EXISTENCE OF S-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS FOR TWO-TIMES FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. Using a generalization of the semigroup theory of linear operators, we prove existence and uniqueness of S-asymptotically ω-periodic mild solutions for a class of linear and semilinear fractional order differential equations of the form

$$D_t^{\alpha+1}u(t)+\mu D_t^{\beta}u(t)-Au(t)=f(t,u(t)),\quad t>0,\quad 0<\alpha\leq\beta\leq 1,\ \mu\geq 0,$$

1. Introduction

Let X be a Banach space. A bounded continuous function $u:[0,\infty)\to X$ is said to be S-asymptotically ω-periodic if

$$\lim_{t\to\infty}[u(t+\omega)-u(t)]=0.$$

This paper is devoted to study the existence of S-asymptotically ω-periodic mild solutions for fractional order differential equations of the form

$$D_t^{\alpha+1}u(t)+\mu D_t^{\beta}u(t)-Au(t)=f(t,u(t)),\quad t>0, \tag{1.1}$$

with prescribed initial conditions $u(0)$ and $u'(0)$, and where $A:D(A)\subset X\to X$ is sectorial of angle $\beta\pi/2$, f is a vector-valued function, and D_t^{γ} denotes the Caputo fractional derivative of order γ.

The literature concerning S-asymptotically ω-periodic functions with values in Banach spaces is very new. Recently some interesting articles were published by Henríquez et al. [10], [11], Nicola and Pierri [19], Cuevas and de Souza [5],[6] and Cuevas and Lizama [7].

On the other hand, fractional order differential equations represent a subject of increasing interest in different contexts and areas of research, see e.g. [2, 4, 12, 13, 15, 20, 21], the survey paper [9] and the references therein. Our motivation to study equation (1.1) comes from recent investigations on the subject. Indeed, in the article [17] the author studied existence and uniqueness of solutions for the abstract equation (1.1) in the special case $\alpha=\beta$ and in the article [22] the authors studied the nonlinear two-term time fractional diffusion wave equation (1.1) with $0<\alpha<\beta-1$ and $A=\frac{d^2}{dx^2}$.

In the paper [14], asymptotic behavior for mild solutions of (1.1) was studied, whereas the recent article [1] analyzed the existence and uniqueness of pseudo

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asymptotic mild solutions which in particular includes the classes of pseudo periodic, pseudo almost periodic and pseudo almost automorphic functions. However, the investigation of existence and uniqueness of S-asymptotically ω-periodic solutions was left open. The main objective of this paper is close this gap, establishing the first results on this qualitative property for equation (1.1).

Our methods are as follows: In [14] the authors proved that it is possible to give an abstract operator approach to equation (1.1) by defining first an ad-hoc solution family of strongly continuous operators $S_{\alpha,\beta}(t)$ for (1.1) in case $f \equiv 0$. It turns out, that it is a particular case of an (a,k)-regularized family [16] and a generalization of the semigroup theory. By means of the use of this class of operator-valued families, the solution for equation (1.1) can be written in terms of a kind of variation of constants formula. It give us the necessary framework to apply an operator theoretical approach in the analysis of S-asymptotically ω-periodic solutions for the abstract fractional order differential equation (1.1).

We outline the plan of the paper as follows. In section 2, we recall the concept of fractional order derivatives and some properties of $(\alpha,\beta)_\mu$-regularized families. In section 3 we consider the linear case, that is $f(t,u(t)) = f(t)$ and show existence and uniqueness of S-asymptotically ω-periodic solutions of our problem. The existence, uniqueness of S-asymptotically ω-periodic solutions for the semi-linear problem is investigated in Section 4. Existence is proved by means of the contraction mapping theorem. Finally, we conclude the paper by giving a concrete example where the situation in the previous sections can be applied.

2. Preliminaries

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $u : [0,\infty) \to X$, where X is a complex Banach space. We denote by $\mathbb{R}_+$ the closed interval $[0,\infty)$. The Caputo fractional derivative of $u \in C(\mathbb{R}_+)$ of order α is defined by

$$D_t^\alpha u(t) := \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds, \quad t > 0,$$

where $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$, and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin. When $\alpha = n$ is integer, we define $D_t^n := \frac{d^n}{dt^n}, n \in \mathbb{N}$.

We denote by

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is continuous}, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

the Banach space of X-valued bounded and continuous functions on $\mathbb{R}$, with natural norm.

Now we turn our attention to the family of function spaces built on X and which will play a key role in our study.

Definition 2.1. [10] A function $f \in BC(\mathbb{R}_+;X)$ is called S-asymptotically ω periodic if there exists $\omega > 0$ such that $lim_{t\to\infty}[f(t+\omega) - f(t)] = 0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.

Denote by $SAP_\omega(X)$ the set of all such functions. We note that $SAP_\omega(X)$, is a Banach space with the supnorm. In [10] is was shown the surprising fact that the property $\lim_{t\to\infty}[f(t+\omega)-f(t)]=0$ does not characterize asymptotically ω-periodic functions, that is, bounded and continuous functions which admits the decomposition $f=g+h$, where g is ω-periodic and $\lim_{t\to\infty} h(t)=0$.

In order to give an operator theoretical approach to equation (1.1) we have the following definition.

Definition 2.2. ([14]) Let $\mu \geq 0$ and $0 \leq \alpha, \beta \leq 1$ be given. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an $(\alpha,\beta)_\mu$-regularized family if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^{\alpha+1}+\mu\lambda^\beta : Re\lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \lambda^\alpha(\lambda^{\alpha+1}+\mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)x dt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness theorem for the Laplace transform, if $\mu=0$ and $\alpha=0$, this corresponds to the case of a C_0-semigroup whereas the case $\mu=0,\ \alpha=1$ corresponds to the concept of cosine family. For more details on the Laplace transform approach to semigroups and cosine functions, we refer to the monograph [3].

Let us recall that a closed and densely defined operator A is said to be ω-sectorial of angle θ if there exists $\theta \in [0,\pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector $\omega + S_\theta := \{\omega+\lambda : \lambda \in \mathbb{C},\ |arg(\lambda)| < \frac{\pi}{2}+\theta\} \setminus \{\omega\}$, and

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda-\omega|}, \quad \lambda \in \omega + S_\theta. \tag{2.1}$$

These are generators of holomorphic semigroups. In case $\omega=0$ we merely say that A is sectorial of angle θ. We should mention that in the general theory of sectorial operators, it is not required that (2.1) holds in a sector of angle $\pi/2$. Our restriction corresponds to the class of operators used in this paper.

Sufficient conditions to obtain generators of an $(\alpha,\beta)_\mu$-regularized family are given in the following result.

Theorem 2.3. *([14]) Let $0<\alpha\leq\beta\leq 1$, $\mu>0$ and A be a ω sectorial operator of angle $\beta\pi/2$. Then A generates a bounded $(\alpha,\beta)_\mu$-regularized family.*

We next consider the linear fractional differential equation

$$D_t^{\alpha+1}u(t)+\mu D_t^\beta u(t) - Au(t) = D_t^\alpha f(t), \quad t \geq 0, \quad 0<\alpha\leq\beta\leq 1,\ \mu\geq 0, \tag{2.2}$$

with initial conditions $u(0)=x,\ u'(0)=y$ and A is a ω-sectorial operator of angle $\beta\pi/2$.

Recall that a function $u \in C^1(\mathbb{R}_+;X)$ is called a strong solution of (2.2) on $\mathbb{R}_+$ if $u(t)\in D(A)$ and (2.2) holds on $\mathbb{R}_+$. We have the following result.

If A is ω-sectorial of angle $\beta\pi/2$ then, by [14, Cor.3.4] and Theorem 2.3, a strong solution for (2.2) always exists and is given by:

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + (S_{\alpha,\beta} * f)(t), \quad (2.3)$$

where $0 < \alpha \leq \beta \leq 1$, $\mu > 0$; $x, y \in D(A)$; $f : \mathbb{R}_+ \to D(A)$ and $S_{\alpha,\beta}(t)$ is the $(\alpha,\beta)_\mu$-regularized family generated by A. If merely $x, y \in X$ and $f : \mathbb{R}_+ \to X$ instead of the domain of A, we say that u given by the formula (2.3) is a *mild solution* of the linear equation (2.2).

In order to study the pseudo asymptotic behavior of mild solutions, we need the following result on the integrability of the $(\alpha,\beta)_\mu$-regularized family generated by A.

Theorem 2.4. *([14]) Let $0 < \alpha \leq \beta \leq 1$, $\mu > 0$ and $\omega < 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$, then A generates an $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ satisfying the estimate*

$$||S_{\alpha,\beta}(t)|| \leq \frac{C}{1 + |\omega|(t^{\alpha+1} + \mu t^\beta)}, \quad t \geq 0, \quad (2.4)$$

for some constant $C > 0$ depending only on α, β.

3. S-asymptotically ω periodic solutions: The linear case.

We can prove the following theorem which is the main result in this section.

Theorem 3.1. *Let $0 < \alpha \leq \beta \leq 1$ and $\mu > 0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega < 0$. Then for each $f \in SAP_\omega(X)$ there exists a unique mild solution u of equation (2.2) such that $u \in SAP_\omega(X)$.*

Proof. Let $f \in SAP_\omega(X)$ be given. By Theorem 2.4, A generates a uniformly integrable $(\alpha,\beta)_\mu$-regularized family $S_{\alpha,\beta}(t)$ on the Banach space X, and the unique mild solution for (2.2) is given by (2.3), that is;

$$u(t) = S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x + (S_{\alpha,\beta} * f)(t),$$

where $0 < \alpha \leq \beta \leq 1; \mu > 0$ and $x, y \in X$. An adaptation of the proof of [8] shows that $S_{\alpha,\beta} * f \in SAP_\omega(X)$. On the other hand, note that by (2.4) we have $\lim_{t\to\infty} \|S_{\alpha,\beta}(t)\| = 0$. Hence $S_{\alpha,\beta} \in SAP_\omega(X)$. We now prove that $g_1 * S_{\alpha,\beta} \in SAP_\omega(X)$. Indeed, by (2.4) we have $\sup_{t>\tau} ||tS_{\alpha,\beta}(t)|| < \infty$, for each $\tau > 0$. Since A is an ω-sectorial of angle $\beta\frac{\pi}{2}$ then $||\hat{S}_{\alpha,\beta}(\lambda)|| \to 0$ as $\lambda \to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [3, Theorem 4.2.9]) we conclude that $||(g_1 * S_{\alpha,\beta})(t)|| \to 0$ as $t \to \infty$ and the claim is proved. It remains only to show that $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in SAP_\omega(X)$ for $\alpha < \beta$. To see this, we estimate $||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)||$ as follows. Let $0 < \epsilon < \beta - \alpha$ be given, then

$$\begin{aligned} ||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)|| &= ||\Gamma(\beta-\alpha-\epsilon) \int_0^t g_{1+\alpha-\beta}(t-\tau) g_{\beta-\alpha-\epsilon}(\tau) \tau^{\alpha-\beta+\epsilon+1} S_{\alpha,\beta}(\tau) d\tau|| \\ &\leq \Gamma(\beta-\alpha-\epsilon) \int_0^t g_{1+\alpha-\beta}(t-\tau) g_{\beta-\alpha-\epsilon}(\tau) \tau^{\alpha-\beta+\epsilon+1} ||S_{\alpha,\beta}(\tau)|| d\tau \end{aligned}$$

where, thanks to (2.4), we have that

$$\Gamma(\beta-\alpha-\epsilon)\tau^{\alpha-\beta+\epsilon+1}||S_{\alpha,\beta}(\tau)|| \le \frac{M\tau^{\alpha-\beta+\epsilon-1}}{1+|\omega|\tau^{\alpha+1}} = \frac{M\tau^{-\beta+\epsilon}}{\frac{1}{\tau^{\alpha+1}}+|\omega|}, \quad \tau>0.$$

Since $\epsilon<\beta$, there exists a constant $C>0$ such that $\tau^{\alpha-\beta+\epsilon+1}||S_{\alpha,\beta}(\tau)|| \le C$. Therefore,

$$||g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)|| \le C\int_0^t g_{1+\alpha-\beta}(t-\tau)g_{\beta-\alpha-\epsilon}(\tau)d\tau = Cg_{1-\epsilon}(t) = Ct^{-\epsilon},$$

which shows that $\|g_{1+\alpha-\beta} * S_{\alpha,\beta}(t)\| \to 0$ as $t\to\infty$. Therefore $g_{1+\alpha-\beta} * S_{\alpha,\beta} \in SAP_\omega(X)$ and finally, we have shown that $u \in SAP_\omega(X)$. □

4. S-asymptotically ω periodic solutions: The semilinear case.

Define the Nemytskii superposition operator $\mathcal{N}(\varphi)(\cdot) := f(\cdot,\varphi(\cdot))$ for $\varphi \in SAP_\omega(X)$. We define the set $SAP_\omega(\mathbb{R}_+\times X;X)$ to consist of all functions $f:\mathbb{R}_+\times X\to X$ such that $f(\cdot,x)\in SAP_\omega(X)$ uniformly for each $x\in K$, where K is any bounded subset of X.

In what follows we study existence and uniqueness of solutions in $SAP_\omega(X)$ for the semi-linear fractional order differential equation

$$D_t^{\alpha+1}u(t)+\mu D_t^{\beta}u(t)-Au(t) = D_t^{\alpha}f(t,u(t)), \quad t\ge 0, \quad 0<\alpha\le\beta\le 1, \quad \mu>0, \tag{4.1}$$

where A is an ω-sectorial operator of angle $\beta\pi/2$ with $\omega<0$, $u(0)=x$ and $u'(0)=y$.

In view of the linear case, the following definition of mild solution is natural. Note that in the borderline case $\mu=0$ and $\alpha=1$ it corresponds to the notion of mild solution for the semi-linear problem $u''(t)=Au(t)+f(t,u(t))$ under the hypothesis that A is the generator of a cosine family $C(t)$. In fact, in this case: $S_{1,0}(t)\equiv C(t)$ and the associate sine family is equal to $(g_1 * S_{1,0})(t)$.

Definition 4.1. Suppose $0<\alpha\le\beta\le 1$, $\mu>0$. A function $u:\mathbb{R}_+\to X$ is said to be a mild solution to Equation (4.1) if it satisfies

$$u(t) = S_{\alpha,\beta}(t)x+(g_1 * S_{\alpha,\beta})(t)y+\mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x+\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \tag{4.2}$$

for each $t\in\mathbb{R}_+$ and $x,y\in X$.

We next give a result on existence of mild solutions for the semi-linear problem.

Theorem 4.2. *Let $0<\alpha\le\beta\le 1$ and $\mu>0$. Assume that A is an ω-sectorial operator of angle $\beta\pi/2$ and $\omega<0$. Let $f:\mathbb{R}_+\times X\to X$ be a function on $SAP_\omega(\mathbb{R}_+\times X;X)$ and assume that there exists a bounded integrable function $L_f:\mathbb{R}_+\to\mathbb{R}_+$ satisfying*

$$||f(t,x)-f(t,y)|| \le L_f(t)||x-y||, \tag{4.3}$$

for all $x,y\in X$ and $t\ge 0$. Then Equation (4.1) has a unique mild solution $u\in SAP_\omega(X)$.

Proof. Let $S_{\alpha,\beta}(t)$ be the $(\alpha,\beta)_\mu$-regularized family generated by A (cf. Theorem 2.4). We define the operator $K_{\alpha,\beta}$ on the space $SAP_\omega(X)$ by

$$\begin{aligned}(K_{\alpha,\beta}u)(t) &= S_{\alpha,\beta}(t)x + (g_1 * S_{\alpha,\beta})(t)y + \mu(g_{1+\alpha-\beta} * S_{\alpha,\beta}(t))x \\ &+ \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds. \end{aligned} \tag{4.4}$$

From the proof of Theorem 3.1, we know that $S_{\alpha,\beta}(t)x+(g_1*S_{\alpha,\beta})(t)y+\mu(g_{1+\alpha-\beta}* S_{\alpha,\beta}(t))x \in SAP_\omega(X)$. Moreover by [18, Theorem 4.1] we conclude that the function $s \to f(s,u(s))$ is in $SAP_\omega(X)$. Then, by hypothesis and in the same way as in the proof of Theorem 3.1, we arrive at the conclusion that $\int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds$ is also in $SAP_\omega(X)$ and thus $K_{\alpha,\beta}$ is well defined. Let u,v be in $SAP_\omega(X)$. Observe that

$$\begin{aligned}||(K_{\alpha,\beta}u)(t)-(K_{\alpha,\beta}v)(t)|| &\le \int_0^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u(s))-f(s,v(s))\|ds \\ &\le \int_0^t \|S_{\alpha,\beta}(t-s)\|L_f(s)\|u(s)-v(s)\|ds \\ &\le \|S_{\alpha,\beta}\|_1\|u-v\|_\infty \int_0^t L_f(s)ds \\ &\le \|S_{\alpha,\beta}\|_1\|u-v\|_\infty\|L_f\|_1.\end{aligned}$$

By induction, we find the following estimate:

$$\begin{aligned}||(K^n_{\alpha,\beta}u)(t)-(K^n_{\alpha,\beta}v)(t)|| &\le \frac{||S_{\alpha,\beta}||_1^n}{(n-1)!}||u-v||_\infty \int_0^t L_f(s)\left(\int_0^s L_f(\tau)d\tau\right)^{n-1} ds \\ &= \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty \left(\int_0^t L_f(\tau)d\tau\right)^n \\ &\le \frac{||S_{\alpha,\beta}||_1^n}{n!}||u-v||_\infty ||L_f||_1^n.\end{aligned}$$

Since $\frac{||S_{\alpha,\beta}||_1^n}{n!}||L_f||_1^n < 1$ for n sufficiently large, applying the contraction principle we conclude that F has a unique fixed point $u \in SAP_\omega(X)$ such that $(K_{\alpha,\beta}u)(t) = u(t)$. □

To finish, we present one example, which do not aim at generality but indicate how our theorems can be applied to concrete problems.

Example 4.3. Suppose that $b \in L^1(\mathbb{R}_+) \cap SAP_\omega(\mathbb{R}_+)$. Then the equation

$$\begin{aligned}D_t^{\alpha+1}u(x,t)+\mu D_t^\beta u(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t)+\tau u(x,t) \\ &+ D_t^\alpha[b(t)\sin(u(t))],\ t>0,\ 0<\alpha\le\beta\le 1\end{aligned} \tag{4.5}$$

where $\tau < 0$ is fixed, with initial and zero boundary conditions has a unique mild solution $u(t,x)$ such that $u(\cdot,x)$ belongs to the space $SAP_\omega(X)$

Indeed, the equation (4.5) is of the form (4.1) with $Au = \frac{\partial^2}{\partial x^2}u + \tau u$ and $f(t,u) = b(t)\sin(u(t))$. Setting the Dirichlet boundary conditions $u(0,t) = u(2\pi,t) = 0$ we consider A with domain $D(A) := \{u \in L^2[0,2\pi] : u'' \in L^2[0,2\pi]; u(0) = u(2\pi) = 0\}$ and $f(t,x) = b(t)\sin(x)$. Then it is wellknown that the operator A is ω sectorial with $\omega = \tau < 0$ and angle $\pi/2$ (and hence of angle $\beta\pi/2$ for all $\beta \leq 1$). On the other hand, since $b \in L^1(\mathbb{R}_+)$ we have

$$\|f(t,u) - f(t,v)\|_2^2 = \int_0^\pi |b(t)|^2 |\sin(u(s)) - \sin(v(s))|^2 ds \leq |b(t)|^2 \|u - v\|_2^2,$$

and the condition (4.3) holds. Hence the hypothesis of Theorem 4.2 are satisfied and thus the conclusion of the example follows.

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