S-ASYMPTOTICALLY ω-PERIODIC SOLUTIONS FOR SEMILINEAR VOLTERRA EQUATIONS

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Abstract. We study S-asymptotically ω-periodic mild solutions of the semilinear Volterra equation $u'(t) = (a * Au)(t) + f(t, u(t))$, considered in a Banach space X, where A is the generator of an (exponentially) stable resolvent family. In particular, we extend recent results for semilinear fractional integro-differential equations considered in [4] and for semilinear Cauchy problems of first order given in [20]. Applications to integral equations arising in vicoelasticity theory are shown.

1. Introduction

The study of existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, compact almost automorphic and pseudo almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due both to its mathematical interest as to their applications in physics and mathematical biology, among other areas. Some recent contributions on existence of these type of solutions for abstract differential equations have been made. Related with this subject, we refer the reader to the extensive bibliography in [2, 5, 6, 7, 9, 10, 13, 18, 22, 23, 29, 30].

A vector-valued function $f \in C_b([0, \infty), X)$ is called S-asymptotically ω-periodic (see Henríquez, et.al. [20]) if there exists $\omega > 0$ such that

$$\lim_{t\to\infty} (f(t + \omega) - f(t)) = 0. \tag{1.1}$$

In [20] it was shown the surprising fact the property (1.1) does not characterize asymptotically ω-periodic functions, that is, bounded and continuous functions which admits the decomposition $f = g + \varphi$, where g is ω-periodic and $\lim_{t\to\infty} \varphi(t) = 0$.

On the other hand, the literature concerning the qualitative behavior (1.1) for evolution equations is incipient and limited essentially to the study of the existence of solutions of ordinary differential equations described on finite dimensional spaces (see [16, 19, 24, 28, 31]). Only recently, has been developed a theory of S-asymptotically ω-periodic functions with values in Banach spaces, and stated the existence of S-asymptotically ω-periodic functions for the first order semilinear Cauchy problem [20]. These results were now used in [4] to establish S-asymptotically ω-solutions of semilinear fractional integro-differential equations.

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We study in this work the existence and uniqueness of S-asymptotically ω-periodic solutions of the semilinear Volterra equation

$$v'(t) = \int_0^t a(t-s)Av(s)ds + f(t, v(t)), \quad t \geq 0, \tag{1.2}$$

$$v(0) = u_0 \in X, \tag{1.3}$$

where $A : D(A) \subset X \to X$ is the generator of a bounded analytic semigroup or, more generally, the generator of a resolvent family on a complex Banach space X, $a \in L^1_{loc}(\mathbb{R}_+)$ and $f : [0, \infty) \times X \to X$ is a continuous function satisfying suitable conditions.

Due to their applications in several fields of science (see [1, 11, 14]), type (1.2) equations are attracting increasing interest as well as their numerical treatment. Properties of the solutions of (1.2) have been extensively studied in the last years. In the infinite-dimensional setting, we refer to the classical monograph [27] and references therein.

The results in the present paper are, on one side, an extension of the results in [20] and [4] and, on the other side, a contribution to the study of qualitative properties for the Volterra equation (1.2), which are new even in the scalar case.

This work is organized as follows: In Section 2, we state and review the main definitions and results of other sources to be used in the article. In Section 3 we study the linear case by means of an integrated version of (1.2). As a consequence of the theory of linear evolution equations for Volterra equations [27] we derive our main result (Theorem 3.4) which states maximal regularity under the conditions that $a(t)$ is 1-regular, A is the generator of an strongly integrable resolvent, and the initial condition belongs to the domain of the operator A (generally unbounded). In Section 4, we study the existence and uniqueness of S-asymptotically ω-periodic mild solutions of the semilinear problem (1.2)-(1.3). To achieve our results, we require on $f(t, x)$ Lipschitz type conditions (Theorems 4.2, 4.6 and 4.7) or compactness (Theorem 4.9). In passing, we give easy to check conditions solely in terms of $a(t)$, f and A (being now the generator of a bounded analytic C_0-semigroup) to guarantee that problem (1.2)-(1.3) has a unique S-asymptotically ω-periodic mild solution (Corollary 4.4). To illustrate our main results, at the end of this paper we examine sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic mild solution to a specific integral equation arising in viscoelasticity theory.

2. Preliminaries

We recall that the Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is given by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

where the integral is absolutely convergent for $Re\lambda > \omega$. Furthermore, we denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators, and the notation $\rho(A)$ stands for the resolvent set of A.

In order to give an operator theoretical approach to equation (1.2) we recall the following definition (cf. [27] and [25]).

Definition 2.1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of a *solution operator* (or *resolvent family*) if there exists $\mu \in \mathbb{R}$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{1}{\hat{a}(\lambda)} : Re\lambda > \mu\} \subset \rho(A)$ and

$$\frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad Re\lambda > \mu, \quad x \in X.$$

In this case, $S(t)$ is called the solution operator generated by A.

In the scalar case there is a large bibliography which studies the concept of resolvent, we refer to the monograph by Gripenberg et al. [17], and references therein. We emphasize the fact that because of the uniqueness of the Laplace transform, in the case $a(t) \equiv 1$ the family $S(t)$ corresponds to a C_0-semigroup whereas that in case $a(t) = t$ a solution operator corresponds to the concept of cosine family, see e.g. [3] or [12]. We note that solution operators, as well as resolvent families, are a particular case of (a, k)-regularized families introduced in [25]. According to [25] a solution operator $S(t)$ corresponds to a $(1, a)$-regularized family.

In this work $C_b([0, \infty), X)$ denotes the space consisting of the continuous and bounded functions from $[0, \infty)$ into X, endowed with the norm of the uniform convergence, which is denoted by $|| \bullet ||_\infty$, and the notation $SAP_\omega(X)$ stands for the subspace of $C_b([0, \infty), X)$ consisting of the S-asymptotically ω-periodic functions. We note that $SAP_\omega(X)$ is a Banach space (see [20], Proposition 3.5).

Definition 2.2. ([20]) A continuous function $f : [0, \infty) \times X \to X$ is said uniformly S-asymptotically ω-periodic on bounded sets if for every bounded subset K of X, the set $\{f(t, x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t, x) - f(t + \omega, x)) = 0$ uniformly in $x \in K$.

Definition 2.3. ([20]) A continuous function $f : [0, \infty) \times X \to X$ is said asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded subset K of X, there exist $L_{\epsilon,K} \geq 0$ and $\delta_{\epsilon,K} > 0$ such that $||f(t, x) - f(t, y)|| \leq \epsilon$, for all $t \geq L_{\epsilon,K}$ and all $x, y \in K$ with $||x - y|| \leq \delta_{\epsilon,K}$.

Lemma 2.4. *([20]) Let $f : [0, \infty) \times X \to X$ be a uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets function and, let $u : [0, \infty) \to X$ be a S-asymptotically ω-periodic function. Then the function $v(t) = f(t, u(t))$ is S-asymptotically ω-periodic.*

Definition 2.5. ([27]) A strongly measurable family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly integrable (or strongly integrable) if $\int_0^\infty ||T(t)||dt < \infty$.

We also recall the following concept, studied for resolvent families in [26]:

Definition 2.6. A strongly continuous family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly stable if $||T(t)|| \to 0$ as $t \to \infty$.

Definition 2.7. ([20]) A strongly continuous family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be strongly S-asymptotically ω-periodic if there is $\omega > 0$ such that $T(\cdot)x$ is S-asymptotically ω-periodic for all $x \in X$.

Definition 2.8. ([27]) Let $a \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The function $a(t)$ is called $k-$regular if there is a constant $C > 0$ such that

$$|\lambda^n \hat{a}^n(\lambda)| \leq C|\hat{a}(\lambda)|, \text{ for all } Re(\lambda) > 0, \; 0 \leq n \leq k.$$

3. The linear case

In this section we consider the linear version for equation (1.2), that is

$$v'(t) = \int_0^t b(t-s)Av(s)ds + f(t), \quad t \geq 0, \tag{3.1}$$

$$v(0) = u_0 \in X. \tag{3.2}$$

or, equivalently, the integrated form

$$v(t) = \int_0^t a(t-s)Av(s)ds + \int_0^t f(s)ds + u_0, \quad t \geq 0, \tag{3.3}$$

where $u_0 \in X$ and $a(t) = \int_0^t b(s)ds$. Recall that a function $v \in C(\mathbb{R}_+; X)$ is called a strong solution of (3.3) on $\mathbb{R}_+$ if $v \in C(\mathbb{R}_+; D(A))$ and (3.3) holds on $\mathbb{R}_+$. If A generates a resolvent family $S(t)$, the variation of parameters formula allows us to write the solution of problem (3.3) as (cf. [27, Proposition 1.2])

$$v(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \; t \geq 0.$$

Moreover, $v(t)$ is a strong solution of (3.3) if $u_0 \in D(A)$, see [27, Proposition 1.2].

Lemma 3.1. *Suppose that A generates an uniformly integrable resolvent family $S(t)$ and let $f \in SAP_\omega(X)$. Then*

$$\int_0^t S(t-s)f(s)ds \in SAP_\omega(X).$$

Proof. Let $v(t) := \int_0^t S(t-s)f(s)ds$. We have

$$\begin{aligned} v(t+\omega) - v(t) &= \int_0^{t+\omega} S(t+\omega-s)f(s)ds - \int_0^t S(t-s)f(s)ds \\ &= \int_0^{\omega} S(t+\omega-s)f(s)ds + \int_\omega^{t+\omega} S(t+\omega-s)f(s)ds - \int_0^t S(t-s)f(s)ds \\ &= \int_0^{\omega} S(t+\omega-s)f(s)ds + \int_0^{t} S(t-s)f(s+\omega)ds - \int_0^t S(t-s)f(s)ds \end{aligned}$$

For each $\epsilon > 0$, there is a positive constant L_ϵ such that $||f(t+\omega) - f(t)|| \leq \epsilon$, for every $t \geq L_\epsilon$. Under these conditions, for $t \geq L_\epsilon$, we can estimate

$$\begin{aligned} ||v(t+\omega) - v(t)|| &\leq \int_0^\omega ||S(t+\omega-s)f(s)||ds \\ &+ \int_0^{L_\epsilon} ||S(t-s)[f(s+\omega) - f(s)]||ds \\ &+ \int_{L_\epsilon}^t ||S(t-s)[f(s+\omega) - f(s)]||ds \\ &\leq ||f||_\infty \int_0^\omega ||S(t+\omega-s)||ds + 2||f||_\infty \int_0^{L_\epsilon} ||S(t-s)||ds \\ &+ \epsilon \int_{L_\epsilon}^t ||S(t-s)||ds \end{aligned}$$

$$
\begin{aligned}
&= ||f||_\infty \int_t^{t+\omega} ||S(s)||ds + 2||f||_\infty \int_{t-L_\epsilon}^{t} ||S(s)||ds \\
&+ \epsilon \int_0^{t-L_\epsilon} ||S(s)||ds \\
&\leq ||f||_\infty \int_t^{\infty} ||S(s)||ds + 2||f||_\infty \int_{t-L_\epsilon}^{\infty} ||S(s)||ds \\
&+ \epsilon \int_0^{\infty} ||S(s)||ds
\end{aligned}
$$

which permit to infer that $v(t+\omega) - v(t) \to 0$ as $t \to \infty$.

□

Direct consequence of the previous lemma is the following theorem.

Theorem 3.2. *Suppose that A generates an uniformly integrable resolvent $S(t)$ which is S-asymptotically ω-periodic, and let $f \in SAP_\omega(X)$. Then for each $u_0 \in D(A)$ the equation (3.3) admits a unique S- asymptotically ω-periodic strong solution.*

We recall the following definition:

Definition 3.3. We say that A generates a *parabolic* resolvent family if the following conditions are satisfied.

(P1) $\hat{a}(\lambda) \neq 0$, $1/\hat{a}(\lambda) \in \rho(A)$ for all $Re\lambda > 0$.
(P2) There exists a constant $M \geq 1$ such that

$$\|(\lambda - \lambda\hat{a}(\lambda)A)^{-1}\| \leq \frac{M}{|\lambda|} \text{ for all } Re\lambda > 0. \tag{3.4}$$

If A generate an analytic resolvent which is bounded on some sector $\Sigma(0,\theta)$ then A generates a parabolic resolvent family. The converse is not true. A standard situation leading with generators of parabolic resolvents is the following: Let $a(t)$ of subexponential growth of positive type, and let A generate a bounded analytic C_0-semigroup in X, then A generate a parabolic resolvent (cf. [27, Proposition 3.1]). With the above definitions, we are ready to state the following result.

Theorem 3.4. *Suppose that $a(t)$ is 1-regular and A generates a parabolic and uniformly integrable resolvent family $\{S(t)\}_{t\geq 0}$, and let $f \in SAP_\omega(X)$. Then for each $u_0 \in D(A)$ the equation (3.3) admits a unique S- asymptotically ω-periodic strong solution.*

Proof. Since $a(t)$ is 1-regular and A generates a parabolic and uniformly integrable resolvent family, we obtain by the main result in [26] that $\{S(t)\}_{t\geq 0}$ is uniformly stable. In particular, $\{S(t)\}_{t\geq 0}$ is S-asymptotically ω-periodic for any $\omega > 0$. The result is now a consequence of Theorem 3.2. □

4. The semilinear case

In this section we consider the existence and uniqueness of S-asymptotically ω-periodic mild solutions of the problem (1.2)-(1.3). The considerations in the linear case motivates the following definition.

Definition 4.1. A function $u \in C_b([0,\infty),X)$ is said S-asymptotically ω-periodic mild solution of problem (1.2)-(1.3) if $u(\cdot)$ is S-asymptotically ω-periodic and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds, \text{ for all } t \geq 0. \tag{4.1}$$

In the case that the resolvent family is differentiable, we can give an alternative definition of mild solution of problem (1.2) (without initial condition) as follows:

$$u(t) = f(t,u(t)) + \int_0^t \dot{S}(t-s)f(s,u(s))ds, \text{ for all } t \geq 0. \tag{4.2}$$

Note that in case of a C_0-semigroup $T(t)$, the last definition is the usually used when the semigroup is, in addition, analytic where $\dot{T}(t) = AT(t)$. Moreover, we observe that if A generates a parabolic resolvent family $S(t)$ (see Definition 3.3) and the kernel $a(t)$ is 2-regular (see Definition 2.8), then $S(t)$ is differentiable (cf. [27, Theorem 3.1]).

Theorem 4.2. *Suppose A generates an uniformly integrable solution operator $S(t)$, which is in addition strongly S-asymptotically ω-periodic. Let $f : [0,\infty)\times X \to X$ be a continuous function such that $f(\cdot,0)$ is integrable in $[0,\infty)$ and there exists a continuous integrable function $L : [0,\infty) \to \mathbb{R}$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \text{ for all } x,y \in X,\ t \geq 0. \tag{4.3}$$

Then the problem (1.2)-(1.3) has a unique S-asymptotically ω-periodic mild solution.

Proof. We define the operator Γ on the space $SAP_\omega(X)$ by

$$(\Gamma u)(t) := S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds = S(t)u_0 + v(t). \tag{4.4}$$

We show initially that $\Gamma u \in SAP_\omega(X)$. In fact, we observe that by hypothesis $S(\cdot)u_0 \in SAP_\omega(X)$. It follows from the inequality $||f(s,u(s))|| \leq L(s)||u(s)|| + ||f(s,0)||$, that the function $s \to f(s,u(s))$ is integrable in $[0,\infty)$. Hence, we obtain that $v(t) \in C_b([0,\infty),X)$ and $\int_a^t S(t-s)f(s,u(s))ds \to 0$, as $a \to \infty$, uniformly for $t \geq a$. In addition, for fixed a, the set $\{f(s,u(s)) : 0 \leq s \leq a\}$ is compact, which implies that $S(t+\omega)f(s,u(s)) - S(t)f(s,u(s)) \to 0$, as $t \to \infty$, uniformly in $s \in [0,a]$. Combining these properties with the decomposition

$$\begin{aligned} v(t+\omega) - v(t) &= \int_0^a [S(t+\omega-s) - S(t-s)]f(s,u(s))ds \\ &+ \int_a^{t+\omega} S(t+\omega-s)f(s,u(s))ds - \int_a^t S(t-s)f(s,u(s))ds. \end{aligned}$$

Hence $v(t+\omega) - v(t) \to 0$ as $t \to \infty$. Furthermore, for $u_1, u_2 \in SAP_\omega(X)$ the inequality $||(\Gamma u_1)(t) - (\Gamma u_2)(t)|| \leq C\int_0^t L(s)||u_1(s) - u_2(s)||ds$ shows that $\Gamma : SAP_\omega(X) \to SAP_\omega(X)$ is a continuous map. On the other hand, we define the linear map $B : C_b([0,\infty)) \to C_b([0,\infty))$ by $(Bg)(t) = C\int_0^t L(s)g(s)ds$, for $t \geq 0$. It is clear that B is continuous. Moreover, B is completely continuous. To establish this assertion, for each $\epsilon > 0$, we take

$a \geq 0$ such that $C\int_a^\infty L(s)ds \leq \epsilon$ and, for each $g \in C_b([0,\infty))$ with $||g||_\infty \leq 1$, we define the functions

$$\Gamma_1(g)(t) := \begin{cases} C\int_0^t L(s)g(s)ds, & 0 \leq t \leq a, \\ C\int_0^a L(s)g(s)ds, & t \geq a, \end{cases} \quad \Gamma_2(g)(t) := \begin{cases} 0, & 0 \leq t \leq a, \\ C\int_0^t L(s)g(s)ds, & t \geq a. \end{cases}$$

It follows from the Ascoli-Arzela Theorem that the set $R_0 := \{\Gamma_1(g) : ||g||_\infty \leq 1\}$ is relatively compact. Since $Bg(t) = \Gamma_1(g)(t) + \Gamma_2(g)(t)$ for all $t \geq 0$, we can infer that $\{Bg : ||g||_\infty \leq 1\} \subset R_0 + \{\beta : \beta \in C_b([0,\infty)), ||\beta||_\infty \leq \epsilon\}$, which shows that the set $\{Bg : ||g||_\infty \leq 1\}$ is relatively compact and, in turn, that B is completely continuous. Moreover, since the point spectrum $\sigma_p(B) = \{0\}$, the spectral radius of B is equal to zero. Let $m : C_b([0,\infty), X) \to C_b([0,\infty))$ be the map defined by $m(u)(t) = \sup_{0\leq s\leq t} ||u(s)||$. It is not difficult to verify that the maps Γ, B and m satisfy all the conditions of Theorem 1 in [21] which implies that Γ has a unique fixed point u. □

Theorem 4.2 together with the argument used in the proof of Theorem 3.4 permits to infer the following consequence.

Corollary 4.3. *Suppose $a(t)$ is 1-regular and A generates a parabolic and uniformly integrable resolvent family $S(t)$. Let $f : [0,\infty) \times X \to X$ be a continuous function such that $f(\cdot, 0)$ is integrable in $[0,\infty)$ and there exists a continuous integrable function $L : [0,\infty) \to \mathbb{R}$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \text{ for all } x, y \in X,\ t \geq 0. \tag{4.5}$$

Then the problem (1.2)-(1.3) has a unique S-asymptotically ω-periodic mild solution.

The following result will be of more practical use.

Corollary 4.4. *Suppose $a(t)$ is 1-regular, of subexponential growth, of positive type, completely monotonic and satisfies $a(\infty) = \lim_{t\to\infty} a(t) > 0$. Assume that A generates a bounded analytic C_0-semigroup and $0 \in \rho(A)$. Let $f : [0,\infty) \times X \to X$ be a continuous function such that $f(\cdot, 0)$ is integrable in $[0,\infty)$ and there exists a continuous integrable function $L : [0,\infty) \to \mathbb{R}$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, \text{ for all } x, y \in X,\ t \geq 0. \tag{4.6}$$

Then the problem (1.2)-(1.3) has a unique S-asymptotically ω-periodic mild solution.

Proof. Under the stated hypothesis, it follows from [27, Corollary 10.1] that A generates an uniformly integrable analytic resolvent. Hence A generates a uniformly integrable parabolic resolvent, which is then also uniformly stable. The result follows. □

Example 4.5. *Let $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0$. Then $a(t)$ satisfies the hypotheses in the previous corollary if and only if $\beta \geq 1$. Note that we esssentially recover* [4, Theorem 3.2], *and* [20, Theorem 4.3] *in case $\beta = 1$. Also observe that in the above cited works, the examples given are always corresponding to generators of analytic semigroups with $0 \in \rho(A)$.*

The case where L in (4.3) is a constant, is analyzed in the following two results.

Theorem 4.6. *Suppose A generates an uniformly integrable resolvent family $S(t)$ which is stronlgy S-asymptotically ω-periodic. Let $f : [0,\infty) \times X \to X$ be a function uniformly S-asymptotically ω-periodic on bounded sets and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|, \text{ for all } x,y \in X,\, t \ge 0. \tag{4.7}$$

If $L < ||S||_1^{-1}$, then the problem (1.2)-(1.3) has a unique S-asymptotically ω-periodic mild solution.

Proof. Proceeding as in the proof of Theorem 4.2, we define the map Γ on the space $SAP_\omega(X)$ by the expression (4.4). We next prove that Γ is a contraction from $SAP_\omega(X)$ into $SAP_\omega(X)$. We show initially that Γ is $SAP_\omega(X)-$valued. Let u be in $SAP_\omega(X)$. By hypothesis the function $S(\cdot)u_0 \in SAP_\omega(X)$ and the problem is reduced to show that the function v given by (4.4) belongs to $SAP_\omega(X)$. In view of that f is asymptotically uniformly continuous on bounded sets and applying Lemma 2.4 and Lemma 3.1 we get that $\Gamma u \in SAP_\omega(X)$. On the other hand, for $u_1, u_2 \in SAP_\omega(X)$, we have the inequality

$$||(\Gamma u_1)(t) - (\Gamma u_2)(t)|| \le L||S||_1||u_1 - u_2||_\infty,$$

which proves that Γ is a contraction. Now, the assertion is consequence of the contraction mapping principle. The proof is complete. □

Theorem 4.7. *Suppose A generates an uniformly bounded and integrable resolvent family $S(t)$ such that $\lim_{t\to\infty}(S(t)x - S(t+n\omega)x) = 0$ uniformly in $n \in \mathbb{N}$, for all $x \in X$. Let Condition (4.7) be holds and assume that $f(\cdot,0)$ is a bounded function and $\lim_{t\to\infty}(f(t,x) - f(t+n\omega,x)) = 0$ uniformly in $x \in K$ and $n \in \mathbb{N}$, for every bounded subset K of X. If $L < ||S||_1^{-1}$, then the problem (1.2)-(1.3) has a unique asymptotically ω-periodic mild solution.*

Proof. Let $\mathcal{S}(X)$ be the space consisting of functions $u \in C_b([0,\infty), X)$ such that $\lim_{t\to\infty}(u(t) - u(t+n\omega)) = 0$ uniformly in $n \in \mathbb{N}$. It is easy to see that $\mathcal{S}(X)$ is a closed subspace of $C_b([0,\infty), X)$ (see [20]). Let u be in $\mathcal{S}(X)$. It follows from our assumptions that for each $\epsilon > 0$, there is a positive constant L_ϵ such that $||f(t+n\omega, u(t+n\omega)) - f(t,u(t))|| \le \epsilon$ for every $t \ge L_\epsilon$ and every $n \in \mathbb{N}$. We consider the map Γ on $\mathcal{S}(X)$ by the expression (4.4). We have the following estimate: $||\Gamma u||_\infty \le ||S||_\infty||u_0|| + [L||u||_\infty + ||f(\cdot,0)||_\infty]||S||_1$. Let $v(t) := \int_0^t S(t-s)f(s)ds$, proceeding as the proof of Lemma 3.1, we have for $t \ge L_\epsilon$

$$\begin{aligned} ||v(t+n\omega) - v(t)|| &\le [L||u||_\infty + ||f(\cdot,0)||_\infty]\big(\int_t^\infty ||S(s)||ds \\ &+ 2\int_{t-L_\epsilon}^\infty ||S(s)||ds\big) + \epsilon||S||_1. \end{aligned}$$

We get that Γ is $\mathcal{S}(X)-$valued. Therefore the fixed point of Γ belongs to $\mathcal{S}(X)$ and the assertion is consequence of Corollary 3.1 in [20]. The proof is complete. □

We next study the existence of S-asymptotically ω-periodic mild solutions of the equation (1.2) when the function f is not Lipschitz continuous. We will consider functions f that satisfies the following boundedness condition.

(B) There exist a continuous nondecreasing function $W : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}_+$ such that $\|f(t,x)\| \le W(\|x\|)$ for all $t \in \mathbb{R}_+$ and $x \in X$.

Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $h(t) \geq 1$ for all $t \in \mathbb{R}_+$, and $h(t) \to \infty$ as $t \to \infty$. We consider the space

$$C_h(X) = \{u \in C(\mathbb{R}_+, X) : \lim_{t\to\infty} \frac{u(t)}{h(t)} = 0\}$$

endowed with the norm

$$\|u\|_h = \sup_{t\geq 0} \frac{\|u(t)\|}{h(t)}.$$

We will use the following result (cf. [8, Lemma 2.2]).

Lemma 4.8. *A subset $K \subseteq C_h(X)$ is a relatively compact set if verifies the following conditions:*

(c-1) The set $K_b = \{u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0, b], X)$ for all $b \geq 0$.
(c-2) $\lim_{t\to\infty} \frac{||u(t)||}{h(t)} = 0$ uniformly for all $u \in K$.

Theorem 4.9. *Assume that A generates an uniformly bounded and integrable resolvent family $\{S(t)\}_{t\geq 0}$ which is in addition strongly S-asymptotically ω-periodic. Let $f : \mathbb{R}_+ \times X \to X$ be an uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets function and that satisfies assumption (B), and the following conditions:*

(a) For each $C \geq 0$, the function $t \mapsto \int_0^t ||S(t-s)||W(Ch(s))ds$ is included in $BC(\mathbb{R}_+)$. We set

$$\beta(C) = \left\| ||S(\cdot)|| ||u_0|| + \int_0^{\cdot} ||S(\cdot - s)||W(Ch(s))ds \right\|_h.$$

(b) For each $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in C_h(X)$, $\|v - u\|_h \leq \delta$ implies that $\int_0^t ||S(t-s)|| \|f(s, v(s)) - f(s, u(s))\| ds \leq \varepsilon$ for all $t \geq 0$.

(c) $\liminf_{\xi\to\infty} \frac{\xi}{\beta(\xi)} > 1$.

(d) For all $a, b \in \mathbb{R}$, $a < b$, and $r > 0$, the set $\{f(s, h(s)x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ is relatively compact in X.

Then the problem (1.2)-(1.3) has an S-asymptotically ω-periodic mild solution.

Proof. We define the operator Γ on $C_h(X)$ as in (4.4). We show that Γ has a fixed point in $SAP_\omega(X)$. We divide the proof in several steps.
(i) For $u \in C_h(X)$, we have that

$$\begin{aligned} \left\|\Gamma u(t)\right\| &\leq ||S(t)|| ||u_0|| + \int_0^t ||S(t-s)||W(\|u(s)\|)ds \\ &\leq ||S||_\infty ||u_0|| + \int_0^t ||S(t-s)||W(\|u\|_h h(s))ds. \end{aligned} \tag{4.8}$$

It follows from condition (a) that $\Gamma : C_h(X) \to C_h(X)$.
(ii) The map Γ is continuous. In fact, for $\varepsilon > 0$, we take δ involved in condition (b). If $u, v \in C_h(X)$ and $\|u - v\|_h \leq \delta$, then

$$\|\Gamma u(t) - \Gamma v(t)\| \leq \int_0^t ||S(t-s)|| \|f(s, u(s)) - f(s, v(s))\| ds \leq \varepsilon,$$

which shows the assertion.

(iii) We show that Γ is completely continuous. To abbreviate the text, we set $B_r(Z)$ for the closed ball with center at 0 and radius r in a space Z. Let $V = \Gamma(B_r(C_h(X)))$ and $v = \Gamma(u)$ for $u \in B_r(C_h(X))$.

Initially, we will prove that $V_b(t)$ is a relatively compact subset of X for each $t \in [0, b]$. We get

$$v(t) = S(t)u_0 + \int_0^t S(s)f(t-s, u(t-s))ds \in S(t)u_0 + t\overline{c(K)},$$

where $c(K)$ denotes the convex hull of K and $K = \{S(s)f(\xi, h(\xi)x) : 0 \le s \le t, 0 \le \xi \le t, \|x\| \le r\}$. Using that $S(\cdot)$ is strongly continuous and the property (d) of f, we infer that K is a relatively compact set, and $V(t) \subseteq S(t)u_0 + t\overline{c(K)}$, which establishes our assertion.

We next show that the set V_b is equicontinuous. In fact, we can decompose

$$\begin{aligned} v(t+s) - v(t) &= (S(t+s) - S(t))u_0 \\ &+ \int_0^s S(\xi)f(t+s-\xi, u(t+s-\xi))d\xi \\ &+ \int_0^t (S(\xi+s) - S(\xi))f(t-\xi, u(t-\xi))d\xi. \end{aligned}$$

For each $\varepsilon > 0$, we can choose $\delta_1 > 0$ such that

$$\Big\| \int_0^s S(\xi)f(t+s-\xi, u(t+s-\xi))d\xi \Big\| \le \int_0^s ||S(\xi)||W(rh(t+s-\xi))d\xi \le \varepsilon/2,$$

for $s \le \delta_1$. Moreover, since $\{f(t-\xi, u(t-\xi)) : 0 \le \xi \le t, u \in B_r(C_h(X))\}$ is a relatively compact set and $S(\cdot)$ is strongly continuous, we can choose $\delta_2 > 0$ and $\delta_3 > 0$ such that $||(S(t+s) - S(t))u_0|| < \varepsilon/4$, for $s \le \delta_2$ and $\|(S(\xi+s) - S(\xi))f(t-\xi, u(t-\xi))\| \le \varepsilon/2(t+1)$ for $s \le \delta_3$. Combining these estimates, we get $\|v(t+s) - v(t)\| \le \varepsilon$ for s enough small and independent of $u \in B_r(C_h(X))$.

Finally, applying condition (a), we can show that

$$\frac{v(t)}{h(t)} \le \frac{1}{h(t)}\Big[||S(t)||||u_0|| + \int_0^t ||S(t-s)||W(rh(s))ds\Big] \to 0,\ t \to \infty,$$

and this convergence is independent of $u \in B_r(C_h(X))$.

Hence V satisfies conditions (c-1), (c-2) of Lemma 4.8, which completes the proof that V is a relatively compact set in $C_h(X)$.

(iv) If $u^\lambda(\cdot)$ is a solution of equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$, from the estimate

$$\begin{aligned} \|u^\lambda(t)\| &= \lambda \Big\| S(t)u_0 + \int_0^t S(t-s)f(s, u^\lambda(s))ds \Big\| \\ &\le ||S(t)||||u_0|| + \int_0^t ||S(t-s)||W(\|u^\lambda\|_h h(s))ds \\ &\le \beta(\|u^\lambda\|_h)h(t), \end{aligned}$$

we get

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \le 1$$

and, combining with condition (c), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0,1)\}$ is bounded.
(v) It follows from Lemma 2.4 and Lemma 3.1 that $\Gamma(\overline{SAP_\omega(X)}) \subseteq \overline{SAP_\omega(X)}$ and, consequently, we can consider $\Gamma : \overline{SAP_\omega(X)} \to \overline{SAP_\omega(X)}$. Using properties (i)-(iii) we have that this map is completely continuous. Applying Leray-Schauder alternative theorem [15, Theorem 6.5.4], we infer that Γ has a fixed point $u \in \overline{SAP_\omega(X)}$. Let $(u_n)_n$ be a sequence in $SAP_\omega(X)$ that converges to u. We see that $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $[0,\infty)$. This implies that $u \in SAP_\omega(X)$, and completes the proof. □

Corollary 4.10. *Assume that A generates an uniformly bounded and integrable resolvent family $\{S(t)\}_{t\geq 0}$ which is in addition strongly S-asymptotically ω-periodic. Let $f : \mathbb{R}_+ \times X \to X$ be an uniformly S-asymptotically ω-periodic on bounded sets that satisfies the Hölder type condition*

$$\|f(t,y) - f(t,x)\| \leq C_1\|y-x\|^\alpha, \quad 0 < \alpha < 1, \tag{4.9}$$

for all $x, y \in X$, $t \geq 0$, where $C_1 > 0$ is a constant. Moreover, assume the following conditions:
(i) $f(t,0) = q$.
(ii) $\sup_{t\geq 0} \int_0^t ||S(t-s)||h(s)^\alpha ds < +\infty$.
(iii) For all $a, b \geq 0$, $a < b$, and $r > 0$, the set $\{f(s, h(s)x) : a \leq s \leq b,\ x \in X, \|x\| \leq r\}$ is relatively compact in X.
Then the problem (1.2)-(1.3) has an S-asymptotically ω-periodic mild solution.

To finish this work, we examine the existence and uniqueness of S-asymptotically ω-periodic mild solution to the integro differential equation

$$\begin{aligned} u_t(t,x) &= \int_0^t da(s)u_{xx}(t-s,x) + f(t,u(t,x)), \quad t \geq 0,\ x \in [0,1], \\ u(t,0) &= u(t,1) = 0, \\ u(0,x) &= u_0(x), \quad x \in [0,1]. \end{aligned} \tag{4.10}$$

Here $a : \mathbb{R} \to \mathbb{R}_+$ is a function of bounded variation on each compact interval $J = [0,T](T > 0)$, with $a(0) = 0$. The above initial-boundary problem is a typical example of one-dimensional problems in viscoelasticity, like simple shearing motions, simple tension, torsion of a rod; see [27, Section 5.4].

To obtain a formulation as an abstract evolutionary integral equation like (1.2)-(1.3), we choose $X = L^2[0,1]$ and define an operator A by means of $Au(x) = u_{xx}(x)$ with domain $D(A) = \{u \in X : u_{xx} \in X, u(0) = u(1) = 0\}$. It is well known that A generates an bounded analytic semigroup with $0 \in \rho(A)$. Then Corollary 4.4 implies the following result.

Corollary 4.11. *Suppose $a(t)$ is 1-regular, of subexponential growth, of positive type, completely monotonic and satisfies $a(\infty) = \lim_{t\to\infty} a(t) > 0$. Let $f : [0,\infty) \times X \to X$ be a continuous function such that $f(\cdot, 0)$ is integrable in $[0,\infty)$ and there exists a continuous integrable function $L : [0,\infty) \to \mathbb{R}$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \text{ for all } x, y \in X,\ t \geq 0. \tag{4.11}$$

Then the problem (4.10) has a unique S-asymptotically ω-periodic mild solution.

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