ON A CONNECTION BETWEEN THE DISCRETE FRACTIONAL LAPLACIAN AND SUPERDIFFUSION

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Abstract. We relate the fractional powers of the discrete Laplacian with a standard time-fractional derivative in the sense of Liouville by encoding the iterative nature of the discrete operator through a time-fractional memory term.

1. Introduction

Let $0 < \alpha < 1$ be given. Our concern in this paper is the study of the connection between two seemingly distinct classes of partial differential equations that have a mixed character, i.e. that can be modeled both in continuous time as well as in discrete space

$$D_t v(n,t) = (-\Delta_{\mathrm{d}})^{\alpha} v(n,t), \quad t > 0, \quad n \in \mathbb{Z}, \tag{1}$$

and

$$D_t^{1/\alpha} u(n,t) = (-\Delta_{\mathrm{d}}) u(n,t), \quad t > 0, \quad n \in \mathbb{Z}. \tag{2}$$

In (1), D_t denotes the continuous derivative in the variable t, $D_t^{1/\alpha}$ denotes the fractional derivative of order $1/\alpha$ in the sense of Liouville (left-sided) and $(-\Delta_{\mathrm{d}})^{\alpha}$ denotes the fractional powers of order α of the unidimensional discrete Laplacian, introduced in [3] (where other operators in Harmonic Analysis, such as the discrete Riesz transform, square functions, conjugate harmonic functions and the Poisson semigroup were also studied). See Section 2 for definitions. For $\beta := \frac{1}{\alpha} > 1$, equation (2) describes superdiffusive phenomena in time. It models anomalous superdiffusion in which a particle cloud spreads faster than the classical diffusion model predicts. The connection between order in time and space for partial differential equations is a surprising phenomena that seems not to be addressed for the discrete fractional Laplacian. It shows that the spatial-Laplacian of fractional order is entirely translated into temporal regularization. First studies on this unexpected property appear in [7] for the unidimensional continuous Laplacian. There, it was observed that ordinary PDEs of first order in space are transformed into PDEs of half-th order in time and second order in space, and that from an applied perspective (Stokes problem) this property shows numerical advantages. For the d-dimensional and continuous bi-Laplacian, the connection appeared in [5, Example 2.14], where also an abstract setting for higher powers is studied. For stochastic processes the relation seems to be more analyzed, but only very recently by H. Allouba [1] and B. Baeumer, M.M. Meerschaert and E. Nane [2, Theorem 3.1 and Theorem 3.9]. See also the references therein.

In this paper we are able to prove that, for appropriate initial values in the Lebesgue space $\ell^{\infty}(\mathbb{Z})$ and $0 < \alpha < 1$, the solution of problems (1) and (2) is the same, and that it admits an explicit representation in convolution form by means of a special kernel. This is shown in Theorem 3, that is the main result of this work.

We define the discrete fractional Laplacian via the discrete Fourier transform, because this way is the most suitable to prove Theorem 3. Moreover, this definition coincides with the genuine definition of the fractional powers of a more general linear second order partial differential operator L by means

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of the semigroup generated by L, as shown by P.R. Stinga and J.L. Torrea in [15]. In particular, we provide the discrete counterpart of the formula for the fractional Laplacian via the semigroup due to Stinga and Torrea [15]; it is presented in Theorem 2 and has its own interest.

2. Preliminaries

In order to establish and clarify the meaning of the equations (1) and (2), and the relationship between their solutions, we need to define several continuous and discrete operators. In particular, in what follows we are going to give precise definitions for several kinds of Fourier transforms and fractional operators on some spaces, and to provide some of their properties.

For a given sequence f, we define the discrete Fourier transform

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T},$$

where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ is the unidimensional torus, that we identify with the interval $(-\pi, \pi]$.

The inverse discrete Fourier transform is obtained by the formula

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z},$$

for a given function φ. Therefore

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\mathbb{Z}}(f)(\theta)e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

It is easily verified that

$$\mathcal{F}_{\mathbb{Z}}(f * g)(\theta) = \mathcal{F}_{\mathbb{Z}}(f)(\theta)\mathcal{F}_{\mathbb{Z}}(g)(\theta),$$

where $*$ denote the usual convolution in $\mathbb{Z}$.

We are going to motivate our definition of the discrete fractional Laplacian; the details can be seen in [3]. Observe that from the discrete Laplacian

$$\Delta_{\mathrm{d}} f(n) := f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z}$$

(such as defined in [4]), and using the identity $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$, we obtain

$$\mathcal{F}_{\mathbb{Z}}(-\Delta_{\mathrm{d}} f)(\theta) = 4\sin^2(\theta/2)\mathcal{F}_{\mathbb{Z}}(f)(\theta).$$

Let $f \in \ell^\infty(\mathbb{Z})$ be given. By taking the inverse discrete Fourier transform, the discrete fractional Laplacian of order $\alpha > 0$ is then defined by

$$(-\Delta_{\mathrm{d}})^\alpha f(n) := \sum_{k\in\mathbb{Z}} K^\alpha(n-k)f(k), \quad n \in \mathbb{Z}, \tag{3}$$

where

$$K^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(4\sin^2(\theta/2)\right)^\alpha e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Then, it is clear that

$$\mathcal{F}_{\mathbb{Z}}(K^\alpha)(\theta) = \left(4\sin^2(\theta/2)\right)^\alpha, \quad \theta \in (-\pi, \pi]. \tag{4}$$

Remark 1. Using [13, formula 2.5.12 (22), p. 402] we obtain the following explicit expression for the kernel $K^\alpha(n)$ in terms of the Gamma function:

$$K^\alpha(n) = \frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}, \quad n \in \mathbb{Z}. \tag{5}$$

In particular, it shows that definition (3) coincides with the generalized fractional difference corresponding to the type 1 central derivative considered by M.D. Ortigueira in [10, 11]. See also [12, formula (2) and formula (36)]. Because of this connection, some properties for the discrete Laplacian can be directly deduced. For example, associativity $(-\Delta_{\mathrm{d}})^\alpha(-\Delta_{\mathrm{d}})^\beta = (-\Delta_{\mathrm{d}})^{\alpha+\beta}$ provided $\alpha + \beta > -1$. Moreover,

$$|K^\alpha(n)| \sim \frac{\Gamma(2\alpha+1)}{\pi}|n|^{-2\alpha-1}, \quad n \to \pm\infty,$$

see [10, formula (4.24)]. In particular, it shows that the series in the right hand side of (3) converges for $f \in \ell^\infty(\mathbb{Z})$. For other properties, we refer to [12, Section 2.2].

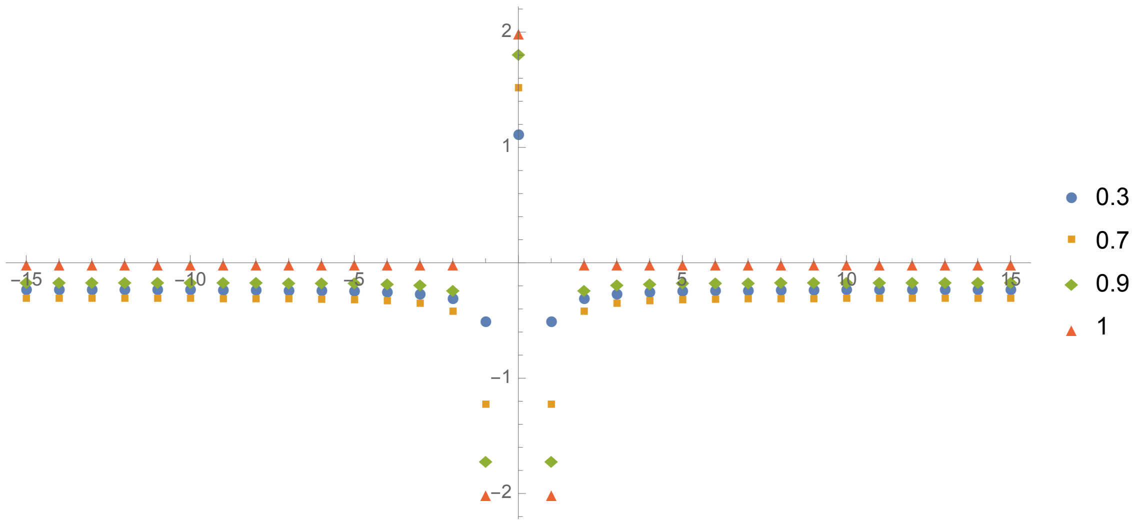

FIGURE 1. Graphical representation of $(1+|n|^{2\alpha+1})K^\alpha(n)$ for $\alpha = 0.3$, 0.7, 0.9 and 1.

In Figure 1 we show the aspect of $K^\alpha(n)$ (for $-15 \le n \le 15$) for several values of the parameter α; actually, to compensate the big decay of $K^\alpha(n)$ when $n \to \pm\infty$ and to get a more significative picture, we represent the kernel multiplied by $1+|n|^{2\alpha+1}$. In particular, we clearly observe that $K^\alpha(n) > 0$ only when $n = 0$ (of course, this can be also proved from (5)). From this and the identity

$$(-\Delta_{\mathrm{d}})^\alpha f(n) := \sum_{k\in\mathbb{Z}} K^\alpha(n-k)f(k) = \sum_{k\neq n} K^\alpha(n-k)f(k) + K^\alpha(0)f(n)$$

we deduce that $(-\Delta_{\mathrm{d}})^\alpha f(n_0) \le 0$ whenever $f(k) \ge 0$ for all $k \neq n_0$ and $f(n_0) \le 0$. It extends [3, Theorem 1].

In what follows, we present the definition and some properties of the Liouville fractional derivative on the whole axis $\mathbb{R}$. More detailed information may be found in the book [6, Chapter II, Section 2.3].

We denote

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0, \tag{6}$$

and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin. Recall also the well-known formula for the Gamma function

$$\int_0^\infty e^{-\lambda t} g_\beta(t)\,dt = \frac{1}{\lambda^\beta}, \qquad \lambda > 0, \quad \beta > 0; \tag{7}$$

see, for instance, [13, formula 2.3.3 (1), p. 322]. The Liouville (left-sided) fractional derivative of order $\alpha > 0$ is defined by

$$D_t^\alpha h(t) = \frac{d^n}{dt^n} \int_{-\infty}^t g_{n-\alpha}(t-s)h(s)\,ds$$

where $n = \lfloor\alpha\rfloor + 1$, $t \in \mathbb{R}$. In particular, when $\alpha = m \in \mathbb{N}_0$, then $D_t^0 h(t) = h(t)$ and $D_t^m h(t) = h^{(m)}(t)$ where $h^{(m)}(t)$ is the usual derivative of $h(t)$ of order m.

Let h be in the N-dimensional Schwartz's class $\mathcal{S}$. The Fourier transform of h is given by

$$\mathcal{F}_{\mathbb{R}^N}(h)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} h(x)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^N.$$

We recall that the continuous fractional Laplacian $(-\Delta)^\alpha$ with $0<\alpha<1$ can be defined in several equivalent ways. It is defined via the Fourier transform as

$$\mathcal{F}_{\mathbb{R}^N}((-\Delta)^\alpha h)(\xi) = |\xi|^{2\alpha}\mathcal{F}_{\mathbb{R}^N}(h)(\xi).$$

An equivalent definition, obtained by computing the inverse Fourier transform (see [8]), is given by the singular integral

$$(-\Delta)^\alpha h(x) = C_{N,\alpha}\, P.V. \int_{\mathbb{R}^N} \frac{h(x)-h(y)}{\|x-y\|^{N+2\alpha}}\,dy,$$

where $C_{N,\alpha}$ is a explicit positive constant. A comparable formula, that avoids the computation of the inverse Fourier transform, and provides the pointwise formula above in a simple way, can be found for example in [14, Lemma 2.1, p. 35]:

$$(-\Delta)^\alpha h(x) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{t\Delta}h(x) - h(x))\frac{dt}{t^{1+\alpha}}$$

where $e^{t\Delta}$ is the heat-diffusion semigroup.

3. The discrete Laplacian and superdiffusion

We recall from [3, Section 2] that the discrete heat semigroup is defined by

$$e^{t\Delta_{\mathrm{d}}}f(n) := \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m),$$

where I_k is the modified Bessel function of the first kind and order $k\in\mathbb{Z}$, defined as

$$I_k(t) = \sum_{m=0}^\infty \frac{1}{m!\,\Gamma(m+k+1)}\left(\frac{t}{2}\right)^{2m+k}.$$

Several properties of I_k are listed in [3, Section 8]. In [3, Proposition 1] it was proved that $\{e^{t\Delta_{\mathrm{d}}}\}_{t\ge 0}$ is a positive Markovian diffusion semigroup. Moreover, for each $\varphi\in\ell^\infty$, the function $u(n,t) = e^{t\Delta_{\mathrm{d}}}\varphi(n)$ is a solution of the discrete heat equation, that is (1) with $\alpha = 1$.

The following result corresponds to the discrete counterpart of the formula with the semigroup for the fractional Laplacian [15, Lemma 5.1].

Theorem 2. *For all $0<\alpha<1$ and $f\in\ell^\infty(\mathbb{Z})$ the following holds:*

$$(-\Delta_{\mathrm{d}})^\alpha f(n) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{t\Delta_{\mathrm{d}}}f(n) - f(n))\frac{dt}{t^{1+\alpha}}, \quad n\in\mathbb{Z}.$$

Proof. By definition (3) and the identity (4) we have

$$\mathcal{F}_{\mathbb{Z}}((-\Delta_{\mathrm{d}})^\alpha f)(\theta) = \mathcal{F}_{\mathbb{Z}}(K^\alpha)(\theta)\mathcal{F}_{\mathbb{Z}}(f)(\theta) = (4\sin^2(\theta/2))^\alpha \mathcal{F}_{\mathbb{Z}}(f)(\theta).$$

On the other hand, we have

$$\mathcal{F}_{\mathbb{Z}}(e^{t\Delta_{\mathrm{d}}}f)(\theta) = e^{-4t\sin^2(\theta/2)}\mathcal{F}_{\mathbb{Z}}(f)(\theta),$$

see the proof of (iii) in [3, Proposition 1]. The claim then follows from the identity

$$\frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{-4t\sin^2(\theta/2)} - 1)\frac{dt}{t^{1+\alpha}} = (4\sin^2(\theta/2))^\alpha,$$

that can be proven using integration by parts and the formula (7). □

Let $\alpha>0$ be given. We define

$$K_t^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha} e^{-in\theta}\,d\theta, \quad t\in\mathbb{R},\ n\in\mathbb{Z}.$$

The following is the main result of this paper.

Theorem 3. *For every α that satisfies $0<\alpha<1$ and $\varphi\in\ell^\infty(\mathbb{Z})$, the function*

$$u(n,t)=\sum_{k\in\mathbb{Z}}K_t^\alpha(n-k)\varphi(k),\quad t\geq 0,\ n\in\mathbb{Z}, \tag{8}$$

solves the problems

$$\begin{cases} D_t u(n,t)=(-\Delta_{\mathrm{d}})^\alpha u(n,t), & t>0,\ n\in\mathbb{Z},\\ u(n,0)=\varphi(n), & n\in\mathbb{Z}, \end{cases} \tag{9}$$

and

$$\begin{cases} D_t^{1/\alpha} u(n,t)=(-\Delta_{\mathrm{d}}) u(n,t), & t>0,\ n\in\mathbb{Z},\\ u(n,0)=\varphi(n), & n\in\mathbb{Z}. \end{cases} \tag{10}$$

Proof. We first prove that (8) solves (9). Indeed, by taking discrete Fourier transform in the variable n, the equation (9) becomes

$$\begin{cases} D_t\mathcal{F}_{\mathbb{Z}}(u(\cdot,t))(\theta)=\left(4\sin^2\frac{\theta}{2}\right)^\alpha\mathcal{F}_{\mathbb{Z}}(u(\cdot,t))(\theta), & t>0,\\ \mathcal{F}_{\mathbb{Z}}(u(\cdot,0))(\theta)=\mathcal{F}_{\mathbb{Z}}(\varphi)(\theta), \end{cases} \tag{11}$$

and a solution to (11) is

$$\mathcal{F}_{\mathbb{Z}}(u(\cdot,t))(\theta)=e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\mathcal{F}_{\mathbb{Z}}(\varphi)(\theta),\quad t>0.$$

Now, by applying inverse discrete Fourier transform, we have

$$\begin{aligned} u(n,t)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}e^{-in\theta}\mathcal{F}_{\mathbb{Z}}(\varphi)(\theta)\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}e^{-in\theta}\sum_{k\in\mathbb{Z}}\varphi(k)e^{ik\theta}\,d\theta\\ &=\sum_{k\in\mathbb{Z}}\varphi(k)\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}e^{-i(n-k)\theta}\,d\theta=\sum_{k\in\mathbb{Z}}\varphi(k)K_t^\alpha(n-k). \end{aligned}$$

Secondly, we prove that (8) solves (10). In fact, we have

$$D_t^{1/\alpha}u(n,t)=\sum_{k\in\mathbb{Z}}D_t^{1/\alpha}K_t^\alpha(n-k)\varphi(k)$$

where, by interchanging the order of integration,

$$D_t^{1/\alpha}K_t^\alpha(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_t^{1/\alpha}e^{(\cdot)\left(4\sin^2\frac{\theta}{2}\right)^\alpha}(t)e^{-in\theta}\,d\theta.$$

Since $0<\alpha<1$, there exists $m\in\mathbb{N}$ such that $\frac{1}{m+1}<\alpha\leq\frac{1}{m}$ for $m\in\mathbb{N}\setminus\{1\}$, or $\frac{1}{m+1}<\alpha<\frac{1}{m}$ for $m=1$. Remembering the notation for g_β in (6), we obtain

$$\begin{aligned} D_t^{1/\alpha}\left(e^{(\cdot)\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\right)(t)&=\frac{d^{m+1}}{dt^{m+1}}\int_{-\infty}^{t}g_{m+1-1/\alpha}(t-s)e^{s\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\,ds\\ &=\frac{d^{m+1}}{dt^{m+1}}\left(e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\right)\int_0^\infty g_{m+1-1/\alpha}(\tau)e^{-\tau\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\,d\tau\\ &=\left(4\sin^2\frac{\theta}{2}\right)^{(m+1)\alpha}e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}\frac{1}{\left((4\sin^2\frac{\theta}{2})^\alpha\right)^{m+1-1/\alpha}}\\ &=\left(4\sin^2\frac{\theta}{2}\right)e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}, \end{aligned}$$

where we used (7) in the third equality. (It is well known that the β-order Liouville (left-sided) fractional derivative D_t^β of $e^{\lambda t}$ is $\lambda^\beta e^{\lambda t}$, for $\lambda>0$; see, for example, [6, formula (2.3.11), p. 88] or [9, Example 2.6]. We give here a direct proof for the sake of completeness.) Therefore

$$D_t^{1/\alpha}u(n,t)=\sum_{k\in\mathbb{Z}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(2-2\cos\theta)e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha}e^{-i(n-k)\theta}\,d\theta\right)\varphi(k). \tag{12}$$

On the other hand, using (8) we have

$$
\begin{aligned}
\Delta_{\mathrm{d}} u(n,t) &= u(n+1,t) - 2u(n,t) + u(n-1,t) \\
&= \sum_{k\in\mathbb{Z}} \big(K_t^\alpha(n+1-k) - 2K_t^\alpha(n-k) + K_t^\alpha(n-1-k)\big)\varphi(k) \\
&= \sum_{k\in\mathbb{Z}} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha} \big(e^{-i(n-k+1)\theta} - 2e^{-i(n-k)\theta} + e^{-i(n-k-1)\theta}\big)\,d\theta\Big)\varphi(k) \qquad (13)\\
&= -\sum_{k\in\mathbb{Z}} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t\left(4\sin^2\frac{\theta}{2}\right)^\alpha} (2\cos\theta - 2) e^{-i(n-k)\theta}\,d\theta\Big)\varphi(k).
\end{aligned}
$$

Combining (12) with (13) we obtain (10). □

Remark 4. We note that if $\alpha \geq \frac{1}{2}$ then $1 < \frac{1}{\alpha} \leq 2$ in (10), and then the equation should have two initial conditions. By using (3), a calculation shows that in such case the second initial condition reads

$$u_t(n,0) = (-\Delta_{\mathrm{d}})^\alpha \varphi(n).$$

More generally, if $\frac{1}{m+1} \leq \alpha < \frac{1}{m}$, $m \in \mathbb{N}$, then we have m extra initial conditions in (10) and they are $u_t(n,0) = (-\Delta_{\mathrm{d}})^\alpha\varphi(n)$, $u_{tt}(n,0) = (-\Delta_{\mathrm{d}})^{2\alpha}\varphi(n), \dots, u_t^{(m)}(n,0) = (-\Delta_{\mathrm{d}})^{m\alpha}\varphi(n)$.

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