BOUNDED MILD SOLUTIONS OF PERTURBED VOLTERRA EQUATIONS WITH INFINITE DELAY

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ABSTRACT. We study existence and regularity of bounded mild solutions on the real line to perturbed integral equations with infinite delay in the space of almost periodic functions (in the Bohr's sense), the space of compact almost automorphic functions, the space of almost automorphic functions and the space of asymptotically almost automorphic functions.

1. Introduction

In this paper, we study the perturbation problem

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)[Au_\epsilon(s) + f(s) + \epsilon g(s, u_\epsilon(s))]ds, \tag{1.1}$$

for the linear Volterra equation of convolution type,

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s)]ds, \tag{1.2}$$

where $\epsilon \geq 0$ is a parameter. Here $u(t)$ denotes the state of the system at time t, A a closed linear operator in the state space, a Banach space X, with domain $D(A)$, the real kernel a belongs to $L^1(\mathbb{R})$, and f, g are given functions.

In the case of finite dimensional spaces, sufficient conditions under which stability of the linear equation implies the corresponding stability of the nonlinear equation has been studied for many authors (see e.g. [15, 16, 26] and references therein). In particular, in [26] conditions are obtained under which the perturbed equation has asymptotically periodic solutions. Perturbation results on integrodifferential equations can be found in [15]. For a careful overview of perturbations of Volterra equations, see [19].

Equation (1.2), and their nonlinear counterpart (1.1), arises in the problem of heat flow with memory (cf. [31]), and have been the object of intensive study during the past years (see e.g. [7, 16, 32, 33] and references therein). Observe that it can be viewed as the *limiting equation* for the Volterra equation

$$u(t) = \int_{0}^{t} a(t-s)[Au(s) + f(s)]ds, \quad t \geq 0, \tag{1.3}$$

see [32, Chapter III, Section 11.5] to obtain details on this assertion. Moreover, under the additional assumption that a is bounded and the first moment of a exists, it was proved in

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[7] that problem (1.2) is equivalent to

$$u(t)+\frac{d}{dt}(\alpha u(t)+\int_{-\infty}^{t}k(t-s)u(s)ds)=(\int_{0}^{\infty}a(\tau)d\tau)(Au(t)+f(t)),\quad t\in\mathbb{R}, \tag{1.4}$$

for some $\alpha > 0$ and $k \in L^1(\mathbb{R}_+)$ nonnegative and nonincreasing. This linear integrodifferential equation was studied in [8] obtaining existence and regularity of solutions when A generates a contraction semigroup on X. Under the stronger assumption that A generates an analytic semigroup, it was studied in [7] where regularity of solutions in spaces $L^p(X), BUC(X), W^{\alpha,p}(X)$, and the space of α-Hölder continuous and bounded functions $BC^{\alpha}(X)$, for $0<\alpha<1$ and $1\leq p<\infty$, was established. These maximal regularity results were then applied to study existence and regularity of solutions for (1.1) in case of several nonlinearities $g(t,u(t))$.

We take an operator theoretical approach to solve our problem, by first defining a natural concept of "resolvent family" associated to the abstract linear equation (1.2). Then, under the assumption that A is the generator of a strongly continuous integral resolvent family (see below for the definition) and under some regularity conditions on the functions f and g, we will show that the nonlinear problem (1.1) has a unique mild solution u_ϵ on some classes of closed subspaces of $BC(X)$, for each $\epsilon > 0$ subject to certain restriction. Moreover, we will prove that u_ϵ converges uniformly to u as $\epsilon \to 0$, the unique mild solution of equation (1.2).

2. Preliminaries

We introduce some notations. We denote by $BC(X)$ the space consisting of continuous and bounded functions $f:\mathbb{R}\to X$ endowed with the norm of uniform convergence

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

We set $AP(X)$, $AA(X)$, $AA_c(X)$ and $AAA(X)$ for the closed subspaces formed by the almost periodic functions, the almost automorphic functions, the compact almost automorphic functions, and the asymptotically almost periodic functions respectively.

Almost automorphic functions were first introduced in the literature by S. Bochner [5] as a natural generalization of the classical concept of almost periodic function. In 1980s, G.M. N' Guérékata [30] defined asymptotically almost automorphic functions as perturbations of almost automorphic functions by functions vanishing at infinity. Compact almost automorphic functions were introduced by A.M. Fink [12]. All this concepts, as well as their application to the field of evolution equations in Banach spaces, has been intensively studied in recent years (see [1, 3, 4, 6, 10, 11, 13, 14, 17, 18, 21, 27, 28, 29] and [35]).

It is well known that

$$AP(X)\subset AA_c(X)\subset AA(X)\subset AAA(X)\subset BC(X).$$

We recall that the Laplace transform of a function $f\in L^1_{loc}(\mathbb{R}_+,X)$ is given by

$$\mathcal{L}(f)(\lambda):=\hat{f}(\lambda):=\int_0^\infty e^{-\lambda t}f(t)dt,\quad Re\lambda>\omega,$$

where the integral is absolutely convergent for $Re\lambda > \omega$. Furthermore, we denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators, and the notation $\rho(A)$ stands for the resolvent set of A. In order to establish an operator theoretical approach to the equations studied in this paper, we consider the following definition (cf. [25]).

Definition 2.1. *Let A be a closed linear operator with domain $D(A) \subseteq X$. We say that A is the generator of an integral resolvent if there exists $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{1/\hat{a}(\lambda) : Re\lambda > \omega\} \subseteq \rho(A)$ and*

$$(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

The concept of integral resolvent, as defined above, is closely related with the concept of resolvent family (see Prüss [32, Chapter I]). A closed but weaker definition was formulated by Prüss [32, Definition 1.6]. The book of Gripenberg, Londen and Staffans [16] contains an overview of the theory for the scalar case.

Because of the uniqueness of the Laplace transform, an integral resolvent family with $a(t) \equiv 1$ is the same as a C_0-semigroup whereas that an integral resolvent family with $a(t) = t$ corresponds to the concept of sine family, see [2, Section 3.15]. However, concerning the equations (1.1)-(1.2) of this paper, the typical kernel will be $a(t) = t^\alpha e^{-\omega t}$, where $\omega > 0$ and $\alpha \geq 0$.

We note that integral resolvent families are a particular case of (a, k)-regularized families introduced in [22]. These are studied in a series of several papers in recent years (see [23, 24, 34]). According to [22] an integral resolvent family $S(t)$ corresponds to a (a, a)-regularized family.

In a similar way as occurs for C_0-semigroups, we can establish several relations between the integral resolvent family and its generator. The following result is a direct consequence of [22, Proposition 3.1 and Lemma 2.2].

Proposition 2.2. *Let $S(t)$ be the integral resolvent family on X with generator A. Then the following properties hold:*

(a) $S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(b) Let $x \in D(A)$ and $t \geq 0$. Then

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds. \tag{2.5}$$

(c) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)S(s)xds \in D(A)$ and

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)xds.$$

In particular, $S(0) = a(0)I$.

If an operator A with domain $D(A)$ is the infinitesimal generator of an integral resolvent family $S(t)$ and $a(t)$ is a continuous, positive and nondecreasing function which satisfies $\limsup_{t \to 0^+} \frac{||S(t)||}{a(t)} < \infty$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - a(t)x}{(a * a)(t)}.$$

For instance, the case $a(t) \equiv 1$ corresponds to the generator of a C_0-semigroup and $a(t) = t$ actually corresponds to the generator of a sine family. We refer the reader to [24, Theorem 2.1] for these properties. Furthermore, a characterization of generators of integral resolvent families, analogous to the Hille-Yosida Theorem for C_0 semigroups, can be directly deduced from [22, Theorem 3.4]. Results on perturbation, approximation, representation as well as ergodic type theorems can be also deduced from the more general context of (a,k) regularized resolvents (see [23, 24] and [34]).

3. Bounded mild solutions

Consider the following two equations

$$u(t) = \int_{-\infty}^{t} a(t-s)\{Au(s)+f(s)\}\,ds \tag{3.6}$$

$$u(t) = \int_{-\infty}^{t} a(t-s)\{Au(s)+f(s)+g(s,u(s))\}\,ds, \tag{3.7}$$

Assume that A is the generator of an integral resolvent family $S(t)$ which is integrable, that is

$$||S|| := \int_0^\infty ||S(\tau)||d\tau < \infty. \tag{3.8}$$

Given $f \in BC(X)$, let $\varphi^*(t)$ be the function given by

$$\varphi^*(t) = \int_{-\infty}^{t} S(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.9}$$

Then, we have

$$||\varphi^*||_\infty \le ||S||||f||_\infty.$$

Suppose that $f(t) \in D(A)$, it then follows that $\varphi^*(t) \in D(A)$ for all $t \in \mathbb{R}$ (see e.g. [32, Proposition 1.2]). Using (2.5) and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{-\infty}^{t} a(t-s)A\varphi^*(s)ds &= \int_{-\infty}^{t} a(t-s)A\int_{-\infty}^{s} S(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{-\infty}^{s} a(t-s)AS(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{\tau}^{t} a(t-s)AS(s-\tau)f(\tau)dsd\tau \\
&= \int_{-\infty}^{t}\int_{0}^{t-\tau} a(t-\tau-p)AS(p)dpf(\tau)d\tau \\
&= \int_{-\infty}^{t} (S(t-\tau)f(\tau)-a(t-\tau)f(\tau))d\tau \\
&= \varphi^*(t) - \int_{-\infty}^{t} a(t-\tau)f(\tau)d\tau
\end{aligned}$$

which establishes that $\varphi^*(\cdot)$ is (strict) solution of equation (3.6). In general, we have only $f(t) \in X$ and hence, in what follows, we will say that $\varphi^*(t)$ defined by (3.9) is a *mild* solution of equation (3.6).

Definition 3.3. *Let A be the generator of an integral resolvent family $\{S(t)\}_{t\geq 0}$. A continuous function $u : \mathbb{R} \to X$ satisfying the integral equation*

$$u(t) = \int_{-\infty}^{t} S(t-s)[f(s) + g(s, u(s))]ds, \quad \forall t \in \mathbb{R}, \tag{3.10}$$

is called a mild solution on $\mathbb{R}$ to the equation (3.7).

We will prove the existence of a bounded mild solution for equations of the form

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)\{Au_\epsilon(s) + f(s) + \epsilon g(s, u_\epsilon(s))\}ds, \tag{3.11}$$

where ϵ is a parameter and the function $g(t, x)$ satisfy certain conditions which will be specified further.

Let us state the principal result of this section:

Theorem 3.4. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying assumption (3.8). If f is a bounded continuous function, g satisfies the Lipschitz condition*

$$\|g(t, x_1) - g(t, x_2)\| \leq L\|x_1 - x_2\|$$

for all $t \in \mathbb{R}$, $x_1, x_2 \in X$, where

$$\int_{\mathbb{R}} \|g(t, 0)\| \, dt =: M < \infty.$$

Then for all $0 < \epsilon < \frac{1}{L||S||}$ there exists a mild solution $\varphi_\epsilon \in BC(X)$ of $(3.11)_\epsilon$ such that $\varphi_\epsilon \to \varphi^*$ uniformly as $\epsilon \to 0$, where φ^* is the mild bounded solution of (3.6).

Proof. Let $0 < \epsilon < \frac{1}{L||S||}$ be fixed. On $BC(X)$ we define an operator F_ϵ by

$$(F_\epsilon\varphi)(t) = \epsilon \int_{-\infty}^{t} S(t-s)g(s, \varphi(s)) \, ds, \quad \varphi \in BC(X). \tag{3.12}$$

We note that F is well-defined. In fact:

$$\begin{aligned} \|F_\epsilon\varphi(t)\| &\leq \epsilon \int_{-\infty}^{t} \|S(t-s)\|[\|(g(s, \varphi(s)) - g(s, 0))\| + \|g(s, 0)\|] \, ds \\ &\leq \epsilon||S|| \left\{ \sup_{t\in\mathbb{R}} \|\varphi(t)\|L + \int_{-\infty}^{t} \|g(s, 0)\| \, ds \right\} \end{aligned}$$

and hence,

$$\|F_\epsilon\varphi\|_\infty = \sup_{t\in\mathbb{R}} \|F_\epsilon\varphi(t)\| \leq \epsilon||S||\{\|\varphi\|_\infty L + M\}. \tag{3.13}$$

We now prove that F_ϵ is a contraction:

$$\|F_\epsilon\varphi_1 - F_\epsilon\varphi_2\|_\infty = \sup_{t\in\mathbb{R}} \|F_\epsilon\varphi_1(t) - F_\epsilon\varphi_2(t)\| \leq \epsilon L||S||\|\varphi_1 - \varphi_2\|_\infty < \|\varphi_1 - \varphi_2\|_\infty.$$

Now, for each $\varphi \in BC(X)$, we define

$$G_\epsilon\varphi(t) := \int_{-\infty}^{t} S(t-s)f(s)ds + F_\epsilon\varphi(t). \tag{3.14}$$

Since $f \in BC(X)$ we have $G_\epsilon \varphi \in BC(X)$. Moreover G_ϵ is a contraction. Hence, by the Banach fixed point theorem, we conclude that there exists a unique bounded mild solution φ_ϵ of $(3.11)_\epsilon$.

Let φ^* be the bounded mild solution of (3.6), that is $\varphi^*(t) = \int_{-\infty}^{t} S(t-s)f(s)ds$. We now have from (3.14) and (3.13),

$$||\varphi_\epsilon - \varphi^*||_\infty = ||F_\epsilon \varphi|| \leq \epsilon ||S|| \{ \|\varphi\|_\infty L + M \},$$

and hence $||\varphi_\epsilon - \varphi^*||_\infty \to 0$ as $\epsilon \to 0$. This completes the proof. □

Remark 3.5. We observe that in the scalar case, i.e. $X = \mathbb{C}^n$, necessary and sufficient conditions for the integrability of $S(t)$ are known. See Example 4.9 below. In the case that X is a Hilbert or Banach space, the problem of integrability of resolvents has been studied by J. Prüss in [32, Chapter III].

4. MILD SOLUTIONS ON SUBSPACES OF $BC(X)$

Recall that we denote by $AP(X)$ the Banach space (with the sup-norm) of all almost periodic functions (in the Bohr's sense). Similarly, $AA_c(X)$ is the space of compact almost automorphic functions, $AA(X)$ is the space of almost automorphic functions, and $AAA(X)$ is the space of asymptotically almost automorphic functions.

The following is our main result on regularity under convolution of the above mentioned spaces. It corresponds to a summary, and in some cases a slight extension and improvement, of recent results given by a number of authors (cf. [1, 10, 17, 27, 28] and [29]).

Theorem 4.6. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded linear operators that satisfies assumption (3.8). If f belongs to one of the spaces $AP(X), AA_c(X), AA(X)$ or $AAA(X)$, and $w(t)$ is given by*

$$w(t) = \int_{-\infty}^{t} S(t-s)f(s)\, ds$$

then w belongs to the same space as f.

Proof. We first consider almost periodic functions. Almost periodicity of f means that for each $\epsilon > 0$ there exists a $T > 0$ such that every subinterval of $\mathbb{R}$ of length T contains at least one point h such that $\sup_{t\in\mathbb{R}} ||f(t+h) - f(t)|| \leq \epsilon$. Now

$$\begin{aligned} \sup_{t\in\mathbb{R}} ||w(t+h) - w(t)|| &= \sup_{t\in\mathbb{R}} || \int_{-\infty}^{t} S(t-s)[f(s+h) - f(s)]ds|| \\ &\leq ||S|| \sup_{t\in\mathbb{R}} ||f(t+h) - f(t)|| \leq \epsilon ||S||, \end{aligned}$$

and therefore, w has the same property as f, i.e., it is almost periodic.

The compact almost automorphic case: Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. $f \in AA_c(X)$ means, by definition, that there exist a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous function $v \in BC(X)$ such that $f(t+s_n)$ converges to $v(t)$ and $v(t-s_n)$ converges to $f(t)$ uniformly on compact subsets of $\mathbb{R}$.

Since

$$w(t+s_n) = \int_{-\infty}^{t+s_n} S(t+s_n-s)f(s)ds = \int_{-\infty}^{t} S(t-s)f(s+s_n)ds, \tag{4.15}$$

using the Lebesgue's dominated convergence theorem, we obtain that $w(t+s_n)$ converges to $z(t) = \int_{-\infty}^{t} S(t-s)v(s)ds$ as $n \to \infty$ for each $t \in \mathbb{R}$.

Furthermore, the preceding convergence is uniform on compact subsets of $\mathbb{R}$. To show this assertion, we take a compact set $K = [-a, a]$. For $\varepsilon > 0$, we choose $L_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} \int_{L_\varepsilon}^{\infty} ||S(s)||ds &\leq \varepsilon, \\ \|f(s+s_n) - v(s)\| &\leq \varepsilon, \;\; n \geq N_\varepsilon, \;\; s \in [-L, L], \end{aligned}$$

where $L = L_\varepsilon + a$. For $t \in K$, we now can estimate

$$\begin{aligned} \|w(t+s_n) - z(t)\| &\leq \int_{-\infty}^{t} ||S(t-s)||\|f(s+s_n) - v(s)\|ds \\ &\leq \int_{-\infty}^{-L} ||S(t-s)||\|f(s+s_n) - v(s)\|ds \\ &\quad + \int_{-L}^{t} ||S(t-s)||\|f(s+s_n) - v(s)\|ds \\ &\leq 2\|f\|_\infty \int_{t+L}^{\infty} ||S(s)||ds + \varepsilon \int_{0}^{\infty} ||S(s)||ds \\ &\leq \varepsilon \left(2\|f\|_\infty + ||S||\right), \end{aligned}$$

which proves that the convergence is independent of $t \in K$. Repeating this argument, one can show that $z(t - s_n)$ converges to $w(t)$ as $n \to \infty$ uniformly for t in compact subsets of $\mathbb{R}$. This completes the proof in case of the space $AA_c(X)$.

We now consider the space of almost automorphic functions. Let $(s'_n) \subset \mathbb{R}$ be an arbitrary sequence. Since $f \in AA(X)$ there exists a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} f(t+s_n) = v(t), \;\; \text{for all } t \in \mathbb{R}$$

and

$$\lim_{n\to\infty} v(t-s_n) = f(t), \;\; \text{for all } t \in \mathbb{R}.$$

From (4.15), note that

$$\|w(t+s_n)\| \leq ||S||\|f\|_\infty$$

and by continuity of $S(\cdot)x$ we have $S(t-\sigma)f(\sigma+s_n) \to S(t-\sigma)g(\sigma)$, as $n \to \infty$ for each $\sigma \in \mathbb{R}$ fixed and any $t \geq \sigma$. Then by the Lebesgue's dominated convergence theorem, we obtain that $w(t+s_n)$ converges to $z(t) = \int_{-\infty}^{t} S(t-s)v(s)ds$ as $n \to \infty$ for each $t \in \mathbb{R}$. In similar way we can show that

$$z(t-s_n) \to w(t) \text{ as } n \to \infty, \;\; \text{for all } t \in \mathbb{R},$$

and the proof of this case is complete.

Finally we consider the case of asymptotically almost automorphic functions. Let $f \in AAA(\mathbb{R}_+, X)$ be given. Then $f = g\,|_{\mathbb{R}_+} + h$; with $g \in AA(\mathbb{R}, X)$ and $||h(t)|| \to 0$ as $t \to \infty$.

We write $w(t)$ as

$$w(t) = \int_{-\infty}^{t} S(t-s)g(s)\,ds + \int_{-\infty}^{t} S(t-s)h(s)\,ds,$$

and define

$$w_1(t) := \int_{-\infty}^{t} S(t-s)g(s)\,ds, \quad w_2(t) := \int_{-\infty}^{t} S(t-s)h(s)\,ds.$$

Since $g \in AA(\mathbb{R}, X)$, we known that $w_1 \in AA(X)$. Let $\varepsilon > 0$, then there exist $T > 0$ such that $\|h(s)\| < \varepsilon$ for all $s > T$ and hence we can write

$$w_2(t) = \int_{-\infty}^{T} S(t-s)h(s)\,ds + \int_{T}^{t} S(t-s)h(s)\,ds.$$

Then

$$\begin{aligned} \|w_2(t)\| &\leq \int_{-\infty}^{T} \|S(t-s)\|\|h(s)\|\,ds + \int_{T}^{t} \|S(t-s)\|\varepsilon\,ds \\ &\leq \|h\|_\infty \int_{t-T}^{\infty} ||S(v)||dv + ||S||\varepsilon. \end{aligned}$$

Hence $||w_2(t)|| \to 0$ as $t \to \infty$. This completes the proof of the Theorem.

□

In what follows, we let $\mathcal{M}(\mathbb{R} \times X; X)$ stand for the space of functions $f : \mathbb{R} \times X \to X$ such that $f(\cdot, x) \in \mathcal{M}(X)$ uniformly for each $x \in K$, where K is any bounded subset of X. The following is the main result of this section.

Theorem 4.7. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying assumption (3.8). Let $\mathcal{M}$ be the space of almost periodic, compact almost automorphic or almost automorphic functions. If $f \in \mathcal{M}(X)$, $g \in \mathcal{M}(\mathbb{R} \times X; X)$ and satisfies the Lipschitz condition*

$$\|g(t, x_1) - g(t, x_2)\| \leq L\|x_1 - x_2\|$$

for all $t \in \mathbb{R}$, $x_1, x_2 \in X$, where

$$\int_{\mathbb{R}} \|g(t, 0)\|\,dt =: M < \infty.$$

Then $\varphi^* \in \mathcal{M}(X)$ and for all $0 < \epsilon < \frac{1}{L||S||}$ there exists a mild solution $\varphi_\epsilon \in \mathcal{M}(X)$ of $(3.11)_\epsilon$ such that $\varphi_\epsilon \to \varphi^*$ uniformly as $\epsilon \to 0$.

Proof. On $\mathcal{M}(X)$ we define the functions F_ϵ and G_ϵ as in (3.12) and (3.14) respectively. Since $f \in \mathcal{M}(X)$ we have by Theorem 4.6 that $\displaystyle\int_{-\infty}^{t} S(t-s)f(s)ds \in \mathcal{M}(X)$. By the composition rule (cf. [28, Theorem 1.69] for almost periodic functions, [20, Lemma 2.2] in case of $AA(X)$ and [10, Lemma 2.1] in case of $AA_c(X)$), we have $G_\epsilon\varphi \in \mathcal{M}(X)$ for all $\varphi \in \mathcal{M}(X)$. We conclude that $\mathcal{M}(X)$ is invariant under F_ϵ and the proof then follows as in Theorem 3.4.

□

Finally, we will consider asymptotically almost periodic functions. This case have a slight different assumption as the previous one, due essentially to the rule of composition (cf. [20, Theorem 2.3]).

We denote $C_0(\mathbb{R} \times X, X)$ the set of all bounded continuous functions $f : \mathbb{R} \times X \to X$ such that $\lim_{t\to\infty} ||f(t,x)|| = 0$ uniformly on any bounded subset of X. We recall that $g \in AAA(\mathbb{R}\times X; X)$ if and only if $g = h+\varphi$ where $h \in AA(\mathbb{R}\times X; X)$ and $\varphi \in C_0(\mathbb{R}\times X, X)$.

Theorem 4.8. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ satisfying assumption (3.8). If $f \in AAA(X)$, $g = h+\varphi \in AAA(\mathbb{R}\times X; X)$ and h satisfies the Lipschitz condition*

$$\|h(t,x_1) - h(t,x_2)\| \leq L\|x_1 - x_2\|$$

for all $t \in \mathbb{R}$, $x_1, x_2 \in X$, where

$$\int_{\mathbb{R}} \|h(t,0)\|\, dt =: M < \infty.$$

Then the same conclusion of Theorem 4.7 holds.

Example 4.9. Let $\mathcal{M}$ be the space of almost periodic, compact almost automorphic or almost automorphic functions. Let $b \in L^1(\mathbb{R})$ be bounded, and $h \in \mathcal{M}(X)$ satisfying a Lipchitz condition:

$$||h(x) - h(y)|| < L||x - y||, \quad x, y \in X.$$

Using the preceding theorem, with $g(t,x) = b(t)h(x)$, we can assert that for all $f \in \mathcal{M}(X)$, the equation

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)\{Au_\epsilon(s) + f(s) + \epsilon b(s)h(u_\epsilon(s))\}ds, \tag{4.16}$$

where $a \in L^1(\mathbb{R}_+)$ and A is the generator of an integral resolvent family satisfying assumption (3.8), has a unique mild solution in $\mathcal{M}(X)$, for sufficiently small $\epsilon > 0$.

In particular, if we set $A \equiv 0$, we have that the integral resolvent family is given by $S(t)x = a(t)x$ and therefore, the equation

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)\{f(s) + \epsilon b(s)h(u_\epsilon(s))\}ds, \tag{4.17}$$

has unique mild solution in $\mathcal{M}(X)$, for sufficiently small $\epsilon > 0$, whenever $f \in \mathcal{M}(X)$.

We now consider some examples in the scalar case:

We set $X = \mathbb{C}^n$, $A = \rho I$, $\rho \in \mathbb{C}$, and $a \in L^1(\mathbb{R}_+)$. Suppose that

$$\rho\hat{a}(\lambda) \neq 1 \text{ for all } Re\lambda \geq 0, \tag{4.18}$$

where the hat $\hat{}$ denotes the Laplace transform of $a(t)$. By the half-line Paley- Wiener Theorem (see [16, p.45]) we get that A generates an integral resolvent $S_\rho \in L^1(\mathbb{R}_+, \mathbb{C}^{n\times n})$, so that assumption (3.8) is satisfied. We conclude that for all

$$0 < \epsilon < \frac{1}{L||S_\rho||} \tag{4.19}$$

the equation

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)\{\rho u_\epsilon(s) + f(s) + \epsilon b(s)h(u_\epsilon(s))\}ds, \tag{4.20}$$

have a unique mild solution that belong to $\mathcal{M}(X)$ whenever $f \in \mathcal{M}(X)$.

As a concrete example, we can take $a(t) = t^\alpha e^{-\omega t}$, $\omega > 0, \alpha \geq 0$. Note that in case $\alpha = 0$ we have

$$S_0(t) = e^{(\omega+\rho)t}, \quad t \geq 0$$

and, in case $\alpha = 1$, we have for $\rho > 0$:

$$S_1(t) = \frac{1}{\sqrt{\rho}} e^{\omega t} \sinh(\sqrt{\rho} t), \quad t \geq 0.$$

In the particular case $\alpha = 0$, we observe that the condition (4.18) is equivalent to say

$$Re(\rho) < -\omega,$$

and hence, clearly $S_0 \in L^1(\mathbb{R}_+, \mathbb{C}^{n\times n})$. More precisely, we have $||S_0|| = \frac{-1}{\omega + Re(\rho)}$. Using (4.19) we conclude that for all ρ such that $Re(\rho) < 0$, and *any* $\epsilon > 0$, there exists a unique mild solution of the equation

$$u_\epsilon(t) = \int_{-\infty}^{t} a(t-s)\{\rho u_\epsilon(s) + f(s) + \epsilon b(s)h(u_\epsilon(s))\}ds, \tag{4.21}$$

Moreover, this solution belongs to $\mathcal{M}(X)$ for every $f \in \mathcal{M}(X)$.

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