SEMILINEAR EVOLUTION EQUATIONS OF SECOND ORDER VIA MAXIMAL REGULARITY

CLAUDIO CUEVAS AND CARLOS LIZAMA

Abstract. This paper deals with the existence and stability of solutions for semilinear second order evolution equations on Banach spaces by using recent characterizations of discrete maximal regularity.

Keywords: Discrete time; Second order difference operator; Semilinear evolution equations; Discrete maximal regularity; Stability.

1. Introduction

Let A be a bounded linear operator defined on a complex Banach space X. In this article we are concerned with the study of existence of bounded solutions and stability for the semilinear problem

$$\Delta^2 x_n - Ax_n = f(n, x_n, \Delta x_n), \qquad n \in \mathbb{Z}_+, \tag{1.1}$$

by means of the knowledge of maximal regularity properties for the vector-valued discrete time evolution equation

$$\Delta^2 x_n - Ax_n = f_n, \qquad n \in \mathbb{Z}_+, \tag{1.2}$$

with initial conditions $x_0 = 0$ and $x_1 = 0$.

The theory of dynamical systems described by the difference equations has attracted a good deal of interest in the last decade due to the various applications of their qualitative properties, see [1, 18, 19, 27] and [28].

In this paper, we prove a very general theorem on the existence of bounded solutions for the semilinear problem (1.1) on $l_p(\mathbb{Z}_+; X)$ spaces. The general framework for the proof of this statement uses a new approach based on discrete maximal regularity.

In the continuous case, it is well known that the study of maximal regularity is very useful for treating semilinear and quasilinear problems. (see for example Amann [2], Denk-Hieber and Prüss [16], Clément-Londen-Simonett [12], the survey by Arendt [3] and the bibliography therein). Maximal regularity has also been studied in the finite difference setting. S. Blunck considered in [6] and [7] maximal regularity for linear difference equations of first order. See also Portal [31, 32]. In [21] maximal regularity on discrete Hölder spaces for finite difference operators subject to Dirichlet boundary conditions in one and two dimensions is proved. Furthermore, the authors investigated maximal regularity in

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discrete Hölder spaces for the Crank- Nicolson scheme. In [20] maximal regularity for linear parabolic difference equations is treated, whereas in [15] a characterization in terms of R-boundedness properties of the resolvent operator for linear second order difference equations was given. See also the recent paper by Kalton and Portal [23], where they discussed maximal regularity of power-bounded operators and relate the discrete to the continuous time problem for analytic semigroups. However, for nonlinear discrete time evolution equations like (1.1), this new approach appears not to be considered in the literature.

The paper is organized as follows. The second section provides an explanation for the basic notations and definitions to be used in the article. In the third section we prove the existence of bounded solutions whose second discrete derivative is in l_p ($1 < p < +\infty$) for the semilinear problem (1.1) by using maximal regularity and a contraction principle. We also get some a priori estimates for the solutions x_n and their discrete derivatives Δx_n and $\Delta^2 x_n$. Such estimates will follows from the discrete Gronwall's inequality [1](see also [30] and [29]). In the fourth section we give a criterion for stability of equation (1.1). Finally, in the last section we deal with local perturbations of the system (1.2).

2. Discrete maximal regularity

Let X be a Banach space. Let $\mathbb{Z}_+$ denote the set of non negative integer numbers, Δ the forward difference operator of the first order, i.e. for each $x : \mathbb{Z}_+ \to X$, and $n \in \mathbb{Z}_+$, $\Delta x_n = x_{n+1} - x_n$. We consider the second order difference equation

$$\Delta^2 x_n - (I-T)x_n = f_n \text{ for all } n \in \mathbb{Z}_+, \quad x_0 = x, \ \ \Delta x_0 = x_1 - x_0 = y. \tag{2.1}$$

where $T \in \mathcal{B}(X), \Delta^2 x_n = \Delta(\Delta x_n)$ and $f : \mathbb{Z}_+ \to X$.

Denote $\mathcal{C}(0) = I$, the identity operator on X, and define

$$\mathcal{C}(n) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (I-T)^k \quad \text{for } n = 1, 2, ... \tag{2.2}$$

and $\mathcal{C}(n) = \mathcal{C}(-n)$ for $n = -1, -2, ...$ We define also $\mathcal{S}(0) = 0$,

$$\mathcal{S}(n) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} (I-T)^k \tag{2.3}$$

for $n = 1, 2, ...$ and $\mathcal{S}(n) = -\mathcal{S}(-n)$ for $n = -1, -2, ...$

Considering the above notations, it was proved in [15] that the (unique) solution of equation (2.1) is given by

$$x_{m+1} = \mathcal{C}(m)x + \mathcal{S}(m)y + (\mathcal{S} * f)_m. \tag{2.4}$$

Moreover,

$$\Delta x_{m+1} = (I-T)\mathcal{S}(m)x + \mathcal{C}(m)y + (\mathcal{C} * f)_m. \tag{2.5}$$

The following definition is the natural extension of the concept of maximal regularity from the continuous case; cf. [15].

Definition 2.1. *Let $1 < p < +\infty$. We say that an operator $T \in \mathcal{B}(X)$ has discrete maximal regularity if $\mathcal{K}_T f := \sum_{k=1}^{n}(I-T)\mathcal{S}(k)f_{n-k}$ defines a bounded operator $\mathcal{K}_T \in \mathcal{B}(l_p(\mathbb{Z}_+, X))$.*

As consequence of the definition, if $T \in \mathcal{B}(X)$ has discrete maximal regularity then T has discrete l_p-maximal regularity, that is, for each $(f_n) \in l_p(\mathbb{Z}_+; X)$ we have $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+; X)$, where (x_n) is the solution of the equation

$$\Delta^2 x_n - (I-T)x_n = f_n \text{ for all } n \in \mathbb{Z}_+, \quad x_0 = 0, \;\; x_1 = 0. \tag{2.6}$$

Moreover,

$$\Delta^2 x_n = \sum_{k=1}^{n-1}(I-T)\mathcal{S}(k)f_{n-1-k} + f_n. \tag{2.7}$$

We introduce the means

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \|\sum_{j=1}^{n} \epsilon_j x_j\|$$

for $x_1, ..., x_n \in X$.

Definition 2.2. *Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there exists a constant $c \geq 0$ such that*

$$\|(T_1 x_1, ..., T_n x_n)\|_R \leq c\|(x_1, ..., x_n)\|_R \tag{2.8}$$

for all $T_1, ..., T_n \in \mathcal{T}, x_1, ..., x_n \in X, n \in \mathbb{N}$. The least c such that (2.8) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

An equivalent definition using the Rademacher functions can be found in [16]. We note that R-boundedness clearly implies boundedness. If $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [4, Proposition 1.17]. Some useful criteria for $R-$boundedness are provided in [4], [16] and [22].

Remark 2.3. a) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded sets, then $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$ is R- bounded.

b) Let $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X,Z)$ is R- bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T}).$$

c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle (see [4, 11] or [16]).

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [8] and Burkholder [9].

Recall that $T \in \mathcal{B}(X)$ is called analytic if the set

$$\{n(T-I)T^n : n \in \mathbb{N}\}, \tag{2.9}$$

is bounded. For recent and related results on analytic operators we refer to [17]. The characterization of discrete maximal regularity for second order difference equations by R-boundedness properties of the resolvent operator T reads as follows (see [15]).

Theorem 2.4. *Let X be a UMD space and let $T \in \mathcal{B}(X)$ be analytic. Then the following assertions are equivalent.*
(i) T has discrete maximal regularity of order 2.
(ii) $\{(\lambda-1)^2 R((\lambda-1)^2,\ I-T) :\ |\lambda| = 1,\ \lambda \neq 1\}$ is R-bounded.

Observe that from the point of view of applications, the above given characterization provides a workable criteria, see Section 4 below. We remark that the concept of R-boundedness plays a fundamental role in recent works by Clément-Da Prato [10], Clément et al. [11], Weis [33, 34], Arendt-Bu [4, 5] and Keyantuo-Lizama [24, 25, 26].

3. Semilinear second order evolution equations

In this section our aim is to investigate the existence of bounded solutions, whose second discrete derivative is in ℓ_p, for semilinear evolution equations via discrete maximal regularity.

Next, we consider the following second order evolution equation:

$$\Delta^2 x_n - Ax_n = f(n, x_n, \Delta x_n), \qquad n \in \mathbb{Z}_+, \quad x_0 = 0, \quad x_1 = 0, \tag{3.1}$$

which is equivalent to:

$$x_{n+2} - 2x_{n+1} + Tx_n = f(n, x_n, \Delta x_n), \quad \text{for all} \quad n \in \mathbb{Z}_+, \quad x_0 = 0, \quad x_1 = 0, \tag{3.2}$$

where $T := I - A$.

To establish the next result, we need to introduce the following assumption:

Assumption (A): Suppose that the following condition holds:
(i) The function $f : \mathbb{Z}_+ \times X \times X \longrightarrow X$ satisfy the Lipschitz condition on $X \times X$, i.e. for all $z, w \in X \times X$ and $n \in \mathbb{Z}_+$, we get $||f(n,z) - f(n,w)||_X \leq \alpha_n ||z-w||_{X\times X}$, where $\alpha := (\alpha_n) \in l_1(\mathbb{Z}_+)$.
(ii) $f(\cdot, 0, 0) \in l_1(\mathbb{Z}_+, X)$.

We remark that the condition $\alpha \in l_1(\mathbb{Z})$ in (i) is satisfied quite often in applications. For example it appear when we study asymptotic behavior of discrete Volterra systems which describe process whose current state is determined by their entire history. These processes are encountered in models of materials with memory, various problems of heredity or epidemics, theory of viscoelasticity and to solve optimal control problems (see for instance [13], [14]).

We began with the following property which will be useful in the proof of our main result.

Lemma 3.1. *Let (α_n) be a sequence of positive real numbers. For all $n, l \in \mathbb{Z}_+$, we have*

$$\sum_{m=0}^{n-1} \alpha_m (\sum_{j=0}^{m-1} \alpha_j)^l \leq \frac{1}{l+1} (\sum_{j=0}^{n-1} \alpha_j)^{l+1}.$$

Proof. Putting $A_m := \sum_{j=0}^{m-1} \alpha_j$, we obtain

$$\begin{aligned}
(l+1)(A_{m+1} - A_m)A_m^l &= (A_{m+1} - A_m)(A_m^l + A_m^{l-1}A_m + ... + A_m A_m^{l-1} + A_m^l) \\
&\leq (A_{m+1} - A_m)(A_{m+1}^l + A_{m+1}^{l-1}A_m + ... + A_{m+1}A_m^{l-1} + A_m^l) \\
&= A_{m+1}^{l+1} - A_m^{l+1}.
\end{aligned}$$

Hence

$$(l+1)\sum_{m=0}^{n-1}(A_{m+1} - A_m)A_m^l \leq \sum_{m=0}^{n-1}(A_{m+1}^{l+1} - A_m^{l+1}) = A_n^{l+1}.$$

■

Denote by $\mathcal{W}_0^{2,p}$ the Banach space of all sequences $V = (V_n)$ belonging to $l_\infty(\mathbb{Z}_+, X)$ such that $V_0 = V_1 = 0$ and $\Delta^2 V \in l_p(\mathbb{Z}_+, X)$ equipped with the norm $|||V||| = ||V||_\infty + ||\Delta^2 V||_p$. We will say that $T \in \mathcal{B}(X)$ is $\mathcal{S}$-bounded if $\mathcal{S} \in l_\infty(\mathbb{Z}_+; X)$. With the above notations we have the following main result:

Theorem 3.2. *Assume that Condition (A) holds. In addition suppose that T is $\mathcal{S}$-bounded and that it has discrete maximal regularity of order 2. Then, there is a unique bounded solution $x = (x_n)$ of equation (3.1) such that $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+, X)$. Moreover, we have the following a priori estimates for the solution:*

$$\sup_{n\in\mathbb{Z}_+} [||x_n||_X + ||\Delta x_n||_X] \leq 3M||f(\cdot, 0, 0)||_1 e^{3M||\alpha||_1}, \tag{3.3}$$

and

$$||\Delta^2 x||_p \leq C||f(\cdot, 0, 0)||_1 e^{6M||\alpha||_1}, \quad 1 < p < +\infty, \tag{3.4}$$

where $M := \sup_{n\in\mathbb{Z}^+} ||S(n)||$ and $C > 0$.

Proof. Let V be a sequence in $\mathcal{W}_0^{2,p}$. Then using Assumption (A) we obtain that the function $g := f(\cdot, V., \Delta V.)$ is in $l_p(\mathbb{Z}_+, X)$. In fact, we have

$$
\begin{aligned}
||g||_p^p &= \sum_{n=0}^{\infty} ||f(n, V_n, \Delta V_n)||_X^p \\
&\leq \sum_{n=0}^{\infty} (||f(n, V_n, \Delta V_n) - f(n,0,0)||_X + ||f(n,0,0)||_X)^p \\
&\leq 2^p \sum_{n=0}^{\infty} ||f(n, V_n, \Delta V_n) - f(n,0,0)||_X^p + 2^p \sum_{n=0}^{\infty} ||f(n,0,0)||_X^p \\
&\leq 2^p \sum_{n=0}^{\infty} \alpha_n^p ||(V_n, \Delta V_n)||_{X\times X}^p + 2^p \sum_{n=0}^{\infty} ||f(n,0,0)||_X^p,
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
\sum_{n=0}^{\infty} ||f(n,0,0)||_X^p &= \sum_{n=0}^{\infty} ||f(n,0,0)||_X^{p-1} ||f(n,0,0)||_X \\
&\leq ||f(\cdot,0,0)||_\infty^{p-1} \sum_{n=0}^{\infty} ||f(n,0,0)||_X \\
&= ||f(\cdot,0,0)||_\infty^{p-1} ||f(\cdot,0,0)||_1.
\end{aligned}
$$

Analogously, we have

$$
\sum_{n=0}^{\infty} \alpha_n^p \leq ||\alpha||_\infty^{p-1} ||\alpha||_1.
$$

On the other hand,

$$
||(V_n, \Delta V_n)||_{X\times X} = ||V_n|||_X + ||V_{n+1} - V_n||_X \leq 2||V_n|||_X + ||V_{n+1}||_X \leq 3||V||_\infty. \tag{3.6}
$$

Hence

$$
\begin{aligned}
||g||_p^p &\leq 6^p ||V||_\infty^p \sum_{n=0}^{\infty} \alpha_n^p + 2^p ||f(\cdot,0,0)||_\infty^{p-1} ||f(\cdot,0,0)||_1 \\
&\leq 6^p ||V||_\infty^p ||\alpha||_\infty^{p-1} ||\alpha||_1 + 2^p ||f(\cdot,0,0)||_\infty^{p-1} ||f(\cdot,0,0)||_1,
\end{aligned}
$$

proving that $g \in l_p(\mathbb{Z}_+, X)$.

Since T has discrete maximal regularity, the Cauchy problem

$$
\begin{cases} z_{n+2} - 2z_{n+1} + Tz_n = g_n, \\ z_0 = z_1 = 0, \end{cases} \tag{3.7}
$$

has a unique solution (z_n) such that $(\Delta^2 z_n) \in l_p(\mathbb{Z}_+, X)$, which is given by

$$z_n = [\mathcal{K}V]_n = \begin{cases} 0 & \text{if } n = 0, 1, \\ \sum_{k=1}^{n-1} S(k) f(n-1-k,\ V_{n-1-k}, \Delta V_{n-1-k}) & \text{if } n \geq 2. \end{cases} \tag{3.8}$$

We now show that the operator $\mathcal{K} : \mathcal{W}_0^{2,p} \longrightarrow \mathcal{W}_0^{2,p}$ has a unique fixed point. To verify that $\mathcal{K}$ is well defined we have only to show that $\mathcal{K}V \in l_\infty(\mathbb{Z}_+, X)$. In fact, we use Assumption (A) as above and $M := \sup_{n\in\mathbb{Z}^+} ||S(n)||$ to obtain

$$\begin{aligned}
&||\sum_{k=1}^{n-1} S(k) f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})||_X \\
&\leq M \sum_{k=1}^{n-1} ||f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})) - f(n-1-k, 0, 0)||_X \\
&+M \sum_{k=1}^{n-1} ||f(n-1-k, 0, 0)||_X \\
&\leq M \sum_{k=1}^{n-1} \alpha_{n-1-k} ||(V_{n-1-k}, \Delta V_{n-1-k})||_{X\times X} + M \sum_{j=0}^{n-2} ||f(j, 0, 0)||_X \\
&\leq 3M ||V||_\infty \sum_{j=0}^{n-2} \alpha_j + M \sum_{j=0}^{n-2} ||f(j, 0, 0)||_X \\
&\leq M[3||V||_\infty ||\alpha||_1 + ||f(\cdot, 0, 0)||_1].
\end{aligned} \tag{3.9}$$

It proves that the space $\mathcal{W}_0^{2,p}$ is invariant under $\mathcal{K}$.

Let V and $\tilde{V}$ be in $\mathcal{W}_0^{2,p}$. In view of Assumption (A) (i) and $M < \infty$, we have initially as in (3.9)

(3.10)

$$||[\mathcal{K}V]_n - [\mathcal{K}\tilde{V}]_n||_X$$

$$= ||\sum_{k=1}^{n-1} S(k)(f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k}) - f(n-1-k,\ \tilde{V}_{n-1-k},\ \Delta\tilde{V}_{n-1-k}))||_X$$

$$\leq M\sum_{k=1}^{n-1} \alpha_{n-1-k}||((V-\tilde{V})_{n-1-k}, \Delta(V-\tilde{V})_{n-1-k})||_{X\times X}$$

$$= M\sum_{j=0}^{n-2} \alpha_j||((V-\tilde{V})_j, \Delta(V-\tilde{V})_j)||_{X\times X} \leq 3M||\alpha||_1||V-\tilde{V}||_\infty.$$

Hence, we obtain

$$||\mathcal{K}V - \mathcal{K}\tilde{V}||_\infty \leq 3M||\alpha||_1|||V-\tilde{V}|||. \tag{3.11}$$

On the other hand, using the fact that $S(1) = I$, we observe first that

$$\begin{aligned}\Delta[\mathcal{K}V]_n &= f(n-1,\ V_{n-1},\ \Delta V_{n-1}) \\ &+ \sum_{k=1}^{n-1}(S(k+1)-S(k))f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k}),\ n\geq 1.\end{aligned}$$

Since $S(2) = 2I$, we get

$$\Delta^2[\mathcal{K}V]_n$$

$$= f(n, V_n, \Delta V_n) - f(n-1, V_{n-1}, \Delta V_{n-1}) + (S(2)-I)f(n-1, V_{n-1}, \Delta V_{n-1})$$

$$+\sum_{k=1}^{n-1}(S(k+2)-2S(k+1)+S(k))f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})$$

$$= f(n, V_n, \Delta V_n) + \sum_{k=1}^{n-1}(S(k+2)-2S(k+1)+TS(k))f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})$$

$$+\sum_{k=1}^{n-1}(I-T)S(k)f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k}).$$

Taking into account that $z_{n+1} = (S*g)_n$ is solution of (3.7) , we get the following identity:

$$\sum_{k=1}^{n-1}(S(k+2)-2S(k+1)+TS(k))f(n-1-k,\ V_{n-1-k},\Delta V_{n-1-k})=0. \tag{3.12}$$

Using (3.12), we obtain for $n \geq 1$

$$\Delta^2[\mathcal{K}V]_n = f(n,\ V_n,\ \Delta V_n)+\sum_{k=1}^{n-1}(I-T)S(k)f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k}), \tag{3.13}$$

whence, for $n \geq 1$

$$\begin{aligned}&\Delta^2[\mathcal{K}V]_n-\Delta^2[\mathcal{K}\tilde{V}]_n\\ &= f(n,\ V_n,\ \Delta V_n)-f(n,\ \tilde{V}_n,\ \Delta\tilde{V}_n)\\ &+\sum_{k=1}^{n-1}(I-T)S(k)(f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k})-f(n-1-k,\ \tilde{V}_{n-1-k},\ \Delta\tilde{V}_{n-1-k})).\end{aligned}$$

Furthermore, using the fact that $\Delta^2[\mathcal{K}V]_0 = f(0,0,0)$, the above identity and then Minkowskii's inequality, we get

$$\begin{aligned}&||\Delta^2\mathcal{K}V-\Delta^2\mathcal{K}\tilde{V}||_p\\ &= (||f(0,0,0)-f(0,0,0)||_X^p+\sum_{n=1}^{\infty}||\Delta^2[\mathcal{K}V]_n-\Delta^2[\mathcal{K}\tilde{V}]_n||_X^p)^{1/p}\\ &\leq [\sum_{n=1}^{\infty}||f(n,V_n,\Delta V_n)-f(n,\tilde{V}_n,\Delta\tilde{V}_n)||_X^p]^{1/p}\\ &+\Big[\sum_{n=1}^{\infty}||\sum_{k=1}^{n-1}(I-T)S(k)(f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k})\\ &-f(n-1-k,\ \tilde{V}_{n-1-k},\ \Delta\tilde{V}_{n-1-k}))||_X^p\Big]^{1/p}.\end{aligned} \tag{3.14}$$

Since K_T is bounded on $l_p(\mathbb{Z}_+, X)$, using Assumption (A), we obtain

$$
\begin{aligned}
&||\Delta^2 \mathcal{K}V - \Delta^2 \mathcal{K}\tilde{V}||_p \\
&\le (1+||K_T||)[\sum_{n=1}^{\infty} ||f(n, V_n, \Delta V_n) - f(n, \tilde{V}_n, \Delta\tilde{V}_n)||_X^p]^{1/p} \\
&\le (1+||K_T||)[\sum_{n=1}^{\infty} \alpha_n^p ||((V-\tilde{V})_n, \Delta(V-\tilde{V})_n)||_{X\times X}^p]^{1/p} \\
&\le 3(1+||K_T||)||\alpha||_1 ||V-\tilde{V}||_\infty.
\end{aligned}
\tag{3.15}
$$

Hence, we obtain from (3.11) and (3.15)

$$
\begin{aligned}
|||\mathcal{K}V - \mathcal{K}\tilde{V}||| &= ||\mathcal{K}V - \mathcal{K}\tilde{V}||_\infty + ||\Delta^2\mathcal{K}V - \Delta^2\mathcal{K}\tilde{V}||_p \\
&\le 3M||\alpha||_1|||V-\tilde{V}||| + 3(1+||K_T||)||\alpha||_1|||V-\tilde{V}||| \\
&= 3(M+1+||K_T||)||\alpha||_1|||V-\tilde{V}||| = ab|||V-\tilde{V}|||,
\end{aligned}
$$

where $a := 3M||\alpha||_1$ and $b := 1 + (1+||K_T||)M^{-1}$.

Next, we consider the iterates of the operator $\mathcal{K}$. Let V and $\tilde{V}$ be in $\mathcal{W}_0^{2,p}$. Taking into account that $S(1) = I$, $S(0) = 0$ and $V_0 = V_1 = \tilde{V}_0 = \tilde{V}_1 = 0$, we observe first that for $n \ge 2$

$$
\begin{aligned}
&\Delta[\mathcal{K}V]_n - \Delta[\mathcal{K}\tilde{V}]_n \\
&= \sum_{k=0}^{n-1}(S(k+1)-S(k))(f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k}) - f(n-1-k,\ \tilde{V}_{n-1-k},\ \Delta\tilde{V}_{n-1-k})) \\
&= \sum_{k=1}^{n-1}(S(n-k)-S(n-k-1))(f(k,\ V_k,\ \Delta V_k) - f(k,\ \tilde{V}_k,\ \Delta\tilde{V}_k)),
\end{aligned}
\tag{3.16}
$$

whence

$$\begin{aligned}
&||\Delta[\mathcal{K}V]_n - \Delta[\mathcal{K}\tilde{V}]_n||_X \\
&\leq 2M\sum_{k=1}^{n-1}||f(k,V_k,\Delta V_k) - f(k,\tilde{V}_k,\Delta\tilde{V}_k)||_X \\
&\leq 2M\sum_{k=1}^{n-1}\alpha_k||((V-\tilde{V})_k,\Delta(V-\tilde{V})_k)||_{X\times X}.
\end{aligned} \tag{3.17}$$

On the other hand, from (3.10), we get

$$||[\mathcal{K}V]_n - [\mathcal{K}\tilde{V}]_n||_X \leq M\sum_{k=1}^{n-2}\alpha_k||((V-\tilde{V})_k,\Delta(V-\tilde{V})_k)||_{X\times X}. \tag{3.18}$$

Using estimates (3.17) and (3.18), we obtain for $n \geq 2$

(3.19)

$$||([\mathcal{K}V - \mathcal{K}\tilde{V}]_n, \Delta[\mathcal{K}V - \mathcal{K}\tilde{V}]_n)||_{X\times X} \leq 3M\sum_{k=1}^{n-1}\alpha_k||((V-\tilde{V})_k,\Delta(V-\tilde{V})_k)||_{X\times X}.$$

Next, using $[\mathcal{K}V]_0 = [\mathcal{K}V]_1 = 0$ and estimates (3.19) and (3.6), we obtain
(3.20)

$$\begin{aligned}
||[\mathcal{K}^2V]_n - [\mathcal{K}^2\tilde{V}]_n||_X &\leq M\sum_{j=0}^{n-2}||f(j,[\mathcal{K}V]_j,\Delta[\mathcal{K}V]_j) - f(j,[\mathcal{K}\tilde{V}]_j,\Delta[\mathcal{K}\tilde{V}]_j)||_X \\
&\leq M\sum_{j=1}^{n-2}\alpha_j||([\mathcal{K}V - \mathcal{K}\tilde{V}]_j,\Delta[\mathcal{K}V - \mathcal{K}\tilde{V}]_j)||_{X\times X}. \\
&\leq 3M^2\sum_{j=1}^{n-1}\alpha_j(\sum_{i=1}^{j-1}\alpha_i||((V-\tilde{V})_i,\Delta(V-\tilde{V})_i)||_{X\times X}) \\
&\leq \tfrac{1}{2}(3M)^2(\sum_{\tau=1}^{n-1}\alpha_\tau)^2||V-\tilde{V}||_\infty.
\end{aligned}$$

Since $[\mathcal{K}^2V]_0 = [\mathcal{K}^2V]_1 = 0$, we get

$$||\mathcal{K}^2V - \mathcal{K}^2\tilde{V}||_\infty \leq \frac{1}{2}(3M||\alpha||_1)^2|||V - \tilde{V}|||. \tag{3.21}$$

Furthermore, using the identity

$$\begin{aligned}
&\Delta^2[\mathcal{K}^2V]_n - \Delta^2[\mathcal{K}^2\tilde{V}]_n \\
&= f(n,\ [\mathcal{K}V]_n, \Delta[\mathcal{K}V]_n) - f(n,\ [\mathcal{K}\tilde{V}]_n,\ \Delta[\mathcal{K}\tilde{V}]_n) \\
&+ \sum_{k=1}^{n-1}(I-T)S(k)(f(n-1-k,\ [\mathcal{K}V]_{n-1-k},\ \Delta[\mathcal{K}V]_{n-1-k}) \\
&- f(n-1-k,\ [\mathcal{K}\tilde{V}]_{n-1-k},\ \Delta[\mathcal{K}\tilde{V}]_{n-1-k})),
\end{aligned}$$

the fact that $\Delta^2[\mathcal{K}^2V]_0 = f(0,0,0)$ for all $V \in \mathcal{W}_0^{2,p}$ and Lemma 3.1 we obtain

$$\begin{aligned}
&||\Delta^2\mathcal{K}^2V - \Delta^2\mathcal{K}^2\tilde{V}||_p \\
&= (||\Delta^2[\mathcal{K}^2V]_0 - \Delta^2[\mathcal{K}^2\tilde{V}]_0||_X^p + \sum_{n=1}^{\infty}||\Delta^2[\mathcal{K}^2V]_n - \Delta^2[\mathcal{K}^2\tilde{V}]_n||_X^p)^{1/p} \\
&\leq (1+||K_T||)[\sum_{n=1}^{\infty}||f(n,[\mathcal{K}V]_n,\Delta[\mathcal{K}V]_n) - f(n,[\mathcal{K}\tilde{V}]_n,\Delta[\mathcal{K}\tilde{V}]_n)||_X^p]^{1/p} \\
&\leq (1+||K_T||)[\sum_{n=1}^{\infty}\alpha_n^p||([\mathcal{K}V-\mathcal{K}\tilde{V}]_n,\Delta[\mathcal{K}V-\mathcal{K}\tilde{V}]_n)||_{X\times X}^p]^{1/p} \\
&\leq 3M(1+||K_T||)[\sum_{n=1}^{\infty}\alpha_n^p(\sum_{k=1}^{n-1}\alpha_k||([V-\tilde{V}]_k,\Delta[V-\tilde{V}]_k)||_{X\times X})^p]^{1/p} \\
&\leq 3^2M(1+||K_T||)[\sum_{n=0}^{\infty}\alpha_n^p(\sum_{k=0}^{n-1}\alpha_k)^p||V-\tilde{V}||_\infty^p]^{1/p} \\
&\leq 3^2M(1+||K_T||)\tfrac{1}{2}(\sum_{j=0}^{\infty}\alpha_j)^2||V-\tilde{V}||_\infty,
\end{aligned}$$

whence

$$||\Delta^2\mathcal{K}^2V - \Delta^2\mathcal{K}^2\tilde{V}||_p \leq \frac{1}{2}(3M||\alpha||_1)^2(1+||K_T||)M^{-1}|||V-\tilde{V}|||. \tag{3.22}$$

From estimates (3.21) and (3.22), we get

$$|||\mathcal{K}^2V - \mathcal{K}^2\tilde{V}||| \leq \frac{b}{2}a^2|||V-\tilde{V}|||, \tag{3.23}$$

with a and b defined as above. Taking into account (3.17), (3.19), (3.20) and (3.6), we can infer

$$||([\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_j,\Delta[\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_j)||_{X\times X}\leq \tfrac{3}{2}(3M)^2(\sum_{\tau=1}^{j-1}\alpha_\tau)^2||V-\tilde{V}||_\infty. \tag{3.24}$$

Next using estimate (3.24) and Lemma 3.1, we get

$$\begin{aligned}||[\mathcal{K}^3V]_n-[\mathcal{K}^3\tilde{V}]_n||_X &\leq M\sum_{j=1}^{n-2}\alpha_j||([\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_j,\Delta[\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_j)||_{X\times X}.\\ &\leq \tfrac{1}{2}(3M)^3\sum_{j=0}^{n-1}\alpha_j(\sum_{\tau=1}^{j-1}\alpha_\tau)^2||V-\tilde{V}||_\infty \qquad (3.25)\\ &\leq \tfrac{1}{6}(3M)^3(\sum_{j=1}^{n-1}\alpha_j)^3||V-\tilde{V}||_\infty.\end{aligned}$$

Hence

$$||\mathcal{K}^3V-\mathcal{K}^3\tilde{V}||_\infty\leq\frac{1}{6}(3M||\alpha||_1)^3|||V-\tilde{V}|||. \tag{3.26}$$

Using (3.24), we get

$$\begin{aligned}&||\Delta^2\mathcal{K}^3V-\Delta^2\mathcal{K}^3\tilde{V}||_p\\ &\leq(1+||K_T||)[\sum_{n=1}^{\infty}\alpha_n^p||([\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_n,\Delta[\mathcal{K}^2V-\mathcal{K}^2\tilde{V}]_n)||_{X\times X}^p]^{1/p}\\ &\leq 3(3M)^2(1+||K_T||)\tfrac{1}{6}(\sum_{j=0}^{\infty}\alpha_j)^3||V-\tilde{V}||_\infty,\end{aligned}$$

whence

$$||\Delta^2\mathcal{K}^3V-\Delta^2\mathcal{K}^3\tilde{V}||_p\leq\frac{1}{6}(3M||\alpha||_1)^3(1+||K_T||)M^{-1}|||V-\tilde{V}|||. \tag{3.27}$$

From estimates (3.26) and (3.27), we get

$$|||\mathcal{K}^3V-\mathcal{K}^3\tilde{V}|||\leq\frac{b}{3!}a^3|||V-\tilde{V}|||. \tag{3.28}$$

An induction argument shows us that:

$$|||\mathcal{K}^nV-\mathcal{K}^n\tilde{V}|||\leq\frac{b}{n!}a^n|||V-\tilde{V}|||. \tag{3.29}$$

Since $ba^n/n! < 1$ for n sufficiently large, by the fixed point iteration method $\mathcal{K}$ has a unique fixed point $V \in \mathcal{W}_0^{2,p}$. Let V be the unique fixed point of $\mathcal{K}$, then by Assumption (A) we have

$$\begin{aligned} ||V_n||_X &= ||\sum_{k=1}^{n-1} S(k) f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})||_X \\ &\leq M\sum_{k=0}^{n-2} ||f(k, V_k, \Delta V_k) - f(k,0,0)||_X + M\sum_{k=0}^{n-2} ||f(k,0,0)||_X \qquad (3.30) \\ &\leq M\sum_{k=0}^{n-2} \alpha_k ||(V_k, \Delta V_k)||_{X\times X} + M||f(\cdot,0,0)||_1, \end{aligned}$$

hence,

$$||V_n||_X \leq M||f(\cdot,0,0)||_1 + M\sum_{k=0}^{n-1} \alpha_k ||(V_k, \Delta V_k)||_{X\times X}. \qquad (3.31)$$

On the other hand, we have

$$\begin{aligned} ||\Delta V_n||_X &= ||\sum_{k=1}^{n-1} (S(k+1) - S(k)) f(n-1-k, V_{n-1-k}, \Delta V_{n-1-k})||_X \\ &\leq 2M\sum_{k=0}^{n-1} \alpha_k ||(V_k, \Delta V_k)||_{X\times X} + 2M\sum_{k=0}^{n-1} ||f(k,0,0)||_X, \end{aligned} \qquad (3.32)$$

hence

$$||\Delta V_n||_X \leq 2M||f(\cdot,0,0)||_1 + 2M\sum_{k=0}^{n-1} \alpha_k ||(V_k, \Delta V_k)||_{X\times X}. \qquad (3.33)$$

From (3.31) and (3.33), we get

$$||(V_n, \Delta V_n)||_{X\times X} \leq 3M||f(\cdot,0,0)||_1 + 3M\sum_{k=0}^{n-1} \alpha_k ||(V_k, \Delta V_k)||_{X\times X}. \qquad (3.34)$$

Then, by application of the discrete Gronwall's inequality [1, Corollary 4.12, p.183], we get

$$\begin{aligned} ||(V_n, \Delta V_n)||_{X\times X} &\leq 3M||f(\cdot,0,0)||_1 \prod_{j=0}^{n-1} (1 + 3M\alpha_j) \leq 3M||f(\cdot,0,0)||_1 \prod_{j=0}^{n-1} e^{3M\alpha_j} \\ &= 3M||f(\cdot,0,0)||_1 e^{3M\sum_{j=0}^{n-1}\alpha_j} \leq 3M||f(\cdot,0,0)||_1 e^{3M||\alpha||_1}. \end{aligned}$$

Then

$$\sup_{n\in\mathbb{Z}_+}[\ ||(V_n,\Delta V_n)||_{X\times X}\ ] \leq 3M||f(\cdot,0,0)||_1 e^{3M||\alpha||_1}. \tag{3.35}$$

Finally, by (3.13) we obtain

$$\Delta^2 V_n = f(n,V_n,\Delta V_n) + \sum_{k=1}^{n-1}(I-T)S(k)f(n-1-k,V_{n-1-k},\Delta V_{n-1-k}). \tag{3.36}$$

Hence, using the fact that $\Delta^2 V_0 = f(0,0,0)$ and proceeding analogously as in (3.15), we get

$$\begin{aligned}
||\Delta^2 V||_p &= (||f(0,0,0)||_X^p + \sum_{n=1}^{\infty}||\Delta^2 V_n||_X^p)^{1/p}\\
&\leq ||f(0,0,0)||_X + (\sum_{n=1}^{\infty}||\Delta^2 V_n||_X^p)^{1/p}\\
&\leq ||f(0,0,0)||_X + (\sum_{n=1}^{\infty}||f(n,V_n,\Delta V_n)||_X^p)^{1/p} + ||K_T||(\sum_{n=1}^{\infty}||f(n,V_n,\Delta V_n)||_X^p)^{1/p}\\
&\leq 2(\sum_{n=0}^{\infty}||f(n,V_n,\Delta V_n)||_X^p)^{1/p} + ||K_T||(\sum_{n=0}^{\infty}||f(n,V_n,\Delta V_n)||_X^p)^{1/p}\\
&\leq (2+||K_T||)\sum_{n=0}^{\infty}||f(n,V_n,\Delta V_n)||_X,
\end{aligned}$$

where, by Assumption (A) and (3.35)

$$\begin{aligned}
\sum_{n=0}^{\infty}||f(n,V_n,\Delta V_n)||_X &\leq \sum_{k=0}^{\infty}\alpha_k||(V_k,\Delta V_k)||_{X\times X} + ||f(\cdot,0,0)||_1\\
&\leq 3M||\alpha||_1||f(\cdot,0,0)||_1 e^{3M||\alpha||_1} + ||f(\cdot,0,0)||_1\\
&\leq ||f(\cdot,0,0)||_1 e^{6M||\alpha||_1}.
\end{aligned}$$

This ends the proof of the theorem. ∎

In view of Theorem 2.4, we obtain the following result valid on UMD spaces.

Corollary 3.3. *Let X be a UMD space. Assume that Assumption (A) holds and suppose $T\in\mathcal{B}(X)$ is an analytic $\mathcal{S}$-bounded operator and such that the set $\{(\lambda-1)^2R((\lambda-1)^2,\ I-T):\ |\lambda|=1,\ \lambda\neq 1\}$ is R-bounded. Then, there is a unique bounded solution $x=(x_n)$ of equation (3.1) such that $(\Delta^2 x_n)\in l_p(\mathbb{Z}_+,X)$. Moreover, the a priori estimates (3.3) and (3.4) hold.*

Example 3.4. Consider the semilinear problem

$$\Delta^2 x_n - (I-T)x_n = q_n f(x_n),\ n\in\mathbb{Z}_+,\ x_0=x_1=0, \tag{3.37}$$

where f is defined and satisfy a Lipschitz condition with constant L on a Hilbert space H. In addition suppose $(q_n) \in l_1(\mathbb{Z}_+)$. Then Assumption (A) is satisfied. In our case, applying the preceding result we obtain that if $T \in \mathcal{B}(H)$ is an analytic $\mathcal{S}$-bounded operator and such that the set $\{(\lambda-1)^2 R((\lambda-1)^2,\ I-T):\ |\lambda|=1,\ \lambda\neq 1\}$ is bounded, then there exists a unique bounded solution $x=(x_n)$ of the equation (3.37) such that $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+, H)$. Moreover,

$$\max\{\sup_{n\in\mathbb{Z}_+}[\ ||x_n||_H + ||\Delta x_n||_H\ ], ||\Delta^2 x||_p\} \leq C||f(0)||_H||q||_1 e^{6LM||q||_1}. \tag{3.38}$$

In particular, taking $T=I$ the identity operator, we obtain the following scalar result which complement those in Drozdowicz- Popenda [18].

Corollary 3.5. *Suppose f is defined and satisfy a Lipschitz condition with constant L on a Hilbert space H. Let $(q_n) \in l_1(\mathbb{Z}_+, H)$, then the equation*

$$\Delta^2 x_n = q_n f(x_n), \tag{3.39}$$

has a unique bounded solution $x=(x_n)$ such that $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+, H)$ and (3.38) holds.

We remark that the above result holds in the finite dimensional case where it is new and covers a wide range of difference equations.

4. A criterion for stability

The following result provides a new criterion to verify the stability of discrete semilinear systems. Note that the characterization of maximal regularity is the key to give conditions based only on the data of a given system.

Theorem 4.1. *Let X be a UMD space. Assume that Assumption (A) holds and suppose $T \in \mathcal{B}(X)$ is analytic and $1 \in \rho(T)$. Then the system (3.1) is stable, that is the solution (x_n) of (3.1) is such that $x_n \to 0$ as $n \to \infty$.*

Proof. It is assumed that T is analytic (which implies that the spectrum is contained in the unit disc and the point 1, see [6]) and that 1 is not in the spectrum, then in view of Proposition 3.6 [33], the set

$$\{(\lambda-1)^2 R((\lambda-1)^2,\ I-T):\ |\lambda|=1,\ \lambda\neq 1\} \tag{4.1}$$

is R-bounded, because $(\lambda-1)^2 R((\lambda-1)^2,\ I-T)$ is an analytic function in a neighborhood of the circle. The $\mathcal{S}-$boundedness assumption of the operator T follows from maximal regularity and the fact that $I-T$ is invertible. In fact, we get the following estimate:

$$\sup_{n\geq 0}||\mathcal{S}(n)|| \leq ||(I-T)^{-1}||||K_T||.$$

By Corollary 3.3, there exists a unique bounded solution x_n of equation (3.1) such that $(\Delta^2 x_n) \in l_p(\mathbb{Z}_+, X)$. Then $\Delta^2 x_n \to 0$ as $n \to \infty$. Next, observe that Assumption (A) and estimate (3.6) imply

$$
\begin{aligned}
||f(n,x_n,\Delta x_n)||_X &\leq ||f(n,x_n,\Delta x_n)-f(n,0,0)||_X+||f(n,0,0)||_X\\
&\leq \alpha_n||(x_n,\Delta x_n)||_{X\times X}+||f(n,0,0)||_X\\
&\leq \alpha_n \sup\nolimits_{n\in\mathbb{Z}_+}||(x_n,\Delta x_n)||_{X\times X}+||f(n,0,0)||_X \qquad (4.2)\\
&\leq 3\alpha_n||x||_\infty+||f(n,0,0)||_X.
\end{aligned}
$$

Since $(f(\cdot,0,0))\in l_1(\mathbb{Z}_+,X)$ and $(\alpha_n)\in l_1(\mathbb{Z}_+)$, we obtain that $f(n,x_n,\Delta x_n)\to 0$ as $n\to\infty$. Then, the result follows from hypothesis and equation (3.1).

■

From the point of view of applications we specialize to Hilbert spaces. The following corollary provide easy to check conditions for stability.

Corollary 4.2. *Let H be a Hilbert space. Let $T\in\mathcal{B}(H)$ such that $||T||<1$. Suppose that Assumption (A) holds in H. Then the system (3.1) is stable.*

Proof. First we note that each Hilbert space is UMD, and then the concept of R-boundedness and boundedness coincide, see [16]. Since $||T||<1$, we get that T is analytic and $1\in\rho(T)$. Furthermore, for $|\lambda|=1$, $\lambda\neq 1$ the inequality

$$
||(\lambda-1)^2R((\lambda-1)^2,I-T)||=||\frac{(\lambda-1)^2}{\lambda(\lambda-2)}\sum_{n=0}^{\infty}(\frac{T}{\lambda(\lambda-2)})^n||\leq\frac{|\lambda-1|^2}{|\lambda-2|-||T||}\leq\frac{4}{1-||T||},
$$

shows that the set (4.1) is bounded.

■

Of course, the same result holds in the finite dimensional case.

5. Local perturbations

In the process of obtaining our next result, we will require the following assumption.

Assumption (A)*: The following conditions hold:
(i)* The function $f(n,z)$ is locally Lipschitz with respect to $z\in X\times X$, i.e. for each positive number R, for all $n\in\mathbb{Z}_+$, and $z,w\in X\times X$, $||z||_{X\times X}\leq R, ||w||_{X\times X}\leq R$

$$
||f(n,z)-f(n,w)||_X\leq l(n,R)||z-w||_{X\times X},
$$

where $\ell:\mathbb{Z}_+\times[0,\infty)\longrightarrow[0,\infty)$ is a nondecreasing function with respect to the second variable.
(ii)* There is a positive number a such that $\sum_{n=0}^{\infty}\ell(n,a)<+\infty$.
(iii)* $f(\cdot,0,0)\in\ell_1(\mathbb{Z}_+,X)$.

We need to introduce some basic notations: We denote by $\mathcal{W}_m^{2,p}$ the Banach space of all sequences $V=(V_n)$ belonging to $\ell_\infty(\mathbb{Z}_+,X)$ such that $V_n=0$ if $0\leq n\leq m$, and $\Delta^2V\in\ell_p(\mathbb{Z}_+,X)$ equipped with the norm $|||\cdot|||$. For $\lambda>0$, denote by $\mathcal{W}_m^{2,p}[\lambda]$ the ball $|||V||| \leq \lambda$ in $\mathcal{W}_m^{2,p}$. Our main result in this section is the following local version of Theorem 3.2.

Theorem 5.1. *Suppose that the following conditions are satisfied:*
(a) Condition (A)* holds.*
(b) T is a $\mathcal{S}$-bounded operator and it has discrete maximal regularity.*

Then, there are a positive constant $m \in \mathbb{N}$ and a unique bounded solution $x = (x_n)$ of equation (3.1) for $n \geq m$ such that $x_n = 0$ if $0 \leq n \leq m$ and the sequence $(\Delta^2 x_n)$ belongs to $\ell_p(\mathbb{Z}_+, X)$. Moreover, we get

$$||x||_\infty + ||\Delta^2 x||_p \leq a, \tag{5.1}$$

where a is the constant of condition (ii).*

Proof. Let $\beta \in (0, 1/3)$. Using (iii)* and (ii)* there are n_1 and n_2 in $\mathbb{N}$ such that:

$$(M + 2 + ||K_T||) \sum_{j=n_1}^{\infty} ||f(j, 0, 0)||_X \leq \beta a, \tag{5.2}$$

and

$$\mathcal{T} := \beta + (M + 2 + ||K_T||) \sum_{j=n_2}^{\infty} \ell(j, a) < \frac{1}{3}, \tag{5.3}$$

where $M := \sup_{n \in \mathbb{Z}^+} ||S(n)||$.

Let V be a sequence in $\mathcal{W}_m^{2,p}[a/3]$, with $m = \max\{n_1, n_2\}$. A short argument similar to (3.5) and involving Assumption (A)* shows that the sequence

$$g_n := \begin{cases} 0 & \text{if } \ 0 \leq n \leq m, \\ f(n, V_n, \Delta V_n) & \text{if } \ n > m, \end{cases} \tag{5.4}$$

belongs to ℓ_p. By the discrete maximal regularity, the Cauchy's problem (3.7) with g_n defined as in (5.4) has a unique solution (z_n) such that $(\Delta^2 z_n) \in l_p(\mathbb{Z}_+, X)$, which is given by

$$z_n = [\tilde{\mathcal{K}} V)]_n = \begin{cases} 0 & \text{if } 0 \leq n \leq m, \\ \displaystyle\sum_{k=0}^{n-1-m} S(k) f(n-1-k,\ V_{n-1-k},\ \Delta V_{n-1-k}) & \text{if } n \geq m+1. \end{cases} \tag{5.5}$$

We will prove that $\tilde{\mathcal{K}} V$ belongs to $\mathcal{W}_m^{2,p}[a/3]$. In fact, since

$$||(V_j, \Delta V_j)||_{X \times X} \leq 3||V|||_\infty \leq 3|||V||| < a, \tag{5.6}$$

we have by Assumption $(A)^*$

$$
\begin{aligned}
||[\tilde{\mathcal{K}}V]_n||_X &= M\sum_{j=m}^{n-2}||f(j,V_j,\Delta V_j)||_X \\
&\le M\sum_{j=m}^{n-2}||f(j,V_j,\Delta V_j)-f(j,0,0)||_X + M\sum_{j=m}^{n-2}||f(j,0,0)||_X \\
&\le M\sum_{j=m}^{n-2}l(j,a)||(V_j,\Delta V_j)||_{X\times X} + M\sum_{j=m}^{n-2}||f(j,0,0)||_X \\
&\le M\sum_{j=m}^{\infty}l(j,a)a + M\sum_{j=m}^{\infty}||f(j,0,0)||_X.
\end{aligned} \tag{5.7}
$$

Proceeding in a way similar to (3.13), we get for $n \ge m$

$$
\Delta^2[\tilde{\mathcal{K}}V]_n = f(n,V_n,\Delta V_n) + \sum_{k=1}^{n-1-m}(I-T)S(k)f(n-1-k,V_{n-1-k},\Delta V_{n-1-k}). \tag{5.8}
$$

Hence

$$
\begin{aligned}
||\Delta^2\tilde{\mathcal{K}}V||_p &= \big[||f(m,V_m,\Delta V_m)||_X^p + \sum_{n=m+1}^{\infty}||\Delta^2[\tilde{\mathcal{K}}V]_n||_X^p\big]^{1/p} \\
&\le ||f(m,V_m,\Delta V_m)||_X + \big[\sum_{n=m+1}^{\infty}||f(n,V_n,\Delta V_n) \\
&+ \sum_{k=1}^{n-1-m}(I-T)S(k)f(n-1-k,V_{n-1-k},\Delta V_{n-1-k})||_X^p\big]^{1/p} \\
&\le ||f(m,V_m,\Delta V_m)||_X + (1+||K_T||)[\sum_{n=m}^{\infty}||f(n,V_n,\Delta V_n)||_X^p]^{1/p} \\
&\le (2+||K_T||)\sum_{n=m}^{\infty}||f(n,V_n,\Delta V_n)||_X.
\end{aligned}
$$

Therefore using (5.7) we get

$$
||\Delta^2\tilde{\mathcal{K}}V||_p \le (2+||K_T||)\big[\sum_{j=m}^{\infty}l(j,a)a + \sum_{j=m}^{\infty}||f(j,0,0)||_X\big]. \tag{5.9}
$$

Then, inequalities (5.7) and (5.9) together with (5.2) and (5.3) implies

$$\begin{aligned} |||\tilde{\mathcal{K}}V||| &\leq (M+2+||K_T||)\sum_{j=m}^{\infty}\ell(j,a)a + (M+2+||K_T||)\sum_{j=m}^{\infty}||f(j,0,0)||_X \\ &\leq (\tfrac{1}{3}-\beta)a + \beta a = \tfrac{1}{3}a, \end{aligned}$$

proving that $\tilde{\mathcal{K}}V$ belongs to $\mathcal{W}_m^{2,p}[a/3]$. In an essentially similar way to the proof of Theorem 3.2, for all V and W in $\mathcal{W}_m^{2,p}[a/3]$, we prove that:

$$||\tilde{\mathcal{K}}V - \tilde{\mathcal{K}}W||_\infty \leq 3M\sum_{j=m}^{\infty}\ell(j,a)|||V-W|||. \tag{5.10}$$

$$||\Delta^2\tilde{\mathcal{K}}V - \Delta^2\tilde{\mathcal{K}}W||_p \leq 3(1+||K_T||)\sum_{j=m}^{\infty}\ell(j,a)|||V-W|||, \tag{5.11}$$

whence

$$|||\tilde{\mathcal{K}}V - \tilde{\mathcal{K}}W||| \leq 3(M+1+||K_T||)\sum_{j=m}^{\infty}\ell(j,a)|||V-W||| = 3(\mathcal{T}-\beta)|||V-W|||. \tag{5.12}$$

Since $3(\mathcal{T}-\beta)<1$, $\tilde{\mathcal{K}}$ is a $3(\mathcal{T}-\beta)$-contraction. This complete the proof of the theorem. ∎

This enable us to prove, as an application, the following corollary.

Corollary 5.2. *Let $B_i : X\times X \longrightarrow X$, $i=1,2$ be two bounded bilinear operators; $y\in\ell_1(\mathbb{Z}_+,X)$ and $\alpha,\beta\in\ell_1(\mathbb{Z}_+,\mathbb{R})$. In addition suppose that T has discrete maximal regularity. Then, there is a unique bounded solution x such that $(\Delta^2 x)\in l_p(\mathbb{Z}_+,X)$ for the equation*

$$x_{n+2}-2x_{n+1}+Tx_n = y_n + \alpha_n B_1(x_n,x_n) + \beta_n B_2(\Delta x_n,\Delta x_n).$$

Proof. Take $l(n,R) := 2R(|\alpha_n|+|\beta_n|)(||B_1||+||B_2||)$. Then $\sum_{n=0}^{\infty}\ell(n,1)<+\infty$. Note also that $f(n,0,0)=y_n$ belongs to $\ell_1(\mathbb{Z}_+,X)$. Hence Assumption $(A)^*$ is satisfied. ∎

Remark 5.3. We observe that under the hypotheses of the above local theorem and corollary, the same type of conclusions on stability of solutions proved in Section 4 remain true.

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Universidade Federal de Pernambuco, Departamento de Matemática, Av. Prof. Luiz Freire, S/N, Recife-PE, CEP. 50540-740, Brazil.
E-mail address: `cch@dmat.ufpe.br`

Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago-Chile.
E-mail address: `clizama@usach.cl`