ALMOST AUTOMORPHIC SOLUTIONS TO A CLASS OF SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We study almost automorphic (mild) solutions of the semilinear fractional equation $\partial_t^\alpha u = Au + \partial_t^{\alpha-1} f(\cdot, u)$, $1 < \alpha < 2$, considered in a Banach space X, where A is a linear operator of sectorial type $\omega < 0$. We prove the existence and uniqueness of an almost automorphic mild solution assuming $f(t,x)$ is almost automorphic in t for each $x \in X$, satisfies some Lipschitz type conditions and takes values on X.

1. INTRODUCTION

We study in this paper some sufficient conditions for the existence and uniqueness of almost automorphic mild solutions to the following semilinear fractional differential equation

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X and $f : \mathbb{R} \times X \to X$ is an almost automorphic function in t for each $x \in X$ and satisfy suitable Lipschitz type conditions in x. The fractional derivative in understood in the Riemann-Liouville's sense.

Due to their applications in fields of science where characteristics of anomalous diffusion is presented, type (1.1) equations are attracting increasing interest (cf. [4], [12], [14] and references therein). For example, anomalous diffusion in fractals [12] or in macroeconomics [1] has been recently studied in the setting of fractional Cauchy problems like (1.1). While the study of mild almost automorphic solutions of (1.1) in the borderline case $\alpha = 1$ was well studied in [13] and [15] to the knowledge of the authors no results yet exist for the class of fractional differential equations considered in this paper. Upon making some appropriate assumptions, some sufficient conditions for the existence and uniqueness of an almost automorphic mild solution to (1.1) are given. In particular, to illustrate our main results, we will examine sufficient conditions for the existence and uniqueness of almost automorphic mild solutions to the fractional relaxation-oscillation equation given by

$$\partial_t^\alpha(t,x) = \partial_x^2(t,x) - \mu u(t,x) + \partial_t^{\alpha-1} g(t, u(t,x)),\ t \in \mathbb{R},\ x \in [0,\pi] \tag{1.2}$$

with boundary conditions $u(t,0) = u(t,\pi) = 0$, $t \in \mathbb{R}$, and where g satisfies some additional conditions.

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2. Preliminaries

A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n), \quad \text{for each } t \in \mathbb{R}.$$

Almost automorphicity is a generalization of the classical concept of an almost periodic function. It was introduced in the literature by S. Bochner and recently studied by several authors, including [2, 3, 5, 8, 13, 15, 19] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [16] and [17] by G. M. N'Guérékata. Almost automorphic functions constitute a Banach space $AA(X)$ when it is endowed with the sup norm:

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

The following result on the almost automorphicity of the convolution is the key for the results of this paper. See [5, Theorem 2.1] or [2, Lemma 3.1] for a detailed proof.

Lemma 2.1. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t)$ for almost all $t \in \mathbb{R}_+$ with $\phi \in L^1(\mathbb{R}_+)$. If $f : \mathbb{R} \to X$ is an almost automorphic function then*

$$\int_{-\infty}^{t} S(t-s)f(s)\,ds \in AA(X).$$

We recall that a closed and linear operator A is said to be *sectorial of type* ω and angle θ if there exists $0 < \theta < \pi/2$, $M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, \quad |arg(-\lambda)| < \theta\},$$

and

$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + S_\theta.$$

Sectorial operators are well studied in the literature, usually in case $\omega = 0$. For a recent reference including several examples and properties we refer to [18]. Note que an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

3. Almost Automorphic Mild Solutions

Let $1 < \alpha < 2$. We first consider the linear version for equation (1.1), that is

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t), \quad t \in \mathbb{R}. \tag{3.1}$$

We observe that equation (3.1) can be viewed as the *limiting equation* for the equation

$$v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s)ds + f(t), \quad t \geq 0, \ v(0) = u_0 \in X, \tag{3.2}$$

in the sense that the solutions $u(t)$ of (3.1) and $v(t)$ of (3.2) are asymptotic of each other as $t \to \infty$. In fact, if we assume that A is sectorial of type ω with $0 \leq \theta < \pi(1 - \alpha/2)$, then problem (3.2) is well posed (cf.[6]) and the variation of parameters formula allows us to write the solution of (3.2) as

$$v(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f(s)ds, \quad t \geq 0,$$

where the family of operators $S_\alpha(t)$, for $t \geq 0$ on X are defined by

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}d\lambda, \quad t \geq 0, \tag{3.3}$$

with γ a suitable path lying outside the sector $\omega + \Sigma_\theta$. Note that in the border case $\alpha = 1$ the family $S_\alpha(t)$ corresponds to a C_0-semigroup. On the other hand, if $S_\alpha(t)$ is integrable, then the solution of (3.1) is given by

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s)ds. \tag{3.4}$$

Hence

$$v(t) - u(t) = S_\alpha(t)u_0 - \int_t^\infty S_\alpha(s)f(t-s)ds,$$

which shows that $v(t) - u(t) \to 0$ as $t \to \infty$ whenever $f \in L^p(\mathbb{R}_+; X)$ for some $p \in [1,\infty)$.

Very recently, Cuesta in [6, Theorem 1] has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \leq \theta < \pi(1-\alpha/2)$, then there exists $C > 0$ such that

$$||S_\alpha(t)|| \leq \frac{CM}{1+|\omega|t^\alpha}, \tag{3.5}$$

for $t \geq 0$. In the border case $\alpha = 1$, this is the analog to say that A is the generator of a exponentially stable C_0-semigroup. The main difference is that in case $\alpha > 1$ the solution family $S_\alpha(t)$ decay like $t^{-\alpha}$. Cuesta's result proves that $S_\alpha(t)$ is, in fact, integrable. The above considerations motivates the following definition.

Definition 3.1. A function $u : \mathbb{R} \to X$ is said to be a mild solution to equation (1.1) if the function $s \to S_\alpha(t-s)f(s,u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s,u(s))ds, \tag{3.6}$$

for each $t \in \mathbb{R}$.

The following are the main results of this paper.

Theorem 3.2. *Assume that A is sectorial of type $\omega < 0$. Let $f : \mathbb{R}\times X \to X$ almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \text{ for all } x,y \in X,\, t \in \mathbb{R}, \tag{3.7}$$

where $L \in L^1(\mathbb{R})$. Then equation (1.1) has a unique almost automorphic mild solution.

Proof. We define the operator $F : AA(X) \mapsto AA(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^t S_\alpha(t-s)f(s,\varphi(s))\, ds, \quad t \in \mathbb{R}. \tag{3.8}$$

Given $\varphi \in AA(X)$, in view of [17, Theorem 2.2.6] we have that the function $s \to f(s,\varphi(s))$ is almost automorphic, and hence bounded in $\mathbb{R}$. Since the function $\frac{1}{1+|\omega|t^\alpha}$ is integrable on $\mathbb{R}_+$ $(\alpha > 1)$, we get that $F\varphi$ exists. Now, by Lemma 2.1 we obtain that $F\varphi \in AA(X)$ and hence F is well defined. It suffices now to show that this operator F has a unique fixed point in $AA(X)$. For this, let φ_1, φ_2 be in $AA(X)$ and denote $C_\alpha := \sup_{t\in\mathbb{R}} ||S_\alpha(t)||$. We have:

$$||(F\varphi_1)(t) - (F\varphi_2)(t)|| = \left\| \int_{-\infty}^{t} S_\alpha(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\|$$

$$\leq \int_{-\infty}^{t} L(s)\|S_\alpha(t-s)\|\|\varphi_1(s) - \varphi_2(s)\|ds \leq C_\alpha\|\varphi_1 - \varphi_2\|_\infty ||L||_1$$

In general we get

$$||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| \leq \frac{C_\alpha^n}{(n-1)!}\left(\int_{-\infty}^{t} L(s)\left(\int_{-\infty}^{s} L(\tau)d\tau\right)^{n-1} ds\right)||\varphi_1 - \varphi_2||_\infty$$

$$\leq \frac{C_\alpha^n}{n!}\left(\int_{-\infty}^{t} L(\tau)d\tau\right)^n ||\varphi_1 - \varphi_2||_\infty \leq \frac{(C_\alpha||L||_1)^n}{n!}||\varphi_1 - \varphi_2||_\infty$$

Hence, since $\frac{(C_\alpha||L||_1)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in AA(X)$. □

We note that conditions of type (3.7) has been previously considered in the literature of almost automorphic functions [5]. Our motivation comes from their use in the study of pseudo almost periodic solutions of semilinear Cauchy problems [7]. Now we consider a more general case of equations introducing a new class of functions L which do not necessarily belong to $L^1(\mathbb{R})$. We have the following result.

Theorem 3.3. *Assume that A is sectorial of type $\omega < 0$. Let $f : \mathbb{R} \times X \to X$ almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition (3.7) where the integral $\int_{-\infty}^{t} L(s)ds$ exists for all $t \in \mathbb{R}$. Then equation (1.1) has a unique almost automorphic mild solution.*

Proof. Define a new norm $|||\varphi|| := \sup_{t\in\mathbb{R}}\{v(t)||\varphi(t)||\}$, where $v(t) := e^{-k\int_{-\infty}^{t} L(s)ds}$ and k is a fixed positive constant greater than $C_\alpha := \sup_{t\in\mathbb{R}} ||S_\alpha(t)||$. Let φ_1, φ_2 be in $AA(X)$, then we have

$$v(t)||(F\varphi_1)(t) - (F\varphi_2)(t)|| = v(t)\left\| \int_{-\infty}^{t} S_\alpha(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\|$$

$$\leq C_\alpha \int_{-\infty}^{t} v(t)L(s)\|\varphi_1(s) - \varphi_2(s)\|ds \leq C_\alpha \int_{-\infty}^{t} v(t)v(s)^{-1}L(s)v(s)\|\varphi_1(s) - \varphi_2(s)\|ds$$

$$\leq C_\alpha|||\varphi_1 - \varphi_2||| \int_{-\infty}^{t} v(t)v(s)^{-1}L(s)ds = \frac{C_\alpha}{k}|||\varphi_1 - \varphi_2||| \int_{-\infty}^{t} ke^{k\int_t^s L(\tau)d\tau}L(s)ds$$

$$= \frac{C_\alpha}{k}|||\varphi_1 - \varphi_2||| \int_{-\infty}^{t} \frac{d}{ds}\left(e^{k\int_t^s L(\tau)d\tau}\right)ds = \frac{C_\alpha}{k}[1 - e^{-k\int_{-\infty}^{t} L(\tau)d\tau}]|||\varphi_1 - \varphi_2|||$$

$$\leq \frac{C_\alpha}{k}|||\varphi_1 - \varphi_2|||.$$

Hence, since $C_\alpha/k < 1$, F has a unique fixed point $u \in AA(X)$. □

Note that the above result does not include the cases where L is a constant.

Theorem 3.4. *Assume that A is sectorial of type $\omega < 0$. Let $f : \mathbb{R} \times X \to X$ almost automorphic in t for each $x \in X$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y \in X,\, t \in \mathbb{R}.$$

Then equation (1.1) has a unique almost automorphic mild solution whenever $CML < \frac{\alpha \sin(\pi/\alpha)}{|\omega|^{-1/\alpha}\pi}$.

Proof. Note that $\displaystyle\int_0^\infty \frac{1}{1+|\omega|t^\alpha}dt = \frac{|\omega|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}$ for $1 < \alpha < 2$. For $\varphi_1, \varphi_2 \in AA(X)$ we have

$$\begin{aligned}
\|F\varphi_1 - F\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^t S_\alpha(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\
&\leq L \sup_{t\in\mathbb{R}} \int_0^\infty \|S_\alpha(\tau)\| \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\| d\tau \\
&\leq L\|\varphi_1 - \varphi_2\|_\infty ||S_\alpha||_1 \leq CML \frac{|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \|\varphi_1 - \varphi_2\|_\infty.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA(X)$. ☐

Taking $A = -\rho^\alpha I$ with $\rho > 0$ in equation (1.1), the above theorem give the following Corollary.

Corollary 3.5. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ almost automorphic in t for each $x \in \mathbb{R}$ and satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \text{ for all } x, y, t \in \mathbb{R}. \tag{3.9}$$

Then equation (1.1) has a unique almost automorphic mild solution whenever $CL < \frac{\alpha\sin(\pi/\alpha)}{\rho\pi}$.

Remark 3.6. It is interesting to note that the function $\alpha \to \frac{\alpha\sin(\pi/\alpha)}{\rho\pi}$ is increasing from 0 to $2/\rho\pi$ in the interval $1 < \alpha < 2$. Therefore, with respect to the Lipschitz condition (3.9), the class of admissible semilinear terms $f(t, u(t))$ is the best in case $\alpha = 2$ and the worst in case $\alpha = 1$. Note the direct relation with the term $\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ in equation (3.2), where the singularity becomes better (smooth) when α goes from 1 to 2.

Example 3.7.

To illustrate the above results we examine the existence and uniqueness of almost automorphic mild solutions to the fractional differential equation given by (1.2). Let $X = L^2[0,\pi]$ and the operator A defined on X by

$$Au = u'' - \mu u, \qquad (\mu > 0)$$

with domain

$$D(A) = \{u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0\}.$$

It is well known that $\Delta u = u''$ is the generator of an analytic semigroup on $L^2[0,\pi]$. Hence, $\mu I - A$ is sectorial of type $\omega = -\mu < 0$. Equation (1.2) can be formulated by the inhomogeneous problem (1.1), where $u(t) = u(t,\cdot)$. Let us consider the nonlinearity $f(t,\varphi)(s) = \beta b(t)\sin(\varphi(s))$ for all $\varphi \in X$ and $s \in [0,\pi]$, $t \in \mathbb{R}$ with $b \in AA(X), \beta \in \mathbb{R}$. We observe that $t \to f(t,\varphi)$ is almost automorphic in t for each $\varphi \in X$ and we have

$$||f(t,\varphi_1) - f(t,\varphi_2)||_2^2 \leq \int_0^\pi \beta^2|b(t)|^2|\sin(\varphi_1(s)) - \sin(\varphi_2(s))|ds \leq \beta^2|b(t)|^2||\varphi_1 - \varphi_2||_2^2.$$

In consequence, the fractional differential equation (1.2) has a unique almost automorphic mild solutions either if $b \in L^1(\mathbb{R})$ (Theorem 3.2) or $\int_{-\infty}^{t} b(s)ds$ exists for all $t \in \mathbb{R}$ (Theorem 3.3). If we assume that $b \in L^\infty(\mathbb{R})$ and $|\beta| < \frac{\alpha \sin(\pi/\alpha)}{CM||b||_\infty |\mu|^{-1/\alpha}\pi}$, then the same conclusion holds by Theorem 3.4.

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