WELL-POSEDNESS OF SECOND ORDER EVOLUTION EQUATION ON DISCRETE TIME

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ABSTRACT. We characterize the well-posedness for second order discrete evolution equations in UMD spaces by means of Fourier multipliers and R-boundedness properties of the resolvent operator which defines the equation. Applications to semilinear problems are given.

1. INTRODUCTION

Let X be a Banach space and let A be a bounded linear operator. Our main objective of this paper is to characterize the well-posedness in weighted spaces $l_p^r(\mathbb{Z}_+; X) := \{(x_n) : (r^{-n}x_n) \in l_p(\mathbb{Z}_+; X)\}$ $(r > 0)$ for the following discrete second order evolution equation:

$$\Delta^2 u_n + Au_n = f_n, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

with zero initial conditions and $f \in l_p^r(\mathbb{Z}_+; X)$.

Beside its theoretical interest, the study of abstract discrete evolution equation together with well-posedness has great importance and there is much interest in developing the qualitative theory for such equations. Indeed, this has been strongly motivated by their natural and widespread applicability in several fields of sciences and technology, see e.g. [1, 25, 35].

For example S. Blunck considered, in [12, 13], maximal regularity of first order evolution equations on discrete time; see also Portal [37, 38], where he discussed discrete analytic semigroup and maximal regularity on discrete time scales, respectively. In [29], maximal regularity on discrete Hölder spaces for finite difference operators subject to Dirichlet boundary conditions in one and two dimensions is proved. Furthermore, the authors investigated maximal regularity in discrete Hölder spaces for the Crank- Nicolson scheme. In [7], well-posedness of difference schemes for abstract elliptic problems in L^p spaces was studied. In [26], maximal regularity for linear parabolic difference equations is treated while in [21, 22] the authors has made a perturbation theory for semilinear evolution equations on discrete time for first and second order using discrete maximal regularity; see also the recent paper by Kalton and Portal [30], where they discussed maximal regularity of power-bounded operators and relate the discrete to the continuous time problem for

analytic semigroups. Recently, discrete maximal regularity for functional difference equations with infinite delay was considered in [20]. There, applications to Volterra difference equations with infinite delay are also shown.

A motivation for our studies in this paper stems in the recent article by Arendt and Bu [4] and Cuevas and Lizama [19]. In the first one the authors consider the operator-valued Marcinkiewich multiplier theorem and maximal regularity, and the second one the authors shown a characterization for the maximal regularity for a second order difference equation by R-boundedness properties of the resolvent operator which defines the equation.

We remark that the concept of R-boundedness was implicity introduced by Bourgain in [14] and later on also by Zimmermann [41]. Explicitly it is due to Berkson and Gillespie [11] and Clément et al. [17]. It plays a fundamental role in recent works by Clément-Da Prato [16], Weis [39, 40], Arendt-Bu [4, 5], Keyantuo-Lizama [31, 32, 33] and Ashyralyev et al. [7, 8].

One of the most important tools to prove well-posedness are Fourier multiplier theorems. They plays a key role in the analysis of elliptic and parabolic problems. In recent years it has become apparent that one need not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. In particular, in [19] the authors used a vector-valued Fourier multiplier theorem due to Blunck (see [12], Theorem 1.3), to study maximal regularity of second order difference equations. These approach will be also our method in this paper.

For an overview of the organization of the work. The second section provides the definitions and preliminary results to be used. In particular to facilitate a comprehensive understanding to the reader we have supplied several basic R-boundedness properties which are a natural tool in our setting. We also introduce the concept of l_p-multiplier by using Z-transform. In the third section, we show how R-boundedness and l_p-multipliers can be used to obtain a characterization about well-posedness for equation (1.1) in UMD Banach spaces (see Theorem 3.4). By using our characterization and the Implicit Function Theorem, we obtain as an application the existence of solutions to the discrete semilinear evolution equation

$$(1.2) \qquad \Delta_r^2 x - r^2(I - T)x = G(x) + \rho f,$$

where G is a Frechét differentiable function satisfying suitable assumptions and $\rho > 0$ is a small parameter (see Theorem 3.7 and Corollary 3.8).

2. Preliminaries

Let X be a Banach space. Let $\mathbb{Z}_+$ denote the set of positive integer numbers, the Δ_r denotes the $r-$difference operator of the first order, i.e. for each $x : \mathbb{Z}_+ \to X$, and $n \in \mathbb{Z}_+$, $\Delta_r x_n = x_{n+1} - r x_n$, $r \in \mathbb{R}_+$. In case $r = 1$ we denote $\Delta \equiv \Delta_1$. Moreover, we denote $\Delta_r^2 x_n = \Delta_r(\Delta_r x_n)$. If we set $(\tau_r x)(n) := r^{-n} x(n)$, then is easy to check that the following identity holds:

$$(2.1) \qquad \Delta_r^2 = r^2 \, \tau_{r^{-1}} \circ \Delta^2 \circ \tau_r.$$

It shows that well-posedness of equation (1.1) in the weighted spaces $l_p^r(\mathbb{Z}_+; X)$ is equivalent to the study of the discrete time evolution equation

$$(2.2) \qquad \Delta_r^2 x_n - r^2(I - T)x_n = f_n, \quad \text{for all } n \in \mathbb{Z}_+, \quad x_0 = 0, \ x_1 = 0.$$

in the usual spaces $l_p(\mathbb{Z}_+; X)$, where $T := I - A \in \mathcal{B}(X)$ and $f : \mathbb{Z}_+ \to X$.

Denote $\mathcal{C}(0) = I$, the identity operator on X, and define

$$(2.3) \qquad \mathcal{C}(n) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (I - T)^k, \quad \text{for } n = 1, 2, \ldots$$

We define also $\mathcal{S}(0) = 0$,

$$(2.4) \qquad \mathcal{S}(n) = \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} (I - T)^k,$$

for $n = 1, 2, \ldots$ The sequences of linear and bounded operators $\mathcal{C}(n)$ and $\mathcal{S}(n)$ were introduced in [19] to represent the solution of (2.2) in the border case $r = 1$. The following result generalize [19, Proposition 2.2].

Proposition 2.1. *Let $T \in \mathcal{B}(X)$ be given, then the (unique) solution of equation (2.2) is given by*

$$(2.5) \qquad x_{m+1} = (r^{\bullet - 1}\mathcal{S} * f)_m.$$

Moreover,

$$(2.6) \qquad \Delta_r x_{m+1} = (r^{\bullet}\mathcal{C} * f)_m.$$

Proof. Let x_n be the solution of equation (2.2) and define $v_n := [x_n, \Delta_r x_n]$, $F_n := [0, f_n]$ and the operator $R_{T,r} \in \mathcal{B}(X \times X)$ by

$$R_{T,r}[x, y] = [rx + y, \ r^2(x - Tx) + ry].$$

Then, we can infer that the equation (2.2) is equivalent to:

$$(2.7) \qquad v_{n+1} - R_{T,r}v_n = F_n, \quad v_0 = [0, 0],$$

which has the solution

$$(2.8) \qquad v_{m+1} = \sum_{n=0}^{m} R_{T,r}^n F_{m-n}.$$

Denote

$$R_{T,r} = \begin{bmatrix} rI & I \\ r^2(I - T) & rI \end{bmatrix}.$$

Then a calculation shows us that

$$R_{T,r}^n = r^n \begin{bmatrix} \mathcal{C}(n) & \frac{1}{r}\mathcal{S}(n) \\ r(I - T)\mathcal{S}(n) & \mathcal{C}(n) \end{bmatrix}.$$

The result is now a consequence of formula (2.8). The uniqueness follows from induction and then the proof is finished.

∎

Definition 2.2. *Let $1 < p < +\infty$. One says that equation (2.2) has discrete maximal regularity if $\mathcal{K}^r f := (I - T)r^{\bullet+1}\mathcal{S} * f$ defines a linear bounded operator $\mathcal{K}^r \in \mathcal{B}(l_p(\mathbb{Z}_+, X))$.*

Note that, in particular, the definition implies that for all $(f_n) \in l_p(\mathbb{Z}_+, X)$ we have $(\Delta_r^2 x_n) \in l_p(\mathbb{Z}_+, X)$ in equation (2.2). This is the main property that allows the use of maximal regularity in the treatment of non-linear problems.

We introduce the means

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \|\sum_{j=1}^n \epsilon_j x_j\|,$$

for $x_1, ..., x_n \in X$.

Let X and Y be Banach spaces, $\mathcal{B}(X, Y)$ be the space of bounded linear operators from X to Y.

Definition 2.3. *A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called R-bounded if there exists a constant $c \geq 0$ such that*

$$(2.9) \qquad \|(T_1 x_1, ..., T_n x_n)\|_R \leq c\|(x_1, ..., x_n)\|_R,$$

for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$. The least c such that (2.9) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

The notion of $R-$boundedness has proved to be a significant tool in the study of abstract multiplier operators. An equivalent definition using the Rademacher functions can be found in [23]. We note that R-boundedness clearly implies boundedness. If $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space, see [4, Proposition 1.17]. Some useful criteria for $R-$boundedness are provided in [4, 23] and [28].

Remark 2.4. a) Any finite family $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded.

b) If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded then it is uniformly bounded, with $\sup\{\|T\| : T \in \mathcal{T}\} \leq R_p(\mathcal{T})$.

c) The definition of R-boundedness is independent of $p \in [1, \infty)$.

d) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

e) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$ be R-bounded sets, then $\mathcal{S} \pm \mathcal{T} := \{S \pm T : S \in \mathcal{S}, T \in \mathcal{T}\}$ are R- bounded.

f) Let $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X, Z)$ is R-bounded and $R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T})$.

g) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded, whenever $\Omega \subset \mathbb{C}$ is bounded (see Example 3.2 in [12]).

Definition 2.5. *A Banach space X is said to have the unconditional martingale difference property (UMD) if for each $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for any martingale $(f_n)_{n \geq 0} \subset L^p(\Omega, \Sigma, \mu; X)$ and any choice of signs $(\xi_n)_{n \geq 0} \subset \{-1, 1\}$ and any $N \in \mathbb{Z}_+$ the following estimate holds*

$$\|f_0 + \sum_{n=1}^N \xi_n(f_n - f_{n-1})\|_{L^p(\Omega, \Sigma, \mu; X)} \leq C_p\|f_N\|_{L^p(\Omega, \Sigma, \mu; X)}.$$

Remark 2.6. A Banach space X is said to be $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here, the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [14] and Burkholder [15]. For more information and details on the Hilbert transform and the UMD Banach spaces we refer to Amann's book [3, Section III.4.3-III.4.5]. The UMD spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega), 1 < p < \infty$ (see [4]), Lebesgue spaces $L^p(\Omega, \mu), L^p(\Omega, \mu; X)$, $1 < p < \infty$, where X is a UMD space and the Schatten-von Neumann classes $C_p(H)$, $1 < p < \infty$, of operators on Hilbert spaces. On the other hand, we can observe that:
a) Every closed subspace of a UMD space is a UMD space.
b) Every UMD space is reflexive.
c) A Banach space X is UMD if and only if its dual X^* is UMD.

We recall now the following operator-valued multiplier theorem on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ due to Blunck, see [12, Theorem 1.3].

Theorem 2.7. *Let $p \in (1, \infty)$ and X be a UMD space. Let $M : (-\pi, \pi) \setminus \{0\} \to \mathcal{B}(X)$ be a differentiable function such that the set*

$$\{M(t), (e^{it} - 1)(e^{it} + 1)M'(t) : t \in (-\pi, \pi) \setminus \{0\}\}$$

is R-bounded. Then there is an operator $T_M \in \mathcal{B}(l_p(\mathbb{Z}; X))$ such that

$$\mathcal{F}(T_M f)(e^{it}) = M(t)\mathcal{F}f(e^{it}), \quad t \in (-\pi, \pi) \setminus \{0\}, \ \mathcal{F}f \in L_\infty(\mathbb{T}; X)$$

of compact support.

Definition 2.8. *An operator $S \in \mathcal{B}(X)$ is called analytic if the set $\{n(S - I)S^n : n \in \mathbb{N}\}$ is bounded.*

This notion is a discrete analogue of the property "$\{tA \exp(tA) : t > 0\}$ is bounded", which characterizes the analyticity of the bounded semigroup $(\exp(tA))_{t \geq 0}$. For recent and related results on analytic operators we refer the reader to [24]. In what follows, we denote $\mathbb{D}(z, a) = \{w \in \mathbb{C} : |w - z| < a\}$.

Proposition 2.9. *Let $T \in \mathcal{B}(X)$ be an analytic operator. Then $\sigma(r^2(I - T)) \subseteq \mathbb{D}(r^2, r^2) \cup \{0\}$. In particular $(z - r)^2 \in \rho(r^2(I - T))$ whenever $|z| = \alpha r$, $\alpha = 1 + \sqrt{2}$, $z \neq \alpha r$.*

Proof. Let $z \notin \mathbb{D}(r^2, r^2) \cup \{0\}$, then $\frac{z}{r^2} \notin \mathbb{D}(1, 1) \cup \{0\}$. By Lemma 2.10 Ref. [19], we get $\frac{z}{r^2} \in \rho(I - T)$, that is, $z \in \rho(r^2(I - T))$. Hence $\sigma(r^2(I - T)) \subseteq \mathbb{D}(r^2, r^2) \cup \{0\}$. For the last assertion, we note that $|(z - r)^2 - r^2| = |z||z - 2r| \geq \alpha(\alpha - 2)r^2 = (\sqrt{2} + 1)(\sqrt{2} - 1)r^2 = r^2$. ∎

For the rest of the article, we will always assume that

$$(2.10) \qquad \alpha = 1 + \sqrt{2}, \ r \geq r_0, \ 1/(1 + \sqrt{2}) < r_0 < 1.$$

We recall that the Z-transform on $\mathbb{T}_r^\alpha := \{z \in \mathbb{C} : |z| = \alpha r\}$ is defined as

$$\mathcal{F}f(z) = \hat{f}(z) = \sum_{j=0}^\infty z^{-j} f(j), \quad z \in \mathbb{T}_r^\alpha.$$

It follows from Hölder inequality that the Z-transform, $\hat{f}(z)$, is well defined for all $f \in l_p(\mathbb{Z}_+; X)$. Moreover, we can relate the Fourier transform of f on $\mathbb{T}_r^\alpha$ with the Fourier transform of $(\alpha r)^{-\bullet} f$ on $\mathbb{T}_1^1$ by the formula

$$\mathcal{F}[f](\alpha r e^{it}) = \mathcal{F}[(\alpha r)^{-\bullet} f](e^{it}).$$

This observation will be very useful in the third section in connection with Theorem 2.7. The preceding proposition enables us to prove the following properties on the Fourier transform of the solution of (2.2).

Proposition 2.10. *Let $T \in \mathcal{B}(X)$ be an analytic operator. Then*

(2.11) $$\mathcal{F}[r^{\bullet-1}\mathcal{S}](z) = zR((z-r)^2, r^2(I-T)), \quad z \in \mathbb{T}_r^\alpha \backslash \{\alpha r\},$$

and

(2.12) $$\mathcal{F}[r^\bullet \mathcal{C}](z) = z(z-r)R((z-r)^2, r^2(I-T)), \quad z \in \mathbb{T}_r^\alpha \backslash \{\alpha r\}.$$

Proof.   Given $x \in X$, we define

$$f_n = \begin{cases} x, & \text{for } n = 0, \\ \\ 0, & \text{for } n \neq 0. \end{cases}$$

A direct calculation shows that $\hat{f}(z) = x$. We consider the following evolution problem

(2.13) $$\Delta_r^2 x_n - r^2(I-T)x_n = f_n \text{ for all } n \in \mathbb{Z}_+, \quad x_0 = x_1 = 0.$$

By Proposition 2.1 the (unique) solution is given by $x_{n+1} = (r^{\bullet-1}\mathcal{S} * f)(n)$. Then $z\hat{x}(z) = \mathcal{F}[r^{\bullet-1}\mathcal{S}](z)x$, $z \in \mathbb{T}_r^\alpha$. On the other hand, we note that a direct calculation gives $\widehat{\Delta_r x}(z) = (z-r)\hat{x}(z)$, for $z \in \mathbb{T}_r^\alpha$. Hence, applying Z-transform in (2.13) and then multiplying the result by z, we obtain $zx = ((z-r)^2 - r^2(I-T))\mathcal{F}[r^{\bullet-1}\mathcal{S}](z)x$, $z \in \mathbb{T}_r^\alpha$, obtaining the first assertion. To prove the second one, we note that by Proposition 2.1 $\Delta_r x_{n+1} = (r^\bullet \mathcal{C} * f)(n)$ and then $z(z-r)\hat{x}(z) = \mathcal{F}[r^\bullet \mathcal{C}](z)x$, $z \in \mathbb{T}_r^\alpha$. Therefore, applying Z-transform in (2.13) and then multiplying the result by $z(z-r)$, we get the second assertion and the proof is finished.

∎

Next, we consider the following sequence spaces; for $r \geq r_0$

$$l_{p,r}^1(\mathbb{Z}_+; X) := \{y = (y_n)/y_0 = 0, (\Delta_r y_n) \in l_p(\mathbb{Z}_+; X)\},$$

$$l_{p,r}^2(\mathbb{Z}_+; X) := \{y = (y_n)/y_0 = y_1 = 0, (\Delta_r^2 y_n) \in l_p(\mathbb{Z}_+; X)\},$$

$$l_{p,I-T}(\mathbb{Z}_+; X) := \{y = (y_n)/((I-T)y_n) \in l_p(\mathbb{Z}_+; X)\}.$$

Proposition 2.11. *Assume that (2.10) is fulfilled. For each $y \in l_{p,r}^i(\mathbb{Z}_+; X)$, $i = 1, 2$, the Z-transform of y is well defined in $\mathbb{T}_r^\alpha$.*

Proof. For $y \in l_{p,r}^1(\mathbb{Z}_+; X)$, we have

$$
\begin{aligned}
||\hat{y}(z)||_X &= \sum_{j=0}^{\infty} |z|^{-j} ||y(j)||_X \\
&= \sum_{j=0}^{\infty} (\frac{1}{\alpha r})^j || \sum_{i=0}^{j-1} r^i \Delta_r y_{j-1-i} ||_X \\
&\leq \sum_{j=0}^{\infty} (\frac{1}{\alpha r})^j \sum_{i=0}^{j-1} r^i ||\Delta_r y_{j-1-i}||_X \\
&\leq C(\alpha, r) ||\Delta_r y_\bullet||_\infty,
\end{aligned}
$$

(2.14)

where $C(\alpha, r)$ is a constant depending on α and r. On the other hand, for $y \in l_{p,r}^2(\mathbb{Z}_+; X)$, we can infer that

$$
\begin{aligned}
||\hat{y}(z)||_X &\leq \sum_{j=0}^{\infty} (\frac{1}{\alpha r})^j \sum_{i=0}^{j-1} r^i \sum_{k=0}^{j-i-2} r^k ||\Delta_r^2 y_{j-i-2-k}||_X \\
&\leq \sum_{j=0}^{\infty} (\frac{1}{\alpha r})^j \sum_{i=0}^{j-1} r^i \sum_{k=0}^{j-i-2} r^k ||\Delta_r^2 y_\bullet||_\infty \\
&\leq C(\alpha, r) ||\Delta_r^2 y_\bullet||_\infty,
\end{aligned}
$$

(2.15)

and the proof is finished. ∎

In the following definition we denote $l_{p,r}^0(\mathbb{Z}_+; X) := l_p(\mathbb{Z}_+; X)$.

Definition 2.12. *Assume that (2.10) is fulfilled. We say that $\{Q(z)\}_{z \in \mathbb{T}_r^\alpha}$ is a $l_p - l_{p,r}^i -$multiplier, $i = 0, 1, 2$, if for each $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $y = (y_n) \in l_{p,r}^i(\mathbb{Z}_+; X)$ such that $\hat{y}(z) = Q(z)\hat{f}(z)$, $z \in \mathbb{T}_r^\alpha$.*

We finish this section with the following proposition.

Proposition 2.13. *Assume that (2.10) is fulfilled. The following assertions are equivalent.*
(i) $\{Q(z)\}_{z \in \mathbb{T}_r^\alpha}$ is a $l_p - l_{p,r}^i -$multiplier, $i = 1, 2$.
(ii) $\{(z - r)^i Q(z)\}_{z \in \mathbb{T}_r^\alpha}$ is a $l_p - l_p -$multiplier, $i = 1, 2$.

Proof. Initially we consider the case $i = 1$. To prove $(i) \Rightarrow (ii)$, for $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $y = (y_n) \in l_{p,r}^1(\mathbb{Z}_+; X)$ such that $\hat{y}(z) = Q(z)\hat{f}(z)$, $z \in \mathbb{T}_r^\alpha$. Putting $x_0 = 0$, and $x_n = \Delta_r y_n$, $n \geq 1$, we have $x = (x_n) \in l_p(\mathbb{Z}_+; X)$ and $\hat{x}(z) = \widehat{\Delta_r y}(z) = (z - r)Q(z)\hat{f}(z)$. To prove $(ii) \Rightarrow (i)$, for $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $y = (y_n) \in l_p(\mathbb{Z}_+; X)$ such that $\hat{y}(z) = (z - r)Q(z)\hat{f}(z)$. Let $x = (x_n) \in l_{p,r}^1(\mathbb{Z}_+; X)$ be a

sequence such that

$$
\begin{cases}
\Delta_r x_n = y_n, \\[2mm]
x_0 = 0.
\end{cases}
$$

Indeed, we observe that $x_n = \sum_{i=0}^{n-1} r^i y_{n-1-i}$. Therefore,

$$
(z-r)\hat{x}(z) = \widehat{\Delta_r x}(z) = (z-r)Q(z)\hat{f}(z).
$$

Hence, $Q(z)$ is a $l_p - l_{p,r}^1$−multiplier. Finally, we treat the case $i = 2$. To prove $(i) \Rightarrow (ii)$, for $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $y = (y_n) \in l_{p,r}^2(\mathbb{Z}_+; X)$ such that $\hat{y}(z) = Q(z)\hat{f}(z)$, $z \in \mathbb{T}_r^\alpha$. Putting $x_n = \Delta_r^2 y_n \in l_p(\mathbb{Z}_+; X)$, we get

$$
\hat{x}(z) = (z-r)\widehat{\Delta_r y}(z) - z(y_1 - ry_0) = (z-r)^2 Q(z)\hat{f}(z).
$$

Hence $(z-r)^2 Q(z)$ is a $l_p - l_p$−multiplier. To prove $(ii) \Rightarrow (i)$, for $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $y = (y_n) \in l_p(\mathbb{Z}_+; X)$ such that $\hat{y}(z) = (z-r)^2 Q(z)\hat{f}(z)$, $z \in \mathbb{T}_r^\alpha$. Let $x = (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X)$ be a sequence such that

$$
\begin{cases}
\Delta_r^2 x_n = y_n, \\[2mm]
x_0 = x_1 = 0.
\end{cases}
$$

Indeed, we observe that $x_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-2} r^{i+j} y_{n-2-i-j}$. Therefore,

$$
(z-r)^2 \hat{x}(z) = \widehat{\Delta_r^2 x}(z) = (z-r)^2 Q(z)\hat{f}(z).
$$

Hence, $Q(z)$ is a $l_p - l_{p,r}^2$−multiplier. This completes the proof of the proposition. ∎

3. A Characterization of Maximal Regularity

Having presented in the previous sections preliminary material on R−boundedness and Fourier multipliers, we will now show how these tools can be used to handle well-posedness of equation (2.2). The following is a natural extension of the concept of well-posedness to the continuous to the discrete case.

Definition 3.1. *One says that problem (2.2) is well-posed if for each $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is an unique solution $x = (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$ of equation (2.2).*

We observe that the space $l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$ becomes a Banach space under the norm

$$
(3.1) \qquad\qquad |||x||| := ||\Delta_r^2 x||_p + ||r^2(I-T)x||_p.
$$

The following result characterizes well-posedness of (2.2) in terms of l_p-multipliers.

Proposition 3.2. *Let X be a Banach space and let $T \in \mathcal{B}(X)$ be an analytic operator. Assume that (2.10) is fulfilled. Then, the following assertions are equivalent.*
(i) Problem (2.2) is well-posed.
(ii) $\{M(z) := (z-r)^2((z-r)^2 - r^2(I-T))^{-1} : z \in \mathbb{T}_r^\alpha, z \neq \alpha r\}$ is a $l_p - l_p$−multiplier.

Proof. $(i) \Rightarrow (ii)$, applying Z-transform to equation (2.2), we get

$$(z - r)^2 \hat{x}(z) - r^2(I - T)\hat{x}(z) = \hat{f}(z), \quad z \in \mathbb{T}_r^\alpha.$$

By Proposition 2.9, we have $\hat{x}(z) = R((z-r)^2, r^2(I-T))\hat{f}(z), z \in \mathbb{T}_r^\alpha, z \neq \alpha r$; consequently $R((z-r)^2, r^2(I-T))$ is a $l_p - l_{p,r}^2$−multiplier. By Proposition 2.13, we conclude the proof of (ii).

$(ii) \Rightarrow (i)$, by Proposition 2.13, $R((z-r)^2, r^2(I-T))$ is a $l_p - l_{p,r}^2$−multiplier. Hence, for all $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ there is a sequence $x = (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X)$ so that $(z - r)^2 \hat{x}(z) - r^2(I - T)\hat{x}(z) = \hat{f}(z)$. By uniqueness of the Z-transform, we conclude that $x = (x_n)$ is solution of equation (2.2). In particular, $r^2(I - T)x_n = \Delta_r^2 x_n - f_n$ belongs to $l_p(\mathbb{Z}_+; X)$. To prove the uniqueness of the solution, we observe that by application of the Z-transform to equation (2.2) with $f_n \equiv 0$, we get $\hat{x}(z) = 0$, and so $x \equiv 0$. ∎

Proposition 3.3. *Let X be a Banach space and let $T \in \mathcal{B}(X)$ be an analytic operator; assume that (2.10) is fulfilled and suppose that problem (2.2) is well-posed. Then $(z-r)^2 \in \rho(r^2(I - T))$ whenever $|z| = \alpha r, z \neq \alpha r$ and the set $\{M(z) : |z| = \alpha r, z \neq \alpha r\}$ is R-bounded.*

Proof. By Proposition 2.9 the first assertion follows. On the other hand, by Proposition 3.2, for all $f = (f_n) \in l_p(\mathbb{Z}_+; X)$, there is a sequence $\mathcal{L}_M f \in l_p(\mathbb{Z}_+; X)$ such that $\widehat{\mathcal{L}_M f}(z) = M(z)\hat{f}(z), z \in \mathbb{T}_r^\alpha, z \neq \alpha r$. By the uniqueness of Z-transform the operator $\mathcal{L}_M : l_p(\mathbb{Z}_+; X) \to l_p(\mathbb{Z}_+; X)$ is well defined. It follows from the Closed Graph Theorem that $\mathcal{L}_M$ is in $\mathcal{B}(l_p(\mathbb{Z}_+; X))$. Now the conclusion is a consequence of [12, Proposition 1.4]. ∎

The following is the main result of this paper. It shows that the converse of the above proposition is valid in UMD spaces.

Theorem 3.4. *Let X be a UMD space and let $T \in \mathcal{B}(X)$ be an analytic operator; assume that (2.10) is fulfilled. Then, the following assertions are equivalent.*
(i) Equation (2.2) is well-posed.
(ii) The set $\{M(z) : |z| = \alpha r, z \neq \alpha r\}$ is R-bounded.
(iii) Equation (2.2) has discrete maximal regularity.

Proof. We note that $(i) \Rightarrow (ii)$ is the preceding proposition. To prove $(ii) \Rightarrow (iii)$, we define $k_r : \mathbb{Z} \to \mathcal{B}(X)$ by

$$k_r(n) = \begin{cases} r^2(I - T)r^{n-1}\mathcal{S}(n), & \text{for } n \in \mathbb{Z}_+, \\ \\ 0, & \text{otherwise,} \end{cases}$$

and the corresponding operator $\mathcal{K}^r : l_p(\mathbb{Z}_+; X) \to l_p(\mathbb{Z}_+; X)$ by

$$[\mathcal{K}^r f](n) = \sum_{j=0}^n k_r(j)f_{n-j} = (k_r * f)(n), \quad n \in \mathbb{Z}_+.$$

By Proposition 2.9, $(z-r)^2 \in \rho(r^2(I-T))$ whenever $|z| = \alpha r, z \neq \alpha r$. Then, by Proposition 2.10, we have

$$\hat{k}_r(z) = r^2(I - T)\mathcal{F}[r^{\bullet-1}\mathcal{S}](z) = r^2(I - T)zR((z - r)^2, r^2(I - T)) = z(M(z) - I),$$

where $z \in \mathbb{T}_r^\alpha$, $z \neq \alpha r$.

Define $M(t) := M(\alpha r e^{it})$, $t \in (-\pi, 0) \cup (0, \pi)$. We note that

$$M'(t) = 2i\alpha e^{it}(\alpha e^{it} - 1)^{-1}[M(t) - M(t)^2].$$

Define $N(t) := \alpha r e^{it}[M(t) - I]$, then $N(t)$ is R-bounded by hypothesis and Remark 2.4(b). A calculation shows that

$$
\begin{aligned}
(e^{it} - 1)(e^{it} + 1)N'(t) &= i\alpha r e^{it}(e^{it} - 1)(e^{it} + 1)[M(t) - I] \\
\\
&+ \frac{2i\alpha^2 r e^{2it}(e^{it} - 1)(e^{it} + 1)}{\alpha e^{it} - 1}[M(t) - M(t)^2].
\end{aligned}
$$

We observe that $|\alpha e^{it} - 1| \geq \sqrt{2}$. Hence, the set

$$\left\{ \frac{2i\alpha^2 r e^{2it}(e^{it} - 1)(e^{it} + 1)}{\alpha e^{it} - 1} \right\},$$

is bounded by $4\alpha^2 r\sqrt{2}$. It shows that the set $\{(e^{it} - 1)(e^{it} + 1)N'(t)\}$ is R−bounded thanks to Remark 2.4 again. We conclude, from Theorem 2.7, that there exists $T_N \in \mathcal{B}(l_p(\mathbb{Z}, X))$ such that $\mathcal{F}[T_N g](e^{it}) = N(t)\mathcal{F}[g](e^{it})$ for all $g \in l_p(\mathbb{Z}_+; X)$. Let $f \in l_p(\mathbb{Z}_+; X)$ and define $g := (\alpha r)^{-\bullet}f$. Then $g \in l_p(\mathbb{Z}_+; X)$ because $\alpha r > 1$, and we have for all $z \in \mathbb{T}_r^\alpha$

$$
\begin{aligned}
\mathcal{F}[(\alpha r)^\bullet T_N((\alpha r)^{-\bullet}f)](z) &= \mathcal{F}[T_N((\alpha r)^{-\bullet}f)](e^{it}) \\
&= N(t)\mathcal{F}[(\alpha r)^{-\bullet}f](e^{it}) \\
&= N(t)\mathcal{F}[f](\alpha r e^{it}) \\
&= \mathcal{F}[k_r](\alpha r e^{it})\mathcal{F}[f](\alpha r e^{it}) \\
&= \mathcal{F}[k_r * f](\alpha r e^{it}) \\
&= \mathcal{F}[\mathcal{K}^r f](z).
\end{aligned}
$$

Then by uniqueness of the Z-transform, we conclude that $(\alpha r)^{-\bullet}\mathcal{K}^r f = T_N((\alpha r)^{-\bullet}f)$ and we have

$$||(\alpha r)^{-\bullet}\mathcal{K}^r f||_p = ||T_N((\alpha r)^{-\bullet}f)||_p \leq M||(\alpha r)^{-\bullet}f||_p \leq M||f||_p.$$

Hence $(\alpha r)^{-\bullet}\mathcal{K}^r$ belongs to $\mathcal{B}(l_p(\mathbb{Z}_+, X))$. It follows from the Closed Graph Theorem that $\mathcal{K}^r \in \mathcal{B}(l_p(\mathbb{Z}_+, X))$ then (iii) follows. To prove $(iii) \Rightarrow (i)$, by Proposition 2.1 we known that $x(n) = (r^{\bullet-1}\mathcal{S} * f)(n-1)$ is the unique solution of equation (2.2). Then $\Delta_r^2 x(n) = r^2(I - T)(r^{\bullet-1}\mathcal{S} * f)(n-1) + f(n) = [\mathcal{K}^r f](n-1) + f(n)$. Hence $x = (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$, and the proof is finished. $\blacksquare$

Since Hilbert spaces are UMD spaces, we obtain as immediate consequence the following corollary.

Corollary 3.5. *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ be an analytic operator; assume that (2.10) is fulfilled. Then, the following assertions are equivalent.*
(i) Equation (2.2) is well-posed.
(ii) $\sup_{z \in \mathbb{T}_r^\alpha, z \neq \alpha r} ||(z - r)^2((z - r)^2 - r^2(I - T))^{-1}|| < \infty$
(iii) Equation (2.2) has discrete maximal regularity.

Remark 3.6. *a) We note that when (2.2) has discrete maximal regularity, then $(x_n) \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$ whenever $(f_n) \in l_p(\mathbb{Z}_+; X)$. It establishes an isomorphism between the set of data (f_n) and the set of solutions (x_n).*

b) For each $f = (f_n) \in l_p(\mathbb{Z}_+; X)$ and under the equivalent conditions of the above theorem, there exists a constant $C > 0$ such that $||\Delta_r^2 x||_p + ||r^2(I - T)x||_p \leq C||f||_p$, for all $x = (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$, that is, the application $R : l_p(\mathbb{Z}_+; X) \to l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$ given by $(f_n) \mapsto (x_n)$ is continuous.

Let $f = (f_n) \in l_p(\mathbb{Z}_+; X)$. We finish this paper with an application to the following semilinear evolution problem

$$(3.2) \qquad \Delta_r^2 x - r^2(I - T)x = G(x) + \rho f,$$

where G is a Frechét differentiable function and $\rho > 0$ is a small parameter.

Theorem 3.7. *Let X be a UMD space and $T \in \mathcal{B}(X)$ be an analytic operator. Assume that (2.10) is fulfilled and assume that*

(i) the set $\{M(z) : |z| = \alpha r, z \neq \alpha r\}$ is R-bounded.

(ii) G maps $l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$ into $l_p(\mathbb{Z}_+; X)$; $G(0) = 0$; G is a continuously (Frechét) differentiable function at $x = 0$ and $G'(0) = 0$.

Then, there exists $\rho^* > 0$ such that the equation (3.2) is solvable for each $\rho \in [0, \rho^*)$, with solution $x_\rho := (x_n) \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X)$.

Proof. Define the operator $L : l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X) \to l_p(\mathbb{Z}_+; X)$ by $L(x) = \Delta_r^2 x - r^2(I - T)x$. We consider for $\rho \in (0, 1)$ the one parameters family

$$H[x, \rho] := -L(x) + G(x) + \rho f.$$

Keeping in mind that $G(0) = 0$ we see that $H[0, 0] = 0$. Also, by hypothesis, H is continuously differentiable at $(0, 0)$. We observe that L is an isomorphism onto. In fact, by uniqueness L is injective. By Theorem 3.4, L is surjective. By definition of the norm in (3.1), L is bounded. Now, the claim follows by the Open Mapping Theorem. Hence the partial Frechét derivative $H_{(0,0)}^1 = -L$ is invertible. The conclusion of the theorem now follows from the Implicit Function Theorem (see [27, Theorem 17.6]). ∎

In the very special case that $T = I$ and $X = \mathbb{R}$ we obtain the following scalar result which, in our best know, is also new.

Corollary 3.8. *Assume that (2.10) is fulfilled, let $f = (f_n)$ be in $l_p(\mathbb{Z}_+; \mathbb{R})$. Assume that G is like in the preceding theorem. Then, there exists $\rho^* > 0$ such that the discrete time evolution equation*

$$(3.3) \qquad \Delta_r^2 x = G(x) + \rho f,$$

is solvable for each $\rho \in [0, \rho^*)$, with solution $x_\rho := (x_n) \in l_{p,r}^2(\mathbb{Z}_+; \mathbb{R}) \cap l_p(\mathbb{Z}_+; \mathbb{R})$.

Next, we exhibit a concrete example of mapping G like in Theorem 3.7.

Example 3.9. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Let G be a map from $l_{p,r}^2(\mathbb{Z}_+; H) \cap l_{p,I-T}(\mathbb{Z}_+; H)$ into $l_p(\mathbb{Z}_+; H)$ defined by

$$G(h)(n) = < \Delta_r^2 h_n, \Delta_r^2 h_n > .$$

We observe that

$$||G(h)||_p \leq ||\Delta_r^2 h||_\infty ||\Delta_r^2 h||_p, \ h \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X),$$

whence G is well defined and $G(0) = 0$. Also, we get

$$G'(x)(h) = 2 < \Delta_r^2 x, \Delta_r^2 h >, \ x, h \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X),$$

so $G'(0) = 0$. On the other hand, we can infer that

$$||G'(x) - G'(x^0)|| \leq 2|||x - x^0|||, \ x, x^0 \in l_{p,r}^2(\mathbb{Z}_+; X) \cap l_{p,I-T}(\mathbb{Z}_+; X),$$

then G is a continuously (Frechét) differentiable function.

Example 3.10. *Assume that (2.10) is fulfilled; let μ be a positive real number and $f = (f_n) \in l_p(\mathbb{Z}_+; \mathbb{R})$. Then, there exists $\rho^* > 0$ such that the discrete time evolution problem*

$$\begin{cases} \Delta_r^2 x_n = [\Delta_r^2 x_n]^\mu + \rho f_n, \ n \in \mathbb{Z}_+, \\ \\ x_0 = x_1 = 0. \end{cases}$$

is solvable for each $\rho \in [0, \rho^*)$, with solution $x_\rho := (x_n) \in l_{p,r}^2(\mathbb{Z}_+; \mathbb{R}) \cap l_p(\mathbb{Z}_+; \mathbb{R})$.

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