \mathbb{R})$ and $|\mu| + |\nu|$ is small enough, then problem (4.1)-(4.3) has an asymptotically almost-periodic mild solution.*

**Example 1** We consider the following partial differential equation

$$u_{tt} + \frac{\nu}{t+1} u_t + \triangle u^2 - \delta \triangle u_t = \mu(cos\ t + cos\ \sqrt{2}t)|u|^{\rho-1}(u\nabla \cdot u + u\triangle u),\ x \in \Omega, t \geq 0, \tag{4.17}$$

$$u = \triangle u = 0,\ x \in \partial\Omega,\ t > 0, \tag{4.18}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega. \tag{4.19}$$

If $|\mu| + |\nu|$ is small enough by Proposition 4.1, problem (4.17)-(4.19) has an asymptotically almost-periodic mild solution. In fact, take $h(u) = g(u) = u|u|^{\rho-1}$, $a(t) = cos\ t + cos\ \sqrt{2}t$ and $b(t) = \frac{1}{t+1}$.

The next application is a modification of the problem (4.1)-(4.3). Let $k$ be a fixed non-negative integer and let $c : \mathbb{R} \to \mathbb{R}^+$ and $b : \mathbb{R}^+ \to \mathbb{R}$ be two bounded continuous functions. Suppose that $f \in AAP(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies the following growth condition

$$|f(t,x,u,w) - f(t,x,\tilde{u},\tilde{w})| \leq c(t)\big[|w-\tilde{w}| + |u-\tilde{u}|(1+|u|^k+|\tilde{u}|^k)\big], \tag{4.20}$$

for each $t \geq 0$, $x \in \mathbb{R}^N$, $u, \tilde{u}, w, \tilde{w} \in \mathbb{R}$. We consider the following partial differential equation

$$u_{tt} + b(t)u_t + \triangle^2 u - \delta\triangle u_t = f(t,x,u,\nabla \cdot u),\ x \in \Omega,\ t \geq 0, \tag{4.21}$$

$$u = \triangle u = 0,\ x \in \partial\Omega,\ t > 0, \tag{4.22}$$

$$u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega. \tag{4.23}$$

We model (4.21)-(4.23) in the abstract form (1.1) in a similar way to problem (4.1)-(4.3). That is we set $\eta = \frac{\delta}{2}$, $p > N$ (hence $H^2_p(\Omega) \hookrightarrow C^1(\Omega)$ which will be useful to get some a priori estimates) and we consider the operator $A$ as defined in (4.6) and (4.7). Suppose that

$$\sup_{t\geq 0}\Big(\int_\Omega |f(t,x,0,0)|^p dx\Big)^{\frac{1}{p}} < \infty. \tag{4.24}$$

We define $\tilde{F}:\mathbb{R}^+\times\Big[L^p(\Omega)\Big]^{\frac{1}{2}}\times L^p(\Omega)\to L^p(\Omega)$ by

$$\tilde{F}(t,\varphi_1,\varphi_2)(x)=f(t,x,\varphi_1(x),\nabla\cdot\varphi_1(x))-b(t)\varphi_2(x). \tag{4.25}$$

We observe that $\tilde{F}$ is well defined. In fact, we get the following estimates

$$\begin{aligned}
&\Big(\int_\Omega|f(t,x,\varphi_1(x),\nabla\cdot\varphi_1(x))|^pdx\Big)^{\frac{1}{p}}\\
&\le c(t)\bigg(\Big(\int_\Omega|\nabla\cdot\varphi_1(x)|^pdx\Big)^{\frac{1}{p}}+\Big(\int_\Omega|\varphi_1(x))|^p(1+|\varphi_1(x)|^k)^pdx\Big)^{\frac{1}{p}}+\Big(\int_\Omega|f(t,x,0,0)|^pdx\Big)^{\frac{1}{p}}\bigg)\\
&\le ||c||_\infty\Big(||\nabla\cdot\varphi_1||_{L^p(\Omega)}+(1+||\varphi_1||^k_{L^\infty(\Omega)})||\varphi_1||_{L^p(\Omega)}+\Big(\int_\Omega|f(t,x,0,0)|^pdx\Big)^{\frac{1}{p}}\Big).
\end{aligned} \tag{4.26}$$

On the other hand for $\varphi_1,\tilde{\varphi}_1\in\Big[L^p(\Omega)\Big]^{\frac{1}{2}}$ and $\varphi_2,\tilde{\varphi}_2\in L^p(\Omega)$ we have the following estimate:

$$\begin{aligned}
&\|\tilde{F}(t,\varphi_1,\varphi_2)-\tilde{F}(t,\tilde{\varphi}_1,\tilde{\varphi}_2)\|_{L^p(\Omega)}\\
&\le 4\max\{||c||_\infty,||b||_\infty\}(1+\|\varphi_1\|^k_{H^2_p(\Omega)}+\|\tilde{\varphi}_1\|^k_{H^2_p(\Omega)})(\|\varphi_1-\tilde{\varphi}_1\|_{H^2_p(\Omega)}+\|\varphi_2-\tilde{\varphi}_2\|_{L_p(\Omega)}),
\end{aligned} \tag{4.27}$$

which means that $\tilde{F}$ satisfies (3.17) with $L_{\tilde{F}}(r)=4\max\{||c||_\infty,||b||_\infty\}(1+2r^k)$.

Since $f\in AAP(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ there are two functions

$$f_{ap}\in AP(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R},\mathbb{R}), \tag{4.28}$$

$$\varphi_f\in C_0(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R},\mathbb{R}), \tag{4.29}$$

so that $f=f_{ap}+\varphi_f$. Let $K$ be a compact subset of $\Big[L^p(\Omega)\Big]^{\frac{1}{2}}\times L^p(\Omega)$. For $\varphi_1\in\Big[L^p(\Omega)\Big]^{\frac{1}{2}}$ and $\varphi_2\in L^p(\Omega)$, we set

$$\tilde{F}_{ap}(t,\varphi_1,\varphi_2)(x)=f_{ap}(t,x,\varphi_1(x),\nabla\cdot\varphi_1(x)),$$

$$\tilde{\Phi}_{\tilde{F}}(t,\varphi_1,\varphi_2)(x)=\varphi_f(t,x,\varphi_1(x),\nabla\cdot\varphi_1(x))-b(t)\varphi_2(x).$$

If $(\varphi_1,\varphi_2)\in K$, then

$$\|\tilde{\Phi}_{\tilde{F}}(t,\varphi_1,\varphi_2)\|_{L^p(\Omega)}\le\Big(\int_\Omega|\varphi_f(t,x,\varphi_1(x),\nabla\cdot\varphi_1(x))|^pdx\Big)^{\frac{1}{p}}+|b(t)|M_K, \tag{4.30}$$

where $M_K$ is given by (4.14).

We note that using Sobolev embedding we have that there is a positive constant $\tilde{c}>0$ depending only on $K$ so that

$$\sup_{x\in\Omega}|\varphi_1(x)|\le\tilde{c}\text{ and }\sup_{x\in\Omega}|\nabla\cdot\varphi_1(x)|\le\tilde{c}.$$

Taking into account that $Y=\overline{\Omega}\times B_{\tilde{c}}(\mathbb{R}^2)$ is a compact subset of $\mathbb{R}^{N+2}$, (4.29) and (4.30). It follows that $\lim_{t\to\infty}\tilde{\Phi}_{\tilde{F}}(t,\varphi_1,\varphi_2)=0$ uniformly for $(\varphi_1,\varphi_2)\in K$. Hence

$$\tilde{\Phi}_{\tilde{F}}\in C_0(\mathbb{R}^+\times[L^p(\Omega)]^{\frac{1}{2}}\times L^p(\Omega),L^p(\Omega)).$$

As an immediate consequence of Definition 2.3 and using the fact that $\Omega$ is bounded we have that

$$\tilde{F}_{ap} \in AP(\mathbb{R} \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega)).$$

Since $\tilde{F} = \tilde{F}_{ap} + \tilde{\Phi}_{\tilde{F}}$ we conclude that

$$\tilde{F} \in AAP(\mathbb{R}^+ \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega)).$$

**Proposition 4.2** *Under the above conditions, if* $\max\{||c||_\infty, ||b||_\infty\}$ *is small enough, then problem (4.21)-(4.23) has an asymptotically almost-periodic mild solution.*

**Proof.** We argue as follows. Let us choose $r > 0$ such that $\frac{K}{rC} \sup_{t\geq 0}(\int_\Omega |f(t,x,0,0)|^p dx)^{\frac{1}{p}}$ is small enough, where $K$ and $C$ are constant given in (1.8). Then condition (3.20) is fulfilled. Now applying Theorem 3.3 we conclude the proof. ■

Let $a : \mathbb{R}^+ \to \mathbb{R}$ be a bounded continuous function and let $\beta$ be in $(0, 1)$ and $\delta > 0$. In a bounded smooth domain $\Omega \subseteq \mathbb{R}^N$ we consider the following partial differential equation.

$$u_{tt} + \triangle^2 u - \delta \triangle u_t = a(t) \left| \int_\Omega \nabla \cdot u(t,x) dx \right|^\beta \Phi_0, \; x \in \Omega, \; t \geq 0, \tag{4.31}$$

$$u = \triangle u = 0, \; x \in \partial\Omega, \; t > 0, \tag{4.32}$$

$$u(0,x) = u_0(x), \; u_t(0,x) = v_0(x), \; x \in \Omega, \tag{4.33}$$

where $\Phi_0 \in L^p(\Omega)$, $p > \frac{N}{2}$. We model (4.31)-(4.33) in the abstract form (1.1), that is we set $\eta = \frac{\delta}{2}$ and we consider the operator $A$ as in (4.6) and (4.7). We define $f : \mathbb{R}^+ \times \left[L^p(\Omega)\right]^{\frac{1}{2}} \times L^p(\Omega) \to L^p(\Omega)$ by

$$f(t, \varphi_1, \varphi_2)(x) = a(t) \left| \int_\Omega \nabla \cdot \varphi_1(\xi) d\xi \right|^\beta \Phi_0(x). \tag{4.34}$$

To establish our next result, we assume the following conditions.

($H_1$) Let $a(\cdot) = a_{ap}(\cdot) + \varphi_a(\cdot)$ be in $AAP(\mathbb{R}^+)$, with $a_{ap}(\cdot) \in AP(\mathbb{R})$ and $\varphi_a(\cdot) \in C_0(\mathbb{R}^+)$.

($H_2$) There is a continuous nondecreasing function $h : [0, \infty) \to [1, \infty)$ such that $h(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{1}{h(t)} \int_0^t e^{-C(t-s)} h(s)^\beta ds = 0 \text{ and } \sup_{t\geq 0} \int_0^t e^{-C(t-s)} |a(t)| h(s)^\beta ds < \infty.$$

**Proposition 4.3** *Under the above conditions, the problem (4.31)-(4.33) has an asymptotically almost-periodic mild solution.*

**Proof.** Let $q$ be so that $\frac{1}{q} + \frac{1}{p} = 1$. We can infer the following estimate:

$$||f(t, \varphi_1, \varphi_2)||_{L^p(\Omega)} \leq ||a||_\infty ||\Phi_0||_{L^p(\Omega)} |\Omega|^{\frac{\beta}{q}} ||\varphi_1||^\beta_{H^2_p(\Omega)}. \tag{4.35}$$

Hence, we can define $W_f$ in $(W)$ by $W_f(\xi) = ||a||_\infty ||\Phi_0||_{L^p(\Omega)}|\Omega|^{\frac{\beta}{q}}\xi^\beta$. Let $K$ be a compact subset of $\Big[L^p(\Omega)\Big]^{\frac{1}{2}} \times L^p(\Omega)$ and let $M_K$ be as in (4.14). We set

$$f_{ap}(t,\varphi_1,\varphi_2)(x) = a_{ap}(t)\left|\int_\Omega \nabla\cdot\varphi_1(\xi)d\xi\right|^\beta \Phi_0(x),$$

$$\Phi_f^\sharp(t,\varphi_1,\varphi_2)(x) = \varphi_a(t)\left|\int_\Omega \nabla\cdot\varphi_1(\xi)d\xi\right|^\beta \Phi_0(x),$$

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \in K.$$

We have the following estimates:

$$||f_{ap}(t+\tau,\varphi_1,\varphi_2) - f_{ap}(t,\varphi_1,\varphi_2)||_{L^p(\Omega)} \le |\Omega|^{\frac{\beta}{q}} M_K^\beta ||\Phi_0||_{L^p(\Omega)}|a_{ap}(t+\tau)-a_{ap}(t)|, \tag{4.36}$$

$$||\Phi_f^\sharp(t,\varphi_1,\varphi_2)||_{L^p(\Omega)} \le |\Omega|^{\frac{\beta}{q}} M_K^\beta ||\Phi_0||_{L^p(\Omega)}|\varphi_a(t)|, \tag{4.37}$$

$$||f(t,\varphi_1,\varphi_2) - f(t,\psi_1,\psi_2)||_{L^p(\Omega)} \le |\Omega|^{\frac{\beta}{q}}||a||_\infty ||\Phi_0||_{L^p(\Omega)}||\varphi_1-\psi_1||^\beta_{H^2_p(\Omega)}. \tag{4.38}$$

It follows from (4.36) and (4.37) that $f \in AAP(\mathbb{R}^+ \times [L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega), L^p(\Omega))$. In addition, from the estimate (4.38) we obtain that $f$ is uniformly continuous on compact sets.

From $(\mathrm{H}_2)$, for $\begin{bmatrix}u_i\\ v_i\end{bmatrix} \in C_h([L^p(\Omega)]^{\frac{1}{2}} \times L^p(\Omega))$, $i=1,2$ we can infer

$$\frac{1}{h(t)}\int_0^t e^{-C(t-s)}W_f(Lh(s))ds = ||a||_\infty||\Phi_0||_{L^p(\Omega)}|\Omega|^{\frac{\beta}{q}}L^\beta\left(\frac{1}{h(t)}\int_0^t e^{-C(t-s)}h(s)^\beta ds\right) \to 0,\ t\to\infty,$$

$$\begin{aligned}
&\int_0^t e^{-C(t-s)}\|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\|_{L^p(\Omega)}ds\\
&\le |\Omega|^{\frac{\beta}{q}}||\Phi_0||_{L^p(\Omega)}\int_0^t e^{-C(t-s)}|a(s)|||u_1(s)-u_2(s)||^\beta_{H^2_p(\Omega)}ds\\
&= |\Omega|^{\frac{\beta}{q}}||\Phi_0||_{L^p(\Omega)}\int_0^t e^{-C(t-s)}|a(s)|h(s)^\beta\left(\frac{||u_1(s)-u_2(s)||_{H^2_p(\Omega)}}{h(s)}\right)^\beta ds\\
&\le |\Omega|^{\frac{\beta}{q}}||\Phi_0||_{L^p(\Omega)}\sup_{t\ge0}\left(\int_0^t e^{-C(t-s)}|a(t)|h(s)^\beta ds\right)\\
&\times\left[\sup_{t\ge0}\left(\frac{1}{h(t)}\Big[\|u_1(t)-u_2(t)\|_{H^2_p(\Omega)} + \|v_1(t)-v_2(t)\|_{L^p(\Omega)}\Big]\right)\right]^\beta.
\end{aligned}$$

Therefore, conditions (c) and (d) of Theorem 3.4 are satisfied. A straightforward computation shows that (e) holds. We can prove that the set $\{f(t,\varphi_1,\varphi_2) : 0\le t\le a, ||\varphi_1||_{H^2_p(\Omega)} + ||\varphi_2||_{L^p(\Omega)} \le r\}$ is relatively compact in $L^p(\Omega)$. In fact, first we denote by $\tilde{u}$ the zero extension of $u$ outside $\Omega$, that is

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\in\mathbb{R}^N\setminus\Omega.\end{cases}$$

For every number $\varepsilon > 0$ there exists a number $\delta > 0$ and a subset $G \subset\subset \Omega$ such that for every $\varrho \in \mathbb{R}^N$ with $|\varrho| < \delta$

$$\int_{\Omega} |\tilde{\Phi}_0(x+\varrho) - \tilde{\Phi}_0(x)|^p dx < \left( \frac{\varepsilon}{r^{\beta} ||a||_{\infty} |\Omega|^{\frac{\beta}{q}}} \right)^p$$

and

$$\int_{\Omega \setminus G} |\Phi_0(x)|^p dx < \left( \frac{\varepsilon}{r^{\beta} ||a||_{\infty} |\Omega|^{\frac{\beta}{q}}} \right)^p .$$

Hence

$$\int_{\Omega} |\tilde{f}(t,\varphi_1,\varphi_2)(x+\varrho) - \tilde{f}(t,\varphi_1,\varphi_2)(x)|^p dx \leq [r^{\beta} ||a||_{\infty} |\Omega|^{\frac{\beta}{q}}]^p \int_{\Omega} |\tilde{\Phi}_0(x+\varrho) - \tilde{\Phi}_0(x)|^p dx \leq \varepsilon$$

and

$$\int_{\Omega \setminus G} |f(t,\varphi_1,\varphi_2)(x)|^p dx \leq [r^{\beta} ||a||_{\infty} |\Omega|^{\frac{\beta}{q}}]^p \int_{\Omega \setminus G} |\Phi_0(x)|^p dx \leq \varepsilon.$$

From [1, Theorem 2.21] it follows the desired assertion. Note finally that using Remark 3.3 we have that (4.31)-(4.33) has an asymptotically almost-periodic mild solution. This ends the proof. ■

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