ALMOST AUTOMORPHIC SOLUTIONS TO INTEGRAL EQUATIONS ON THE LINE

CLAUDIO CUEVAS AND CARLOS LIZAMA

ABSTRACT. Given $a \in L^1(\mathbb{R})$ and A the generator of an L^1-integrable family of bounded and linear operators defined on a Banach space X, we prove the existence of almost automorphic solution to the semilinear integral equation $u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))]ds$ for each $f : \mathbb{R} \times X \to X$ almost automorphic in t, uniformly in $x \in X$, and satisfying diverse Lipschitz type conditions. In the scalar case, we prove that $a \in L^1(\mathbb{R})$ positive, nonincreasing and log-convex is already sufficient.

1. INTRODUCTION

We study in this paper the almost automorphicity of semilinear integral equations of the form

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + s^n g(s, u(s))]ds, \quad t \in \mathbb{R},\, n \in \mathbb{Z}_+ \tag{1.1}$$

where $a \in L^1(\mathbb{R})$, $A : D(A) \subset X \to X$ is the generator of an integral resolvent family defined on a complex Banach space X and $g : \mathbb{R} \times X \to X$ is an almost automorphic function satisfying suitable Lipschitz conditions.

A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n), \quad \text{for each } t \in \mathbb{R}.$$

Almost automorphicity is a generalization of the classical concept of an almost periodic function. It was introduced in the literature by S. Bochner and recently studied by several authors, including [4, 6, 8, 10, 14, 19] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [20] and [21] by G. M. N'Guérékata. Observe that equation (1.1) can be viewed as the *limiting equation* for the Volterra equation

$$u(t) = \int_{0}^{t} a(t-s)[Au(s) + s^n g(s, u(s))]ds, \quad t \geq 0,\, n \in \mathbb{Z}_+, \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 43A60; Secondary 47D60, 45N05, 47J35.

Key words and phrases. Almost automorphic function, integral resolvent family, semilinear integro-differential equations.

The first author is partially supported by CNPQ/Brazil under Grant 300068/2005-0 .

The second author is partially financed by Laboratorio de Análisis Estocástico, Proyecto Anillo PBCT-ACT-13.

see [23, Chapter III, Section 11.5] to obtain details on this assertion.

Sufficient conditions for the existence of almost automorphic solutions of linear and nonlinear evolution equations have been studied in several papers in recent years (see [19, 10, 9, 14, 1, 4] and [6]).

Equation (1.1) arises in the study of heat flow in materials of fading memory type (see [7] and [22]). While the study of the almost automorphic solutions of (1.1) in the particular case $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ with $1 \leq \alpha \leq 2$ was studied in [1], to the knowledge of the authors no results yet exist for the general class of integral equations considered in this paper.

Our plan is as follows: In Section 2, we introduce some preliminaries on integral resolvent families. In section 3, we treat the linear case

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + s^n f(s)]ds, \quad t \in \mathbb{R},\, n \in \mathbb{Z}_+ \tag{1.3}$$

and prove our first main result. Section 4 is devoted to the semilinear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s,u(s))]ds, \quad t \in \mathbb{R}. \tag{1.4}$$

We prove existence of a unique almost automorphic mild solution to (1.4) under the assumption f is almost automorphic and that some Lipchitz condition on f is satisfied.

2. Preliminaries

Recall that the Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

if the integral is absolutely convergent for $Re\lambda > \omega$. In order to give an operator theoretical approach to equation (1.1) we recall the following definition (cf. [18]).

Definition 2.1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an *integral resolvent* if there exists $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{1/\hat{a}(\lambda) : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt, \quad Re\lambda > \omega, \quad x \in X.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The concept of integral resolvent, as defined above, is closely related with the concept of resolvent family (see Prüss [23, Chapter I]). A closed but weaker definition was formulated by Prüss [23, definition 1.6]. For the scalar case, where there is a large bibliography, we refer to the monograph by Gripenberg, Londen and Staffans [12], and references therein.

Because of the uniqueness of the Laplace transform, an integral resolvent family with $a(t) \equiv 1$ is the same as a C_0-semigroup whereas an integral resolvent family with $a(t) = t$ corresponds to the concept of sine family, see [2, Section 3.15].

We note that integral resolvent families are a particular case of (a,k)-regularized families introduced in [15]. These are studied in a series of several papers in recent years (see [16], [17], [24]). According to [15] an integral resolvent family $S(t)$ corresponds to a (a,a)-regularized family.

Notably, integral resolvent families are also present in [5, p.62] (see formula (4.33)) in the particular case $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ and where some properties are studied in the context of vector-valued $L^p(\mathbb{R}, X)$ spaces.

As in the situation of C_0-semigroups we have diverse relations of an integral resolvent and its generator. The following result is a direct consequence of [15, Proposition 3.1 and Lemma 2.2].

Proposition 2.2. *Let $S(t)$ be an integral resolvent family on X with generator A. Then the following holds:*

(a) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A), t \geq 0$;

(b) Let $x \in D(A)$ and $t \geq 0$. Then

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds. \tag{2.1}$$

(c) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$ and

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)xds.$$

In particular, $S(0) = a(0)$.

If an operator A with domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function which satisfies $\overline{\lim_{t\to 0^+}} \frac{||S(t)||}{a(t)} < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - a(t)x}{(a*a)(t)},$$

see [17, Theorem 2.1]. For example, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup and $a(t) = t$ actually corresponds to the generator of a sine family.

A characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for C_0 semigroups, can be directly deduced from [15, Theorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be also deduced from the more general context of (a, k) regularized resolvents (see [16, 17] and [24]).

3. Almost Automorphic Solutions

In this section we consider the existence and uniqueness of almost automorphic solutions to the evolution equation

$$u(t) = \int_{-\infty}^t a(t-s)[Au(s) + s^n g(s)]ds, \quad t \in \mathbb{R}, n \in \mathbb{Z}_+ \tag{3.1}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R})$.

As a consequence of the definition of an almost automorphic function given in the introduction, the following properties hold (cf [21]) : let $f, g : \mathbb{R} \to X$ be almost automorphic functions and let $\lambda \in \mathbb{R}$, then $f+g, \lambda f$ and f_λ are almost automorphic, where $f_\lambda(t) := f(t+\lambda)$. Moreover, the range $R(f)$ of f is relatively compact, therefore it is bounded. Almost automorphic functions constitute a Banach space $AA(X)$ when it is endowed with the sup norm:

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

The following result on the almost automorphicity of the convolution is the key for the results of this paper. It can be proved by the simple argument of [5, Proposition 2.3]. See also [6, Theorem 2.1] or [1, Lemma 3.1] for a detailed proof.

Lemma 3.1. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded and linear operators such that*

$$\|S(t)\| \leq \phi(t) \text{ for all } t \in \mathbb{R}_+ \text{ with } \phi \in L^1(\mathbb{R}_+).$$

If $f : \mathbb{R} \to X$ is an almost automorphic function then

$$\int_{-\infty}^t S(t-s)f(s)\,ds \in AA(X).$$

Proposition 3.2. *Let $a \in L^1(\mathbb{R})$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ on X, which is in addition integrable. If f is almost automorphic and takes values on $D(A)$ then the unique bounded solution of the problem*

$$u(t) = \int_{-\infty}^t a(t-s)[Au(s)+f(s)]ds, \quad t \in \mathbb{R}, \tag{3.2}$$

is almost automorphic and is given by

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Proof. Since $f(t) \in D(A)$ for all $t \in \mathbb{R}$, we obtain $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [23, Proposition 1.2]). Then by (2.1) and Fubini's theorem we obtain $\int_{-\infty}^t a(t-s)\,Au(s)\,ds = u(t) - \int_{-\infty}^t a(t-s)\,f(\tau)\,d\tau$. The statement follows by Lemma 3.1. □

Remark 3.3. A complete discussion on integrable integral resolvents is given in [23, Chapter 3, section 10].

If $w_n(t) := (1+|t|)^n, n \in \mathbb{Z}_+, t \in \mathbb{R}$ and $f : \mathbb{R} \to X$ we set

$$||f||_{w_n,\infty} := ||f/w_n||_\infty.$$

We consider the following weighted classes (see [4])

$$AA_{w_n}(X) := \{w_n f : f \in AA(X)\} \text{ and } C_{w_n,0}(\mathbb{R}, X) := \{w_n f : f \in C_0(\mathbb{R}; X)\}.$$

Then one can check that $AA_{w_n}(X)$ and $C_{w_n,0}(\mathbb{R}, X)$ are Banach spaces endowed with the norm $|| \cdot ||_{w_n,\infty}$. Moreover, it was proved in [4] that $AA_{w_n}(X) + C_{w_n,0}(\mathbb{R}, X)$ is a closed subspace of $BC_{w_n}(\mathbb{R}, X)$ and that the sum is (topologically) direct, see [4, Theorem 1.6] and the remark before Definition 3.2 in the cited paper.

The following is the main result of this section. Their proof follows the same lines of [4, Theorem 4.3] (see also [1]), but we give it here for the sake of completeness.

Theorem 3.4. *Let $n \in \mathbb{Z}_+$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying*

$$\|t^k S(t)\| \leq \phi(t),\ t \in \mathbb{R}_+,\ \text{with } \phi \in L^1(\mathbb{R}_+) \tag{3.3}$$

for all $k = 0, 1, ..., n$. Let $f \in AA(X)$ taking values on $D(A)$. Then the equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + s^n f(s)]ds, \quad t \in \mathbb{R},$$

has a unique solution $u \in AA_{w_n}(X) \oplus C_{w_n,0}(\mathbb{R}, X)$.

Proof. Let $u(t) = \int_0^\infty S(s)(t-s)^n f(t-s)\, ds$ then

$$\begin{aligned} \int_0^\infty S(s)(t-s)^n f(t-s)\, ds &= t^n \int_0^\infty S(s) f(t-s)\, ds \\ &+ \sum_{k=1}^{n} (-1)^k \binom{n}{k} t^{n-k} \int_0^\infty s^k S(s) f(t-s)\, ds \\ &=: u_1(t) + u_2(t). \end{aligned}$$

(in case $n = 0$ we take $u_2(t) \equiv 0$). By Lemma 3.1, $u_1 \in AA_{w_n}(X)$. We will show that $u_2 \in C_{w_n,0}(\mathbb{R}, X)$. Indeed, by (3.3) we have $s^k S(\cdot) \in L^1(\mathbb{R}_+, \mathcal{B}(X))$ for all $k = 0, 1, ..., n$. So

$$\left\| \int_0^\infty s^k S(s) f(t-s)\, ds \right\| \leq \int_0^\infty \|s^k S(s) f(t-s)\|\, ds \leq \|f\|_\infty ||\phi_k||_1$$

for all $k = 0, 1, ..., n$. Since $\lim_{|t|\to\infty} \frac{t^r}{(1+|t|)^n} = 0$, we have

$$t^r \int_0^\infty s^k S(s) f(t-s)\, ds \in C_{w_n,0}(\mathbb{R}, X), \quad 0 \leq r < n$$

and this shows $u_2 \in C_{w_n,0}(\mathbb{R}, X)$. □

Taking $n = 0$ and $X = \mathbb{R}$ we obtain the following result for the scalar case.

Corollary 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost automorphic function, $a \in L^1(\mathbb{R})$ and let $\rho > 0$ be a real number. Then the equation*

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{3.4}$$

has an almost automorphic solution given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s) f(s) ds, \quad t \in \mathbb{R},$$

whenever $S_\rho(t)$, being the solution of the one dimensional equation

$$S_\rho(t) = a(t) - \rho \int_0^t a(t-s) S_\rho(s) ds, \tag{3.5}$$

satisfy $|S_\rho(t)| \leq \phi_\rho(t)$, with $\phi_\rho \in L^1(\mathbb{R}_+)$.

Example 3.6. *Consider* $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta t}$ *where* $\beta > 0$ *and* $1 < \alpha < 2$. *Then we can check that*

$$S_\rho(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\rho t) e^{-\beta t}$$

where $E_{\alpha,\alpha}$ *denotes the generalized Mittag-Leffler function (see e.g.* [11]*) which is defined by*

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad \alpha, \beta > 0, \quad z \in \mathbb{C}.$$

Using the explicit description of $S_\rho(t)e^{\beta t}$ *given in* [1, Corollary 3.7] *we can show that* $|S_\rho(t)| \leq \phi_\rho(t)$, *with* $\phi_\rho \in L^1(\mathbb{R}_+)$. *We conclude that the equation*

$$u(t) = \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta(t-s)} [-\rho u(s) + f(s)] ds, \quad t \in \mathbb{R}, \tag{3.6}$$

has an almost automorphic solution whenever f *is almost automorphic.*

Notably, the following result provide a wide class of kernels $a(t)$ such that the condition $|S_\rho(t)| \leq \phi_\rho(t)$, with $\phi_\rho \in L^1(\mathbb{R}_+)$ of Corollary (3.5) holds.

Corollary 3.7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an almost automorphic function and let* $\rho > 0$ *be a real number. Suppose* $a \in L^1(\mathbb{R})$ *is positive, nonincreasing and log-convex, then*

a) There is $S_\rho \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ *such that equation (3.5) is satisfied;*

b) The equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s)] ds, \quad t \in \mathbb{R}, \tag{3.7}$$

has a mild almost automorphic solution given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s) f(s) ds, \quad t \in \mathbb{R}.$$

Proof. Part a) follows by [23, Lemma 4.1, p.98] and then part b) is a consequence of the previous corollary.

□

4. Semilinear Integral Equations on the Line

In this section we study the semilinear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))] ds, \quad t \in \mathbb{R}. \tag{4.1}$$

Definition 4.1. Let A be the generator of an integral resolvent family $\{S(t)\}_{t \geq 0}$. A continuous function $u : \mathbb{R} \to X$ satisfying the integral equation

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s, u(s)) ds, \text{ for all } t \in \mathbb{R}, \tag{4.2}$$

is called a *mild* solution on $\mathbb{R}$ to the equation (4.1).

Theorem 4.2. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0, \text{ with } \phi \in L^1(\mathbb{R}_+).$$

Let $f : \mathbb{R} \times X \to X$ be almost automorphic in t uniformly in $x \in X$ and satisfy a Lipschitz condition in x uniformly in t, that is,

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y \in X.$$

Then

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s,u(s))]ds, \quad t \in \mathbb{R}, \tag{4.3}$$

has a unique almost automorphic mild solution whenever $L < ||\phi||_1^{-1}$.

Proof. We define the operator $F : AA(X) \mapsto AA(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s) f(s,\varphi(s))\, ds, \quad t \in \mathbb{R}. \tag{4.4}$$

In view of [13, Lemma 2.2] (see also [21]) and Lemma 3.1, F is well defined. Then for $\varphi_1, \varphi_2 \in AA(X)$ we have:

$$\begin{aligned}
\|F\varphi_1 - F\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\
&\leq L \sup_{t\in\mathbb{R}} \int_0^\infty \|S(\tau)\| \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\| d\tau \\
&\leq L\|\varphi_1 - \varphi_2\|_\infty \int_0^\infty \phi(\tau) d\tau.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA(X)$, such that $Fu = u$, that is $u(t) = \displaystyle\int_{-\infty}^{t} S(t-s) f(s,u(s)) ds.$ □

An immediate consequence of Theorem 4.2 and Corollary 3.7 is the following remarkable result.

Corollary 4.3. *Let $\rho > 0$ be a real number. Suppose $a \in L^1(\mathbb{R})$ is positive, nonincreasing and log-convex and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be almost automorphic in the first variable uniformly with respect to the second variable, and satisfies a Lipschitz condition in the second variable, that is,*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y \in \mathbb{R}.$$

Then there is $S_\rho \in L^1(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ satisfying the linear equation (3.5). Moreover, the semilinear equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s,u(s))]ds, \quad t \in \mathbb{R},$$

has a unique almost automorphic mild solution whenever $L < ||S_\rho||_1^{-1}$.

A different Lipschitz condition is provided in the following result.

Theorem 4.4. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0, \text{ with } \phi \in L^1(\mathbb{R}_+).$$

Let $f : \mathbb{R} \times X \to X$ be almost automorphic in t uniformly in $x \in X$ and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \text{ for all } x, y \in X,\ t \in \mathbb{R}, \tag{4.5}$$

where $L \in L^1(\mathbb{R})$. Then equation (4.1) has a unique almost automorphic mild solution.

Proof. We define the operator F as in (4.4). Let φ_1, φ_2 be in $AA(X)$ and denote $C := \sup_{t\in\mathbb{R}} ||S(t)||$. We have:

$$\begin{aligned} ||(F\varphi_1)(t) - (F\varphi_2)(t)|| &= \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\ &\leq \int_{-\infty}^{t} L(s)\|S(t-s)\|\|\varphi_1(s) - \varphi_2(s)\|ds \\ &\leq C\|\varphi_1 - \varphi_2\|_\infty ||L||_1. \end{aligned}$$

In general we get

$$\begin{aligned} ||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| &\leq \frac{C^n}{(n-1)!}\left(\int_{-\infty}^{t} L(s) \left(\int_{-\infty}^{s} L(\tau)d\tau \right)^{n-1} ds \right) ||\varphi_1 - \varphi_2||_\infty \\ &\leq \frac{C^n}{n!}\left(\int_{-\infty}^{t} L(\tau)d\tau \right)^{n} ||\varphi_1 - \varphi_2||_\infty \\ &\leq \frac{(C||L||_1)^n}{n!} ||\varphi_1 - \varphi_2||_\infty. \end{aligned}$$

Hence, since $\frac{(C||L||_1)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in AA(X)$. ☐

We notice that in (4.5) different type of conditions can be considered for $L(t)$. This fact is studied in the following results.

Theorem 4.5. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0, \text{ with } \phi \in L^1(\mathbb{R}_+).$$

Let $f : \mathbb{R} \times X \to X$ be almost automorphic in t uniformly in $x \in X$ and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \text{ for all } x, y \in X,\ t \in \mathbb{R},$$

where the integral $\displaystyle\int_{-\infty}^{t} L(s)ds$ exists for all $t \in \mathbb{R}$. Then equation (4.1) has a unique almost automorphic mild solution.

Proof. Define a new norm $|||\varphi|| := \sup_{t\in\mathbb{R}}\{v(t)||\varphi(t)||\}$, where $v(t) := e^{-k\int_{-\infty}^{t} L(s)ds}$ and k is a fixed positive constant greater than $C := \sup_{t\in\mathbb{R}} ||S(t)||$. Let φ_1, φ_2 be in $AA(X)$, then we have

$$\begin{aligned}
v(t)||(F\varphi_1)(t)-(F\varphi_2)(t)|| &= v(t)\left\|\int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s))-f(s,\varphi_2(s))]ds\right\| \\
&\le C\int_{-\infty}^{t} v(t)L(s)\|\varphi_1(s)-\varphi_2(s)\|ds \\
&\le C\int_{-\infty}^{t} v(t)v(s)^{-1}L(s)v(s)\|\varphi_1(s)-\varphi_2(s)\|ds \\
&\le C|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} v(t)v(s)^{-1}L(s)ds \\
&= \frac{C}{k}|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} ke^{k\int_t^s L(\tau)d\tau}L(s)ds \\
&= \frac{C}{k}|||\varphi_1-\varphi_2|||\int_{-\infty}^{t} \frac{d}{ds}\left(e^{k\int_t^s L(\tau)d\tau}\right)ds \\
&= \frac{C}{k}[1-e^{-k\int_{-\infty}^{t} L(\tau)d\tau}]|||\varphi_1-\varphi_2||| \\
&\le \frac{C}{k}|||\varphi_1-\varphi_2|||.
\end{aligned}$$

Hence, since $C/k<1$, F has a unique fixed point $u\in AA(X)$.

□

Theorem 4.6. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\ge 0}$ such that*

$$\|S(t)\|\le \phi(t),\ \textit{for all } t\ge 0,$$

where $\phi:\mathbb{R}_+\to\mathbb{R}_+$ is a decreasing function such that $\phi_0:=\sum_{m=0}^{\infty}\phi(m)<\infty$. Let $f:\mathbb{R}\times X\to X$ almost automorphic in t uniformly in $x\in X$ satisfying the Lipschitz condition

$$\|f(t,x)-f(t,y)\|\le L(t)\|x-y\|,\ \textit{for all } x,y\in X,\ t\in\mathbb{R},$$

where $L:=\sup_{t\in\mathbb{R}}\int_t^{t+1}L(s)ds<\infty$ Then equation (4.1) has a unique almost automorphic mild solution whenever $L\phi_0<1$.

Proof. Since ϕ is a decreasing function such that $\sum_{m=0}^{\infty}\phi(m)<\infty$ we have that $\phi\in L^1(\mathbb{R}_+)$ and hence $S(t)$ is integrable. Let φ_1,φ_2 be in $AA(X)$, then for the same operator

F defined in (4.4), we have

$$\begin{aligned}
||(F\varphi_1)(t)-(F\varphi_2)(t)|| &= \left\|\int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s))-f(s,\varphi_2(s))]ds\right\| \\
&\le \int_{-\infty}^{t} L(s)\|S(t-s)\|\|\varphi_1(s)-\varphi_2(s)\|ds \\
&\le (\sum_{m=0}^{\infty}\int_{t-(m+1)}^{t-m} L(s)\|S(t-s)\|ds)||\varphi_1-\varphi_2||_\infty \\
&\le (\sum_{m=0}^{\infty}\int_{t-(m+1)}^{t-m} L(s)\phi(t-s)ds)||\varphi_1-\varphi_2||_\infty \\
&\le (\sum_{m=0}^{\infty}\phi(m)\int_{t-(m+1)}^{t-m} L(s)ds)||\varphi_1-\varphi_2||_\infty \\
&\le L(\sum_{m=0}^{\infty}\phi(m))||\varphi_1-\varphi_2||_\infty = L\phi_0||\varphi_1-\varphi_2||_\infty,
\end{aligned}$$

which finish the proof.

□

References

[1] D. Araya, C. Lizama. *Almost automorphic mild solutions to fractional differential equations,* Nonlinear Analysis Series A: Theory Methods & Appl., **69** (11) (2008), 3692–3705.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* Monographs in Mathematics. **96**, Birkhäuser, Basel, 2001.

[3] B. Basit. *Harmonic Analysis and Asymptotic Behavior of Solutions to the Abstract Cauchy Problem.* Semigroup Forum **54,** (1997), 58-74.

[4] B. Basit, A.J. Pryde. *Asymptotic behavior of orbits of C_0-semigroups and solutions of linear and semilinear abstract differential equations,* Russ. J. Math. Phys. **13** (1) (2006), 13–30.

[5] E. Bazhlekova. *Fractional Evolution Equations in Banach Spaces,* Ph.D. Thesis, Eindhoven University of Technology, 2001.

[6] D. Bugajewski, T. Diagana. *Almost automorphy of the convolution operator and applications to differential and functional differential equations*, Nonlinear Stud., **13** (2) (2006), 129-140.

[7] Ph. Clément, G. Da Prato. *Existence and regularity results for an integral equation with infinite delay in a Banach space*, Integral Equations and Operator Theory, **11** (1988), 480-500.

[8] T. Diagana. *Some remarks on some second-order hyperbolic differential equations*, Semigroup Forum **68** (2004), 357-364.

[9] T. Diagana, G.M. N'Guérékata. *Almost automorphic solutions to semilinear evolution equations,* Funct. Differ. Equ. **13** (2) (2006), 195-206.

[10] J.A. Goldstein, G.M.N'Guérékata. *Almost automorphic solutions of semilinear evolution equations,* Proc. Amer. Math. Soc. **133** (8) (2005), 2401-2408.

[11] R. Gorenflo, F. Mainardi. *On Mittag-Leffler-type functions in fractional evolution processes,* J. Comp. Appl. Math. **118** (2000), 283-299.

[12] G. Gripenberg, S-O Londen, O. Staffans. *Volterra Integral and Functional Equations.* Encyclopedia of Mathematics and Applications, **34**, Cambridge University Press, Cambridge-New York, 1990.

[13] J. Liang; J. Zhang; T.J. Xiao, *Composition of pseudo almost automorphic and asymptotically almost automorphic functions.* J. Math. Anal. Appl. 340 (2008), no. 2, 1493–1499.

[14] J. Liu, G.M. N'Guérékata, N. van Minh. *Almost automorphic solutions of second order evolution equations,* Appl. Anal. **84** (11) (2005), 1173-1184.

[15] C. Lizama. *Regularized solutions for abstract Volterra equations,* J. Math. Anal. Appl. **243** (2000), 278-292.
[16] C. Lizama. *On approximation and representation of k-regularized resolvent families,* Integral Equations Operator Theory **41** (2), (2001), 223-229.
[17] C. Lizama, J. Sánchez. *On perturbation of k-regularized resolvent families,* Taiwanese J. Math. **7** (2), (2003), 217-227.
[18] C. Lizama, V. Poblete. *On multiplicative perturbation of integral resolvent families,* J. Math. Anal. Appl., **327** (2) (2007), 1335-1359.
[19] G.M. N'Guérékata. *Existence and Uniqueness of Almost Automorphic Mild Solutions of Some Semilinear Abstract Differential Equations,* Semigroup Forum **69** (2004), 80-86.
[20] G.M. N'Guérékata. *Topics in almost automorphy,* Springer Verlag, New York, 2005.
[21] G.M. N'Guerekata. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces,* Kluwer Acad/Plenum, New York-Boston-Moscow-London, 2001.
[22] J.W. Nunziato. *On heat conduction in materials with memory*, Quart. Appl. Math. **29** (1971), 187-204.
[23] J. Prüss. *Evolutionary Integral Equations and Applications.* Monographs Math., **87**, Birkhäuser Verlag, 1993.
[24] S.Y.Shaw, J.C. Chen. *Asymptotic behavior of (a, k)-regularized families at zero,* Taiwanese J. Math. **10** (2) (2006), 531-542.

Universidade Federal de Pernambuco, Departamento de Matemática, Av. Prof. Luiz Freire, S/N, Recife-PE, CEP. 50540-740, Brazil.
E-mail address: cch@dmat.ufpe.br

Departamento de Matemática, Facultad de Ciencias. Universidad de Santiago de Chile, Casilla 307-Correo 2, Santiago, Chile
E-mail address: carlos.lizama@usach.cl