ON THE EXISTENCE OF ALMOST AUTOMORPHIC SOLUTIONS OF VOLTERRA DIFFERENCE EQUATIONS

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Abstract. Given a complex summable sequence $a(n)$ and a discrete almost automorphic function $f(n)$ with values in a complex Banach space X, we give criteria for the existence of discrete almost automorphic solutions of the linear Volterra difference equation $u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n)$, $n \in \mathbb{Z}$, for λ in a distinguished subset of the complex plane, determined by $a(n)$. We prove also the existence of a discrete almost automorphic solution of the nonlinear Volterra difference equation $u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u(n))$, $n \in \mathbb{Z}$, where $f(n, x)$ is a discrete almost automorphic in n for each $x \in X$ function that satisfies a global Lipschitz condition and takes values in X. Finally, we treat the same type of problem for Volterra functional difference equations.

1. Introduction

Volterra difference equations can be considered as natural generalization of difference equations. During the last few years Volterra difference equations have emerged vigorously in several applied fields and nowadays there is a wide interest in developing the qualitative theory for such equations. Volterra difference equations mainly arise to model many real phenomena, like the study of competitive species in population dynamics and the study of motions of interacting bodies, or by applying numerical methods for solving Volterra integral or integrodifferential equations. It would be noted that Volterra systems describe process whose current state is determined by their entire prehistory. These processes are encountered in models of propagation of perturbation in materials with memory, various models to describe the evolution of epidemics, the theory of viscoelasticity, and the study of optimal control problems (cf. [9, 10, 13, 14, 20, 21, 22] and references therein).

This paper deals with the existence of discrete almost automorphic solutions to linear and nonlinear Volterra difference equations of convolution type

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

where λ is a complex number and $\sum_{n=0}^{\infty} |a(n)| < +\infty$.

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Discrete almost automorphic functions, a class of functions which are more general than discrete almost periodic ones, was considered in [28, Definition 2.6] in connection with the study of (continuous) almost automorphic bounded mild solutions of differential equations. See also [15], [23] and [2]. The subject has also been systematically studied in the recent paper [2] where the authors review their main properties and discuss the existence of almost automorphic solutions of first order difference equations in linear and nonlinear cases. This paper is a continuation of this study, and deals with existence of discrete almost automorphic solutions of linear and nonlinear Volterra difference- as well as functional difference- equations of convolution type.

The theory of continuous almost automorphic functions was introduced by S. Bochner, in relation to some aspects of differential geometry [6, 4, 5, 3]. A unified and homogeneous exposition of the theory and its applications was first given by N'Guérékata in his book [24]. Important contributions to the theory of almost automorphic functions have been obtained, for example, in the papers [1, 19, 35, 36, 37, 38, 39, 40], in the books [24, 25, 39] (concerning almost automorphic functions with values in Banach spaces), and in [33] (concerning almost automorphy on groups). Also, the theory of almost automorphic functions with values in fuzzy-number-type spaces was developed in [16] (see also Chapter 4 in [25]). Note that, in [17] and [18], the theory of almost automorphic functions with values in a locally convex space (Fréchet space) and a p-Fréchet space has been developed.

The range of applications of almost automorphic functions include at present linear and nonlinear evolution equations, integro-differential and functional-differential equations, dynamical systems, etc. A recent reference is the book [25]. However, as we pointed out before, the study of almost automorphy for Volterra difference equations does not exist at this time and should be developed, so that to produce a progress in the qualitative theory of Volterra difference equations.

This paper is organized as follows. In Section 2, we give a characterization of almost automorphic sequences using the almost automorphic functions of a real variable. This new result answer in the affirmative an open problem posed in [2], and brings to the theory of almost automorphic sequences the analog to the same property valid in the case of the theory of almost periodic functions [7, p.46]. Then we review from [2] some basic and related properties needed for our purposes. In Section 3 we are able, based on an auxiliary recursive sequence, to show the existence of almost automorphic solutions of linear Volterra difference equations of scalar type, under some geometrical assumptions. In Section 4, we examine the existence of almost automorphic solutions of nonlinear Volterra difference equations of the form (1.1), under two types of Lipschitz conditions. A more general case when instead of λ we have a bounded operator T defined on a Banach space X, is treated in analogous way. Section 5 is devoted to the study of almost automorphic solutions of semilinear Volterra functional difference equations. Volterra functional difference equations have been considered by several authors ([29, 11, 22, 27, 30, 26]). In particular, the existence of periodic and almost periodic solutions has been studied recently in ([31, 32, 12]). We show the existence of almost automorphic solutions for the equation

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n, u_n), \quad n \in \mathbb{Z}, \tag{1.2}$$

where $u_n(k) := u(n+k)$, extending to this new setting the results presented in Section 4.

2. Preliminaries

Let X be a complex Banach space. We recall that a function $f : \mathbb{Z} \to X$ is said to be discrete almost automorphic if for every integer sequence (k'_n), there exist a subsequence (k_n) such that

$$\lim_{n\to\infty} f(k+k_n) =: \overline{f}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \overline{f}(k-k_n) = f(k)$$

for each $k \in \mathbb{Z}$.

Note that if the convergence in the above definition is uniform on $\mathbb{Z}$, then we get discrete almost periodicity. Also, if f is a continuous almost automorphic function in $\mathbb{R}$, then $f\mid_{\mathbb{Z}}$ is discrete almost automorphic. It is an open problem (see [2, Section 5]) to prove the reciprocal, and then to conclude that this result characterizes the discrete almost automorphic functions, as is in fact the case for the class of almost periodic functions (see [7, Theorem 1.27]). In our next result, we answer positively this question and we show how to obtain continuous almost automorphic functions from discrete almost automorphic ones.

Theorem 2.1. *Let $x : \mathbb{Z} \to X$ be a discrete almost automorphic function. Then there exists an almost automorphic function $f : \mathbb{R} \to X$ such that $x(n) = f(n)$ for all $n \in \mathbb{Z}$.*

Proof. Let $\alpha(\cdot)$ be a numerical almost automorphic function such that $\alpha(t) \neq 0$ for all $t \in \mathbb{R}$. We define $f : \mathbb{R} \to X$ by

$$f(t) = \alpha(t)\left[\frac{(m+1-t)}{\alpha(m)}x(m) + \frac{(t-m)}{\alpha(m+1)}x(m+1)\right], \tag{2.1}$$

for $m \leq t < m+1$ and $m \in \mathbb{Z}$. We will show that f is an almost automorphic function. Let $(t'_n)_n$ be a sequence in $\mathbb{R}$. We set $k'_n = [t'_n]$. It follows from our definition that there exists a subsequence $(t_n)_n$ of $(t'_n)_n$ and a subsequence $(k_n)_n$ of $(k'_n)_n$ such that the following conditions hold:

(i) Since $t_n = k_n + s_n$ for some $0 \leq s_n < 1$, we can assume that $s_n \to s$ as $n \to \infty$. It is clear that $s \in [0,1]$.

(ii) $\lim_{n\to\infty} \alpha(t+t_n) \to \overline{\alpha}(t)$ and $\lim_{n\to\infty} \overline{\alpha}(t-t_n) \to \alpha(t)$, for all $t \in \mathbb{R}$.

(iii) $\lim_{n\to\infty} \alpha(m+k_n) \to \beta(m)$ and $\lim_{n\to\infty} \beta(m-k_n) \to \alpha(m)$, for all $m \in \mathbb{Z}$.

(iv) there exists $\overline{x} : \mathbb{Z} \to X$ such that $x(i+k_n) \to \overline{x}(i)$ and $\overline{x}(i-k_n) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{Z}$.

Next we set $t = [t] + r$ with $0 \leq r < 1$. We define $\overline{f} : \mathbb{R} \to X$ by

$$\overline{f}(t) = \begin{cases} \overline{\alpha}(t)\left[\frac{(1-r-s)}{\beta(m)}\overline{x}(m) + \frac{(r+s)}{\beta(m+1)}\overline{x}(m+1)\right] & r \leq 1-s, \\ \overline{\alpha}(t)\left[\frac{(2-r-s)}{\beta(m+1)}\overline{x}(m+1) + \frac{(r+s-1)}{\beta(m+2)}\overline{x}(m+2)\right] & r \geq 1-s. \end{cases}$$

Hence, if $r+s<1$, then

$$\begin{aligned} f(t+t_n) &= f(m+r+k_n+s_n) \\ &= \alpha(t+t_n)\left[\frac{(1-r-s_n)}{\alpha(m+k_n)}x(m+k_n)+\frac{(r+s_n)}{\alpha(m+k_n+1)}x(m+k_n+1)\right] \\ &\to \overline{\alpha}(t)\left[\frac{(1-r-s)}{\beta(m)}\overline{x}(m)+\frac{(r+s)}{\beta(m+1)}\overline{x}(m+1)\right],\ n\to\infty, \\ &= \overline{f}(t). \end{aligned}$$

In a similar way, if $r+s>1$, then

$$\begin{aligned} f(t+t_n) &= f(m+k_n+1+r+s_n-1) \\ &= \alpha(t+t_n)\left[\frac{(2-r-s_n)}{\alpha(m+k_n+1)}x(m+k_n+1)+\frac{(r+s_n-1)}{\alpha(m+k_n+2)}x(m+k_n+2)\right] \\ &\to \overline{\alpha}(t)\left[\frac{(2-r-s)}{\beta(m+1)}\overline{x}(m+1)+\frac{(r+s-1)}{\beta(m+2)}\overline{x}(m+2)\right],\ n\to\infty, \\ &= \overline{f}(t). \end{aligned}$$

Now, in order to calculate $\overline{f}(t-t_n)$ we analyze the possible alternatives:

(a) First we assume that $r+s<1$ and $r>s$. Consequently, for n sufficiently large, we have

$$\begin{aligned} &\overline{f}(m+r-k_n-s_n) \\ &= \overline{\alpha}(t-t_n)\left[\frac{(1-r+s_n-s)}{\beta(m-k_n)}\overline{x}(m-k_n)+\frac{(r-s_n+s)}{\beta(m-k_n+1)}\overline{x}(m-k_n+1)\right] \\ &\to \alpha(t)\left[\frac{(1-r)}{\alpha(m)}x(m)+\frac{r}{\alpha(m+1)}x(m+1)\right],\quad n\to\infty, \\ &= f(t). \end{aligned}$$

(b) Assume now that $r+s<1$ and $r<s$. For n sufficiently large, we have

$$\begin{aligned} \overline{f}(t-t_n) &= \overline{f}(m-k_n-1+1+r-s_n) \\ &= \overline{\alpha}(t-t_n)\left[\frac{(1-r+s_n-s)}{\beta(m-k_n)}\overline{x}(m-k_n)+\frac{(r-s_n+s)}{\beta(m-k_n+1)}\overline{x}(m-k_n+1)\right] \\ &\to \alpha(t)\left[\frac{(1-r)}{\alpha(m)}x(m)+\frac{r}{\alpha(m+1)}x(m+1)\right],\quad n\to\infty, \\ &= f(t). \end{aligned}$$

(c) We assume that $r+s>1$ and $r>s$. Since $r-s+s=r<1$, for n sufficiently large, we have

$$\begin{aligned} &\overline{f}(m+r-k_n-s_n) \\ &= \overline{\alpha}(t-t_n)\left[\frac{(1-r+s_n-s)}{\beta(m-k_n)}\overline{x}(m-k_n)+\frac{(r-s_n+s)}{\beta(m-k_n+1)}\overline{x}(m-k_n+1)\right] \\ &\to \alpha(t)\left[\frac{(1-r)}{\alpha(m)}x(m)+\frac{r}{\alpha(m+1)}x(m+1)\right],\quad n\to\infty, \\ &= f(t). \end{aligned}$$

(d) In the case $r+s>1$ and $r<s$, since $1+r-s_n+s>1$, proceeding as before, we obtain

$$\begin{aligned}\overline{f}(t-t_n) &= \overline{f}(m-k_n-1+1+r-s_n)\\ &= \overline{\alpha}(t-t_n)\left[\frac{(1-r+s_n-s)}{\beta(m-k_n)}\overline{x}(m-k_n)+\frac{(r-s_n+s)}{\beta(m-k_n+1)}\overline{x}(m-k_n+1)\right]\\ &\to \alpha(t)\left[\frac{(1-r)}{\alpha(m)}x(m)+\frac{r}{\alpha(m+1)}x(m+1)\right] \quad n\to\infty,\\ &= f(t).\end{aligned}$$

We can verify easily that the property also holds in the limit cases such as $r+s=1$ or $r=s$, which completes the proof. □

Remark 2.2. We note that for almost periodic functions, is enough to define $f(t)=x(n)+(t-n)[x(n+1)-x(n)]$, $t\in[n,n+1)$, $n\in\mathbb{Z}$, in contrast with (2.1).

We denote by $AA_d(X)$ the set of discrete almost automorphic functions. Such as in the continuous case we have that discrete almost automorphicity is a more general concept than discrete almost periodicity, that is

$$AP_d(X)\subset AA_d(X).$$

Examples of discrete almost automorphic functions which are not discrete almost periodic were first constructed by Veech [34]. See also [2] for concrete examples.

Discrete almost automorphic functions have the following fundamental properties.

Theorem 2.3. *([2]) Let u,v be discrete almost automorphic functions. Then the following assertions are valid:*

(i) *$u+v$ is discrete almost automorphic.*
(ii) *cu is discrete almost automorphic for every scalar c.*
(iii) *For each fixed l in $\mathbb{Z}$, the function $u_l:\mathbb{Z}\to X$ defined by $u_l(k):=u(k+l)$ is discrete almost automorphic.*
(iv) *The function $\hat{u}:\mathbb{Z}\to X$ defined by $\hat{u}(k):=u(-k)$ is discrete almost automorphic.*
(v) $\sup_{k\in\mathbb{Z}}\|u(k)\|<\infty$, *that is, u is a bounded function.*
(vi) *Let $\overline{u}:\mathbb{Z}\to X$ be a function such that*

$$\lim_{n\to\infty}u(k+k_n)=\overline{u}(k) \text{ and } \lim_{n\to\infty}\overline{u}(k-k_n)=u(k),$$

for all $k\in\mathbb{Z}$. Then $\sup_{k\in\mathbb{Z}}\|\overline{u}(k)\|=\sup_{k\in\mathbb{Z}}\|u(k)\|$.

As a consequence of the above theorem, the space of discrete almost automorphic functions provided with the norm

$$\|u\|_d:=\sup_{k\in\mathbb{Z}}\|u(k)\|,$$

becomes a Banach space.

Theorem 2.4. *([2]) Let X,Y be Banach spaces and let $u:\mathbb{Z}\to X$ be a discrete almost automorphic function. If $\phi:X\to Y$ is a continuous function, then the composite function $\phi\circ u:\mathbb{Z}\to Y$ is discrete almost automorphic.*

Corollary 2.5. *([2]) If A is a bounded linear operator in X and $u : \mathbb{Z} \to X$ a discrete almost automorphic function, then $Au(k), k \in \mathbb{Z}$ is also discrete almost automorphic.*

Theorem 2.6. *([2]) Let $u : \mathbb{Z} \to \mathbb{C}$ and $f : \mathbb{Z} \to X$ be discrete almost automorphic functions. Then $uf : \mathbb{Z} \to X$ defined by $(uf)(k) = u(k)f(k), k \in \mathbb{Z}$ is also a discrete almost automorphic function.*

For applications to nonlinear difference equations, the following concept of discrete almost automorphic function depending on parameters will be useful.

Definition 2.7. A function $u : \mathbb{Z} \times X \to X$ is said to be discrete almost automorphic in k for each $x \in X$, if for every sequence of integers numbers (k'_n), there exist a subsequence (k_n) such that

$$\lim_{n\to\infty} u(k + k_n, x) =: \overline{u}(k, x)$$

is well defined for each $k \in \mathbb{Z}$, $x \in X$, and

$$\lim_{n\to\infty} \overline{u}(k - k_n, x) = u(k, x)$$

for each $k \in \mathbb{Z}$ and $x \in X$.

We will denote by $AA_d(\mathbb{Z} \times X)$ the space consisting of all discrete almost automorphics functions in $k \in \mathbb{Z}$ for each x in X.

Theorem 2.8. *([2]) If $u, v : \mathbb{Z} \times X \to X$ are discrete almost automorphic functions in k for each x in X, then the following are assertions are fulfilled:*

(i) *$u + v$ is discrete almost automorphic in k for each x in X.*
(ii) *cu is discrete almost automorphic in k for each x in X, where c is an arbitrary scalar.*
(iii) *$\sup_{k\in\mathbb{Z}} \|u(k, x)\| = M_x < \infty$, for each x in X.*
(iv) *$\sup_{k\in\mathbb{Z}} \|\overline{u}(k, x)\| = M_x < \infty$, for each x in X, where $\overline{u}$ is a function involved in the Definition 2.7.*

The following result will be used to study the almost automorphy of solutions of nonlinear difference equations.

Theorem 2.9. *([2]) Let $f : \mathbb{Z} \times X \to X$ be a discrete almost automorphic function in k for each x in X that satisfies a Lipschitz condition in x uniformly in k, that is, there is a constant $L \geq 0$ such that*

$$\|f(k, x) - f(k, y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in X, k \in \mathbb{Z}.$$

Suppose $\varphi : \mathbb{Z} \to X$ is a discrete almost automorphic function. Then the function $U : \mathbb{Z} \to X$ defined by $U(k) = f(k, \varphi(k))$ is discrete almost automorphic.

The following result is the essential property to study the existence of discrete almost automorphic solutions of linear and nonlinear Volterra difference equations of convolution type.

Theorem 2.10. *([2]) Let $b : \mathbb{Z}_+ \to \mathbb{C}$ be a summable function, i.e.*

$$\sum_{k=0}^{\infty} |b(k)| < \infty.$$

Then for any discrete almost automorphic function $u : \mathbb{Z} \to X$, the function $w(\cdot)$ defined by

$$w(k) = \sum_{l=-\infty}^{k} b(k-l)u(l), \quad k \in \mathbb{Z}$$

is also discrete almost automorphic.

3. Almost automorphic solutions of linear Volterra difference equations

In this section we establish the existence of discrete almost automorphic solutions for linear Volterra difference equations on a Banach space X and described by

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j)u(j) + f(n), \;\; n \in \mathbb{Z}, \tag{3.1}$$

where λ is a complex number or, more generally, a bounded linear operator defined on X, $a : \mathbb{N} \to \mathbb{C}$ is summable, and f is in $AA_d(X)$.

For a given $\lambda \in \mathbb{C}$, let $s(\lambda, k) \in \mathbb{C}$ be the solution of the difference equation

$$\begin{aligned} s(\lambda, k+1) &= \sum_{j=0}^{k} \lambda a(k-j)s(\lambda, j), \;\; k = 0, 1, 2 \ldots \\ s(\lambda, 0) &= 1. \end{aligned} \tag{3.2}$$

We define the set

$$\Omega_s := \{\lambda \in \mathbb{C} : \sum_{k=0}^{\infty} |s(\lambda, k)| < \infty\}. \tag{3.3}$$

For example, for $a(k) = p^k$ where $|p| < 1$ we obtain, after a calculation using in (3.2) the unilateral Z-transform:

$$s(\lambda, k) = (\lambda + p)^k - p(\lambda + p)^{k-1} = \lambda(\lambda + p)^{k-1}, \;\; k \geq 1,$$

and hence $\mathbb{D}(-p, 1) := \{z \in \mathbb{C} : |z + p| < 1\} \subset \Omega_s$. The following is the main result in this section.

Theorem 3.1. *Let X be a Banach space and $\lambda \in \Omega_s$. If $f : \mathbb{Z} \to X$ is a discrete almost automorphic function, then there is a discrete almost automorphic solution of (3.1) given by*

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k)f(k). \tag{3.4}$$

Proof. Let $u(n)$ be the function given by (3.4). Taking into account that $a(\cdot)$ is a summable function, we obtain that

$$\begin{aligned}
\sum_{j=-\infty}^{n} a(n-j)\lambda u(j) &= \lambda \sum_{j=-\infty}^{n} a(n-j)\Big(\sum_{\tau=-\infty}^{j-1} s(\lambda, j-1-\tau)f(\tau)\Big) \\
&= \lambda \sum_{j=-\infty}^{n-1} \sum_{\tau=-\infty}^{j} a(n-1-j)s(\lambda, j-\tau)f(\tau) \\
&= \lambda \sum_{\tau=-\infty}^{n-1} \sum_{j=\tau}^{n-1} a(n-1-j)s(\lambda, j-\tau)f(\tau) \\
&= \sum_{\tau=-\infty}^{n-1} \Big(\sum_{j=0}^{n-1-\tau} \lambda a(n-1-\tau-j)s(\lambda, j)\Big)f(\tau) \\
&= \sum_{\tau=-\infty}^{n-1} s(\lambda, n-\tau)f(\tau) \\
&= \sum_{\tau=-\infty}^{n} s(\lambda, n-\tau)f(\tau) - s(\lambda, 0)f(n) \\
&= u(n+1) - f(n),
\end{aligned}$$

which establishes that $u(\cdot)$ is the solution of the equation (3.1). Applying Theorem 2.8, we infer that u is a discrete almost automorphic function. □

Let $\mathcal{B}(X)$ the Banach space consisting of bounded linear operators from X into X. For $T \in \mathcal{B}(X)$, we can define $s(T, k) \in \mathcal{B}(X)$ as the solution of the difference equation

$$\begin{aligned}
s(T, k+1) &= \sum_{j=0}^{k} Ta(k-j)s(T, j), \quad k = 0, 1, 2 \dots \\
s(T, 0) &= I.
\end{aligned}$$

Proceeding as before, we can prove the following result.

Theorem 3.2. *Let $T \in \mathcal{B}(X)$ so that $s(T, \cdot)$ is a summable function. If $f : \mathbb{Z} \to X$ is a discrete almost automorphic function, then there is a discrete almost automorphic solution of the equation*

$$u(n+1) = \sum_{j=-\infty}^{n} a(n-j)Tu(j) + f(n), \quad n \in \mathbb{Z}, \tag{3.5}$$

given by

$$u(n+1) = \sum_{k=-\infty}^{n} s(T, n-k)f(k). \tag{3.6}$$

We finish this section with the following example.

Example 3.3. Let $|p| < 1$ be fixed and take $\lambda \in \mathbb{D}(-p, 1)$. We consider the following difference equation

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} p^{n-j} u(j) + \sin\Big(\frac{1}{2 - \sin(n) - \sin(\sqrt{2}n)}\Big), \quad n \in \mathbb{Z}, \tag{3.7}$$

By Theorem 3.1, there is a discrete almost automorphic solution $u(n)$ of (3.7) given by

$$u(n+1) = \lambda \sum_{k=-\infty}^{n} (p+\lambda)^{n-k} \sin\Big(\frac{1}{2 - \sin(k) - \sin(\sqrt{2}k)}\Big), \quad n \in \mathbb{Z}.$$

4. Almost automorphic solutions of semilinear Volterra difference equations

In this section, we present conditions under which it is possible to ensure existence of discrete almost automorphic solutions to the equation

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j) u(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{4.1}$$

where $\lambda \in \mathbb{C}$, $a(\cdot)$ is summable and $f \in AA_d(\mathbb{Z} \times X)$. For $\lambda \in \Omega_s$, we define

$$N(\lambda) := \sum_{k=1}^{\infty} |s(\lambda, k)|.$$

Next we establish the main result of this section.

Theorem 4.1. *Let $\lambda \in \Omega_s$ and let $f : \mathbb{Z} \times X \to X$ be a discrete almost automorphic function in k for each $x \in X$. Suppose there exists $L \geq 0$ such that f satisfies the Lipschitz condition*

$$||f(k, x) - f(k, y)|| \leq L||x - y||, \tag{4.2}$$

for all $x, y \in X$ and $k \in \mathbb{Z}$. If

$$LN(\lambda) < 1, \tag{4.3}$$

then equation (4.1) has a unique discrete almost automorphic solution satisfying

$$u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k, u(k)).$$

Proof. We define the operator $F : AA_d(X) \to AA_d(X)$ by

$$F(u)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n-1-k) f(k, u(k)). \tag{4.4}$$

Since $u \in AA_d(X)$ and $f(k, x)$ satisfies (4.2), we obtain by Theorem 2.9 that $f(\cdot, u(\cdot))$ is in $AA_d(X)$. It follows from Theorem 2.10 that F is well-defined. Now, given $u, v \in AA_d(X)$, we have

$$
\begin{aligned}
||F(u)-F(v)||_d &\le \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)|\, \|f(k,u(k))-f(k,v(k))\| \\
&\le \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} L|s(\lambda, n-1-k)|\, \|u(k)-v(k)| \\
&\le L||u-v||_d \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} |s(\lambda, n-1-k)| \\
&= LN(\lambda)||u-v||_d.
\end{aligned}
$$

Since (4.3) holds, we obtain that the function F is a contraction. Then there is a unique function u in $AA_d(X)$ such that $Fu = u$. That is, u satisfies

$$
u(n+1) = \sum_{k=-\infty}^{n} s(\lambda, n-k) f(k, u(k))
$$

and, applying the Theorem 3.1, we obtain that u is solution of the equation (4.1). □

The general case, with an operator T instead of the scalar λ, can be treated in a similar way. In this case we define

$$
N_T = \sum_{j=0}^{\infty} \|s(T,j)\|. \tag{4.5}
$$

We obtain the following result.

Theorem 4.2. *Let $T \in \mathcal{B}(X)$ and let $f : \mathbb{Z} \times X \to X$ be a discrete almost automorphic function in k for each $x \in X$. Suppose that f satisfies the condition (4.2) and $LN_T < 1$ Then the equation*

$$
u(n+1) = \sum_{j=-\infty}^{n} a(n-j)Tu(j) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{4.6}
$$

has a unique discrete almost automorphic solution satisfying

$$
u(n+1) = \sum_{k=-\infty}^{n} s(T, n-k) f(k, u(k)).
$$

The following statement provides an alternative formulation for the Lipschitz condition used in the previous results.

Theorem 4.3. *Let $\lambda \in \Omega_s$ and let $f : \mathbb{Z} \times X \to X$ be a discrete almost automorphic function in k for each $x \in X$. Suppose that f satisfies the following Lipschitz type condition*

$$
||f(k,x) - f(k,y)|| \le L(k)||x-y||, \tag{4.7}
$$

for all $x, y \in X$ and $k \in \mathbb{Z}$, where $L : \mathbb{Z} \to \mathbb{R}_+$ is a summable function. Then equation (4.1) has a unique discrete almost automorphic solution.

Proof. We define the operator $F : AA_d(X) \to AA_d(X)$ by the expression (4.4). Using the Theorem 2.9 and the Theorem 2.10 we conclude that F is well-defined. Now, for $u_1, u_2 \in AA_d(X)$, we have

$$\|F(u_1)(n) - F(u_2)(n)\| \leq \sum_{k=-\infty}^{n-1} L(k)|s(\lambda, n-1-k)| \, \|u_1(k) - u_2(k)\| \tag{4.8}$$
$$\leq \|s(\lambda,\cdot)\|_\infty \Big(\sum_{k=-\infty}^{n-1} L(k) \Big) \|u_1 - u_2\|_d.$$

Hence,

$$||F(u_1) - F(u_2)||_d \leq \|s(\lambda,\cdot)\|_\infty \|L\|_1 \|u_1 - u_2\|_d.$$

Next, we consider the iterates of the operator F. Using (4.8) and taking into account [8, Lemma 3.2], we obtain the following estimate

$$\|F^2(u_1)(n) - F^2(u_2)(n)\| \leq \|s(\lambda,\cdot)\|_\infty \sum_{k=-\infty}^{n-1} L(k) \|(Fu_1)(k) - (Fu_2)(k)\| \tag{4.9}$$
$$\leq \|s(\lambda,\cdot)\|_\infty^2 \Big(\sum_{k=-\infty}^{n-1} L(k) \Big(\sum_{j=-\infty}^{k-1} L(j) \Big) \Big) \|u_1 - u_2\|_d$$
$$\leq \frac{\|s(\lambda,\cdot)\|_\infty^2}{2} \Big(\sum_{k=-\infty}^{n-1} L(k) \Big)^2 \|u_1 - u_2\|_d.$$

Therefore,

$$\|F^2(u_1) - F^2(u_2)\|_d \leq \frac{\big[\|s(\lambda,\cdot)\|_\infty \|L\|_1\big]^2}{2} \|u_1 - u_2\|_d.$$

Using again (4.9) we infer that

$$\|F^3(u_1)(n) - F^3(u_2)(n)\| \leq \frac{\|s(\lambda,\cdot)\|_\infty^3}{2} \Big(\sum_{k=-\infty}^{n-1} L(k) \Big(\sum_{j=-\infty}^{k-1} L(j) \Big)^2 \Big) \|u_1 - u_2\|_d$$
$$\leq \frac{\|s(\lambda,\cdot)\|_\infty^3}{6} \Big(\sum_{k=-\infty}^{n-1} L(k) \Big)^3 \|u_1 - u_2\|_d,$$

which implies that

$$\|F^3(u_1) - F^3(u_2)\|_d \leq \frac{\big[\|s(\lambda,\cdot)\|_\infty \|L\|_1\big]^3}{3!} \|u_1 - u_2\|_d.$$

An inductive argument shows us that

$$\|F^n(u_1) - F^n(u_2)\|_d \leq \frac{\big[\|s(\lambda,\cdot)\|_\infty \|L\|_1\big]^n}{n!} \|u_1 - u_2\|_d.$$

Since $\frac{[\|s(\lambda,\cdot)\|_\infty \|L\|_1]^n}{n!} < 1$ for n sufficiently large, F has a unique fixed point. This completes the proof of the theorem. □

Arguing in a similar way, we obtain the following result.

Theorem 4.4. *Let $T \in \mathcal{B}(X)$ such that $s(T,\cdot)$ is a summable function and let $f : \mathbb{Z} \times X \to X$ be a discrete almost automorphic in k for each $x \in X$. Suppose that f satisfies the condition (4.7). Then equation (4.6) has a unique discrete almost automorphic solution.*

The following example is an application of the main result in this section.

Example 4.5. Let $|p| < 1$, $\lambda \in \mathbb{D}(-p, 1)$ and $\nu \in \mathbb{R}$ be given. We consider the following difference equation

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} p^{n-j} u(j) + \nu \cos\Big(\frac{1}{2 - \sin(\sqrt{2}n) - \sin(n)}\Big) u(n), \quad n \in \mathbb{Z}. \tag{4.10}$$

If $|\nu|$ is small enough, by Theorem 4.1 the equation (4.10) has a unique discrete almost automorphic solution.

5. Almost automorphic solutions of semilinear Volterra functional difference equations

Our objective in this section is to extend the results presented in Section 4 to include Volterra functional difference equations of type

$$u(n+1) = \lambda \sum_{j=-\infty}^{n} a(n-j) u(j) + f(n, u_n), \quad n \in \mathbb{Z}. \tag{5.1}$$

In this equation $a(n)$ is a summable function and $u_n : \mathbb{Z}_- \to X$ is the function given by $u_n(i) = u(n+i)$. We assume that $u_n \in \mathcal{B}$, where $\mathcal{B}$ is the phase space for the equation, and $f : \mathbb{Z} \times \mathcal{B} \to X$ is an appropriate function.

Following ([29, 11, 27]) we will define the phase space $\mathcal{B}$ axiomatically. Specifically, $\mathcal{B}$ will denote a vector space of functions defined from $\mathbb{Z}_-$ into X endowed with a norm denoted $\|\cdot\|_{\mathcal{B}}$ so that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space and the following axiom hold:

- **(A):** There exist a positive constant J, and nonnegative functions $K(\cdot), M(\cdot)$ defined on $\mathbb{Z}_-$ having the following property: If $x : \mathbb{Z} \to X$ is a function such that $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{Z}_+$ the following conditions are fulfilled:
  - (i) $x_n \in \mathcal{B}$.
  - (ii) $J\|x(n)\| \le H\|x_n\|_{\mathcal{B}} \le K(n) \sup_{0\le i\le n} \|x(i)\| + M(n)\|x_0\|_{\mathcal{B}}$.

To obtain our results we consider also the following axiom:

- **(C):** If $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $\mathcal{B}$ which converges pointwise to φ, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Next, the notation $B(\mathbb{Z}_-, X)$ stands for the space consisting of bounded functions from $\mathbb{Z}_-$ into X endowed with the norm $\|\cdot\|_\infty$ of uniform convergence. It is clear that if axiom (C) holds, then $B(\mathbb{Z}_-, X)$ is continuously included in $\mathcal{B}$. Throughout the rest of the paper, $\rho > 0$ denotes a constant such that $\|\varphi\|_{\mathcal{B}} \le \rho\|\varphi\|_\infty$ for every $\varphi \in B(\mathbb{Z}_-, X)$.

Example 5.1. *Let $\gamma > 0$. We define $B^\gamma(X)$ as the space consisting of all functions $\varphi : \mathbb{Z}_- \to X$ such that $\sup_{n\in\mathbb{Z}_-} \|\varphi(n)\| e^{\gamma n} < \infty$, endowed with the norm*

$$\|\varphi\|_\gamma = \sup_{n\in\mathbb{Z}_-} \|\varphi(n)\| e^{\gamma n}.$$

It is well known ([29]) that this space satisfies axiom (A). Moreover, if $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $B^\gamma(X)$ and $\varphi^n(i) \to \varphi(i)$ as $n \to \infty$ for all $i \in \mathbb{Z}_-$, then φ is bounded, so that $\varphi \in B^\gamma(X)$, and $\|\varphi^n - \varphi\|_\gamma \to 0$ as $n \to \infty$.

Example 5.2. *Let $1 \leq p < \infty$ and let $g : \mathbb{Z}_- \to \mathbb{R}$ be a positive function such that $\sum_{i=-\infty}^{0} g(i) < \infty$ and $g(0) > 0$. We define $S_g^p(X)$ as the space consisting of all functions $\varphi : \mathbb{Z}_- \to X$ such that $\sum_{i=-\infty}^{0} \|\varphi(i)\|^p g(i) < \infty$, endowed with the norm*

$$\|\varphi\|_{S_g^p} = \left(\sum_{i=-\infty}^{0} \|\varphi(i)\|^p g(i) \right)^{1/p}.$$

It is easy to see that this space satisfies axiom (A). Moreover, if $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $S_g^p(X)$ and $\varphi^n(i) \to \varphi(i)$ as $n \to \infty$ for all $i \in \mathbb{Z}_-$, then φ is bounded, so that $\varphi \in S_g^p(X)$, and $\|\varphi^n - \varphi\|_\gamma \to 0$ as $n \to \infty$.

Lemma 5.3. *Assume that $\mathcal{B}$ is a phase space that satisfies axiom (C). If $u : \mathbb{Z} \to X$ is a discrete almost automorphic function, then $\mathbb{Z} \to \mathcal{B}$, $n \mapsto u_n$ is also a discrete almost automorphic function.*

Proof. Let $(k_n')_n$ be a sequence in $\mathbb{Z}$. Then there exists a subsequence $(k_n)_n$ such that $v(k) = \lim_{n\to\infty} u(k + k_n)$ exists for each $k \in \mathbb{Z}$ and $u(k) = \lim_{n\to\infty} v(k - k_n)$. It follows from Theorem 2.3(v) that u and v are bounded functions. Applying axiom (C) we infer that $u_{k+k_n} \to v_k$ and that $v_{k-k_n} \to u_k$ as $n \to \infty$ for every $k \in \mathbb{Z}$. □

Keeping the notations introduced in Section 4, we can establish the following result.

Theorem 5.4. *Assume that $\mathcal{B}$ is a phase space that satisfies axiom (C). Let $\lambda \in \Omega_s$ and let $f : \mathbb{Z} \times \mathcal{B} \to X$ be a discrete almost automorphic in k for each $x \in X$ function. Suppose that f satisfies the following Lipschitz condition*

$$||f(k, \varphi) - f(k, \psi)|| \leq L\|\varphi - \psi\|_{\mathcal{B}}, \tag{5.2}$$

for all $\varphi, \psi \in \mathcal{B}$ and all $k \in \mathbb{Z}$, and

$$\rho L N(\lambda) < 1. \tag{5.3}$$

Then equation (5.1) has a unique discrete almost automorphic solution satisfying

$$u(n + 1) = \sum_{k=-\infty}^{n} s(\lambda, n - k) f(k, u_k).$$

Proof. We define the operator $F : AA_d(X) \to AA_d(X)$ by

$$F(\varphi)(n) = \sum_{k=-\infty}^{n-1} s(\lambda, n - 1 - k) f(k, \varphi_k). \tag{5.4}$$

Since $\varphi \in AA_d(X)$ and $f(\cdot)$ satisfies (5.2), it follows from Lemma 5.3 and Theorem 2.9 that the function $k \mapsto f(k, \varphi_k)$ is included in $AA_d(X)$. We complete the proof arguing as in the proof of Theorem 4.1. □

Similar to what has been done in Theorem 4.3, in this case we also can avoid the condition (5.3).

Theorem 5.5. *Assume that $\mathcal{B}$ is a phase space that satisfies axiom (C). Let $\lambda \in \Omega_s$ and let $f : \mathbb{Z} \times \mathcal{B} \to X$ be a discrete almost automorphic in k for each $x \in X$ function. Suppose that f satisfies the following Lipschitz condition*

$$|f(k,\varphi) - f(k,\psi)| \leq L(k)\|\varphi - \psi\|_{\mathcal{B}}, \tag{5.5}$$

for all $\varphi, \psi \in \mathcal{B}$ and $k \in \mathbb{Z}$, where $L : \mathbb{Z} \to \mathbb{R}_+$ is a summable function. Then the equation (5.1) has a unique discrete almost automorphic solution.

Arguing as in the sections 3 and 4, we can also establish similar existence results for the equation

$$u(n+1) = T\sum_{j=-\infty}^{n} a(n-j)u(j) + f(n,u_n), \quad n \in \mathbb{Z},$$

where $T \in \mathcal{B}(X)$.

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