SEMILINEAR EVOLUTION EQUATIONS ON DISCRETE TIME AND MAXIMAL REGULARITY

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Abstract. This paper deals with the existence and stability of solutions for semilinear equations on Banach spaces by using recent characterizations of discrete maximal regularity. As application we examine the asymptotic behavior of discrete control systems.

Keywords: Discrete time; Forward difference operator; Semilinear evolution equations; Discrete maximal regularity; Stability.

1. Introduction

Let A be a bounded linear operator defined on a complex Banach space X. In this article we are concerned with the study of existence of bounded solutions and stability for the semilinear problem

$$\Delta x_n - Ax_n = f(n, x_n), \qquad n \in \mathbb{Z}_+, \tag{1.1}$$

by means of the knowledge of maximal regularity properties for the vector-valued discrete time evolution equation

$$\Delta x_n - Ax_n = f_n, \qquad n \in \mathbb{Z}_+, \tag{1.2}$$

with initial condition $x_0 = 0$.

The theory of dynamical systems described by the difference equations has attracted a good deal of interest in the last decade due to the various applications of their qualitative properties, see [1], [17] , [25] and [26]. For example, in the finite-dimensional case, equation (1.1) appears in the study of stabilization for digital communication channels, see [29].

In this paper, we prove a very general theorem on the existence of bounded solutions for the semilinear problem (1.1) on $l_p(\mathbb{Z}_+; X)$ spaces. The general framework for the proof of this statement uses a new approach based on discrete maximal regularity. Then we give a general stability criterion which, in particular, provide us with simple conditions to guarantee that the solutions converges to zero for the nonlinear control system

$$x_{n+1} = Ax_n + Bu_n + F(x_n, u_n), \quad n \in \mathbb{Z}_+, \tag{1.3}$$

where A and B are constant matrices, F is a nonlinear function and u_n a control input.

In the continuous case, it is well known that the study of maximal regularity is very useful for treating semilinear and quasilinear problems. (see for example Amann [2],

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Denk-Hieber and Prüss [15], Clément-Londen-Simonett [12], the survey by Arendt [3] and the bibliography therein). Maximal regularity has also been studied in the finite difference setting. S. Blunck considered in [6] and [7] maximal regularity for the linear equation (1.2). See also Portal [31], [32]. In [19] maximal regularity on discrete Hölder spaces for finite difference operators subject to Dirichlet boundary conditions in one and two dimensions is proved. Furthermore, the authors investigated maximal regularity in discrete Hölder spaces for the Crank- Nicolson scheme. In [18] maximal regularity for linear parabolic difference equations is treated, whereas in [13] a characterization in terms of R-boundedness properties of the resolvent operator for linear second order difference equations was given. See also the recent paper by Kalton and Portal [21], where they discussed maximal regularity of power-bounded operators and relate the discrete to the continuous time problem for analytic semigroups. However, for nonlinear discrete time evolution equations, this new approach appears not to be considered in the literature.

The paper is organized as follows. The second section provides an explanation for the basic notations and definitions. In the third section we prove the existence of bounded solutions whose first discrete derivative is in l_p ($1 < p < +\infty$) for the semilinear problem (1.1) by using maximal regularity and contraction principle. We also get some a priori estimates for the solutions x_n and its discrete derivative Δx_n. Such estimates will follows from the discrete Gronwall's inequality [1](see also [30] and [27]). In the fourth section we give a criterion for stability of equation (1.1). Application on control systems are shown. Finally, in the last section we deal with local perturbations of the system (1.2).

2. Discrete maximal regularity

Let X be a Banach space. Let $\mathbb{Z}_+$ denote the set of non negative integer numbers, Δ the forward difference operator of the first order, i.e. for each $x : \mathbb{Z}_+ \to X$, and $n \in \mathbb{Z}_+$, $\Delta x_n = x_{n+1} - x_n$.

For $T \in \mathcal{B}(X)$ be given, define $\mathcal{T} : \mathbb{Z}_+ \to \mathcal{B}(X)$ by $\mathcal{T}(n) = T^n$ and consider the discrete time evolution equation

$$\Delta x_n - (T - I)x_n = f_n, \quad x_0 = 0, \quad n \in \mathbb{Z}_+, \tag{2.1}$$

or, equivalently,

$$x_{n+1} - Tx_n = f_n, \quad x_0 = 0, \quad n \in \mathbb{Z}_+, \tag{2.2}$$

where the sequence $f = (f_n)$ is given. Then the (unique) solution x is given by $x_{n+1} = (\mathcal{T} * f)_n$ where

$$(\mathcal{T} * f)_n := \sum_{k=0}^{n} \mathcal{T}(k) f_{n-k}, \quad n \in \mathbb{Z}_+.$$

Recall that $T \in \mathcal{B}(X)$ is said to be power bounded if $\sup_{n\in\mathbb{Z}_+} \|T^n\| < \infty$ and it is called analytic if the set

$$\{n(T - I)T^n : n \in \mathbb{N}\}, \tag{2.3}$$

is bounded. For recent and related results on power bounded and analytic operators we refer to [16]. The following definition was introduced by Blunck [7, p.212].

Definition 2.1. *Let $1 < p < +\infty$. Let $T \in \mathcal{B}(X)$ be a power bounded operator. We say that T has discrete maximal regularity if $(K_T f)_n := \sum_{k=0}^{n}(T-I)T^k f_{n-k}$ defines a bounded operator $K_T \in \mathcal{B}(l_p(\mathbb{Z}_+, X))$.*

This property of $T \in \mathcal{B}(X)$ has been proved to be independent of $p \in (0,\infty)$. It was shown by Blunck [6] that a necessary condition for T to have l_p-maximal regularity for some p is that T satisfies Ritt's condition (see [21]). We introduce the means

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \|\sum_{j=1}^{n} \epsilon_j x_j\|$$

for $x_1, ..., x_n \in X$.

Definition 2.2. *Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there exists a constant $c \geq 0$ such that*

$$\|(T_1 x_1, ..., T_n x_n)\|_R \leq c\|(x_1, ..., x_n)\|_R \tag{2.4}$$

for all $T_1, ..., T_n \in \mathcal{T}, x_1, ..., x_n \in X, n \in \mathbb{N}$. The least c such that (2.4) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

An equivalent definition using the Rademacher functions can be found in [15]. We note that R-boundedness clearly implies boundedness. If $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [4, Proposition 1.17]. Some useful criteria for $R-$boundedness are provided in [4], [15] and [20].

Remark 2.3. a) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded sets, then $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$ is R- bounded.

b) Let $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X,Z)$ is R- bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T}).$$

c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle (see [4], [11] or [15]).

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [8] and Burkholder [9].

In [7], Blunck characterize the discrete maximal regularity for first order difference equations by R-boundedness properties of the resolvent operator T as follows.

Theorem 2.4. *Let X be a UMD space and let $T \in \mathcal{B}(X)$ be power bounded and analytic. Then the following assertions are equivalent.*

(i) T has discrete maximal regularity of order 1.
(ii) $\{(\lambda-1)R(\lambda,\ T):\ |\lambda|=1,\ \lambda\neq 1\}$ is R-bounded.

Observe that from the point of view of applications, Blunck's characterization provide a workable criteria, see Section 4 below. We remark that the concept of R-boundedness play a fundamental role in recent works by Clément-Da Prato [10], Clément et al. [11], Weis [33, 34], Arendt-Bu [4, 5] and Keyantuo-Lizama [22, 23, 24].

3. Semilinear difference equations

In this section our aim is to investigate the existence of bounded solutions, whose first discrete derivative is in ℓ_p, for semilinear difference equations via discrete maximal regularity.

Next, we consider the following first order difference equation:

$$\Delta x_n - Ax_n = f(n,x_n), \quad \text{for all} \quad n\in\mathbb{Z}_+, \quad x_0=0, \tag{3.1}$$

which is equivalent to:

$$x_{n+1} - Tx_n = f(n,x_n), \quad \text{for all} \quad n\in\mathbb{Z}_+, \quad x_0=0, \tag{3.2}$$

where $T := I + A$.

To establish the next result, we need to introduce the following assumption:

Assumption (A): Suppose that the following condition holds:
(i) The function $f:\mathbb{Z}_+\times X\longrightarrow X$ satisfy the Lipschitz condition on X, i.e. for all $z,w\in X$ and $n\in\mathbb{Z}_+$, we get $||f(n,z)-f(n,w)||_X\leq \alpha_n||z-w||_X$, where $\alpha:=(\alpha_n)\in l_1$.
(ii) $f(\cdot,0)\in l_1$.

Denote by $\mathcal{W}_0^{\infty,p}$ the Banach space of all sequences $V=(V_n)$ belonging to $l_\infty(\mathbb{Z}_+,X)$ such that $V_0=0$ and $\Delta V\in l_p(\mathbb{Z}_+,X)$ equipped with the norm $|||V|||=||V||_\infty+||\Delta V||_p$.

We remark that the above defined space contains the set of all bounded solutions of equation (3.2). With the above notations we have the following main result:

Theorem 3.1. *Assume that Condition (A) holds. In addition suppose that T has discrete maximal regularity. Then, there is a unique bounded solution $x=(x_n)$ of equation (3.1) such that $(\Delta x)\in l_p(\mathbb{Z}_+,X)$. Moreover, we have the following a priori estimates for the solution:*

$$||x||_\infty \leq M||f(\cdot,0)||_1 e^{M||\alpha||_1}, \tag{3.3}$$

and

$$||\Delta x||_p \leq C||f(\cdot,0)||_1 e^{2M||\alpha||_1}, \quad 1<p<+\infty, \tag{3.4}$$

where $M:=\sup_{n\in\mathbb{Z}^+}||T^n||$ and $C>0$.

Proof. Let V be a sequence in $\mathcal{W}_0^{\infty,p}$. Then using Assumption (A) we obtain that the function $g := f(\cdot, V.)$ is in $l_p(\mathbb{Z}_+, X)$. In fact, we have

$$
\begin{aligned}
||g||_p^p &= \sum_{n=0}^{\infty} ||f(n, V_n)||_X^p \\
&\leq \sum_{n=0}^{\infty} (||f(n, V_n) - f(n, 0)||_X + ||f(n, 0)||_X)^p \\
&\leq 2^p \sum_{n=0}^{\infty} ||f(n, V_n) - f(n, 0)||_X^p + 2^p \sum_{n=0}^{\infty} ||f(n, 0)||_X^p \\
&\leq 2^p \sum_{n=0}^{\infty} \alpha_n^p ||V_n||_X^p + 2^p \sum_{n=0}^{\infty} ||f(n, 0)||_X^p,
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
\sum_{n=0}^{\infty} ||f(n, 0)||_X^p &= \sum_{n=0}^{\infty} ||f(n, 0)||_X^{p-1} ||f(n, 0)||_X \\
&\leq ||f(\cdot, 0)||_\infty^{p-1} \sum_{n=0}^{\infty} ||f(n, 0)||_X \\
&= ||f(\cdot, 0)||_\infty^{p-1} ||f(\cdot, 0)||_1.
\end{aligned}
$$

Analogously, we have

$$\sum_{n=0}^{\infty} \alpha_n^p \leq ||\alpha||_\infty^{p-1} ||\alpha||_1.$$

Hence

$$
\begin{aligned}
||g||_p^p &\leq 2^p ||V||_\infty^p \sum_{n=0}^{\infty} \alpha_n^p + 2^p ||f(\cdot, 0)||_\infty^{p-1} ||f(\cdot, 0)||_1 \\
&\leq 2^p ||V||_\infty^p ||\alpha||_\infty^{p-1} ||\alpha||_1 + 2^p ||f(\cdot, 0)||_\infty^{p-1} ||f(\cdot, 0)||_1,
\end{aligned}
$$

proving that $g \in l_p(\mathbb{Z}_+, X)$.

Since T has discrete maximal regularity, the Cauchy's problem

$$
\begin{cases} z_{n+1} - T z_n = g_n, \\ z_0 = 0, \end{cases} \tag{3.6}
$$

has a unique solution (z_n) such that $\Delta z_n \in l_p(\mathbb{Z}_+, X)$, which is given by

$$
z_n = (\mathcal{K}V)_n = \begin{cases} 0 & \text{if } n = 0, \\ \displaystyle\sum_{k=0}^{n-1} T^k f(n-1-k,\ V_{n-1-k}) & \text{if } n \geq 1. \end{cases} \tag{3.7}
$$

It suffices now to show that the operator $\mathcal{K} : \mathcal{W}_0^{\infty,p} \longrightarrow \mathcal{W}_0^{\infty,p}$ has a unique fixed point. To verify that $\mathcal{K}$ is well defined we have only to show that $\mathcal{K}V \in l_\infty(\mathbb{Z}_+, X)$. In fact, we use Assumption (A) as above to obtain

(3.8)

$$\begin{aligned}
||\sum_{k=0}^{n-1} T^k f(n-1-k,\ V_{n-1-k})||_X &\leq M\sum_{k=0}^{n-1} ||f(n-1-k, V_{n-1-k}) - f(n-1-k, 0)||_X \\
&+ M\sum_{k=0}^{n-1} ||f(n-1-k, 0)||_X \\
&\leq M\sum_{k=0}^{n-1} \alpha_{n-1-k}||V_{n-1-k}||_X + M\sum_{j=0}^{n-1} ||f(j,0)||_X \\
&\leq M||V||_\infty \sum_{j=0}^{n-1} \alpha_j + M\sum_{j=0}^{n-1} ||f(j,0)||_X \\
&\leq M[||V||_\infty ||\alpha||_1 + ||f(\cdot,0)||_1].
\end{aligned}$$

It proves that the space $\mathcal{W}_0^{\infty,p}$ is invariant under $\mathcal{K}$.

Now, we associate with T the $\mathcal{B}(X)$−valued kernel $k_T : \mathbb{Z} \to \mathcal{B}(X)$ defined by

$$k_T(n) = \begin{cases} (I-T)T^n & \text{for } n \in \mathbb{Z}_+, \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding operator on $\mathbb{Z}_+$, $K_T : l_p(\mathbb{Z}_+; X) \to l_p(\mathbb{Z}_+; X)$ by

$$(K_T f)(n) = \sum_{j=0}^{n} k_T(j) f_{n-j} \quad n \in \mathbb{Z}_+. \tag{3.9}$$

By the discrete maximal regularity, K_T is well defined and bounded on $l_p(\mathbb{Z}_+; X)$.

Let V and $\tilde{V}$ be in $\mathcal{W}_0^{\infty,p}$. In view of Assumption (A) (i), we have initially as in (3.8)

(3.10)

$$\begin{aligned}
||[\mathcal{K}V]_n - [\mathcal{K}\tilde{V}]_n||_X &= ||\sum_{k=0}^{n-1} T^k (f(n-1-k,\ V_{n-1-k}) - f(n-1-k,\ \tilde{V}_{n-1-k}))||_X \\
&\leq M\sum_{j=0}^{n-1} \alpha_j ||V_j - \tilde{V}_j||_X \leq M||\alpha||_1 ||V - \tilde{V}||_\infty.
\end{aligned}$$

On the other hand, we observe first that

$$\begin{aligned}\Delta[\mathcal{K}V]_n - \Delta[\mathcal{K}\tilde{V}]_n &= f(n, V_n) - f(n, \tilde{V}_n)\\ &+ \sum_{k=0}^{n-1}(T-I)T^k(f(n-1-k,\ V_{n-1-k}) - f(n-1-k,\ \tilde{V}_{n-1-k})),\end{aligned}$$

then using Minkowskii's inequality and taking into account that K_T is bounded on $l_p(\mathbb{Z}_+, X)$ and Assumption (A) we get
(3.11)

$$\begin{aligned}||\Delta\mathcal{K}V - \Delta\mathcal{K}\tilde{V}||_p &\leq [\sum_{n=1}^{\infty}||f(n, V_n) - f(n, \tilde{V}_n)||_X^p]^{1/p}\\ &+ \ [\sum_{n=1}^{\infty}||\sum_{k=0}^{n-1}(T-I)T^k(f(n-1-k,\ V_{n-1-k}) - f(n-1-k,\ \tilde{V}_{n-1-k}))||_X^p\Big]^{1/p}\\ &\leq \ [\sum_{n=1}^{\infty}||f(n, V_n) - f(n, \tilde{V}_n)||_X^p]^{1/p} + ||K_T||[\sum_{n=0}^{\infty}||f(n, V_n) - f(n, \tilde{V}_n)||_X^p]^{1/p}\\ &\leq \ (1 + ||K_T||)[\sum_{n=1}^{\infty}\alpha_n^p||V_n - \tilde{V}_n||_X^p]^{1/p} \leq (1 + ||K_T||)||\alpha||_1||V - \tilde{V}||_\infty.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}|||\mathcal{K}V - \mathcal{K}\tilde{V}||| &= ||\mathcal{K}V - \mathcal{K}\tilde{V}||_\infty + ||\Delta\mathcal{K}V - \Delta\mathcal{K}\tilde{V}||_p\\ &\leq M||\alpha||_1|||V - \tilde{V}||| + (1 + ||K_T||)||\alpha||_1|||V - \tilde{V}|||\\ &= (M + 1 + ||K_T||)||\alpha||_1|||V - \tilde{V}||| = ab|||V - \tilde{V}|||,\end{aligned}$$

where $a := M||\alpha||_1$ and $b := 1 + (1 + ||K_T||)M^{-1}$.

Next, we consider the iterates of the operator $\mathcal{K}$. We observe from (3.10) that

$$\begin{aligned}||[\mathcal{K}^2V]_n - [\mathcal{K}^2\tilde{V}]_n||_X &\leq M\sum_{j=0}^{n-1}\alpha_j||[\mathcal{K}V]_j - [\mathcal{K}\tilde{V}]_j||_X\\ &\leq M^2\sum_{j=0}^{n-1}\alpha_j(\sum_{i=0}^{j-1}\alpha_i||V_i - \tilde{V}_i||_X)\\ &\leq \tfrac{1}{2}M^2(\sum_{\tau=0}^{n-1}\alpha_\tau)^2||V - \tilde{V}||_\infty.\end{aligned}$$

Therefore,

$$||\mathcal{K}^2V - \mathcal{K}^2\tilde{V}||_\infty \leq \frac{1}{2}(M||\alpha||_1)^2|||V - \tilde{V}|||. \tag{3.12}$$

Furthermore, using (3.11) and then (3.10), we have

$$\begin{aligned} ||\Delta\mathcal{K}^2V - \Delta\mathcal{K}^2\tilde{V}||_p &\leq (1 + ||K_T||)[\textstyle\sum_{n=0}^\infty \alpha_n^p||[\mathcal{K}V]_n - [\mathcal{K}\tilde{V}]_n||_X^p]^{1/p} \\ &\leq M(1 + ||K_T||)[\textstyle\sum_{n=0}^\infty \alpha_n^p(\sum_{k=0}^{n-1}\alpha_k||V_k - \tilde{V}_k||_X)^p]^{1/p} \\ &\leq M(1 + ||K_T||)[\textstyle\sum_{n=0}^\infty \alpha_n^p(\sum_{j=0}^{n-1}\alpha_j)^p||V - \tilde{V}||_\infty^p]^{1/p} \\ &\leq M(1 + ||K_T||)\tfrac{1}{2}(\textstyle\sum_{j=0}^\infty \alpha_j)^2||V - \tilde{V}||_\infty \\ &\leq \tfrac{1}{2}(M||\alpha||_1)^2(1 + ||K_T||)M^{-1}|||V - \tilde{V}|||, \end{aligned}$$

whence

$$|||\mathcal{K}^2V - \mathcal{K}^2\tilde{V}||| \leq \frac{b}{2}a^2|||V - \tilde{V}|||, \tag{3.13}$$

with a and b defined as above. An induction argument show us that:

$$|||\mathcal{K}^nV - \mathcal{K}^n\tilde{V}||| \leq \frac{b}{n!}a^n|||V - \tilde{V}|||. \tag{3.14}$$

Since $ba^n/n! < 1$ for n sufficiently large, by the contraction principle $\mathcal{K}$ has a unique fixed point $V \in \mathcal{W}_0^{\infty,p}$. Let V the unique fixed point of $\mathcal{K}$, then by Assumption (A) we have

$$\begin{aligned} ||V_{n+1}||_X &= ||\sum_{k=0}^n T^k f(n-k,\ V_{n-k})||_X \\ &\leq M\sum_{k=0}^n ||f(n-k, V_{n-k}) - f(n-k,0)||_X + M\sum_{k=0}^n ||f(n-k,0)||_X \\ &\leq M\sum_{j=0}^n \alpha_j||V_j||_X + M\sum_{j=0}^n ||f(j,0)||_X \\ &\leq M\sum_{j=0}^n \alpha_j||V_j||_X + M||f(\cdot,0)||_1. \end{aligned} \tag{3.15}$$

Then, by application of the discrete Gronwall's inequality [1, Corollary 4.12, p.183], we get

$$\begin{aligned} ||V_{n+1}||_X &\leq M||f(\cdot,0)||_1\prod_{j=0}^n(1 + M\alpha_j) \leq M||f(\cdot,0)||_1\prod_{j=0}^n e^{M\alpha_j} \\ &= M||f(\cdot,0)||_1e^{M\sum_{j=0}^n\alpha_j} \leq M||f(\cdot,0)||_1e^{M||\alpha||_1}. \end{aligned}$$

Then

$$||V||_\infty \leq M||f(\cdot,0)||_1e^{M||\alpha||_1}. \tag{3.16}$$

Finally, by (2.4) we obtain

$$\Delta V_n = f(n, V_n) + \sum_{k=0}^{n-1}(T-I)T^k(f(n-1-k,\ V_{n-1-k})).$$

Hence, using the fact that $\Delta V_0 = f(0,0)$ and proceeding analogously as in (3.11), we get

$$\begin{aligned}
||\Delta V||_p &= (||f(0,0)||_X^p + \sum_{n=1}^{\infty}||\Delta V_n||_X^p)^{1/p} \\
&\leq ||f(0,0)||_X + (\sum_{n=1}^{\infty}||\Delta V_n||_X^p)^{1/p} \\
&\leq ||f(0,0)||_X + (\sum_{n=1}^{\infty}||f(n,V_n)||_X^p)^{1/p} + ||K_T||(\sum_{n=0}^{\infty}||f(n,V_n)||_X^p)^{1/p} \\
&\leq 2(\sum_{n=0}^{\infty}||f(n,V_n)||_X^p)^{1/p} + ||K_T||(\sum_{n=0}^{\infty}||f(n,V_n)||_X^p)^{1/p} \\
&\leq (2+||K_T||)\sum_{n=0}^{\infty}||f(n,V_n)||_X,
\end{aligned}$$

where, by Assumption (A) and (3.16)

$$\begin{aligned}
\sum_{n=0}^{\infty}||f(n,V_n)||_X &= \sum_{n=0}^{\infty}||f(n,V_n)-f(n,0)||_X + \sum_{n=0}^{\infty}||f(n,0)||_X \\
&\leq \sum_{n=0}^{\infty}\alpha_n||V_n||_X + ||f(\cdot,0)||_1 \\
&\leq ||\alpha||_1||V||_\infty + ||f(\cdot,0)||_1 \\
&\leq ||\alpha||_1 M||f(\cdot,0)||_1 e^{M||\alpha||_1} + ||f(\cdot,0)||_1 \\
&\leq ||f(\cdot,0)||_1 e^{2M||\alpha||_1}.
\end{aligned}$$

This ends the proof of the theorem. ■

In view of Blunck's theorem, we obtain the following result valid on UMD spaces.

Corollary 3.2. *Let X be a UMD space. Assume that Assumption (A) holds and suppose $T \in \mathcal{B}(X)$ is power bounded, analytic and satisfy that the set*

$$\{(\lambda-1)R(\lambda,\ T):\ |\lambda|=1,\ \lambda\neq 1\}$$

is R-bounded. Then, there is a unique bounded solution $x=(x_n)$ of equation (3.1) such that $(\Delta x)\in l_p(\mathbb{Z}_+, X)$. Moreover, the a priori estimates (3.3) and (3.4) hold.

Example 3.3. Consider the semilinear problem

$$\Delta x_n - (T-I)x_n = q_n f(x_n), \tag{3.17}$$

where f is defined and satisfy a Lipschitz condition with constant L on a Hilbert space H. In addition suppose $(q_n)\in l_1$. Then Assumption (A) is satisfied . In our case, applying the preceding result we obtain that if $T \in \mathcal{B}(H)$ is power bounded, analytic and satisfy that the set $\{(\lambda-1)R(\lambda,\ T):\ |\lambda|=1,\ \lambda\neq 1\}$ is bounded, then there exists a unique bounded solution $x=(x_n)$ of the equation (3.17) such that $(\Delta x_n)\in l_p(\mathbb{Z}_+,H)$. Moreover,

$$||x||_\infty \leq M||f(0)||_H||q||_1 e^{LM||q||_1}. \tag{3.18}$$

In particular, taking $T=I$ the identity operator, we obtain the following result.

Corollary 3.4. *Suppose f is defined and satisfy a Lipschitz condition with constant L on a Hilbert space H. Let $(q_n)\in l_1(\mathbb{Z}_+,H)$, then the equation*

$$\Delta x_n = q_n f(x_n), \tag{3.19}$$

has a unique bounded solution $x=(x_n)$ such that $(\Delta x_n)\in l_p(\mathbb{Z}_+,H)$ and (3.18) holds with $M=1$.

We remark that the above result is new even in the finite dimensional case and cover a wide range of difference equations.

4. A criterion for stability

The following result provides a new criterion to verify the stability of discrete semilinear systems. Note that the characterization of maximal regularity is the key to give conditions based only in the data of a given system.

Theorem 4.1. *Let X be a UMD space. Assume that Assumption (A) holds and suppose $T\in\mathcal{B}(X)$ is power bounded, analytic and the set*

$$\{(\lambda-1)R(\lambda,\ T):\ |\lambda|=1,\ \lambda\neq 1\} \tag{4.1}$$

is R-bounded. In addition, assume that $1\in\rho(T)$. Then the system (3.1) is stable, that is the solution (x_n) of (3.1) is such that $x_n\to 0$ as $n\to\infty$.

Proof. By Corollary 3.2, there exists a unique solution x_n of equation (3.1) such that $\Delta x_n\in l_p(\mathbb{Z}_+,X)$. Then $\Delta x_n\to 0$ as $n\to\infty$. Next, observe that Assumption (A) imply

$$\begin{aligned} ||f(n,x_n)||_X &\leq ||f(n,x_n)-f(n,0)||_X+||f(n,0)||_X \\ &\leq \alpha_n||x_n||_X+||f(n,0)||_X. \end{aligned} \tag{4.2}$$

Since $(x_n)\in l_\infty(\mathbb{Z}_+,X)$, $(f(n,0))\in l_1(\mathbb{Z}_+,X)$ and $(\alpha_n)\in l_1(\mathbb{Z}_+)$, we obtain that $f(n,x_n)\to 0$ as $n\to\infty$. Then, the result follows from hypothesis and equation (3.1). ∎

From the point of view of applications, the following corollary provide easy to check conditions for stability.

Corollary 4.2. *Let H be a Hilbert space. Let $T\in\mathcal{B}(H)$ such that $||T||<1$. Suppose that the following conditions hold:*

(i) The function $f:\mathbb{Z}_+\times H\longrightarrow H$ satisfy: for all $z,w\in H$ and $n\in\mathbb{Z}_+$, we get $||f(n,z)-f(n,w)||_H\leq\alpha_n||z-w||_H$, where $\alpha:=(\alpha_n)\in l_1(\mathbb{Z}_+)$.

(ii) $f(\cdot, 0) \in l_1(\mathbb{Z}_+, H)$.

Then the system (3.1) is stable.

Proof. First we note that each Hilbert space is UMD, and then the concept of R-boundedness and boundedness coincide, see [15]. Since $||T|| < 1$, we get that T is power bounded and analytic and $1 \in \rho(T)$. Furthermore, for $|\lambda| = 1$, $\lambda \neq 1$ the inequality

$$||(\lambda - 1)R(\lambda, T)|| = ||(\lambda - 1)\sum_{n=0}^{\infty}(\frac{T}{\lambda})^n|| \leq \frac{|\lambda - 1||\lambda|}{1 - ||T||} < \frac{2}{1 - ||T||},$$

shows that the set (4.1) is bounded.

■

Of course, the same result holds in the finite dimensional case. As an application, consider a semilinear discrete control system of the form

$$x_{n+1} = Ax_n + Bu_n + F(x_n, u_n), \quad n \in \mathbb{Z}_+, \tag{4.3}$$

where A and B are constant matrices, F is a nonlinear function and u_n a control input.

The system (4.3) was considered in [29] and consists of a linear discrete-time system and a linearly bounded nonlinear perturbation. Based on the state space quantization method used in [14] and [28], they established sufficient conditions for the global stabilizability of the semilinear discrete-time system under an appropriate growth condition on the nonlinear perturbation. In contrast with this approach, and applying Corollary 4.2 to $f(n, z) = Bu_n + F(z, u_n)$ we directly obtain the following remarkable result.

Proposition 4.3. *Suppose that the following conditions hold:*

(i) The function $F : \mathbb{Z}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *satisfy: for all* $z, w \in \mathbb{R}^N$ *and* $n \in \mathbb{Z}_+$, *we have* $||F(z, u_n) - F(w, u_n)|| \leq \alpha_n ||z - w||$, *where* $\alpha := (\alpha_n) \in l_1(\mathbb{Z}_+)$.

(ii) $Bu_n + F(0, u_n) \in l_1(\mathbb{Z}_+)$.

(iii) $||A|| < 1$.

Then system (4.3) is stable.

Example 4.4. Let α, β, c, d be positive real numbers. The following semilinear discrete control system is considered in [29]:

$$x_{n+1} = \alpha x_n + \beta u_n + F(x_n, u_n), \quad n \in \mathbb{Z}_+, \tag{4.4}$$

where

$$F(x_n, u_n) = cx_n \sin u_n + du_n \cos^2 x_n, \quad n \in \mathbb{Z}_+.$$

Here we have $\alpha_n = (c+2d)|u_n|$. As a consequence of Proposition 4.3, we obtain that for all $0 < \alpha < 1$ and each control input u_n in $l_1(\mathbb{Z}_+)$, the solution of system (4.4) converges to zero as $n \to \infty$.

5. Local perturbations

Note that Theorem 3.1 is not general enough to include perturbations like the following

$$f(n,z) = y_n + \alpha_n B(z,z), \tag{5.1}$$

where $B : X \times X \to X$ is a bounded bilinear operator, and α_n, y_n are fixed sequences. In fact, it does not satisfy condition (A)(i). This yields to us to study locally Lipschitzian perturbations of equation (3.1) (see Corollary 5.2 below).

In the process to obtaining our next result, we will require the following assumption.

Assumption (A)*: The following conditions hold:
(i)* The function $f(n,z)$ is locally Lipschitz with respect to $z \in X$, i.e. for each positive number R, for all $n \in \mathbb{Z}_+$, and $z, w \in X$, $||z||_X \le R, ||w||_X \le R$

$$||f(n,z) - f(n,w)||_X \le l(n,R)||z-w||_X,$$

where $\ell : \mathbb{Z}_+ \times [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing function with respect to second variable.
(ii)* There is a positive number a such that $\sum_{n=0}^{\infty} \ell(n,a) < +\infty$.
(iii)* $f(\cdot,0) \in \ell_1$.

We need to introduce some basic notations: We denote by $\mathcal{W}_m^{\infty,p}$ the Banach space of all sequences $V = (V_n)$ belonging to $\ell_\infty(\mathbb{Z}_+, X)$ such that $V_n = 0$ if $0 \le n \le m$, and $\Delta V \in \ell_p(\mathbb{Z}_+, X)$ equipped with the norm $|||\cdot|||$. For $\lambda > 0$, denote by $\mathcal{W}_m^{\infty,p}[\lambda]$ the ball $|||V||| \le \lambda$ in $\mathcal{W}_m^{\infty,p}$. Our main result in this section is the following local version of Theorem 3.1.

Theorem 5.1. *Suppose that the following conditions are satisfied:*
(a)* The Condition (A)* holds.
(b)* T has discrete maximal regularity.

Then, there are a positive constant $m \in \mathbb{N}$ and a unique bounded solution x_n of equation (3.1) for $n \ge m$ such that $x_n = 0$ if $0 \le n \le m$ and the sequence (Δx_n) belongs to $\ell_p(\mathbb{Z}_+, X)$. Moreover, we get

$$||x||_\infty + ||\Delta x||_p \le a, \tag{5.2}$$

where a is the constant of condition (ii)*.

Proof. We will apply the contraction principle in $\mathcal{W}_m^{\infty,p}[a]$ for m suitable. Let $\beta \in (0,1)$. Using (iii)* and (ii)* there are n_1 and n_2 in $\mathbb{N}$ such that:

$$(M + 2 + ||K_T||) \sum_{j=n_1}^{\infty} ||f(j,0)||_X \le \beta a, \tag{5.3}$$

and

$$\mathcal{T} := \beta + (M + 1 + ||K_T||) \sum_{j=n_2}^{\infty} \ell(j,a) < 1. \tag{5.4}$$

Let V be a sequence in $\mathcal{W}_m^{\infty,p}[a]$, with $m = \max\{n_1, n_2\}$. A short argument similar to (3.5) and involving Assumption (A)* shows that the sequence

$$g_n := \begin{cases} 0 & \text{if} \quad 0 \le n \le m, \\ f(n, V_n) & \text{if} \quad n > m, \end{cases} \tag{5.5}$$

belongs to ℓ_p. By the discrete maximal regularity, the Cauchy's problem (3.6) with g_n defined as in (5.5) has a unique solution (z_n) such that $\Delta z_n \in l_p(\mathbb{Z}_+, X)$, which is given by

$$z_n = (\tilde{\mathcal{K}}V)_n = \begin{cases} 0 & \text{if } 0 \le n \le m, \\ \displaystyle\sum_{k=0}^{n-1-m} T^k f(n-1-k,\ V_{n-1-k}) & \text{if } n \ge m+1. \end{cases} \tag{5.6}$$

We will prove that $(\tilde{\mathcal{K}}V)$ belongs to $\mathcal{W}_m^{\infty,p}[a]$. In fact, since $||V_j||_X \le ||V||_\infty \le |||V||| < a$, we have by Assumption $(A)^*$
(5.7)

$$\begin{aligned} ||[\tilde{\mathcal{K}}V]_n||_X = M\sum_{j=m}^{n-1} ||f(j,V_j)||_X &\le M\sum_{j=m}^{n-1} ||f(j,V_j) - f(j,0)||_X + M\sum_{j=m}^{n-1} ||f(j,0)||_X \\ &\le M\sum_{j=m}^{n-1} l(j,a)||V_j||_X + M\sum_{j=m}^{n-1} ||f(j,0)||_X \\ &\le M\sum_{j=m}^{\infty} l(j,a)a + M\sum_{j=m}^{\infty} ||f(j,0)||_X. \end{aligned}$$

Using that $V_m = 0$, we can proceed analogously as in (3.11) to obtain

$$\begin{aligned} ||\Delta\tilde{\mathcal{K}}V||_p &= \big[||\Delta\tilde{\mathcal{K}}V_m||_X^p + \sum_{n=m+1}^{\infty} ||\Delta\tilde{\mathcal{K}}V_n||_X^p\big]^{1/p} \\ &= \big[||f(m,0)||_X^p + \sum_{n=m+1}^{\infty} ||\tilde{\mathcal{K}}V_{n+1} - \tilde{\mathcal{K}}V_n||_X^p\big]^{1/p} \\ &\le ||f(m,0)||_X \\ &+ \big[\sum_{n=m+1}^{\infty} ||f(n,V_n) + \sum_{j=0}^{n-1-m} (T-I)T^j(f(n-1-j,\ V_{n-1-j}))||_X^p\big]^{1/p} \\ &\le ||f(m,0)||_X + (1+||K_T||)[\sum_{n=m+1}^{\infty} ||f(n,V_n)||_X^p]^{1/p} \\ &\le \sum_{n=m}^{\infty} ||f(n,0)||_X + (1+||K_T||)\sum_{n=m}^{\infty} ||f(n,V_n||_X. \end{aligned}$$

Therefore using (5.7) we get

$$
\begin{aligned}
||\Delta\tilde{\mathcal{K}}V||_p &\le \sum_{j=m}^{\infty} ||f(j,0)||_X + (1+||K_T||)\Big[\sum_{j=m}^{\infty} l(j,a)a + \sum_{j=m}^{\infty} ||f(j,0)||_X\Big] \\
&= (1+||K_T||)\sum_{j=m}^{\infty} l(j,a)a + (2+||K_T||)\sum_{j=m}^{\infty} ||f(j,0)||_X.
\end{aligned} \tag{5.8}
$$

Then, inequalities (5.7) and (5.8) together with (5.3) and (5.4) implies

$$
\begin{aligned}
|||\tilde{\mathcal{K}}V||| &\le (M+1+||K_T||)\sum_{j=m}^{\infty} \ell(j,a)a + (M+2+||K_T||)\sum_{j=m}^{\infty} ||f(j,0)||_X \\
&\le (1-\beta)a + \beta a = a,
\end{aligned}
$$

proving that $(\tilde{\mathcal{K}}V)$ belongs to $\mathcal{W}_m^{\infty,p}[a]$. In a essentially similar way to the proof of Theorem 3.1, for all V and W in $\mathcal{W}_m^{\infty,p}[a]$, we prove that:

$$
\begin{aligned}
&|||\tilde{\mathcal{K}}V - \tilde{\mathcal{K}}W||| \\
&\le M\sum_{j=m}^{\infty} \ell(j,a)|||V-W||| + (1+||K_T||)\sum_{j=m}^{\infty} \ell(j,a)|||V-W||| \\
&= (\mathcal{T}-\beta)|||V-W|||.
\end{aligned}
$$

Hence $\tilde{\mathcal{K}}$ is a $(\mathcal{T}-\beta)$-contraction. This complete the proof of the theorem. ■

This enable us to prove, as an application, the following corollary.

Corollary 5.2. *Let $B : X \times X \longrightarrow X$ be a bounded, bilinear operator; $y \in \ell_1(\mathbb{Z}_+, X)$ and $\alpha \in \ell_1(\mathbb{Z}_+, \mathbb{R})$. In addition suppose that T has discrete maximal regularity. Then, there is a unique bounded solution x such that $(\Delta x) \in l_p(\mathbb{Z}_+, X)$ for the equation*

$$x_{n+1} - Tx_n = y_n + \alpha_n B(x_n, x_n).$$

Proof. Take $l(n,R) := 2R\alpha_n||B||$. Then $\sum_{n=0}^{\infty} \ell(n,1) < +\infty$. Note also that $f(n,0) = y_n$ belongs to $\ell_1(\mathbb{Z}_+, \mathbb{R})$. Hence Assumption $(A)^*$ is satisfied. ■

Remark 5.3. We observe that under the hypothesis of the above local theorem and corollary, the same type of conclusions on stability of solutions proved in Section 4 remain true.

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